Ann. Funct. Anal. (2024) 15:11
https://doi.org/10.1007/s43034-023-00312-x

Tusi Mathematical Research Group

ORIGINAL PAPER

Projective and injective tensor products of Banach L^0-modules

Enrico Pasqualetto[1]

Received: 16 October 2023 / Accepted: 30 November 2023 / Published online: 8 January 2024

Abstract
We study projective and injective tensor products of Banach L^0-modules over a σ-finite measure space. En route, we extend to Banach L^0-modules several technical tools of independent interest, such as quotient operators, summable families, and Schauder bases.

Keywords Banach module · Tensor product · Schauder basis

Mathematics Subject Classification 46M05 · 47A80 · 53C23 · 16D90 · 18F15

1 Introduction

As of now, the language of normed modules introduced by Gigli in [11] has become an indispensable tool in analysis on metric measure spaces, especially on those verifying synthetic lower Ricci curvature bounds (the so-called RCD spaces). Normed modules allow to define several spaces of measurable tensor fields, whose investigation has remarkable analytic and geometric consequences. In this respect, three constructions are particularly important: duals, pullbacks, and (in the case of Hilbert modules) tensor products. For example, the dual of the pullback is important for constructing the differential of a map of bounded deformation or the velocity of a test plan (cf. with the introduction of [13]), while the tensor product of Hilbert modules is a fundamental tool when studying the second order differential calculus on RCD spaces (see [11, Section 3]). However, since many spaces of interest are 'non-Riemannian', it would be interesting to study tensor products of non-Hilbert normed modules, as well as to understand their relation with duals and pullbacks: this is the main goal of this paper.

Communicated by Zinaida Lykova.

✉ Enrico Pasqualetto
enrico.e.pasqualetto@jyu.fi

1 Department of Mathematics and Statistics, University of Jyvaskyla, P.O. Box 35 (MaD), 40014 Jyväskylä, Finland

We assume the reader is familiar with (projective and injective) tensor products of Banach spaces, for which we refer e.g. to the authoritative monograph [21].

Let us briefly describe the content of the paper. Fix a σ-finite measure space $\mathbb{X} = (\mathrm{X}, \Sigma, \mathfrak{m})$, i.e. X is a set, Σ is a σ-algebra on X, and $\mathfrak{m}\colon \Sigma \to [0, +\infty]$ is a σ-finite (countably-additive) measure. We consider the class of *Banach* $L^0(\mathbb{X})$*-modules*, i.e. modules over the commutative ring $L^0(\mathbb{X})$ that are endowed with a complete pointwise norm operator (cf. with Definition 2.4). Even though we are mostly interested in their applications to metric measure geometry, we consider Banach L^0-modules over general measure spaces. Our choice is due to the fact that Banach L^0-modules play an important role also in other research areas, see for example [15], as well as [14] and the references therein. The only results where we need to require an additional assumption on the base measure space (verified in the case of metric measure spaces) are Theorems 4.13 and 5.13. Given two Banach $L^0(\mathbb{X})$-modules $\mathscr{M}$ and $\mathscr{N}$, we first provide a useful criterion to detect the null tensors of the algebraic tensor product $\mathscr{M} \otimes \mathscr{N}$; see Lemma 3.19. Its proof is quite subtle, one reason being the fact that the algebraic dual of a module might not separate the points (differently from duals of vector spaces); cf. with Remark 3.20. Having Lemma 3.19 at our disposal, we can:

- Define and study the *projective tensor product* $\mathscr{M}\hat{\otimes}_\pi\mathscr{N}$, see Sect. 4.
- Define and study the *injective tensor product* $\mathscr{M}\hat{\otimes}_\varepsilon\mathscr{N}$, see Sect. 5.

Motivated by the analysis on metric spaces, our attention is focussed on the following results:

- The *dual* of $\mathscr{M}\hat{\otimes}_\pi\mathscr{N}$ can be identified with the space $\mathrm{B}(\mathscr{M}, \mathscr{N})$ of bounded $L^0(\mathbb{X})$-bilinear maps from $\mathscr{M} \times \mathscr{N}$ to $L^0(\mathbb{X})$ (see Theorem 4.11), while the dual of $\mathscr{M}\hat{\otimes}_\varepsilon\mathscr{N}$ is a quotient of the dual of the space $\mathrm{C}_{\mathrm{pb}}(\mathbb{D}^{w^*}_{\mathscr{M}^*} \times \mathbb{D}^{w^*}_{\mathscr{N}^*}; L^0(\mathbb{X}))$ (see Definition 3.16 and Theorem 5.12).
- The operation of taking *pullbacks* of Banach $L^0(\mathbb{X})$-modules commutes both with projective tensor products (Theorem 4.13) and with injective tensor products (Theorem 5.13).

While some of the concepts and results we presented above are natural extensions of their version for Banach spaces, other ones are non-trivial generalisations (see e.g. the two different notions of a continuous module-valued map in Sect. 3.4) or have no counterpart in the Banach space setting (as in the case of pullback modules). We conclude the introduction by mentioning that a significant portion of the paper is devoted to the development of several technical tools (new in the setting of Banach $L^0(\mathbb{X})$-modules), which are needed in Sects. 4 and 5, and can be useful in the future research concerning normed modules: we study *quotient operators* (Sect. 3.1), *summable families* in Banach $L^0(\mathbb{X})$-modules (Sect. 3.2), and *local Schauder bases* (Sect. 3.3).

2 Preliminaries

Given an arbitrary set $I \neq \varnothing$, we denote by $\mathscr{P}(I)$ its power set (i.e. the set of its subsets) and

$$\mathscr{P}_f(I):=\big\{F \in \mathscr{P}(I) \;\big|\; F \text{ is finite}\big\}.$$

Given any couple of indexes $i, j \in I$, we define $\delta_{ij} \in \{0, 1\}$ as $\delta_{ij}:=1$ if $i = j$ and $\delta_{ij}:=0$ if $i \neq j$. Moreover, if X is a set, then the characteristic function $\mathbb{1}_E : \mathrm{X} \to \{0, 1\}$ of a subset $E \subseteq \mathrm{X}$ is

$$\mathbb{1}_E(x):=\begin{cases} 1 & \text{for every } x \in E, \\ 0 & \text{for every } x \in \mathrm{X}\setminus E. \end{cases}$$

For any map $\varphi : \mathrm{X} \to \mathrm{Y}$ between two sets X and Y, we denote by $\varphi[\mathrm{X}] \subseteq \mathrm{Y}$ the image of φ.

2.1 Tensor products of modules

In this section, we recall the basics of the theory of tensor products of modules, which is originally due to [5]. See also [7] and the references indicated therein. Our standing convention is that all rings are assumed to have a multiplicative identity.

Theorem 2.1 (Tensor products of modules) *Let R be a commutative ring. Let M and N be modules over R. Then there exists a unique couple $(M \otimes N, \otimes)$, where $M \otimes N$ is an R-module and $\otimes : M \times N \to M \otimes N$ is an R-bilinear map, such that the following universal property holds: given any R-module Q and any R-bilinear map $b : M \times N \to Q$, there exists a unique R-linear map $\tilde{b} : M \otimes N \to Q$, called the R-**linearisation** of b, for which the diagram*

$$\begin{tikzcd} M \times N \arrow[r, "b"] \arrow[d, "\otimes"'] & Q \\ M \otimes N \arrow[ur, "\tilde{b}"'] & \end{tikzcd}$$

commutes. The couple $(M \otimes N, \otimes)$ is unique up to a unique isomorphism: given any $(T, \tilde{\otimes})$ with the same properties, there exists a unique isomorphism of R-modules $\Phi : M \otimes N \to T$ such that

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[dr, "\tilde{\otimes}"'] & M \otimes N \arrow[d, "\Phi"] \\ & T \end{tikzcd}$$

*commutes. We say that $(M \otimes N, \otimes)$, or just $M \otimes N$, is the **tensor product** of M and N.*

Those elements of $M \otimes N$ of the form $v \otimes w$ are called **elementary tensors**. Any $\alpha \in M \otimes N$ is a sum of elementary tensors: $\alpha = \sum_{i=1}^{n} v_i \otimes w_i$ for some $v_1, \ldots, v_n \in M$ and $w_1, \ldots, w_n \in N$.

Let us recall the following criterion, which allows us to detect when a given element $\sum_{i=1}^{n} v_i \otimes w_i \in M \otimes N$ is null: $\sum_{i=1}^{n} v_i \otimes w_i = 0$ if and only if

$$\sum_{i=1}^{n} b(v_i, w_i) = 0 \quad \begin{array}{l}\text{whenever} \quad Q \quad \text{is an} \quad R\text{-module and} \\ b\colon M \times N \to Q \text{ is an } R\text{-bilinear map.}\end{array} \tag{2.1}$$

Differently from the case of tensor products of vector spaces, in (2.1) one has to consider R-bilinear maps b with values into an arbitrary R-module Q (taking $Q = R$ is not sufficient). Indeed, it can happen that no non-null bilinear map $b\colon M \times N \to R$ exists even if M, N are non-trivial; see [18].

Lemma 2.2 (Tensor products of R-linear maps) *Let R be a commutative ring. Let $T\colon M \to \tilde{M}$ and $S\colon N \to \tilde{N}$ be R-linear maps between R-modules. Then there exists a unique R-linear map $T \otimes S\colon M \otimes N \to \tilde{M} \otimes \tilde{N}$ such that $(T \otimes S)(v \otimes w) = T(v) \otimes S(w)$ for every $v \in M$ and $w \in N$.*

Each commutative ring R is an R-module. Moreover, each R-module M is canonically isomorphic (as an R-module) to $R \otimes M$ via the R-linear map $M \ni v \mapsto 1_R \otimes v \in R \otimes M$. In particular, $R \otimes R \cong R$.

2.2 The space $L^0(\mathbb{X})$

Let $\mathbb{X} = (\mathrm{X}, \Sigma, \mathfrak{m})$ be a σ-finite measure space. We denote by $L^0(\mathbb{X})$ the space of all real-valued measurable functions from X to $\mathbb{R}$, quotiented up to $\mathfrak{m}$-a.e. identity. The equivalence class in $L^0(\mathbb{X})$ of a given measurable function $\bar{f}\colon \mathrm{X} \to \mathbb{R}$ will be denoted by $[\bar{f}]_{\mathfrak{m}}$. The space $L^0(\mathbb{X})$ is a vector space and a commutative ring if endowed with the natural pointwise operations. Moreover, fixed a probability measure $\tilde{\mathfrak{m}}$ on (X, Σ) with $\mathfrak{m} \ll \tilde{\mathfrak{m}} \ll \mathfrak{m}$, we have that

$$\mathsf{d}_{L^0(\mathbb{X})}(f, g) := \int |f - g| \wedge 1 \, \mathrm{d}\tilde{\mathfrak{m}} \quad \text{for every } f, g \in L^0(\mathbb{X})$$

is a complete distance, and $L^0(\mathbb{X})$ becomes a topological vector space and a topological ring if endowed with $\mathsf{d}_{L^0(\mathbb{X})}$. The distance $\mathsf{d}_{L^0(\mathbb{X})}$ depends on the chosen auxiliary measure $\tilde{\mathfrak{m}}$, but its induced topology does not. We also have that a given sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^0(\mathbb{X})$ converges to a limit function $f \in L^0(\mathbb{X})$ with respect to $\mathsf{d}_{L^0(\mathbb{X})}$ if and only if there exists a subsequence $(n_i)_{i\in\mathbb{N}} \subseteq \mathbb{N}$ such that $f(x) = \lim_i f_{n_i}(x)$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. Finally, $L^0(\mathbb{X})$ is a Riesz space if endowed with the natural partial order defined in the following way: given $f, g \in L^0(\mathbb{X})$, we declare that $f \leq g$ if and only if $f(x) \leq g(x)$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. The **positive cone** of $L^0(\mathbb{X})$ is then denoted by

$$L^0(\mathbb{X})^+ := \big\{ f \in L^0(\mathbb{X}) \;\big|\; f \geq 0 \big\}.$$

We also point out that $L^0(\mathbb{X})$ is **Dedekind complete**, i.e. every subset $\{f_i\}_{i\in I}$ of $L^0(\mathbb{X})$ that is **order-bounded** (which means that there exists $g \in L^0(\mathbb{X})^+$ such that $|f_i| \leq g$ for every $i \in I$) has both a supremum $\bigvee_{i\in I} f_i \in L^0(\mathbb{X})$ and an infimum $\bigwedge_{i\in I} f_i \in L^0(\mathbb{X})$. Let us recall that the supremum $\bigvee_{i\in I} f_i$ is the unique element of $L^0(\mathbb{X})$ having the following properties:

- $f_j \leq \bigvee_{i\in I} f_i$ for every $j \in I$.
- If a function $g \in L^0(\mathbb{X})$ satisfies $f_j \leq g$ for every $j \in I$, then $\bigvee_{i\in I} f_i \leq g$.

The infimum is given by $\bigwedge_{i\in I} f_i := -\bigvee_{i\in I}(-f_i)$. Furthermore, $L^0(\mathbb{X})$ has both the **countable sup property** and the **countable inf property**, i.e. for any order-bounded set $\{f_i\}_{i\in I} \subseteq L^0(\mathbb{X})$ one can find $C \subseteq I$ countable with $\bigvee_{i\in C} f_i = \bigvee_{i\in I} f_i$ and $\bigwedge_{i\in C} f_i = \bigwedge_{i\in I} f_i$. More generally, the space $L^0_{\rm ext}(\mathbb{X})$ of measurable functions from X to $[-\infty, +\infty]$, quotiented up to $\mathfrak{m}$-a.e. identity, is a Dedekind complete Riesz space with the countable sup/inf properties. Notice that every set in $L^0_{\rm ext}(\mathbb{X})$ is order-bounded and that $L^0(\mathbb{X})$ is a solid Riesz subspace of $L^0_{\rm ext}(\mathbb{X})$.

Given any measurable set $E \in \Sigma$ with $\mathfrak{m}(E) > 0$, we will use the following shorthand notation:

$$\mathbb{X}|_E := (\mathrm{X}, \Sigma, \mathfrak{m}|_E),$$

where $\mathfrak{m}|_E$ stands for the restriction of $\mathfrak{m}$ to E, i.e. we set $\mathfrak{m}|_E(F) := \mathfrak{m}(E \cap F)$ for every $F \in \Sigma$.

Remark 2.3 Let $\mathbb{X} = (\mathrm{X}, \Sigma, \mathfrak{m})$ be σ-finite and $\{f_i\}_{i\in I} \subseteq L^0_{\rm ext}(\mathbb{X})$. Fix a representative $\bar{f}_i$ of f_i for any $i \in I$. Suppose there exists a measurable function $\bar{g} \colon \mathrm{X} \to [-\infty, +\infty]$ such that $\sup_{i\in I} \bar{f}_i(x) \leq \bar{g}(x)$ for all $x \in \mathrm{X}$; we do not require that $x \mapsto \sup_{i\in I} \bar{f}_i(x)$ is measurable. Then $\bigvee_{i\in I} f_i \leq [\bar{g}]_{\mathfrak{m}}$. Indeed, we can find a countable set $C \subseteq I$ such that $\bigvee_{i\in C} f_i = \bigvee_{i\in I} f_i$. As $\sup_{i\in C} \bar{f}_i(x) \leq \bar{g}(x)$ for every $x \in \mathrm{X}$ and $\bigvee_{i\in C} f_i = \big[\sup_{i\in C} \bar{f}_i\big]_{\mathfrak{m}}$, we deduce that $\bigvee_{i\in I} f_i \leq [\bar{g}]_{\mathfrak{m}}$.

We also point out that the metric space $(L^0(\mathbb{X}), \mathsf{d}_{L^0(\mathbb{X})})$ is separable if and only if the measure space $(\mathrm{X}, \Sigma, \mathfrak{m})$ is **separable**, which means that we can find a sequence $(E_n)_{n\in\mathbb{N}} \subseteq \Sigma$ such that

$$\inf_{n\in\mathbb{N}} \mathfrak{m}(E_n \Delta E) = 0 \quad \text{for every } E \in \Sigma \text{ such that } \mathfrak{m}(E) < +\infty.$$

We refer e.g. to [12, Section 1.1.2] for a more detailed discussion on $L^0(\mathbb{X})$ spaces. See also [1, 4].

2.3 Banach spaces

We briefly recall some definitions and results concerning Banach spaces.

Given an index set $I \neq \varnothing$ and an exponent $p \in [1, \infty]$, we denote by $\ell_p(I)$ the vector space

$$\ell_p(I) := \big\{ a = (a_i)_{i\in I} \in \mathbb{R}^I \;\big|\; \|a\|_{\ell_p(I)} < +\infty \big\},$$

where for any $a = (a_i)_{i\in I} \in \mathbb{R}^I$ we define the quantity $\|a\|_{\ell_p(I)} \in [0, +\infty]$ as

$$\|a\|_{\ell_p(I)} := \begin{cases} \big(\sum_{i\in I} |a_i|^p\big)^{1/p} & \text{if } p < \infty, \\ \sup_{i\in I} |a_i| & \text{if } p = \infty, \end{cases}$$

where $\sum_{i\in I}|a_i|^p := \sup_{F\in\mathscr{P}_f(I)}\sum_{i\in F}|a_i|^p$. It holds that $(\ell_p(I), \|\cdot\|_{\ell_p(I)})$ is a Banach space.

An **(unconditional) Schauder basis** of a Banach space $\mathbb{B}$ is a family of vectors $\{v_i\}_{i\in I} \subseteq \mathbb{B}$ such that for any $v \in \mathbb{B}$ there is a unique $(\lambda_i)_{i\in I} \subseteq \mathbb{R}^I$ for which $\{\lambda_i v_i\}_{i\in I} \subseteq \mathbb{B}$ is summable and

$$v = \sum_{i\in I} \lambda_i v_i .$$

We recall that this means that for any $\varepsilon > 0$ there exists $F_\varepsilon \in \mathscr{P}_f(I)$ such that

$$\Big\| v - \sum_{i\in F} \lambda_i v_i \Big\|_{\mathbb{B}} < \varepsilon \quad \text{for every } F \in \mathscr{P}_f(I) \text{ with } F_\varepsilon \subseteq F.$$

We point out that, letting $\overline{\mathrm{span}}(S)$ denote the closure of the linear span of a set $S \subseteq \mathbb{B}$, we have

$$v_i \notin \overline{\mathrm{span}}\big\{v_j \bigm| j \in I\setminus\{i\}\big\} \quad \text{for every } i \in I. \tag{2.2}$$

We also recall that the canonical elements $(\mathsf{e}_i)_{i\in I} \subseteq \ell_1(I)$, which are given by

$$\mathsf{e}_i := (\delta_{ij})_{j\in I} \in \ell_1(I), \tag{2.3}$$

form an (unconditional) Schauder basis of $\ell_1(I)$. See e.g. [9] for an account of Schauder bases.

A separable Banach space $\mathbb{B}$ is said to be a **universal separable Banach space** if every separable Banach space can be embedded linearly and isometrically in $\mathbb{B}$. The Banach–Mazur theorem states that universal separable Banach spaces exist; for instance, the space C([0, 1]) endowed with the supremum norm has this property. See e.g. [3] for a proof of this result.

2.4 Banach L^0-modules

The notion of normed/Banach $L^0(\mathbb{X})$-module we are going to recall was introduced in [11], but the axiomatisation we will present is taken from [10] (with a slight difference in the terminology, since here we distinguish between non-complete normed modules and Banach modules). Unless otherwise specified, the discussion is essentially taken from [2, 10, 11].

Definition 2.4 (*Banach L^0-module*) Let $\mathbb{X}$ be a σ-finite measure space, $\mathscr{M}$ a module over $L^0(\mathbb{X})$. Then we say that $\mathscr{M}$ is a **normed $L^0(\mathbb{X})$-module** if it is endowed with a map $|\cdot|\colon \mathscr{M} \to L^0(\mathbb{X})^+$, which is said to be a **pointwise norm** on $\mathscr{M}$, verifying the following properties:

$$
\begin{aligned}
|v| \geq 0 &\quad \text{for every } v \in \mathscr{M}, \text{ with equality if and only if } v = 0, \\
|v + w| \leq |v| + |w| &\quad \text{for every } v, w \in \mathscr{M}, \\
|f \cdot v| = |f||v| &\quad \text{for every } f \in L^0(\mathbb{X}) \text{ and } v \in \mathscr{M}.
\end{aligned}
$$

Moreover, we say that $\mathscr{M}$ is a **Banach $L^0(\mathbb{X})$-module** when the following distance is complete:

$$
\mathsf{d}_{\mathscr{M}}(v, w) := \mathsf{d}_{L^0(\mathbb{X})}(|v - w|, 0) \quad \text{for every } v, w \in \mathscr{M}.
$$

In the case where $\mathbb{X}_{\mathsf{o}} = (\{\mathsf{o}\}, \delta_{\mathsf{o}})$ is the one-point probability space, the normed $L^0(\mathbb{X}_{\mathsf{o}})$-modules (resp. the Banach $L^0(\mathbb{X}_{\mathsf{o}})$-modules) can be identified with the normed spaces (resp. the Banach spaces), with the only caveat that the distance $\mathsf{d}_{\mathscr{M}}$ associated with a normed $L^0(\mathbb{X}_{\mathsf{o}})$-module $\mathscr{M}$ is not induced by the norm $\|\cdot\|_{\mathscr{M}}$ of $\mathscr{M}$. However, one has that $\mathsf{d}_{\mathscr{M}}(v, 0) = \|v\|_{\mathscr{M}} \wedge 1$ for every $v \in \mathscr{M}$.

Next, we recall/introduce a number of definitions related to normed and Banach $L^0(\mathbb{X})$-modules. Let $\mathbb{X} = (\mathrm{X}, \Sigma, \mathfrak{m})$ be a σ-finite measure space. Let $\mathscr{M}$ and $\mathscr{N}$ be normed $L^0(\mathbb{X})$-modules. Given any measurable set $E \in \Sigma$, we can consider the 'localisation' of $\mathscr{M}$ on E, i.e. the space

$$
\mathscr{M}|_E := \mathbb{1}_E \cdot \mathscr{M} = \left\{ v \in \mathscr{M} \;\middle|\; \mathbb{1}_{\mathrm{X} \setminus E} \cdot v = 0 \right\} = \left\{ \mathbb{1}_E \cdot v \;\middle|\; v \in \mathscr{M} \right\}.
$$

We can regard $\mathscr{M}|_E$ either as a normed $L^0(\mathbb{X}|_E)$-module or as a normed $L^0(\mathbb{X})$-submodule of $\mathscr{M}$. We say that some elements $v_1, \ldots, v_n \in \mathscr{M}$ are **independent** on E provided the mapping

$$
L^0(\mathbb{X}|_E)^n \ni (f_1, \ldots, f_n) \mapsto \sum_{i=1}^n f_i \cdot v_i \in \mathscr{M}|_E
$$

is injective, while a vector subspace $\mathcal{V} \subseteq \mathscr{M}$ is said to **generate** $\mathscr{M}$ on E if it holds that $\mathscr{M}|_E = \mathrm{cl}_{\mathscr{M}}\big(\mathbb{1}_E \cdot \mathscr{G}(\mathcal{V})\big)$, where we denote

$$
\mathscr{G}(\mathcal{S}) := \left\{ \sum_{i=1}^n \mathbb{1}_{E_i} \cdot v_i \in \mathscr{M} \;\middle|\; n \in \mathbb{N},\ (E_i)_{i=1}^n \subseteq \Sigma \text{ partition of } \mathrm{X},\ (v_i)_{i=1}^n \subseteq \mathcal{S} \right\}
$$

for every subset $\mathcal{S} \subseteq \mathscr{M}$. The module $\mathscr{M}$ is said to be **finitely-generated** if there exists a finite-dimensional vector subspace $\mathcal{V} \subseteq \mathscr{M}$ that generates $\mathscr{M}$ (on X). A **local basis** for $\mathscr{M}$ on E is a collection of elements $v_1, \ldots, v_n \in \mathscr{M}$ that are independent on E and have the property that their linear span generates $\mathscr{M}$ on E. In this case, $L^0(\mathbb{X}|_E)^n \ni (f_1, \ldots, f_n) \mapsto \sum_{i=1}^n f_i \cdot v_i \in \mathscr{M}|_E$ is bijective. Since two local bases on E must have the same cardinality, one can unambiguously say that $\mathscr{M}$ has **local dimension** n on E. Local bases do exist, whence it follows that $\mathscr{M}$ admits a ($\mathfrak{m}$-a.e. essentially unique) **dimensional decomposition** $(D_n)_{n \in \mathbb{N} \cup \{\infty\}}$, which means that $(D_n)_{n \in \mathbb{N} \cup \{\infty\}} \subseteq \Sigma$ is a partition of X with the following property: $\mathscr{M}$ has local dimension n on D_n for all $n \in \mathbb{N}$, and $\mathscr{M}|_E$ is not finitely-generated if $E \in \Sigma$ satisfies $E \subseteq D_\infty$ and $\mathfrak{m}(E) > 0$.

The **support** of $\mathscr{M}$ is the 'biggest' subset $\mathrm{S}(\mathscr{M})$ of X where some element of $\mathscr{M}$ is not null, i.e.

$$\mathrm{S}(\mathscr{M}) \in \Sigma \quad \text{is } \mathfrak{m}\text{-a.e. characterised by } \mathbb{1}_{\mathrm{S}(\mathscr{M})} := \bigvee_{v \in \mathscr{M}} \mathbb{1}_{\{|v|>0\}}.$$

The space $L^0(\mathbb{X})$ itself is a Banach $L^0(\mathbb{X})$-module with $\mathrm{S}(L^0(\mathbb{X})) = \mathrm{X}$. The **unit sphere** of $\mathscr{M}$ is

$$\mathbb{S}_{\mathscr{M}} := \big\{ v \in \mathscr{M} \backslash \{0\} \;\big|\; |v|(x) \in \{0, 1\} \text{ for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X} \big\}.$$

The **signum map** $\mathrm{sgn} \colon \mathscr{M} \to \mathbb{S}_{\mathscr{M}} \cup \{0\}$ on $\mathscr{M}$ is defined as

$$\mathrm{sgn}(v) := \frac{\mathbb{1}_{\{|v|>0\}}}{|v|} \cdot v \in \mathbb{S}_{\mathscr{M}} \cup \{0\} \quad \text{for every } v \in \mathscr{M}.$$

Notice that $v = |v| \cdot \mathrm{sgn}(v)$ for every $v \in \mathscr{M}$. Moreover, we define the **unit disc** of $\mathscr{M}$ as

$$\mathbb{D}_{\mathscr{M}} := \big\{ v \in \mathscr{M} \;\big|\; |v|(x) \le 1 \text{ for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X} \big\}.$$

A map $T \colon \mathscr{M} \to \mathscr{N}$ is said to be a **homomorphism of normed $L^0(\mathbb{X})$-modules** provided it is $L^0(\mathbb{X})$-linear and continuous, or equivalently if it is linear and there exists $g \in L^0(\mathbb{X})^+$ such that

$$|T(v)| \le g|v| \quad \text{for every } v \in \mathscr{M}. \tag{2.4}$$

We denote by $\textsc{Hom}(\mathscr{M}; \mathscr{N})$ the space of all homomorphisms of normed $L^0(\mathbb{X})$-modules from $\mathscr{M}$ to $\mathscr{N}$. It is a normed $L^0(\mathbb{X})$-module if endowed with the natural pointwise operations and the following pointwise norm:

$$|T| := \bigvee_{v \in \mathscr{M}} \frac{\mathbb{1}_{\{|v|>0\}} |T(v)|}{|v|} = \bigwedge \big\{ g \in L^0(\mathbb{X})^+ \;\big|\; g \text{ satisfies } 2.4) \big\}$$

for every $T \in \textsc{Hom}(\mathscr{M}; \mathscr{N})$. If $\mathscr{N}$ is complete, then $\textsc{Hom}(\mathscr{M}; \mathscr{N})$ is a Banach $L^0(\mathbb{X})$-module. By an **isomorphism of normed $L^0(\mathbb{X})$-modules** we mean a bijective homomorphism of normed $L^0(\mathbb{X})$-modules $T \colon \mathscr{M} \to \mathscr{N}$ that preserves the pointwise norm, i.e. $|T(v)| = |v|$ holds for every $v \in \mathscr{M}$. Whenever an isomorphism between $\mathscr{M}$ and $\mathscr{N}$ exists, we write $\mathscr{M} \cong \mathscr{N}$. The **kernel** $\ker(T)$ of any $T \in \textsc{Hom}(\mathscr{M}; \mathscr{N})$, which is given by $\ker(T) := \big\{ v \in \mathscr{M} : T(v) = 0 \big\}$, is a closed normed $L^0(\mathbb{X})$-submodule of $\mathscr{M}$.

The **dual** of a normed $L^0(\mathbb{X})$-module $\mathscr{M}$ is defined as

$$\mathscr{M}^* := \textsc{Hom}(\mathscr{M}; L^0(\mathbb{X})).$$

If $\mathscr{M}$ is a Banach $L^0(\mathbb{X})$-module and $\mathscr{V}$ is a Banach $L^0(\mathbb{X})$-submodule of $\mathscr{M}$, then we have that the **quotient module** $\mathscr{M}/\mathscr{V}$ is a Banach $L^0(\mathbb{X})$-module if endowed with the pointwise norm

$$|w+\mathscr{V}|:=\bigwedge_{v\in\mathscr{V}}|w+v| \quad \text{for every } w+\mathscr{V}\in\mathscr{M}/\mathscr{V}.$$

Any normed $L^0(\mathbb{X})$-module $\mathscr{M}$ has a unique **completion** $(\bar{\mathscr{M}},\iota)$, i.e. $\bar{\mathscr{M}}$ is a Banach $L^0(\mathbb{X})$-module and $\iota\colon\mathscr{M}\to\bar{\mathscr{M}}$ is a pointwise norm preserving homomorphism of normed $L^0(\mathbb{X})$-modules such that $\iota[\mathscr{M}]$ is dense in $\bar{\mathscr{M}}$. Uniqueness is in this sense: given any $(\tilde{\mathscr{M}},\tilde{\iota})$ having the same properties as $(\bar{\mathscr{M}},\iota)$, there is a unique isomorphism of Banach $L^0(\mathbb{X})$-modules $\phi\colon\bar{\mathscr{M}}\to\tilde{\mathscr{M}}$ with $\tilde{\iota}=\phi\circ\iota$.

Definition 2.5 (*Categories of Banach* $L^0(\mathbb{X})$-modules) Let $\mathbb{X}$ be a σ-finite measure space. Then:

(i) We denote by $\mathbf{BanMod}_{\mathbb{X}}$ the category whose objects are the Banach $L^0(\mathbb{X})$-modules, and the morphisms between Banach $L^0(\mathbb{X})$-modules $\mathscr{M}$ and $\mathscr{N}$ are given by $\textsc{Hom}(\mathscr{M};\mathscr{N})$.

(ii) We denote by $\mathbf{BanMod}^1_{\mathbb{X}}$ the category whose objects are the Banach $L^0(\mathbb{X})$-modules, and the morphisms between Banach $L^0(\mathbb{X})$-modules $\mathscr{M}$ and $\mathscr{N}$ are given by $\mathbb{D}_{\textsc{Hom}(\mathscr{M};\mathscr{N})}$.

Notice that $\mathbf{BanMod}^1_{\mathbb{X}}$ is a lluf subcategory of $\mathbf{BanMod}_{\mathbb{X}}$, i.e. a subcategory containing all the objects of $\mathbf{BanMod}_{\mathbb{X}}$. It is proved in [19, Theorem 3.13] that $\mathbf{BanMod}^1_{\mathbb{X}}$ is a bicomplete category (i.e. it admits all small limits and colimits), while it is observed in [19, Remark 3.1] that $\mathbf{BanMod}_{\mathbb{X}}$ admits all finite limits and colimits.

Theorem 2.6 (Hahn–Banach) *Let $\mathbb{X}$ be a σ-finite measure space and $\mathscr{M}$ a normed $L^0(\mathbb{X})$-module. Then for any given $v\in\mathscr{M}$ there exists an element $\omega_v\in\mathbb{S}_{\mathscr{M}^*}\cup\{0\}$ such that $\omega_v(v)=|v|$.*

Theorem 2.6 appeared in [11] and was obtained as a consequence of the classical Hahn–Banach theorem. For a more direct proof tailored to normed modules, we refer to [17, Theorem 3.30].

A **norming subset** of $\mathscr{M}^*$ is a set $\mathcal{T}\subseteq\mathbb{D}_{\mathscr{M}^*}$ satisfying $|v|=\bigvee_{\omega\in\mathcal{T}}\omega(v)$ for every $v\in\mathscr{M}$. The Hahn–Banach theorem ensures that the unit disc $\mathbb{D}_{\mathscr{M}^*}$ itself is a norming subset of $\mathscr{M}^*$.

Definition 2.7 (*Weak* topology*) Let $\mathbb{X}$ be a σ-finite measure space and let $\mathscr{M}$ be a Banach $L^0(\mathbb{X})$-module. Then we define the **weak* topology** on $\mathscr{M}^*$ as the coarsest topology induced by the family

$$\{\delta_v\,:\,v\in\mathscr{M}\},$$

where $\delta_v\colon\mathscr{M}^*\to L^0(\mathbb{X})$ is given by $\delta_v(\omega):=\omega(v)$ for every $\omega\in\mathscr{M}^*$.

Remark 2.8 Similarly, one could define a weak topology on $\mathscr{M}$. Moreover, the weak and weak* topologies on Banach L^0-modules verify properties that generalise the corresponding ones for Banach spaces. However, in this paper we will not investigate further in this direction.

2.4.1 Examples of Banach L^0-modules

We recall two key examples of Banach $L^0(\mathbb{X})$-modules.

Definition 2.9 (*The space* $L^0(\mathbb{X};\mathbb{B})$) Let $\mathbb{X}=(\mathrm{X},\Sigma,\mathfrak{m})$ be a σ-finite measure space, $\mathbb{B}$ a Banach space. Then we denote by $L^0(\mathbb{X};\mathbb{B})$ the space of all measurable maps from X to $\mathbb{B}$ taking values into a separable subset of $\mathbb{B}$ (which depends on the map itself), quotiented up to $\mathfrak{m}$-a.e. equality.

The L^0-*Lebesgue–Bochner space* $L^0(\mathbb{X};\mathbb{B})$ is a Banach $L^0(\mathbb{X})$-module if endowed with

$$|v|(x):=\|v(x)\|_{\mathbb{B}} \quad \text{for every } v\in L^0(\mathbb{X};\mathbb{B}) \text{ and } \mathfrak{m}\text{-a.e. } x\in \mathrm{X}.$$

Given a vector $\mathsf{v}\in\mathbb{B}$, we denote by $\underline{\mathsf{v}}\in L^0(\mathbb{X};\mathbb{B})$ the vector field that is a.e. equal to v, i.e. we set

$$\underline{\mathsf{v}}(x):=\mathsf{v}\in\mathbb{B} \quad \text{for } \mathfrak{m}\text{-a.e. } x\in\mathrm{X}. \tag{2.5}$$

We also recall the following definition of *module-valued space of generalised sequences*:

Definition 2.10 (*The space* $\ell_p(I,\mathscr{M})$) Let I be a non-empty index set and $p\in[1,\infty)$. Let $\mathbb{X}$ be a σ-finite measure space and $\mathscr{M}$ a Banach $L^0(\mathbb{X})$-module. Given any $v=(v_i)_{i\in I}\in\mathscr{M}^I$, we set

$$|v|_p:=\bigvee_{F\in\mathscr{P}_f(I)}\left(\sum_{i\in F}|v_i|^p\right)^{1/p}.$$

Notice that $|v|_p\in L^0_{\mathrm{ext}}(\mathbb{X})^+$ for every $v\in\mathscr{M}^I$. Then we define the space $\ell_p(I,\mathscr{M})$ as

$$\ell_p(I,\mathscr{M}):=\left\{v\in\mathscr{M}^I \;\middle|\; |v|_p\in L^0(\mathbb{X})^+\right\}.$$

The space $\big(\ell_p(I,\mathscr{M}),|\cdot|_p\big)$ is a Banach $L^0(\mathbb{X})$-module, as it follows from [19, Proposition 3.10] (notice indeed that $\ell_p(I,\mathscr{M})$ is a particular example of ℓ_p-sum in the sense of [19, Definition 3.9]).

2.4.2 Fiberwise representation of a Banach L^0-module

One can easily check that the space of measurable sections of a measurable Banach bundle is a Banach $L^0(\mathbb{X})$-module, a particular example being given by the L^0-Lebesgue–Bochner space $L^0(\mathbb{X};\mathbb{B})$, which corresponds to the constant bundle $\mathbb{B}$. On the other hand, it is much more difficult to show the converse, i.e. that any Banach $L^0(\mathbb{X})$-module can be represented as the space of sections of some measurable Banach bundle. Results in this direction have been obtained in [8, 16]. We will use one such result (i.e. Theorem 2.11 below) to prove Lemma 2.12, which in turn will be essential in order to obtain Lemma 3.19, and accordingly to introduce projective and injective tensor products of Banach L^0-modules.

Given a σ-finite measure space $\mathbb{X} = (\mathrm{X}, \Sigma, \mathfrak{m})$, a separable Banach space $\mathbb{B}$, and measurable maps $v_1, \ldots, v_n \colon \mathrm{X} \to \mathbb{B}$, we say that the multivalued map $\mathrm{X} \ni x \mapsto \mathbf{E}(x) \subseteq \mathbb{B}$, which we define as

$$\mathbf{E}(x) := \mathrm{span}\{v_1(x), \ldots, v_n(x)\} \subseteq \mathbb{B} \quad \text{for every } x \in \mathrm{X},$$

is a **measurable Banach bundle** on $\mathbb{X}$. Notice that each $\mathbf{E}(x)$ is a closed vector subspace of $\mathbb{B}$. The space $\Gamma_{\mathfrak{m}}(\mathbf{E})$ of all $\mathfrak{m}$-measurable **sections** of $\mathbf{E}$ is then defined as the set of all measurable maps $v \colon \mathrm{X} \to \mathbb{B}$ satisfying $v(x) \in \mathbf{E}(x)$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$, quotiented up to $\mathfrak{m}$-a.e. equality. It turns out that $\Gamma_{\mathfrak{m}}(\mathbf{E})$ is a Banach $L^0(\mathbb{X})$-module if endowed with the natural pointwise operations.

Theorem 2.11 (Fiberwise representation of Banach L^0-modules) *Let $\mathbb{X}$ be a σ-finite measure space and $\mathscr{M}$ a Banach $L^0(\mathbb{X})$-module. Let $\mathbb{B}$ be a universal separable Banach space. Suppose that $\mathscr{M}$ has local dimension $n \in \mathbb{N}$ on a set $E \in \Sigma$. Then there exist measurable maps $v_1, \ldots, v_n \colon \mathrm{X} \to \mathbb{B}$ such that $\mathscr{M}|_E \cong \Gamma_{\mathfrak{m}}(\boldsymbol{E})$, where we set $\boldsymbol{E}(x) := \mathrm{span}\{v_1(x), \ldots, v_n(x)\}$ for every $x \in \mathrm{X}$.*

Theorem 2.11 was first proved in [16], but we preferred to present its reformulation from [8].

Lemma 2.12 *Let $\mathbb{X}$ be a σ-finite measure space and let $\mathscr{M}$ be a finitely-generated Banach $L^0(\mathbb{X})$-module. Let $T \colon \mathscr{M} \to L^0(\mathbb{X})$ be an $L^0(\mathbb{X})$-linear operator. Then it holds that $T \in \mathscr{M}^*$.*

Proof Let $D_0, \ldots, D_{\bar{n}}$ be the dimensional decomposition of $\mathscr{M}$. Fix any $n = 1, \ldots, \bar{n}$. Thanks to Theorem 2.11, we can find measurable vector fields $v_1, \ldots, v_n \colon \mathrm{X} \to \mathbb{B}$, where $\mathbb{B}$ is any given universal separable Banach space, such that $v_1(x), \ldots, v_n(x) \in \mathbb{B}$ are linearly independent for every $x \in D_n$ and $\Gamma_{\mathfrak{m}}(\mathbf{E})$ is isomorphic to $\mathscr{M}|_{D_n}$, where we set $\mathbf{E}(x) := \mathrm{span}\{v_1(x), \ldots, v_n(x)\}$ for every $x \in \mathrm{X}$. For any $i = 1, \ldots, n$, we choose a measurable representative $\phi_i \colon \mathrm{X} \to \mathbb{R}$ of the function $T(v_i) \in L^0(\mathbb{X})$, where we are identifying $\mathscr{M}|_{D_n}$ with $\Gamma_{\mathfrak{m}}(\mathbf{E})$. Given any point $x \in D_n$, the unique linear operator from $\mathbf{E}(x)$ to $\mathbb{R}$ sending each $v_i(x)$ to $\phi_i(x)$ is continuous, thus

$$g_n(x) := \sup\left\{ \frac{\left|\sum_{i=1}^n q_i \phi_i(x)\right|}{\left\|\sum_{i=1}^n q_i v_i(x)\right\|_{\mathbb{B}}} \;\middle|\; (q_1, \ldots, q_n) \in \mathbb{Q}^n \setminus \{0\} \right\} < +\infty.$$

Notice that g_n is measurable by construction. Moreover, any $v \in \mathscr{M}|_{D_n}$ can be written (in a unique way) as $v = \sum_{i=1}^n f_i \cdot v_i$ for some $f_1, \ldots, f_n \in L^0(\mathbb{X})$, so that we can estimate

$$|T(v)|(x) = \left|\sum_{i=1}^n f_i(x)\phi_i(x)\right| \le g_n(x) \left\|\sum_{i=1}^n f_i(x) v_i(x)\right\|_{\mathbb{B}} = g_n(x)|v|(x)$$

for $\mathfrak{m}$-a.e. $x \in D_n$. Therefore, letting $g := \sum_{n=1}^{\bar{n}} \mathbb{1}_{D_n} g_n \in L^0(\mathbb{X})$, we conclude that $|T(v)| \le g|v|$ for every $v \in \mathscr{M}$. □

2.4.3 Pullback modules

Let $\mathbb{X} = (\mathrm{X}, \Sigma_{\mathrm{X}}, \mathfrak{m}_{\mathrm{X}})$, $\mathbb{Y} = (\mathrm{Y}, \Sigma_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}})$ be σ-finite measure spaces. Let $\varphi\colon \mathrm{X} \to \mathrm{Y}$ be a measurable map such that $\varphi_\# \mathfrak{m}_{\mathrm{X}} \ll \mathfrak{m}_{\mathrm{Y}}$. Notice that the map φ induces via pre-composition a ring homomorphism $L^0(\mathbb{Y}) \ni f \mapsto f \circ \varphi \in L^0(\mathbb{X})$ that is also a Riesz homomorphism.

This is an instance of a more general phenomenon: given any Banach $L^0(\mathbb{Y})$-module $\mathscr{M}$, there is a unique couple $(\varphi^*\mathscr{M}, \varphi^*)$, where $\varphi^*\mathscr{M}$ is a Banach $L^0(\mathbb{X})$-module, $\varphi^*\colon \mathscr{M} \to \varphi^*\mathscr{M}$ is linear, and

$$\begin{aligned} |\varphi^* v| = |v| \circ \varphi &\quad \text{for every } v \in \mathscr{M}, \\ \varphi^*[\mathscr{M}] &\quad \text{generates } \varphi^*\mathscr{M} \text{ on X.} \end{aligned}$$

We say that $\varphi^*\mathscr{M}$ is the **pullback module** of $\mathscr{M}$ under φ. Uniqueness is in the sense of the following universal property: given any couple $(\mathscr{N}, T)$ having the same properties as $(\varphi^*\mathscr{M}, \varphi^*)$, there exists a unique isomorphism of Banach $L^0(\mathbb{X})$-modules $\phi\colon \varphi^*\mathscr{M} \to \mathscr{N}$ with $T = \phi \circ \varphi^*$.

The pullback of the dual $\varphi^*\mathscr{M}^*$ is isomorphic to a Banach $L^0(\mathbb{X})$-submodule of the dual of the pullback $(\varphi^*\mathscr{M})^*$, but in general the two spaces do not coincide. More precisely, the unique homomorphism of Banach $L^0(\mathbb{X})$-modules $\mathsf{I}_\varphi\colon \varphi^*\mathscr{M}^* \to (\varphi^*\mathscr{M})^*$ satisfying

$$\mathsf{I}_\varphi(\varphi^*\omega)(\varphi^* v) = \omega(v) \circ \varphi \quad \text{for every } \omega \in \mathscr{M}^* \text{ and } v \in \mathscr{M} \tag{2.6}$$

preserves the pointwise norm, but in general is not surjective. However, the following fact holds:

Theorem 2.13 (Sequential weak* density of $\varphi^*\mathscr{M}^*$ in $(\varphi^*\mathscr{M})^*$) *Let $\mathbb{X}$, $\mathbb{Y}$ be separable, σ-finite measure spaces. Let $\varphi\colon \mathrm{X} \to \mathrm{Y}$ be a measurable map such that $\varphi_\# \mathfrak{m}_{\mathrm{X}} \ll \mathfrak{m}_{\mathrm{Y}}$. Let $\mathscr{M}$ be a Banach $L^0(\mathbb{Y})$-module. Let $\Theta \in (\varphi^*\mathscr{M})^*$ be given. Then there exists a sequence $(\Theta_n)_{n\in\mathbb{N}} \subseteq \varphi^*\mathscr{M}^*$ such that $\mathsf{I}_\varphi(\Theta_n) \to \Theta$ with respect to the weak* topology of $(\varphi^*\mathscr{M})^*$ introduced in* Definition 2.7.

Theorem 2.13 was proved in [13, Theorem B.1]. It is unclear whether the separability assumption on $\mathbb{X}$ and $\mathbb{Y}$, which is due only to the proof strategy of [13, Theorem B.1], can be dropped.

2.4.4 Bounded L^0-bilinear operators

In Sects. 4 and 5 we will need to use the space $\mathrm{B}(\mathscr{M}, \mathscr{N})$:

Definition 2.14 (*The space $B(\mathscr{M}, \mathscr{N}; \mathscr{Q})$*) Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}, \mathscr{N}, \mathscr{Q}$ be normed $L^0(\mathbb{X})$-modules. Then we denote by $\mathrm{B}(\mathscr{M}, \mathscr{N}; \mathscr{Q})$ the space of all those $L^0(\mathbb{X})$-bilinear operators $b\colon \mathscr{M} \times \mathscr{N} \to \mathscr{Q}$ that are also continuous. We also set $\mathrm{B}(\mathscr{M}, \mathscr{N}) := \mathrm{B}(\mathscr{M}, \mathscr{N}; L^0(\mathbb{X}))$.

One can readily check that a bilinear map $b\colon \mathscr{M} \times \mathscr{N} \to \mathscr{Q}$ is $L^0(\mathbb{X})$-bilinear and continuous (i.e. it belongs to $\mathrm{B}(\mathscr{M}, \mathscr{N}; \mathscr{Q})$) if and only if there exists a function $g \in L^0(\mathbb{X})^+$ such that

$$|b(v, w)| \leq g|v||w| \quad \text{for every } (v, w) \in \mathscr{M} \times \mathscr{N}. \tag{2.7}$$

Moreover, $\mathrm{B}(\mathscr{M}, \mathscr{N}; \mathscr{Q})$ is a normed $L^0(\mathbb{X})$-module if endowed with the pointwise operations and

$$|b| := \bigvee_{(v,w)\in\mathscr{M}\times\mathscr{N}} \frac{\mathbb{1}_{\{|v||w|>0\}}|b(v, w)|}{|v||w|} = \bigwedge \left\{ g \in L^0(\mathbb{X})^+ \;\middle|\; g \text{ satisfies } 2.7 \right\}$$

for every $b \in \mathrm{B}(\mathscr{M}, \mathscr{N}; \mathscr{Q})$. If $\mathscr{Q}$ is complete, then $\mathrm{B}(\mathscr{M}, \mathscr{N}; \mathscr{Q})$ is a Banach $L^0(\mathbb{X})$-module.

If $\mathscr{M}$, $\mathscr{N}$, $\mathscr{Q}$ are normed $L^0(\mathbb{X})$-modules, then each $b \in \mathrm{B}(\mathscr{M}, \mathscr{N}; \mathscr{Q})$ can be uniquely extended to $\bar{b} \in \mathrm{B}(\bar{\mathscr{M}}, \bar{\mathscr{N}}; \bar{\mathscr{Q}})$, where $\bar{\mathscr{M}}$, $\bar{\mathscr{N}}$, $\bar{\mathscr{Q}}$ are the completions of $\mathscr{M}$, $\mathscr{N}$, $\mathscr{Q}$, respectively, and $|\bar{b}| = |b|$.

3 Auxiliary results on Banach L^0-modules

3.1 Quotient operators between Banach L^0-modules

We begin with the key definition:

Definition 3.1 (*Quotient operator*) Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$ and $\mathscr{N}$ be normed $L^0(\mathbb{X})$-modules. Then we say that a homomorphism $T : \mathscr{M} \to \mathscr{N}$ of normed $L^0(\mathbb{X})$-modules is a **quotient operator** provided it is surjective and it satisfies

$$|w| = \bigwedge_{v\in T^{-1}(w)} |v| \quad \text{for every } w \in \mathscr{N}.$$

Notice that each quotient operator $T : \mathscr{M} \to \mathscr{N}$ verifies $|T| \leq 1$. More precisely, it holds that

$$|T| = \mathbb{1}_{\mathsf{S}(\mathscr{N})}. \tag{3.1}$$

If $\mathscr{M}$, $\mathscr{N}$ are two Banach $L^0(\mathbb{X})$-modules and $T : \mathscr{M} \to \mathscr{N}$ is a homomorphism of Banach $L^0(\mathbb{X})$-modules, then T is a quotient operator if and only if the unique map $\hat{T} : \mathscr{M}/\ker(T) \to \mathscr{N}$ satisfying

$$\begin{tikzcd} \mathscr{M} \arrow[r, "T"] \arrow[d, "\pi"'] & \mathscr{N} \\ \mathscr{M}/\ker(T) \arrow[ur, "\hat{T}"'] & \end{tikzcd}$$

is an isomorphism of Banach $L^0(\mathbb{X})$-modules, with $\pi : \mathscr{M} \to \mathscr{M}/\ker(T)$ the canonical projection.

Remark 3.2 If $T : \mathscr{M} \to \mathscr{N}$ is a quotient operator between normed $L^0(\mathbb{X})$-modules, then its unique linear continuous extension $\bar{T} : \bar{\mathscr{M}} \to \bar{\mathscr{N}}$ to the completions is a quotient operator.

The glueing property of $\mathscr{M}$ ensures that if $T : \mathscr{M} \to \mathscr{N}$ is a quotient operator and $w \in \mathscr{N}$ is given, then for every $\varepsilon > 0$ we can find an element $v \in \mathscr{M}$ such that $T(v) = w$ and $|v| \leq |w| + \varepsilon$.

Lemma 3.3 *Let $\mathbb{X}$ be a σ-finite measure space and let $\mathscr{M}$ be a Banach $L^0(\mathbb{X})$-module. Given any Banach $L^0(\mathbb{X})$-submodule $\mathscr{V}$ of $\mathscr{M}$, we define the* ***annihilator*** *$\mathscr{V}^\perp$ of $\mathscr{V}$ in $\mathscr{M}^*$ as*

$$\mathscr{V}^\perp := \big\{\omega \in \mathscr{M}^* \bigm| \omega(v) = 0 \text{ for every } v \in \mathscr{V}\big\}.$$

Then $\mathscr{V}^\perp$ is a Banach $L^0(\mathbb{X})$-submodule of $\mathscr{M}^*$. Moreover, it holds that

$$\mathscr{V}^* \cong \mathscr{M}^*/\mathscr{V}^\perp,$$

an isomorphism of Banach $L^0(\mathbb{X})$-modules being given by the map

$$\mathscr{M}^*/\mathscr{V}^\perp \ni \omega + \mathscr{V}^\perp \mapsto \omega|_{\mathscr{V}} \in \mathscr{V}^*.$$

Proof It is straightforward to check that $\mathscr{V}^\perp$ is a Banach $L^0(\mathbb{X})$-submodule of $\mathscr{M}^*$. Consider the homomorphism of Banach $L^0(\mathbb{X})$-modules $T : \mathscr{M}^* \to \mathscr{V}^*$ given by $T(\omega) := \omega|_{\mathscr{V}}$ for all $\omega \in \mathscr{M}^*$. Observe that $|T| \leq 1$. Moreover, the Hahn–Banach theorem ensures that for any $\eta \in \mathscr{V}^*$ we can find $\omega \in \mathscr{M}^*$ such that $T(\omega) = \eta$ and $|\omega| = |\eta|$. This shows that T is a quotient operator. Since $\ker(T) = \mathscr{V}^\perp$, we conclude that the operator $\mathscr{M}^*/\mathscr{V}^\perp \ni \omega + \mathscr{V}^\perp \mapsto \omega|_{\mathscr{V}} \in \mathscr{V}^*$ is an isomorphism of Banach $L^0(\mathbb{X})$-modules. Therefore, the proof of the statement is complete. □

We conclude with a sufficient condition for a given homomorphism to be a quotient operator:

Lemma 3.4 *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}, \mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Let $\mathscr{W}$ be a dense vector subspace of $\mathscr{N}$. Let $T : \mathscr{M} \to \mathscr{N}$ be a homomorphism of Banach $L^0(\mathbb{X})$-modules with $|T| \leq 1$ satisfying the following property: given any $w \in \mathscr{N}$ and $\varepsilon > 0$, there exists $v \in \mathscr{M}$ such that $\mathsf{d}_{\mathscr{N}}(T(v), w) < \varepsilon$ and $\mathsf{d}_{L^0(\mathbb{X})}(|v|, |w|) < \varepsilon$. Then T is a quotient operator.*

Proof Let $w \in \mathscr{N}$ and $k \in \mathbb{N}$ be given. Set $u_0^k := 0 \in \mathscr{M}$ and find recursively $u_n^k \in \mathscr{M}$ for $n \in \mathbb{N}$ such that $\mathsf{d}_{\mathscr{N}}\big(T(u_n^k), w - \sum_{i=0}^{n-1} T(u_i^k)\big) < 2^{-k-n-1}$ and $\mathsf{d}_{L^0(\mathbb{X})}\big(|u_n^k|, \big|w - \sum_{i=0}^{n-1} T(u_i^k)\big|\big) < 2^{-k-n-1}$. Now define $v_n^k := \sum_{i=1}^n u_i^k$ for every $n \in \mathbb{N}$. Then we have that $\sum_{n\in\mathbb{N}} \mathsf{d}_{\mathscr{M}}(v^k_{n+1}, v^k_n) < +\infty$, since

$$
\begin{aligned}
&\mathsf{d}_{\mathscr{M}}(v^k_{n+1}, v^k_n)\\
&\quad\leq \mathsf{d}_{L^0(\mathbb{X})}\left(|u^k_{n+1}|, \left|w - \sum_{i=0}^{n} T(u^k_i)\right|\right) + \mathsf{d}_{\mathscr{N}}\left(T(u^k_n), w - \sum_{i=0}^{n-1} T(u^k_i)\right)\\
&\quad< \frac{3}{2^{k+n+2}}.
\end{aligned}
$$

It follows that $(v^k_n)_{n\in\mathbb{N}} \subseteq \mathscr{M}$ is Cauchy, thus it makes sense to define $v^k \in \mathscr{M}$ as $v^k := \lim_n v^k_n$. Since

$$
\mathsf{d}_{\mathscr{N}}(T(v^k_n), w) = \mathsf{d}_{\mathscr{N}}\left(T(u^k_n), w - \sum_{i<n} T(u^k_i)\right) \leq \frac{1}{2^{k+n+1}} \to 0 \quad \text{as } n \to \infty,
$$

the continuity of the map T ensures that $T(v^k) = w$. Moreover, we can estimate

$$
\begin{aligned}
|v^k| &\leq \sum_{n=1}^{\infty} |u^k_n|\\
&\leq |w| + \underbrace{\sum_{n=1}^{\infty}\left||u^k_n| - \left|w - \sum_{i<n} T(u^k_i)\right|\right| + \sum_{n=2}^{\infty}\left|T(u^k_{n-1}) - \left(w - \sum_{i<n-1} T(u^k_i)\right)\right|}_{=:r_k}.
\end{aligned}
$$

Given that $\mathsf{d}_{L^0(\mathbb{X})}(r_k, 0) < 2^{-k}$ and $|w| = |T(v^k)| \leq |v^k|$, we can extract a subsequence $(k_j)_{j\in\mathbb{N}} \subseteq \mathbb{N}$ such that $|v^{k_j}| \to |w|$ in the $\mathfrak{m}$-a.e. sense. This implies that $|w| = \bigwedge_{v\in T^{-1}(w)} |v|$, as desired. □

3.2 Summability in Banach L^0-modules

First, we introduce a notion of *summable family* in a normed L^0-module. Recall that a family $\{\mathsf{v}_i\}_{i\in I}$ in a normed space $\mathbb{B}$ is said to be *summable*, with sum $\mathsf{v} \in \mathbb{B}$, if for every $\varepsilon > 0$ there exists $F \in \mathscr{P}_f(I)$ such that $\left\|\mathsf{v} - \sum_{i\in F\cup G} \mathsf{v}_i\right\|_{\mathbb{B}} \leq \varepsilon$ for every $G \in \mathscr{P}_f(I\setminus F)$. We propose the following generalisation of this notion to normed L^0-modules:

Definition 3.5 (*Summable family in a normed L^0-module*) Let $\mathbb{X}$ be a σ-finite measure space and $\mathscr{M}$ a normed $L^0(\mathbb{X})$-module. Then we say that a family $\{v_i\}_{i\in I} \subseteq \mathscr{M}$ is **summable** in $\mathscr{M}$ if

$$
\bigwedge_{F\in\mathscr{P}_f(I)} \bigvee_{G\in\mathscr{P}_f(I\setminus F)} \left|v - \sum_{i\in F\cup G} v_i\right| = 0 \quad \text{for some } v \in \mathscr{M}. \tag{3.2}
$$

The element $v \in \mathscr{M}$ is unique, is called the **sum** of $\{v_i\}_{i\in I}$ in $\mathscr{M}$, and is denoted by $\sum_{i\in I} v_i$.

In a Banach space $\mathbb{B}$, the *Cauchy summability criterion* states that a family $\{\mathsf{v}_i\}_{i\in I} \subseteq \mathbb{B}$ is summable if and only if for every $\varepsilon > 0$ there exists $F \in \mathscr{P}_f(I)$ such that $\big\|\sum_{i\in G} \mathsf{v}_i\big\|_{\mathbb{B}} \leq \varepsilon$ for every $G \in \mathscr{P}_f(I\backslash F)$. This summability criterion can be generalised to Banach L^0-modules:

Proposition 3.6 (Cauchy summability criterion) *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$ be a Banach $L^0(\mathbb{X})$-module. Then it holds that a family $\{v_i\}_{i\in I} \subseteq \mathscr{M}$ is summable if and only if*

$$\bigwedge_{F\in\mathscr{P}_f(I)} \bigvee_{G\in\mathscr{P}_f(I\backslash F)} \Big|\sum_{i\in G} v_i\Big| = 0. \tag{3.3}$$

In this case, $J := \big\{i \in I \,:\, v_i \neq 0\big\}$ is at most countable. Moreover, given any increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of J satisfying $J = \bigcup_{n\in\mathbb{N}} F_n$, we have that

$$\sum_{i\in F_n} v_i \to \sum_{i\in I} v_i \quad \text{as } n \to \infty. \tag{3.4}$$

Proof Suppose $\{v_i\}_{i\in I}$ is summable and set $v := \sum_{i\in I} v_i \in \mathscr{M}$ for brevity. We have that

$$\Big|\sum_{i\in G} v_i\Big| \leq \Big|v - \sum_{i\in F} v_i\Big| + \Big|v - \sum_{i\in F\cup G} v_i\Big| \quad \text{if } F \in \mathscr{P}_f(I) \text{ and } G \in \mathscr{P}_f(I\backslash F),$$

which yields $\bigvee_{G\in\mathscr{P}_f(I\backslash F)} \big|\sum_{i\in G} v_i\big| \leq 2\bigvee_{G\in\mathscr{P}_f(I\backslash F)} \big|v - \sum_{i\in F\cup G} v_i\big|$ and accordingly

$$\bigwedge_{F\in\mathscr{P}_f(I)} \bigvee_{G\in\mathscr{P}_f(I\backslash F)} \Big|\sum_{i\in G} v_i\Big| \leq 2 \bigwedge_{F\in\mathscr{P}_f(I)} \bigvee_{G\in\mathscr{P}_f(I\backslash F)} \Big|v - \sum_{i\in F\cup G} v_i\Big| = 0.$$

Conversely, suppose (3.3) holds. Then we can find an increasing sequence $(\tilde{F}_k)_{k\in\mathbb{N}}$ of finite subsets of J such that $\psi_k := \bigvee_{G\in\mathscr{P}_f(I\backslash\tilde{F}_k)} \big|\sum_{i\in G} v_i\big| \searrow 0$ holds $\mathfrak{m}$-a.e. Notice that $J = \bigcup_{k\in\mathbb{N}} \tilde{F}_k$. Up to a non-relabelled subsequence, we can also assume that $\mathsf{d}_{L^0(\mathbb{X})}(\psi_k \wedge 1, 0) \leq k^{-1}$ for every $k \in \mathbb{N}$. Define $J_k := \big\{i \in I \,:\, \mathsf{d}_{\mathscr{M}}(v_i, 0) > k^{-1}\big\}$ for every $k \in \mathbb{N}$. Given that $|v_i| \leq \psi_k$ for every $i \in I\backslash\tilde{F}_k$, we deduce that $\mathsf{d}_{\mathscr{M}}(v_i, 0) \leq \mathsf{d}_{L^0(\mathbb{X})}(\psi_k \wedge 1, 0) \leq k^{-1}$, so that $i \notin J_k$. This shows that $J_k \subseteq \tilde{F}_k$, thus in particular J_k is finite. Since $J = \bigcup_{k\in\mathbb{N}} J_k$, we deduce that J is at most countable. Moreover,

$$\Big|\sum_{i\in\tilde{F}_m} v_i - \sum_{i\in\tilde{F}_k} v_i\Big| = \Big|\sum_{i\in\tilde{F}_m\backslash\tilde{F}_k} v_i\Big| \leq \psi_k \quad \text{for every } k, m \in \mathbb{N} \text{ with } m \geq k$$

implies that $\left(\sum_{i\in\tilde F_k} v_i\right)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{M}$. Denoting by $v\in\mathscr{M}$ its limit, we claim that $\{v_i\}_{i\in I}$ is summable and $v=\sum_{i\in I} v_i$. First, letting $\phi_k := \left|v-\sum_{i\in\tilde F_k} v_i\right| \in L^0(\mathbb{X})^+$, we have that $\phi_k\to 0$ in $L^0(\mathbb{X})$ as $k\to\infty$, thus in particular $\bigwedge_{k\in\mathbb{N}}\phi_k=0$. Therefore, we deduce that

$$\begin{aligned}\bigwedge_{F\in\mathscr{P}_f(I)}\bigvee_{G\in\mathscr{P}_f(I\setminus F)}\left|v-\sum_{i\in F\cup G} v_i\right| &\le \bigwedge_{k\in\mathbb{N}}\bigvee_{G\in\mathscr{P}_f(I\setminus\tilde F_k)}\left|v-\sum_{i\in\tilde F_k\cup G} v_i\right|\\ &\le \bigwedge_{k\in\mathbb{N}}\phi_k+\bigwedge_{k\in\mathbb{N}}\psi_k=0,\end{aligned}$$

which shows that $\{v_i\}_{i\in I}$ is summable with sum v, as we claimed. Finally, given an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of J with $J=\bigcup_{n\in\mathbb{N}}F_n$, we can extract a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $\tilde F_k\subseteq F_{n_k}$ for every $k\in\mathbb{N}$, so that $\left|v-\sum_{i\in F_{n_k}} v_i\right|\le\phi_k+\psi_k$ for every $k\in\mathbb{N}$, whence it follows that $\sum_{i\in F_{n_k}} v_i\to v$ as $k\to\infty$. Given that the limit v does not depend on the specific choice of the sequence $(F_n)_{n\in\mathbb{N}}$, we can conclude that (3.4) is verified. The proof is complete. □

Furthermore, given any family $\{f_i\}_{i\in I}\subseteq L^0(\mathbb{X})^+$, we define

$$\sum_{i\in I} f_i := \bigvee_{F\in\mathscr{P}_f(I)}\sum_{i\in F} f_i\in L^0_{\mathrm{ext}}(\mathbb{X})^+.$$

This is consistent with Definition 3.5, since $\bigvee_{F\in\mathscr{P}_f(I)}\sum_{i\in F} f_i\in L^0(\mathbb{X})^+$ if and only if $\{f_i\}_{i\in I}\subseteq L^0(\mathbb{X})$ is summable. In this case, its sum coincides with $\bigvee_{F\in\mathscr{P}_f(I)}\sum_{i\in F} f_i$. Moreover,

$$|v|_p=\left(\sum_{i\in I}|v_i|^p\right)^{1/p}\qquad\text{for every } v=(v_i)_{i\in I}\in\ell_p(I,\mathscr{M})$$

holds whenever $\mathscr{M}$ is a Banach $L^0(\mathbb{X})$-module and $p\in[1,\infty)$. Let us also observe that

$$v=\sum_{i\in I}(\delta_{ij}v_i)_{j\in I}\qquad\text{for every } v=(v_i)_{i\in I}\in\ell_1(I,\mathscr{M}).\tag{3.5}$$

Indeed, using the summability of $\{|v_i|\}_{i\in I}$ in $L^0(\mathbb{X})$ and Proposition 3.6 we obtain that

$$\begin{aligned}&\bigwedge_{F\in\mathscr{P}_f(I)}\bigvee_{G\in\mathscr{P}_f(I\setminus F)}\left|v-\sum_{i\in F\cup G}(\delta_{ij}v_i)_{j\in I}\right|_1\\ &=\bigwedge_{F\in\mathscr{P}_f(I)}\bigvee_{G\in\mathscr{P}_f(I\setminus F)}\bigvee_{H\in\mathscr{P}_f(I\setminus(F\cup G))}\sum_{i\in H}|v_i|=0,\end{aligned}$$

whence the claimed identity (3.5) follows.

Remark 3.7 Let $\{v_i\}_{i\in I}$ be a summable family in a given Banach $L^0(\mathbb{X})$-module $\mathscr{M}$. Then it holds that

$$\left|\sum_{i\in I} v_i\right| \leq \sum_{i\in I} |v_i|. \tag{3.6}$$

Indeed, thanks to Proposition 3.6 we can find a sequence $(F_n)_{n\in\mathbb{N}} \subseteq \mathscr{P}_f(I)$ for which $\left|\sum_{i\in F_n} v_i - \sum_{i\in I} v_i\right| \to 0$ in the $\mathfrak{m}$-a.e. sense as $n \to \infty$, so that $\left|\sum_{i\in I} v_i\right| = \lim_n \left|\sum_{i\in F_n} v_i\right| \leq \lim_n \sum_{i\in F_n} |v_i| \leq \sum_{i\in I} |v_i|$. Also,

$$\{v_i\}_{i\in I} \subseteq \mathscr{M} \text{ is summable} \quad \text{for every } v = (v_i)_{i\in I} \in \ell_1(I, \mathscr{M}).$$

Indeed, arguing as in the proof of (3.5) we deduce that (3.3) is verified, so that $\{v_i\}_{i\in I}$ is summable by Proposition 3.6. Notice also that $\left|\sum_{i\in I} v_i\right| \leq |v|_1$ holds by (3.6).

Lemma 3.8 *Let $\mathbb{X}$ be a σ-finite measure space. Let $\varphi \colon \mathscr{M} \to \mathscr{N}$ be a homomorphism of Banach $L^0(\mathbb{X})$-modules $\mathscr{M}, \mathscr{N}$. Let $\{v_i\}_{i\in I} \subseteq \mathscr{M}$ be summable. Then $\{\varphi(v_i)\}_{i\in I} \subseteq \mathscr{N}$ is summable and*

$$\varphi\left(\sum_{i\in I} v_i\right) = \sum_{i\in I} \varphi(v_i). \tag{3.7}$$

Proof Set $v := \sum_{i\in I} v_i$ for brevity. If $F \in \mathscr{P}_f(I)$ and $G \in \mathscr{P}_f(I\setminus F)$, then

$$\left|\varphi(v) - \sum_{i\in F\cup G} \varphi(v_i)\right| = \left|\varphi\left(v - \sum_{i\in F\cup G} v_i\right)\right| \leq |\varphi| \left|v - \sum_{i\in F\cup G} v_i\right|.$$

By taking first the supremum over G and then the infimum over F, we thus obtain (3.7). □

3.3 Local Schauder bases

We propose a notion of (unconditional) Schauder basis in a Banach L^0-module. The term 'unconditional' will be often omitted, as no other kind of basis is considered.

Definition 3.9 (*Local Schauder basis*) Let $\mathbb{X} = (\mathrm{X}, \Sigma, \mathfrak{m})$ be a σ-finite measure space and $\mathscr{M}$ a Banach $L^0(\mathbb{X})$-module. Let $E \in \Sigma$ satisfy $\mathfrak{m}(E) > 0$. Then we say that a family $\{v_i\}_{i\in I} \subseteq \mathscr{M}$ is an **(unconditional) local Schauder basis** of $\mathscr{M}$ on E provided for any given $v \in \mathscr{M}|_E$ there exists a unique $(f_i)_{i\in I} \in L^0(\mathbb{X}|_E)^I$ such that the family $\{f_i \cdot v_i\}_{i\in I}$ is summable in $\mathscr{M}$ and

$$v = \sum_{i\in I} f_i \cdot v_i.$$

In the case where $E = \mathrm{X}$, we say that $\{v_i\}_{i\in I}$ is an (unconditional) local Schauder basis of $\mathscr{M}$.

Lemma 3.10 *Let $\mathbb{X}$ be a σ-finite measure space, $\mathbb{B}$ a Banach space with a Schauder basis $\{\mathsf{v}_i\}_{i\in I}$. Then it holds that the family $\{\underline{\mathsf{v}}_i\}_{i\in I}$ defined as in (2.5) is a local Schauder basis of $L^0(\mathbb{X};\mathbb{B})$.*

Proof Let $v \in L^0(\mathbb{X};\mathbb{B})$ be given. Fix a measurable representative $\bar{v}\colon \mathrm{X} \to \mathbb{B}$ of v. Since $\{\mathsf{v}_i\}_{i\in I}$ is a Schauder basis of $\mathbb{B}$, for any point $x \in \mathrm{X}$ we can find a unique $(\bar{f}_i(x))_{i\in I} \in \mathbb{R}^I$ such that

$$\bar{v}(x) = \sum_{i\in I} \bar{f}_i(x)\mathsf{v}_i. \tag{3.8}$$

Thanks to (2.2) and the classical Hahn–Banach theorem, for any index $i \in I$ we can find $\omega_i \in \mathbb{B}'$ (where $\mathbb{B}'$ stands for the topological dual of $\mathbb{B}$) with $\omega_i(\mathsf{v}_j) = 0$ for every $j \in I\backslash\{i\}$ and $\omega_i(\mathsf{v}_i) = 1$. Hence, Lemma 3.8 gives

$$\omega_i(\bar{v}(x)) = \omega_i\left(\sum_{j\in I} \bar{f}_j(x)\mathsf{v}_j\right) = \sum_{j\in I} \bar{f}_j(x)\,\omega_i(\mathsf{v}_j) = \bar{f}_i(x) \quad \text{for every } x \in \mathrm{X},$$

whence it follows that $\bar{f}_i\colon \mathrm{X} \to \mathbb{R}$ is measurable. Define $f_i := [\bar{f}_i]_{\mathfrak{m}} \in L^0(\mathbb{X})$ for every $i \in I$. Since

$$\inf_{F\in\mathscr{P}_f(I)} \sup_{G\in\mathscr{P}_f(I\backslash F)} \left\| \bar{v}(x) - \sum_{i\in F\cup G} \bar{f}_i(x)\mathsf{v}_i \right\|_{\mathbb{B}} = 0 \quad \text{for every } x \in \mathrm{X}$$

by (3.8), taking into account also Remark 2.3 (as well as its natural variants) we deduce that

$$\bigwedge_{F\in\mathscr{P}_f(I)} \bigvee_{G\in\mathscr{P}_f(I\backslash F)} \left| v - \sum_{i\in F\cup G} f_i\cdot \underline{\mathsf{v}}_i \right| = 0.$$

This proves that $\{f_i\cdot\underline{\mathsf{v}}_i\}_{i\in I}$ is summable in $L^0(\mathbb{X};\mathbb{B})$ and $v = \sum_{i\in I} f_i\cdot\underline{\mathsf{v}}_i$. Finally, let us check that $(f_i)_{i\in I} \in L^0(\mathbb{X})^I$ is the unique family with this property. Suppose $(g_i)_{i\in I} \in L^0(\mathbb{X})^I$ satisfies the identity $\sum_{i\in I} g_i\cdot\underline{\mathsf{v}}_i = v$ in $L^0(\mathbb{X};\mathbb{B})$. By virtue of Proposition 3.6, the set $J := \big\{i \in I \,:\, g_i \neq 0\big\}$ is at most countable. Fix a measurable representative $\bar{g}_i\colon \mathrm{X} \to \mathbb{R}$ of g_i for every $i \in J$. Since the family of all couples (F, G) with $F \in \mathscr{P}_f(J)$ and $G \in \mathscr{P}_f(J\backslash F)$ is at most countable, we can find a set $N \in \Sigma$ such that $\mathfrak{m}(N) = 0$ and

$$\inf_{F\in\mathscr{P}_f(J)} \sup_{G\in\mathscr{P}_f(J\backslash F)} \left\| \bar{v}(x) - \sum_{i\in F\cup G} \bar{g}_i(x)\mathsf{v}_i \right\|_{\mathbb{B}} = 0 \quad \text{for every } x \in \mathrm{X}\backslash N.$$

It follows that $\bar{g}_i(x) = \bar{f}_i(x)$ for every $i \in J$ and $x \in \mathrm{X}\setminus N$, as well as $\bar{f}_i(x) = 0$ for every $i \in I\setminus J$ and $x \in \mathrm{X}\setminus N$. Hence, we conclude that $(g_i)_{i\in I} = (f_i)_{i\in I}$, so that the statement is achieved. □

3.3.1 Applications to spaces of generalised sequences and to L^0-Lebesgue–Bochner spaces

Fix an arbitrary index set $I \neq \varnothing$. Given any $p \in [1,\infty)$ and any index $i \in I$, we define

$$p_i\big((a_j)_{j\in I}\big) := a_i \quad \text{for every } (a_j)_{j\in I} \in \ell_p(I).$$

The resulting map $p_i : \ell_p(I) \to \mathbb{R}$ is a 1-Lipschitz linear operator. Hence, it makes sense to define

$$a(\cdot)_i := p_i \circ a \in L^0(\mathbb{X}) \quad \text{for every } i \in I \text{ and } a \in L^0(\mathbb{X}; \ell_p(I))$$

whenever $\mathbb{X}$ is a σ-finite measure space. Moreover, recall that any element $\mathsf{a} \in \ell_p(I)$ is associated with the a.e. constant vector field $\underline{\mathsf{a}} \in L^0(\mathbb{X}; \ell_p(I))$, which is given by $\underline{\mathsf{a}}(x) := \mathsf{a}$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$.

Lemma 3.11 *Let $\mathbb{X}$ be a σ-finite measure space and $I \neq \varnothing$ an index set. Let $a \in L^0(\mathbb{X}; \ell_1(I))$ be given. Let $(\mathsf{e}_i)_{i\in I}$ be as in (2.3). Then the family $\{a(\cdot)_i \cdot \underline{\mathsf{e}}_i\}_{i\in I}$ is summable in $L^0(\mathbb{X}; \ell_1(I))$ and*

$$a = \sum_{i\in I} a(\cdot)_i \cdot \underline{\mathsf{e}}_i. \tag{3.9}$$

In particular, the family $\big\{|a(\cdot)_i|\big\}_{i\in I}$ is summable in $L^0(\mathbb{X})$ and it holds that

$$|a| = \sum_{i\in I} |a(\cdot)_i|. \tag{3.10}$$

Proof Fix a measurable representative $\bar{a} : \mathrm{X} \to \ell_1(I)$ of a. As $\{\mathsf{e}_i\}_{i\in I}$ is a Schauder basis of $\ell_1(I)$,

$$\inf_{F\in\mathscr{P}_f(I)} \sup_{G\in\mathscr{P}_f(I\setminus F)} \bigg\| \bar{a}(x) - \sum_{i\in F\cup G} \bar{a}(x)_i\, \mathsf{e}_i \bigg\|_{\ell_1(I)} = 0 \quad \text{for every } x \in \mathrm{X}.$$

Using that $\bar{a}(x)_i = (p_i \circ \bar{a})(x)$ and taking into account Remark 2.3, we can thus conclude that

$$\bigwedge_{F\in\mathscr{P}_f(I)} \bigvee_{G\in\mathscr{P}_f(I\setminus F)} \bigg| a - \sum_{i\in F\cup G} a(\cdot)_i \cdot \underline{\mathsf{e}}_i \bigg| = 0,$$

which gives the first claim (3.9). Finally, (3.10) follows from (3.9) together with the fact that

$$\left||a|-\sum_{i\in F\cup G}|a(\cdot)_i|\right|=\left||a|-\left|\sum_{i\in F\cup G}a(\cdot)_i\cdot\underline{\mathsf{e}}_i\right|\right|\leq\left|a-\sum_{i\in F\cup G}a(\cdot)_i\cdot\underline{\mathsf{e}}_i\right|$$

for every $F\in\mathscr{P}_f(I)$ and $G\in\mathscr{P}_f(I\setminus F)$. All in all, the proof of the statement is achieved. □

Finally, the Banach $L^0(\mathbb{X})$-modules $L^0(\mathbb{X};\ell_1(I))$ and $\ell_1(I,L^0(\mathbb{X}))$ can be canonically identified:

Corollary 3.12 *Let $\mathbb{X}$ be a σ-finite measure space and $I\neq\varnothing$ an index family. Let us define*

$$\phi(a):=\big(a(\cdot)_i\big)_{i\in I}\in\ell_1(I,L^0(\mathbb{X}))\quad\textit{for every } a\in L^0(\mathbb{X};\ell_1(I)).\tag{3.11}$$

Then the operator $\phi\colon L^0(\mathbb{X};\ell_1(I))\to\ell_1(I,L^0(\mathbb{X}))$ is an isomorphism of Banach $L^0(\mathbb{X})$-modules.

Proof The fact that ϕ is a homomorphism of Banach $L^0(\mathbb{X})$-modules satisfying $|\phi(a)|_1=|a|$ for every $a\in L^0(\mathbb{X};\ell_1(I))$ follows from Lemma 3.11. Therefore, it remains to check only that ϕ is surjective. To this aim, fix $f=(f_i)_{i\in I}\in\ell_1(I,L^0(\mathbb{X}))$. We know that $J:=\big\{i\in I\,:\,f_i\neq 0\big\}$ is at most countable. Take a measurable representative $\bar{f}_i\colon \mathrm{X}\to\mathbb{R}$ of f_i for every $i\in I$, with $\bar{f}_i\equiv 0$ for every $i\in I\setminus J$. Since $\sum_{i\in I}|f_i|=\sum_{i\in J}|f_i|\in L^0(\mathbb{X})$, we can also assume (up to modifying the functions $\bar{f}_i$ for $i\in J$ on a null set) that $\big\{|\bar{f}_i(x)|\big\}_{i\in I}\subseteq\mathbb{R}$ is summable for every $x\in\mathrm{X}$. Then the mapping $\mathrm{X}\ni x\mapsto\bar{a}(x):=\big(\bar{f}_i(x)\big)_{i\in I}\in\ell_1(I)$ is well-defined, is measurable, and takes values into a separable subset of $\ell_1(I)$ (namely, the closure of the vector subspace generated by $\{\mathsf{e}_i\}_{i\in J}$). Letting $a\in L^0(\mathbb{X};\ell_1(I))$ be the equivalence class of $\bar{a}\colon\mathrm{X}\to\ell_1(I)$, we have that $\phi(a)=f$ by construction. This proves the surjectivity of ϕ, thus accordingly the statement is achieved. □

3.4 Some notions of continuous module-valued maps

When dealing with injective tensor products of Banach spaces, a special role is played by the Banach space $\mathrm{C}(K)$, where K is a compact, Hausdorff topological space; cf. with the first paragraph of Sect. 5.2. It seems that in the more general setting of Banach L^0-modules there is no ‘canonical’ counterpart of $\mathrm{C}(K)$. Rather, we will propose two generalisations of $\mathrm{C}(K)$ in Definitions 3.13 and 3.16, respectively.

Let (Ω,Φ) be a uniform space (see [6]). Given an entourage $\mathcal{U}\in\Phi$ and any $p\in\Omega$, we define

$$\mathcal{U}[p]:=\big\{q\in\Omega\;\big|\;(p,q)\in\mathcal{U}\big\}.$$

Recall that the uniform structure Φ induces a topology τ_Φ on Ω, which is defined as follows:

$$\tau_\Phi := \big\{ U \subseteq \Omega \,\big|\, \forall p \in U \; \exists \mathcal{U} \in \Phi : \; \mathcal{U}[p] \subseteq U \big\}.$$

We then regard every uniform space (Ω, Φ) as a topological space, endowed with τ_Φ.

Definition 3.13 (*Uniform order-continuity*) Let (Ω, Φ) be a uniform space, $\mathbb{X}$ a σ-finite measure space, and $\mathscr{M}$ a Banach $L^0(\mathbb{X})$-module. Then we say that a map $v\colon \Omega \to \mathscr{M}$ is **order-bounded** if

$$|v| := \bigvee_{p \in \Omega} |v(p)| \in L^0(\mathbb{X})^+, \tag{3.12}$$

or equivalently if the family $\{|v(p)|\}_{p\in\Omega}$ is an order-bounded subset of $L^0(\mathbb{X})$. Moreover, we say that $v\colon \Omega \to \mathscr{M}$ is **uniformly order-continuous** provided

$$\bigwedge_{\mathcal{U}\in\Phi} \mathrm{Var}(v;\mathcal{U}) = 0, \quad \text{where we define } \mathrm{Var}(v;\mathcal{U}) := \bigvee_{(p,q)\in\mathcal{U}} |v(p) - v(q)|.$$

We denote by $\mathrm{UC}_{\mathrm{ord}}(\Omega;\mathscr{M})$ the space of all order-bounded, uniformly order-continuous maps.

Given any $v, w \in \mathrm{UC}_{\mathrm{ord}}(\Omega;\mathscr{M})$ and $f \in L^0(\mathbb{X})$, we define $v + w\colon \Omega \to \mathscr{M}$ and $f \cdot v\colon \Omega \to \mathscr{M}$ as

$$\begin{aligned} (v+w)(p) &:= v(p) + w(p) \quad \text{for every } p \in \Omega, \\ (f\cdot v)(p) &:= f \cdot v(p) \quad \text{for every } p \in \Omega, \end{aligned}$$

respectively. It can be readily checked that $v + w, f \cdot v \in \mathrm{UC}_{\mathrm{ord}}(\Omega;\mathscr{M})$, that $\big(\mathrm{UC}_{\mathrm{ord}}(\Omega;\mathscr{M}), +, \cdot\big)$ is a module over $L^0(\mathbb{X})$, and that the map

$$|\cdot|\colon \mathrm{UC}_{\mathrm{ord}}(\Omega;\mathscr{M}) \to L^0(\mathbb{X})^+$$

defined in (3.12) is a pointwise norm on $\mathrm{UC}_{\mathrm{ord}}(\Omega;\mathscr{M})$. All in all, the couple $\big(\mathrm{UC}_{\mathrm{ord}}(\Omega;\mathscr{M}), |\cdot|\big)$ is a normed $L^0(\mathbb{X})$-module. Moreover:

Lemma 3.14 *Let (Ω, Φ) be a uniform space. Let $\mathbb{X}$ be a σ-finite measure space and $\mathscr{M}$ a Banach $L^0(\mathbb{X})$-module. Let $|\cdot|\colon \mathrm{UC}_{\mathrm{ord}}(\Omega;\mathscr{M}) \to L^0(\mathbb{X})^+$ be defined as in (3.12). Then $\big(\mathrm{UC}_{\mathrm{ord}}(\Omega;\mathscr{M}), |\cdot|\big)$ is a Banach $L^0(\mathbb{X})$-module.*

Proof It only remains to check that $\mathrm{UC}_{\mathrm{ord}}(\Omega;\mathscr{M})$ is complete. To this aim, let $(v_n)_{n\in\mathbb{N}} \subseteq \mathrm{UC}_{\mathrm{ord}}(\Omega;\mathscr{M})$ be a given Cauchy sequence. Up to a non-relabelled subsequence, we can assume that $\mathsf{d}_{L^0(\mathbb{X})}(|v_n - v_{n+1}|, 0) \leq 2^{-n}$ for every $n \in \mathbb{N}$. For any $p \in \Omega$ we can estimate

$$\mathsf{d}_{\mathscr{M}}(v_n(p), v_{n+1}(p)) \leq \mathsf{d}_{L^0(\mathbb{X})}(|v_n - v_{n+1}|, 0) \leq \frac{1}{2^n}.$$

It follows that $(v_n(p))_{n\in\mathbb{N}} \subseteq \mathscr{M}$ is a Cauchy sequence, so that the limit $v(p):=\lim_n v_n(p) \in \mathscr{M}$ exists. To prove that $v\colon \Omega \to \mathscr{M}$ is order-bounded, notice that $\big||v_n| - |v_{n+1}|\big| \leq |v_n - v_{n+1}|$ implies $\mathsf{d}_{L^0(\mathbb{X})}(|v_n|, |v_{n+1}|) \leq 2^{-n}$, so that the sequence $(|v_n|)_{n\in\mathbb{N}} \subseteq L^0(\mathbb{X})$ is Cauchy. Define $g:=\lim_n |v_n| \in L^0(\mathbb{X})$. Given $p \in \Omega$, we can extract a subsequence $(n_i)_{i\in\mathbb{N}}$ such that $|v_{n_i}(p)| \to |v(p)|$ and $|v_{n_i}| \to g$ $\mathfrak{m}$-a.e. as $i \to \infty$. Hence,

$$|v(p)|(x) = \lim_{i\to\infty} |v_{n_i}(p)|(x) \leq \lim_{i\to\infty} |v_{n_i}|(x) = g(x) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X},$$

which implies $|v| = \bigvee_{p\in\Omega} |v(p)| \leq g$. We pass to the verification of the uniform order-continuity of v. For any $n \in \mathbb{N}$ we can find a sequence of entourages $(\mathcal{U}_i^n)_{i\in\mathbb{N}} \subseteq \Phi$ with $\bigwedge_{i\in\mathbb{N}} \mathrm{Var}(v_n; \mathcal{U}_i^n) = 0$. With no loss of generality, we can also require that $\mathcal{U}_{i+1}^n \subseteq \mathcal{U}_i^n$ for every $i \in \mathbb{N}$, whence it follows that $\mathsf{d}_{L^0(\mathbb{X})}\big(\mathrm{Var}(v_n; \mathcal{U}_i^n), 0\big) \to 0$ as $i \to \infty$. Define $h_n:=\sum_{k=n}^{\infty} |v_k - v_{k+1}| \wedge 1$ for every $n \in \mathbb{N}$. Then

$$\begin{aligned}\int h_n \,\mathrm{d}\tilde{\mathfrak{m}} &= \sum_{k=n}^{\infty} \int |v_k - v_{k+1}| \wedge 1 \,\mathrm{d}\tilde{\mathfrak{m}} = \sum_{k=n}^{\infty} \mathsf{d}_{L^0(\mathbb{X})}(|v_k - v_{k+1}|, 0) \leq \sum_{k=n}^{\infty} \frac{1}{2^k} \\ &= \frac{1}{2^{n-1}}\end{aligned}$$

by monotone convergence theorem, thus $h_n \in L^1(\tilde{\mathfrak{m}})$ and $\|h_n\|_{L^1(\tilde{\mathfrak{m}})} \leq 2^{-n+1}$. Notice that

$$\begin{aligned}|v(p) - v(q)| \wedge 1 &\leq |v(p) - v_n(p)| \wedge 1 + |v_n(p) - v_n(q)| \wedge 1 + |v_n(q) - v(q)| \wedge 1 \\ &\leq 2h_n + |v_n(p) - v_n(q)| \wedge 1\end{aligned}$$

for every $p, q \in \Omega$ and $n \in \mathbb{N}$. Fixing $i \in \mathbb{N}$ and passing to the supremum over all $(p, q) \in \mathcal{U}_i^n$, we deduce that $\mathrm{Var}(v; \mathcal{U}_i^n) \wedge 1 \leq 2h_n + \mathrm{Var}(v_n; \mathcal{U}_i^n) \wedge 1$. Integrating with respect to $\tilde{\mathfrak{m}}$, we thus get

$$\begin{aligned}\mathsf{d}_{L^0(\mathbb{X})}\big(\mathrm{Var}(v; \mathcal{U}_i^n), 0\big) &\leq 2\|h_n\|_{L^1(\tilde{\mathfrak{m}})} + \mathsf{d}_{L^0(\mathbb{X})}\big(\mathrm{Var}(v_n; \mathcal{U}_i^n), 0\big) \\ &\leq \frac{1}{2^{n-2}} + \mathsf{d}_{L^0(\mathbb{X})}\big(\mathrm{Var}(v_n; \mathcal{U}_i^n), 0\big).\end{aligned}$$

Given any $k \in \mathbb{N}$, we first choose $n_k \in \mathbb{N}$ such that $2^{-n_k+2} < 1/(2k)$, then we choose $i_k \in \mathbb{N}$ such that $\mathsf{d}_{L^0(\mathbb{X})}\big(\mathrm{Var}(v_{n_k}; \mathcal{U}_k), 0\big) < 1/(2k)$, where we set $\mathcal{U}_k:=\mathcal{U}_{i_k}^{n_k}$. Hence, $\mathsf{d}_{L^0(\mathbb{X})}\big(\mathrm{Var}(v; \mathcal{U}_k), 0\big) < 1/k$ for every $k \in \mathbb{N}$, so that $\underline{\lim}_k \mathrm{Var}(v; \mathcal{U}_k)(x) = 0$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. In particular, we conclude that

$$\bigwedge_{\mathcal{U}\in\Phi} \mathrm{Var}(v; \mathcal{U}) \leq \bigwedge_{k\in\mathbb{N}} \mathrm{Var}(v; \mathcal{U}_k) \leq \varliminf_{k\to\infty} \mathrm{Var}(v; \mathcal{U}_k) = 0,$$

which shows that $v\colon \Omega \to \mathscr{M}$ is uniformly order-continuous. All in all, v belongs to $\mathrm{UC}_{\mathrm{ord}}(\Omega; \mathscr{M})$.

In order to conclude, it remains to check $\lim_n \mathsf{d}_{\mathrm{UC}_{\mathrm{ord}}(\Omega;\mathscr{M})}(v_n, v) \to 0$. Fix $p \in \Omega$. Take a subsequence $(n_j)_{j\in\mathbb{N}}$ with $|v_{n_j}(p) - v_n(p)| \to |v(p) - v_n(p)|$ in the $\mathfrak{m}$-a.e. sense. Then

$$|v(p) - v_n(p)| \wedge 1 = \lim_j |v_{n_j}(p) - v_n(p)| \wedge 1 \leq \varliminf_j |v_{n_j} - v_n| \wedge 1 \leq h_n$$

holds $\mathfrak{m}$-a.e., whence it follows that $|v - v_n| \wedge 1 \leq h_n$. Therefore, we can conclude that $\lim_n \mathsf{d}_{\mathrm{UC}_{\mathrm{ord}}(\Omega;\mathscr{M})}(v, v_n) \leq \lim_n \|h_n\|_{L^1(\tilde{\mathfrak{m}})} = 0$, as desired. □

Remark 3.15 Given any point $p \in \Omega$, let us consider the **evaluation functional** $\delta_p^{\mathscr{M}} : \mathrm{UC}_{\mathrm{ord}}(\Omega; \mathscr{M}) \to \mathscr{M}$, which we define as

$$\delta_p^{\mathscr{M}}(v) := v(p) \quad \text{for every } v \in \mathrm{UC}_{\mathrm{ord}}(\Omega; \mathscr{M}).$$

Observe that $\delta_p^{\mathscr{M}} \in \mathrm{H{\scriptsize OM}}\big(\mathrm{UC}_{\mathrm{ord}}(\Omega; \mathscr{M}); \mathscr{M}\big)$ and $|\delta_p^{\mathscr{M}}| \leq 1$. In particular, we have that $\delta_p := \delta_p^{L^0(\mathbb{X})}$ satisfies

$$\delta_p \in \mathrm{UC}_{\mathrm{ord}}(\Omega; L^0(\mathbb{X}))^* \quad \text{and} \quad |\delta_p| \leq 1.$$

Furthermore, $\{\delta_p : p \in \Omega\}$ is a norming subset of $\mathrm{UC}_{\mathrm{ord}}(\Omega; L^0(\mathbb{X}))^*$. Indeed, thanks to (3.12) we have that $|f| = \bigvee_{p\in\Omega} |f(p)| = \bigvee_{p\in\Omega} |\delta_p(f)|$ holds for every $f \in \mathrm{UC}_{\mathrm{ord}}(\Omega; L^0(\mathbb{X}))$.

Definition 3.16 (*Pointwise bounded continuous maps)* Let (Ω, τ) be a topological space. Let $\mathbb{X}$ be a σ-finite measure space and $\mathscr{M}$ a Banach $L^0(\mathbb{X})$-module. Then we define $\mathrm{C}_{\mathrm{pb}}(\Omega; \mathscr{M})$ as

$$\mathrm{C}_{\mathrm{pb}}(\Omega; \mathscr{M}) := \big\{ v : \Omega \to \mathscr{M} \;\big|\; v \text{ is continuous and order-bounded} \big\}.$$

We say that $\mathrm{C}_{\mathrm{pb}}(\Omega; \mathscr{M})$ is the space of **pointwise bounded continuous maps** from Ω to $\mathscr{M}$.

The space $\mathrm{C}_{\mathrm{pb}}(\Omega; \mathscr{M})$ is a Banach $L^0(\mathbb{X})$-module if endowed with the pointwise norm in (3.12). This claim can be proved by repeating almost verbatim the arguments for Lemma 3.14, the main difference being in the verification of the completeness, where one can use the following remark:

Remark 3.17 Take a sequence $(v_n)_{n\in\mathbb{N}} \subseteq \mathrm{C}_{\mathrm{pb}}(\Omega; \mathscr{M})$ and an order-bounded map $v : \Omega \to \mathscr{M}$ such that

$$\delta_n := \sup_{p\in\Omega} \mathsf{d}_{\mathscr{M}}(v_n(p), v(p)) \to 0 \quad \text{as } n \to \infty.$$

Then it holds that $v \in \mathrm{C}_{\mathrm{pb}}(\Omega; \mathscr{M})$. Indeed, given any $p \in \Omega$ and $\varepsilon > 0$, we can fix $n_0 \in \mathbb{N}$ such that $\delta_{n_0} < \varepsilon/4$ and choose a neighbourhood U of p such that

$\mathsf{d}_{\mathscr{M}}(v_{n_0}(q), v_{n_0}(p)) < \varepsilon/2$ for every $q \in U$. Then

$$\begin{aligned} \mathsf{d}_{\mathscr{M}}(v(q), v(p)) &\leq \mathsf{d}_{\mathscr{M}}(v(q), v_{n_0}(q)) + \mathsf{d}_{\mathscr{M}}(v_{n_0}(q), v_{n_0}(p)) + \mathsf{d}_{\mathscr{M}}(v_{n_0}(p), v(p)) \\ &< 2\delta_{n_0} + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

for every $q \in U$, which implies that v is continuous at each point $p \in \Omega$, as we claimed.

We also point out that if (Ω, Φ) is a uniform space, then we have that

$$\mathrm{UC}_{\mathrm{ord}}(\Omega; \mathscr{M}) \quad \text{is a Banach } L^0(\mathbb{X})\text{-submodule of } \mathrm{C}_{\mathrm{pb}}(\Omega; \mathscr{M}). \tag{3.13}$$

Indeed, if $v \in \mathrm{UC}_{\mathrm{ord}}(\Omega; \mathscr{M})$, $p \in \Omega$, and $\varepsilon > 0$ are given, then we can find an entourage $\mathcal{U} \in \Phi$ such that $\mathsf{d}_{L^0(\mathbb{X})}\big(\mathrm{Var}(v; \mathcal{U}), 0\big) < \varepsilon$. Hence, for every point q in the open set $\mathcal{U}[p]$ we have that

$$\mathsf{d}_{\mathscr{M}}(v(q), v(p)) = \mathsf{d}_{L^0(\mathbb{X})}\big(|v(q) - v(p)|, 0\big) \leq \mathsf{d}_{L^0(\mathbb{X})}\big(\mathrm{Var}(v; \mathcal{U}), 0\big) < \varepsilon.$$

Remark 3.18 If (K, Φ) is compact and $\mathbb{B}$ is Banach (so that $\mathbb{B}$ is a Banach $L^0(\mathbb{X}_\mathsf{o})$-module, where $\mathbb{X}_\mathsf{o}$ is the one-point probability space), then $\mathrm{UC}_{\mathrm{ord}}(K; \mathbb{B}) = \mathrm{C}_{\mathrm{pb}}(K; \mathbb{B}) = \mathrm{C}(K; \mathbb{B})$. Indeed, since the topology of $\mathbb{B}$ as a Banach space and the one as a Banach $L^0(\mathbb{X}_\mathsf{o})$-module coincide, we have that $\mathrm{C}_{\mathrm{pb}}(K; \mathbb{B}) \subseteq \mathrm{C}(K; \mathbb{B})$. Moreover, if $v \in \mathrm{C}(K; \mathbb{B})$ and $k \in \mathbb{N}$ are given, then (by compactness of K) we can find $n \in \mathbb{N}$, $p_1, \ldots, p_n \in K$, and $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \Phi$ such that $K = \bigcup_{i=1}^n \mathcal{U}_i[p_i]$ and

$$\|v(p) - v(p_i)\|_{\mathbb{B}} \leq \frac{1}{k} \quad \text{for every } i = 1, \ldots, n \text{ and } p \in \mathcal{U}_i[p_i].$$

Then $\bigvee_{p \in K} \|v(p)\|_{\mathbb{B}} \leq \max\big\{\|v(p_i)\|_{\mathbb{B}} + 1 \,:\, i = 1, \ldots, n\big\} < +\infty$, so that v is an order-bounded map. Moreover, we have that

$$\bigwedge_{\mathcal{U} \in \Phi} \mathrm{Var}(v; \mathcal{U})(\mathsf{o}) \leq \bigwedge_{i=1}^{n} \mathrm{Var}(v; \mathcal{U}_i)(\mathsf{o}) \leq \frac{2}{k}.$$

Since $k \in \mathbb{N}$ is arbitrary, we deduce that v is uniformly order-continuous and $v \in \mathrm{UC}_{\mathrm{ord}}(K; \mathbb{B})$. All in all, we proved $\mathrm{C}_{\mathrm{pb}}(K; \mathbb{B}) \subseteq \mathrm{C}(K; \mathbb{B}) \subseteq \mathrm{UC}_{\mathrm{ord}}(K; \mathbb{B})$. Recalling also (3.13), the claim follows.

3.5 Algebraic tensor products of normed L^0-modules

In order to define tensor products of Banach L^0-modules, the following criterion to detect null tensors will play a fundamental role:

Lemma 3.19 (Null tensors in normed L^0-modules) *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$, $\mathscr{N}$ be normed $L^0(\mathbb{X})$-modules. Fix any $\alpha \in \mathscr{M} \otimes \mathscr{N}$, say that $\alpha = \sum_{i=1}^n v_i \otimes w_i$. Then it holds that $\alpha = 0$ if and only if*

$$\sum_{i=1}^n \omega(v_i)\eta(w_i) = 0 \quad \textit{for every } \omega \in \mathscr{M}^* \textit{ and } \eta \in \mathscr{N}^*. \tag{3.14}$$

Proof First, assume that $\alpha = 0$. For any $\omega \in \mathscr{M}^*$ and $\eta \in \mathscr{N}^*$, the mapping $b_{\omega,\eta} \colon \mathscr{M} \times \mathscr{N} \to L^0(\mathbb{X})$, which we define as $b_{\omega,\eta}(v, w) := \omega(v)\eta(w)$ for every $(v, w) \in \mathscr{M} \times \mathscr{N}$, is $L^0(\mathbb{X})$-bilinear. Therefore, we deduce from (2.1) that $\sum_{i=1}^n \omega(v_i)\eta(w_i) = \sum_{i=1}^n b_{\omega,\eta}(v_i, w_i) = 0$, which proves that (3.14) holds.

Conversely, assume (3.14) holds. Fix an arbitrary $L^0(\mathbb{X})$-bilinear map $b \colon \mathscr{M} \times \mathscr{N} \to Q$, for some $L^0(\mathbb{X})$-module Q. Denote by $\mathscr{V}$ (resp. by $\mathscr{W}$) the $L^0(\mathbb{X})$-submodule of $\mathscr{M}$ (resp. of $\mathscr{N}$) that is generated by $v_1, \ldots, v_n$ (resp. by $w_1, \ldots, w_n$). Given that the modules $\mathscr{V}$ and $\mathscr{W}$ are finitely-generated, they are Banach $L^0(\mathbb{X})$-modules. Now let $D_0^{\mathscr{V}}, \ldots, D_{\bar{m}}^{\mathscr{V}}$ and $D_0^{\mathscr{W}}, \ldots, D_{\bar{q}}^{\mathscr{W}}$ be the dimensional decompositions of $\mathscr{V}$ and $\mathscr{W}$, respectively. To prove that $\sum_{i=1}^n b(v_i, w_i) = 0$ amounts to showing that $\mathbb{1}_{D_{m,q}} \cdot \sum_{i=1}^n b(v_i, w_i) = 0$ holds for all $m = 1, \ldots, \bar{m}$ and $q = 1, \ldots, \bar{q}$, where $D_{m,q} := D_m^{\mathscr{V}} \cap D_q^{\mathscr{W}}$. To this aim, fix a local basis $x_1, \ldots, x_m$ of $\mathscr{V}$ on $D_{m,q}$ and a local basis $y_1, \ldots, y_q$ of $\mathscr{W}$ on $D_{m,q}$. Given any $v \in \mathscr{V}|_{D_{m,q}}$, we can find (uniquely) functions $\tilde{\omega}_1(v), \ldots, \tilde{\omega}_m(v) \in L^0(\mathbb{X})|_{D_{m,q}}$ so that $v = \sum_{j=1}^m \tilde{\omega}_j(v) \cdot x_j$. Moreover, each mapping $\tilde{\omega}_j \colon \mathscr{V}|_{D_{m,q}} \to L^0(\mathbb{X})$ is $L^0(\mathbb{X})$-linear, thus it is also continuous thanks to Lemma 2.12. An application of the Hahn–Banach theorem for normed L^0-modules ensures the existence of some $\omega_1, \ldots, \omega_m \in \mathscr{M}^*$ such that $\omega_j|_{\mathscr{V}|_{D_{m,q}}} = \tilde{\omega}_j$ for every $j = 1, \ldots, m$. Similarly, we can find elements $\eta_1, \ldots, \eta_q \in \mathscr{N}^*$ such that $w = \sum_{k=1}^q \eta_k(w) \cdot y_k$ for every $w \in \mathscr{W}|_{D_{m,q}}$. Therefore, the $L^0(\mathbb{X})$-bilinearity of b yields

$$\begin{aligned} \mathbb{1}_{D_{m,q}} \cdot \sum_{i=1}^n b(v_i, w_i) &= \sum_{i=1}^n b\big(\mathbb{1}_{D_{m,q}} \cdot v_i, \mathbb{1}_{D_{m,q}} \cdot w_i\big) \\ &= \sum_{j=1}^m \sum_{k=1}^q \left(\sum_{i=1}^n \omega_j(v_i)\eta_k(w_i) \right) \cdot b(x_j, y_k) \overset{(3.14)}{=} 0. \end{aligned}$$

This implies $\sum_{i=1}^n b(v_i, w_i) = 0$, whence it follows that $\alpha = \sum_{i=1}^n v_i \otimes w_i = 0$ by (2.1). □

Remark 3.20 We stress that Lemma 3.19 shows that, in the case of normed $L^0(\mathbb{X})$-modules, a null tensor can be detected by checking only against (a class of) $L^0(\mathbb{X})$-bilinear maps taking values into the ring $L^0(\mathbb{X})$. It is not clear whether this happens for arbitrary $L^0(\mathbb{X})$-modules that are not equipped with a pointwise norm; cf. with the discussion after (2.1). In other words, the proof of Lemma 3.19 is heavily relying on the fact that we are considering normed $L^0(\mathbb{X})$-modules.

Corollary 3.21 *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$ and $\mathscr{N}$ be normed $L^0(\mathbb{X})$-modules. Fix any tensor $\alpha = \sum_{i=1}^n v_i \otimes w_i \in \mathscr{M} \otimes \mathscr{N}$. Then the following conditions are equivalent:*

(i) $\alpha = 0$.
(ii) $\sum_{i=1}^n \omega(v_i) \cdot w_i = 0$ *for every* $\omega \in \mathscr{M}^*$.
(iii) $\sum_{i=1}^n \eta(w_i) \cdot v_i = 0$ *for every* $\eta \in \mathscr{N}^*$.

Proof We prove only the equivalence between i) and ii); the proof of the equivalence between i) and iii) is very similar. Assuming ii), we deduce that $\sum_{i=1}^n \omega(v_i)\eta(w_i) = \eta\big(\sum_{i=1}^n \omega(v_i) \cdot w_i\big) = 0$ for every $\omega \in \mathscr{M}^*$ and $\eta \in \mathscr{N}^*$, so that $\alpha = 0$ by Lemma 3.19. Conversely, if $\alpha = 0$, then the same computation as above shows that $\eta\big(\sum_{i=1}^n \omega(v_i) \cdot w_i\big) = 0$ for every $\omega \in \mathscr{M}^*$ and $\eta \in \mathscr{N}^*$, so that $\sum_{i=1}^n \omega(v_i) \cdot w_i = 0$ for every $\omega \in \mathscr{M}^*$ by the Hahn–Banach theorem, which gives ii). □

4 Projective tensor products of Banach L^0-modules

4.1 Definition and main properties

We begin by introducing the projective pointwise norm:

Theorem 4.1 *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$ and $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Define $|\alpha|_\pi \in L^0(\mathbb{X})^+$ as*

$$|\alpha|_\pi := \bigwedge \left\{ \sum_{i=1}^n |v_i||w_i| \;\middle|\; n \in \mathbb{N},\ (v_i)_{i=1}^n \subseteq \mathscr{M},\ (w_i)_{i=1}^n \subseteq \mathscr{N},\ \alpha = \sum_{i=1}^n v_i \otimes w_i \right\} \tag{4.1}$$

for every $\alpha \in \mathscr{M} \otimes \mathscr{N}$. Then $|\cdot|_\pi : \mathscr{M} \otimes \mathscr{N} \to L^0(\mathbb{X})^+$ is a pointwise norm on $\mathscr{M} \otimes \mathscr{N}$. Moreover,

$$|v \otimes w|_\pi = |v||w| \quad \textit{for every } v \in \mathscr{M} \textit{ and } w \in \mathscr{N}. \tag{4.2}$$

Proof To prove that $|\cdot|_\pi$ is a pointwise norm on $\mathscr{M} \otimes \mathscr{N}$ amounts to showing that:

i) If $\alpha \in \mathscr{M} \otimes \mathscr{N}$ satisfies $|\alpha|_\pi = 0$, then $\alpha = 0$.
ii) $|\alpha + \beta|_\pi \leq |\alpha|_\pi + |\beta|_\pi$ for every $\alpha, \beta \in \mathscr{M} \otimes \mathscr{N}$.
iii) $|f \cdot \alpha|_\pi = |f||\alpha|_\pi$ for every $f \in L^0(\mathbb{X})$ and $\alpha \in \mathscr{M} \otimes \mathscr{N}$.

Let us first check the validity of i). Assume $|\alpha|_\pi = 0$. Let $\omega \in \mathscr{M}^*$ and $\eta \in \mathscr{N}^*$ be given. Then we define $\theta_{\omega,\eta} \in L^0(\mathbb{X})$ as $\theta_{\omega,\eta} := \sum_{i=1}^n \omega(v_i)\eta(w_i)$ for any $v_1, \dots, v_n \in \mathscr{M}$ and $w_1, \dots, w_n \in \mathscr{N}$ satisfying $\alpha = \sum_{i=1}^n v_i \otimes w_i$; thanks to Lemma 3.19, the function $\theta_{\omega,\eta}$ is independent of the chosen representation $\sum_{i=1}^n v_i \otimes w_i$ of α. Now fix $\varepsilon > 0$. Then there exists a partition $(E_k)_{k\in\mathbb{N}} \subseteq \Sigma$ of X and $v_1^k, \dots, v_{n_k}^k \in \mathscr{M}$, $w_1^k, \dots, w_{n_k}^k \in \mathscr{N}$ such that $\alpha = \sum_{i=1}^{n_k} v_i^k \otimes w_i^k$ and $\mathbb{1}_{E_k} \sum_{i=1}^{n_k} |v_i^k||w_i^k| \leq \varepsilon$ for every

$k \in \mathbb{N}$. Therefore, can estimate

$$|\theta_{\omega,\eta}| = \sum_{k\in\mathbb{N}} \mathbb{1}_{E_k} \left| \sum_{i=1}^{n_k} \omega(v_i^k)\eta(w_i^k) \right| \leq \sum_{k\in\mathbb{N}} \mathbb{1}_{E_k} \sum_{i=1}^{n_k} |\omega(v_i^k)||\eta(w_i^k)|$$
$$\leq |\omega||\eta| \sum_{k\in\mathbb{N}} \sum_{i=1}^{n_k} |v_i^k||w_i^k| \leq \varepsilon|\omega||\eta|.$$

Thanks to the arbitrariness of $\varepsilon > 0$, we deduce that $\theta_{\omega,\eta} = 0$, so that $\alpha = 0$ by Lemma 3.19.

In order to prove ii), let us write $\alpha = \sum_{i=1}^{n} v_i \otimes w_i$ and $\beta = \sum_{j=1}^{m} \tilde{v}_j \otimes \tilde{w}_j$. Then we have that

$$|\alpha + \beta|_\pi \leq \sum_{i=1}^{n} |v_i||w_i| + \sum_{j=1}^{m} |\tilde{v}_j||\tilde{w}_j|,$$

where we used the fact that $\alpha + \beta = \sum_{i=1}^{n} v_i \otimes w_i + \sum_{j=1}^{m} \tilde{v}_j \otimes \tilde{w}_j$. By passing to the infimum over all the possible representations of α and β, we conclude that $|\alpha + \beta|_\pi \leq |\alpha|_\pi + |\beta|_\pi$.

We now pass to the verification of iii). If $\alpha = \sum_{i=1}^{n} v_i \otimes w_i$, then we have that $f \cdot \alpha = \sum_{i=1}^{n} (f \cdot v_i) \otimes w_i$. It follows that

$$|f \cdot \alpha|_\pi \leq \sum_{i=1}^{n} |f \cdot v_i||w_i| = |f| \sum_{i=1}^{n} |v_i||w_i|.$$

By passing to the infimum over all the representations of α, we obtain that $|f \cdot \alpha|_\pi \leq |f||\alpha|_\pi$. Moreover, the same estimates yield

$$|f||\alpha|_\pi = |f| \left| \frac{\mathbb{1}_{\{f\neq 0\}}}{f} \cdot (f \cdot \alpha) \right|_\pi \leq |f| \frac{\mathbb{1}_{\{f\neq 0\}}}{|f|} |f \cdot \alpha|_\pi = \mathbb{1}_{\{f\neq 0\}} |f \cdot \alpha|_\pi \leq |f \cdot \alpha|_\pi.$$

All in all, we have shown that $|f \cdot \alpha|_\pi = |f||\alpha|_\pi$.

Finally, let us check that (4.2) holds. The inequality $|v \otimes w|_\pi \leq |v||w|$ is trivially verified. For the converse inequality, choose elements $\omega \in \mathscr{M}^*$ and $\eta \in \mathscr{N}^*$ such that $|\omega|, |\eta| \leq 1$, $\omega(v) = |v|$, and $\eta(w) = |w|$. Given that the $L^0(\mathbb{X})$-linearisation T of $\mathscr{M} \times \mathscr{N} \ni (\tilde{v}, \tilde{w}) \mapsto \omega(\tilde{v})\eta(\tilde{w}) \in L^0(\mathbb{X})$ satisfies

$$|T(\alpha)| \leq \sum_{i=1}^{n} |T(v_i \otimes w_i)| = \sum_{i=1}^{n} |\omega(v_i)||\eta(w_i)| \leq \sum_{i=1}^{n} |v_i||w_i|$$

for every $\alpha = \sum_{i=1}^{n} v_i \otimes w_i \in \mathscr{M} \otimes \mathscr{N}$, it follows that $|T(\alpha)| \leq |\alpha|_\pi$ for every $\alpha \in \mathscr{M} \otimes \mathscr{N}$. In particular, we have that

$$|v||w| = \big|\omega(v)\eta(w)\big| = \big|T(v \otimes w)\big| \leq |v \otimes w|_\pi.$$

All in all, we have shown that $|v \otimes w|_\pi = |v||w|$, thus accordingly (4.2) is proved. $\square$

Definition 4.2 (*Projective tensor product*) Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$, $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Then we denote by $\mathscr{M} \otimes_\pi \mathscr{N}$ the normed $L^0(\mathbb{X})$-module $(\mathscr{M} \otimes \mathscr{N}, |\cdot|_\pi)$, where the pointwise norm $|\cdot|_\pi$ is defined as in (4.1). Moreover, the **projective tensor product** of $\mathscr{M}$, $\mathscr{N}$ is the Banach $L^0(\mathbb{X})$-module $\mathscr{M} \hat{\otimes}_\pi \mathscr{N}$ that is defined as the $L^0(\mathbb{X})$-completion of $\mathscr{M} \otimes_\pi \mathscr{N}$.

Let us now consider the projective tensor product of homomorphisms of Banach $L^0(\mathbb{X})$-modules:

Proposition 4.3 (Projective tensor products of homomorphisms) *Let $\mathbb{X}$ be a σ-finite measure space. Let $T : \mathscr{M} \to \tilde{\mathscr{M}}$ and $S : \mathscr{N} \to \tilde{\mathscr{N}}$ be homomorphisms of Banach $L^0(\mathbb{X})$-modules. Then there exists a unique homomorphism of Banach $L^0(\mathbb{X})$-modules $T \otimes_\pi S : \mathscr{M} \hat{\otimes}_\pi \mathscr{N} \to \tilde{\mathscr{M}} \hat{\otimes}_\pi \tilde{\mathscr{N}}$ with*

$$(T \otimes_\pi S)(v \otimes w) = T(v) \otimes S(w) \quad \textit{for every } v \in \mathscr{M} \textit{ and } w \in \mathscr{N}.$$

Moreover, it holds that $|T \otimes_\pi S| = |T||S|$.

Proof By virtue of Lemma 2.2, there exists a unique $L^0(\mathbb{X})$-linear operator $T \otimes S : \mathscr{M} \otimes \mathscr{N} \to \tilde{\mathscr{M}} \otimes \tilde{\mathscr{N}}$ such that $(T \otimes S)(v \otimes w) = T(v) \otimes S(w)$ for every $(v, w) \in \mathscr{M} \times \mathscr{N}$. If $\alpha = \sum_{i=1}^n v_i \otimes w_i \in \mathscr{M} \otimes \mathscr{N}$, then

$$\big|(T \otimes S)(\alpha)\big|_\pi = \left|\sum_{i=1}^n T(v_i) \otimes S(w_i)\right|_\pi \leq \sum_{i=1}^n |T(v_i)||S(w_i)| \leq |T||S| \sum_{i=1}^n |v_i||w_i|.$$

By passing to the infimum over all possible representations of α, we obtain that $\big|(T \otimes S)(\alpha)\big|_\pi \leq |T||S||\alpha|_\pi$. It follows that the operator $T \otimes S$ can be uniquely extended to a homomorphism of Banach $L^0(\mathbb{X})$-modules

$$T \otimes_\pi S : \mathscr{M} \hat{\otimes}_\pi \mathscr{N} \to \tilde{\mathscr{M}} \hat{\otimes}_\pi \tilde{\mathscr{N}}$$

satisfying $|T \otimes_\pi S| \leq |T||S|$. Finally, we have that

$$\begin{aligned}
|T||S| &= \bigvee_{v \in \mathbb{S}_{\mathscr{M}}} \bigvee_{w \in \mathbb{S}_{\mathscr{N}}} |T(v)||S(w)| = \bigvee_{v \in \mathbb{S}_{\mathscr{M}}} \bigvee_{w \in \mathbb{S}_{\mathscr{N}}} \big|T(v) \otimes S(w)\big|_\pi \\
&= \bigvee_{v \in \mathbb{S}_{\mathscr{M}}} \bigvee_{w \in \mathbb{S}_{\mathscr{N}}} \big|(T \otimes_\pi S)(v \otimes w)\big|_\pi \leq |T \otimes_\pi S| \bigvee_{v \in \mathbb{S}_{\mathscr{M}}} \bigvee_{w \in \mathbb{S}_{\mathscr{N}}} |v \otimes w|_\pi \\
&= |T \otimes_\pi S| \bigvee_{v \in \mathbb{S}_{\mathscr{M}}} \bigvee_{w \in \mathbb{S}_{\mathscr{N}}} |v||w| \leq |T \otimes_\pi S|.
\end{aligned}$$

Consequently, the identity $|T \otimes_\pi S| = |T||S|$ is proved. $\square$

One can easily check that $L^0(\mathbb{X})\hat{\otimes}_\pi L^0(\mathbb{X}) = L^0(\mathbb{X})\otimes_\pi L^0(\mathbb{X}) \cong L^0(\mathbb{X})$ as Banach $L^0(\mathbb{X})$-modules via the isomorphism

$$L^0(\mathbb{X}) \otimes_\pi L^0(\mathbb{X}) \ni \sum_{i=1}^n f_i \otimes g_i \mapsto \sum_{i=1}^n f_i g_i \in L^0(\mathbb{X}).$$

In particular, up to this identification, we have that

$$\omega \otimes_\pi \eta \in (\mathscr{M}\hat{\otimes}_\pi \mathscr{N})^*, \quad |\omega \otimes_\pi \eta| = |\omega||\eta| \quad \text{for every } \omega \in \mathscr{M}^* \text{ and } \eta \in \mathscr{N}^*.$$

Lemma 4.4 *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}, \tilde{\mathscr{M}}, \mathscr{N}, \tilde{\mathscr{N}}$ be Banach $L^0(\mathbb{X})$-modules. Let $T\colon \mathscr{M} \to \tilde{\mathscr{M}}$ and $S\colon \mathscr{N} \to \tilde{\mathscr{N}}$ be quotient operators. Then $T \otimes_\pi S\colon \mathscr{M}\hat{\otimes}_\pi \mathscr{N} \to \tilde{\mathscr{M}}\hat{\otimes}_\pi \tilde{\mathscr{N}}$ is a quotient operator.*

Proof By Remark 3.2, it suffices to prove that $T \otimes S\colon \mathscr{M} \otimes_\pi \mathscr{N} \to \tilde{\mathscr{M}} \otimes_\pi \tilde{\mathscr{N}}$ is a quotient operator. Given any $\beta = \sum_{i=1}^n \tilde{v}_i \otimes \tilde{w}_i \in \tilde{\mathscr{M}} \otimes \tilde{\mathscr{N}}$, we can exploit the surjectivity of T and S to find $(v_i)_{i=1}^n \subseteq \mathscr{M}$ and $(w_i)_{i=1}^n \subseteq \mathscr{N}$ such that $\tilde{v}_i = T(v_i)$ and $\tilde{w}_i = S(w_i)$ for all $i = 1, \ldots, n$, whence it follows that $\beta = \sum_{i=1}^n T(v_i) \otimes S(w_i) = (T \otimes S)\big(\sum_{i=1}^n v_i \otimes w_i\big)$. This shows that $T \otimes S$ is a surjective operator. Moreover, for any tensor $\beta \in \tilde{\mathscr{M}} \otimes_\pi \tilde{\mathscr{N}}$ we can estimate

$$\begin{aligned}|\beta|_\pi &= \bigwedge_{\alpha\in(T\otimes S)^{-1}(\beta)} |(T \otimes S)(\alpha)|_\pi \leq |T||S| \bigwedge_{\alpha\in(T\otimes S)^{-1}(\beta)} |\alpha|_\pi \\ &\overset{(3.1)}{\leq} \bigwedge_{\alpha\in(T\otimes S)^{-1}(\beta)} |\alpha|_\pi .\end{aligned}$$

In order to prove the converse inequality, fix $\varepsilon \in (0, 1)$. We can thus find a partition $(E_k)_{k\in\mathbb{N}} \subseteq \Sigma$ of X and, for any $k \in \mathbb{N}$, a number $n_k \in \mathbb{N}$ and elements $(\tilde{v}_i^k)_{i=1}^{n_k} \subseteq \tilde{\mathscr{M}}$, $(\tilde{w}_i^k)_{i=1}^{n_k} \subseteq \tilde{\mathscr{N}}$ such that

$$\beta = \sum_{i=1}^{n_k} \tilde{v}_i^k \otimes \tilde{w}_i^k, \qquad \mathbb{1}_{E_k} \sum_{i=1}^{n_k} |\tilde{v}_i^k||\tilde{w}_i^k| \leq \mathbb{1}_{E_k}(|\beta|_\pi + \varepsilon).$$

Moreover, we can find $(v_i^k)_{i=1}^{n_k} \subseteq \mathscr{M}$ and $(w_i^k)_{i=1}^{n_k} \subseteq \mathscr{N}$, with $T(v_i^k) = \tilde{v}_i^k$ and $S(w_i^k) = \tilde{w}_i^k$ for every $i = 1, \ldots, n_k$, such that $|v_i^k| \leq (1+\varepsilon)|\tilde{v}_i^k|$ and $|w_i^k| \leq (1+\varepsilon)|\tilde{w}_i^k|$. Therefore, we have that

$$\begin{aligned}\mathbb{1}_{E_k}\left|\sum_{i=1}^{n_k} v_i^k \otimes w_i^k\right|_\pi &\leq \mathbb{1}_{E_k} \sum_{i=1}^{n_k} |v_i^k||w_i^k| \leq (1+\varepsilon)^2 \mathbb{1}_{E_k}(|\beta|_\pi + \varepsilon) \\ &\leq \mathbb{1}_{E_k}|\beta|_\pi + \mathbb{1}_{E_k}(3|\beta|_\pi + 4)\varepsilon.\end{aligned}$$

Since $(T \otimes S)\big(\mathbb{1}_{E_k} \cdot \sum_{i=1}^{n_k} v_i^k \otimes w_i^k\big) = \mathbb{1}_{E_k} \cdot \beta$ for every $k \in \mathbb{N}$, we deduce that

$$\bigwedge_{\alpha \in (T\otimes S)^{-1}(\beta)} |\alpha|_\pi \leq \sum_{k\in\mathbb{N}} \mathbb{1}_{E_k} \left| \sum_{i=1}^{n_k} v_i^k \otimes w_i^k \right|_\pi \leq |\beta|_\pi + (3|\beta|_\pi + 4)\varepsilon.$$

By the arbitrariness of ε, we can conclude that $\bigwedge_{\alpha \in (T\otimes S)^{-1}(\beta)} |\alpha|_\pi \leq |\beta|_\pi$. □

Lemma 4.5 *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$, $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Let $G \subseteq \mathscr{M}$ and $H \subseteq \mathscr{N}$ be generating subsets. Then it holds that the set $\{v \otimes w \mid v \in G,\ w \in H\}$ generates $\mathscr{M} \hat{\otimes}_\pi \mathscr{N}$.*

Proof As the linear span of the elementary tensors is dense in $\mathscr{M} \hat{\otimes}_\pi \mathscr{N}$, it suffices to check that any given $v \otimes w$ with $v \in \mathscr{M}$ and $w \in \mathscr{N}$ can be approximated by elements of the $L^0(\mathbb{X})$-module generated by

$$\big\{\tilde{v} \otimes \tilde{w} \;\big|\; \tilde{v} \in G,\ \tilde{w} \in H\big\}.$$

Since G and H generate $\mathscr{M}$ and $\mathscr{N}$, respectively, we can find $(v_n)_{n\in\mathbb{N}} \subseteq \mathscr{M}$ and $(w_n)_{n\in\mathbb{N}} \subseteq \mathscr{N}$ that are $L^0(\mathbb{X})$-linear combinations of elements of G and H, respectively, such that $|v_n - v| \to 0$ and $|w_n - w| \to 0$ in the $\mathfrak{m}$-a.e. sense. Then

$$\begin{aligned} |v \otimes w - v_n \otimes w_n|_\pi &\leq |v \otimes w - v_n \otimes w|_\pi + |v_n \otimes w - v_n \otimes w_n|_\pi \\ &\overset{(4.2)}{=} |v - v_n||w| + |v_n||w - w_n| \to 0 \end{aligned}$$

in the $\mathfrak{m}$-a.e. sense. In particular, $v_n \otimes w_n \to v \otimes w$ in $\mathscr{M} \hat{\otimes}_\pi \mathscr{N}$. The statement follows. □

Generalising the fact that $\ell_1(I)\hat{\otimes}_\pi \mathbb{B} \cong \ell_1(I, \mathbb{B})$ holds for every Banach space $\mathbb{B}$, we have the following:

Theorem 4.6 *Let $\mathbb{X}$ be a σ-finite measure space, $\mathscr{M}$ a Banach $L^0(\mathbb{X})$-module, and $I \neq \varnothing$ an index family. Then the unique linear continuous operator $\mathfrak{i}\colon L^0(\mathbb{X}; \ell_1(I))\hat{\otimes}_\pi \mathscr{M} \to \ell_1(I, \mathscr{M})$ satisfying*

$$\mathfrak{i}(a \otimes v) = \big(a(\cdot)_i \cdot v\big)_{i\in I} \quad \textit{for every } a \in L^0(\mathbb{X}; \ell_1(I)) \textit{ and } v \in \mathscr{M} \tag{4.3}$$

is an isomorphism of Banach $L^0(\mathbb{X})$-modules.

Proof First, notice that $L^0(\mathbb{X}; \ell_1(I)) \times \mathscr{M} \ni (a, v) \mapsto \big(a(\cdot)_i \cdot v\big)_{i\in I} \in \ell_1(I, \mathscr{M})$ is well-defined and $L^0(\mathbb{X})$-bilinear, thus we can consider its $L^0(\mathbb{X})$-linearisation $\mathfrak{i}\colon L^0(\mathbb{X}; \ell_1(I)) \otimes \mathscr{M} \to \ell_1(I, \mathscr{M})$, i.e.

$$\mathfrak{i}(\alpha) = \left(\sum_{j=1}^{n} a_j(\cdot)_i \cdot v_j\right)_{i\in I} \qquad \text{for every } \alpha = \sum_{j=1}^{n} a_j \otimes v_j \in L^0(\mathbb{X}; \ell_1(I)) \otimes \mathscr{M}.$$

Observe that $\mathfrak{i}$ is the unique linear operator from $L^0(\mathbb{X};\ell_1(I))\otimes\mathscr{M}$ to $\ell_1(I,\mathscr{M})$ satisfying (4.3).

On the one hand, given any tensor $\alpha=\sum_{j=1}^n a_j\otimes v_j\in L^0(\mathbb{X};\ell_1(I))\otimes\mathscr{M}$ we can estimate

$$
\begin{aligned}
|\mathfrak{i}(\alpha)|_1 &= \sum_{i\in I}\left|\sum_{j=1}^n a_j(\cdot)_i\cdot v_j\right| \leq \sum_{i\in I}\sum_{j=1}^n |a_j(\cdot)_i||v_j| = \sum_{j=1}^n\left(\sum_{i\in I}|a_j(\cdot)_i|\right)|v_j| \\
&\overset{(3.10)}{=} \sum_{j=1}^n |a_j||v_j|.
\end{aligned}
$$

By passing to the infimum over all representations of α, we deduce that $|\mathfrak{i}(\alpha)|_1\leq|\alpha|_\pi$. On the other hand, if α is written as $\sum_{j=1}^n a_j\otimes v_j$, then we claim that the elements $w_i:=\sum_{j=1}^n a_j(\cdot)_i\cdot v_j\in\mathscr{M}$ satisfy the following property: the family $\{\underline{\mathsf{e}}_i\otimes w_i\}_{i\in I}$ is summable in $L^0(\mathbb{X};\ell_1(I))\hat{\otimes}_\pi\mathscr{M}$ and

$$
\sum_{i\in I}\underline{\mathsf{e}}_i\otimes w_i=\alpha. \tag{4.4}
$$

In order to prove it, let us first notice that

$$
\underline{\mathsf{e}}_i\otimes w_i=\underline{\mathsf{e}}_i\otimes\left(\sum_{j=1}^n a_j(\cdot)_i\cdot v_j\right)=\sum_{j=1}^n a_j(\cdot)_i\cdot(\underline{\mathsf{e}}_i\otimes v_j)=\sum_{j=1}^n\big(a_j(\cdot)_i\cdot\underline{\mathsf{e}}_i\big)\otimes v_j. \tag{4.5}
$$

Since $L^0(\mathbb{X};\ell_1(I))\ni s\mapsto s\otimes v_j\in L^0(\mathbb{X};\ell_1(I))\hat{\otimes}_\pi\mathscr{M}$ is a homomorphism of Banach $L^0(\mathbb{X})$-modules,

$$
\begin{aligned}
\alpha\sum\nolimits_{j=1}^{=\ n} a_j\otimes v_j &\overset{(3.9)}{=} \sum_{j=1}^n\left(\sum_{i\in I}a_j(\cdot)_i\cdot\underline{\mathsf{e}}_i\right)\otimes v_j \\
&\overset{(3.7)}{=} \sum_{i\in I}\left(\sum_{j=1}^n\big(a_j(\cdot)_i\cdot\underline{\mathsf{e}}_i\big)\otimes v_j\right)\overset{(4.5)}{=}\sum_{i\in I}\underline{\mathsf{e}}_i\otimes w_i.
\end{aligned}
$$

This proves the validity of the claim (4.4). By taking Remark 3.7 into account, we conclude that

$$
\begin{aligned}
|\alpha|_\pi &= \left|\sum_{i\in I}\underline{\mathsf{e}}_i\otimes w_i\right|_\pi \leq\sum_{i\in I}|\underline{\mathsf{e}}_i\otimes w_i|_\pi\overset{(4.2)}{=}\sum_{i\in I}|\underline{\mathsf{e}}_i||w_i|=\sum_{i\in I}|w_i|=\big|(w_i)_{i\in I}\big|_1 \\
&=|\mathfrak{i}(\alpha)|_1.
\end{aligned}
$$

All in all, we have shown that $|\mathfrak{i}(\alpha)|_1 = |\alpha|_\pi$ for every $\alpha \in L^0(\mathbb{X}; \ell_1(I)) \otimes \mathscr{M}$. Therefore, the map $\mathfrak{i}$ can be uniquely extended to a homomorphism of Banach $L^0(\mathbb{X})$-modules from $L^0(\mathbb{X}; \ell_1(I))\hat{\otimes}_\pi \mathscr{M}$ to $\ell_1(I, \mathscr{M})$, which we still denote with the symbol $\mathfrak{i}$. Notice that the extension $\mathfrak{i}$ preserves the pointwise norm.

To conclude, it remains to check that $\mathfrak{i}\colon L^0(\mathbb{X}; \ell_1(I))\hat{\otimes}_\pi \mathscr{M} \to \ell_1(I, \mathscr{M})$ is surjective. Let $v = (v_i)_{i\in I} \in \ell_1(I, \mathscr{M})$ be fixed. Thanks to Proposition 3.6, it follows from the estimates

$$\bigwedge_{F\in\mathscr{P}_f(I)} \bigvee_{G\in\mathscr{P}_f(I\setminus F)} \left|\sum_{i\in G} \underline{\mathsf{e}}_i \otimes v_i\right| \le \bigwedge_{F\in\mathscr{P}_f(I)} \bigvee_{G\in\mathscr{P}_f(I\setminus F)} \sum_{i\in G} |v_i| = 0$$

that $\{\underline{\mathsf{e}}_i \otimes v_i\}_{i\in I}$ is summable in $L^0(\mathbb{X}; \ell_1(I))\hat{\otimes}_\pi \mathscr{M}$. Letting $\alpha := \sum_{i\in I} \underline{\mathsf{e}}_i \otimes v_i$, we have that

$$\begin{aligned} v &\overset{(3.5)}{=} \sum_{i\in I} (\delta_{ij} v_i)_{j\in I} = \sum_{i\in I} (\underline{\mathsf{e}}_i(\cdot)_j \cdot v_i)_{j\in I} \overset{(4.3)}{=} \sum_{i\in I} \mathfrak{i}(\underline{\mathsf{e}}_i \otimes v_i) \overset{(3.7)}{=} \mathfrak{i}\left(\sum_{i\in I} \underline{\mathsf{e}}_i \otimes v_i\right) \\ &= \mathfrak{i}(\alpha), \end{aligned}$$

whence it follows that $\mathfrak{i}$ is surjective. Consequently, the proof of the statement is complete. $\square$

Remark 4.7 Under the assumptions of Theorem 4.6, for any $i \in I$ we define the operator ι_i as

$$\begin{aligned} \iota_i : \mathscr{M} &\longrightarrow L^0(\mathbb{X}; \ell_1(I))\hat{\otimes}_\pi \mathscr{M} \\ v &\longmapsto \underline{\mathsf{e}}_i \otimes v. \end{aligned}$$

Combining Theorem 4.6 with [19, Theorem 3.12], we obtain that

$$\big(L^0(\mathbb{X}; \ell_1(I))\hat{\otimes}_\pi \mathscr{M}, \{\iota_i\}_{i\in I}\big)$$

is the coproduct of $\{\mathscr{M}_i\}_{i\in I}$, where $\mathscr{M}_i := \mathscr{M}$ for every $i \in I$, in the category $\mathbf{BanMod}^1_{\mathbb{X}}$.

Lemma 4.8 *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$ be a Banach $L^0(\mathbb{X})$-module. Define*

$$\varphi(f) := \sum_{v\in\mathbb{S}_{\mathscr{M}}} f_v \cdot v \in \mathscr{M} \quad \textit{for every } f = (f_v)_{v\in\mathbb{S}_{\mathscr{M}}} \in \ell_1(\mathbb{S}_{\mathscr{M}}, L^0(\mathbb{X})). \tag{4.6}$$

Then $\varphi\colon \ell_1(\mathbb{S}_{\mathscr{M}}, L^0(\mathbb{X})) \to \mathscr{M}$ is a quotient operator. In particular, it holds that $\mathscr{M} \cong \ell_1(\mathbb{S}_{\mathscr{M}}, L^0(\mathbb{X}))/\ker(\varphi)$.

Proof First of all, by using Proposition 3.6 we obtain that

$$\bigwedge_{F\in\mathscr{P}_f(\mathbb{S}_{\mathscr{M}})}\ \bigvee_{G\in\mathscr{P}_f(\mathbb{S}_{\mathscr{M}}\setminus F)}\left|\sum_{v\in G} f_v\cdot v\right| \leq \bigwedge_{F\in\mathscr{P}_f(\mathbb{S}_{\mathscr{M}})}\ \bigvee_{G\in\mathscr{P}_f(\mathbb{S}_{\mathscr{M}}\setminus F)}\left|\sum_{v\in G} f_v\right| = 0$$

and thus that $(f_v\cdot v)_{v\in\mathbb{S}_{\mathscr{M}}}$ is summable in $\mathscr{M}$. Since

$$\left|\sum_{v\in\mathbb{S}_{\mathscr{M}}} f_v\cdot v\right| \leq \sum_{v\in\mathbb{S}_{\mathscr{M}}} |f_v| = |f|_1,$$

we have that φ is a well-defined linear operator satisfying $|\varphi(f)|\leq|f|_1$ for every $f\in\ell_1(\mathbb{S}_{\mathscr{M}},L^0(\mathbb{X}))$, thus in particular it is a homomorphism of Banach $L^0(\mathbb{X})$-modules. Moreover, if $w\in\mathscr{M}$ is given, then

$$f^w=(f^w_v)_{v\in\mathbb{S}_{\mathscr{M}}}\in\ell_1(\mathbb{S}_{\mathscr{M}},L^0(\mathbb{X})),\quad f^w_v:=\begin{cases}|w| & \text{if } v=\mathrm{sgn}(w),\\ 0 & \text{otherwise}\end{cases}$$

satisfies $\varphi(f^w)=|w|\cdot\mathrm{sgn}(w)=w$ and $|\varphi(f^w)|=|w|=|f^w|_1$. Hence, φ is a quotient operator, thus it induces an isomorphism of Banach $L^0(\mathbb{X})$-modules between $\ell_1(\mathbb{S}_{\mathscr{M}},L^0(\mathbb{X}))/\ker(\varphi)$ and $\mathscr{M}$. □

We conclude this section with a useful representation formula for the projective pointwise norm:

Theorem 4.9 (Characterisation of the projective pointwise norm) *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$, $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Then for every $\alpha\in\mathscr{M}\hat{\otimes}_\pi\mathscr{N}$ it holds that*

$$|\alpha|_\pi=\bigwedge\left\{\sum_{n\in\mathbb{N}}|v_n||w_n|\;\middle|\;(v_n\otimes w_n)_{n\in\mathbb{N}}\in\ell_1(\mathbb{N},\mathscr{M}\hat{\otimes}_\pi\mathscr{N}),\ \alpha=\sum_{n\in\mathbb{N}}v_n\otimes w_n\right\}. \tag{4.7}$$

Proof For brevity, we denote by $q(\alpha)$ the right-hand side of (4.7). On the one hand, if $(v_n\otimes w_n)_{n\in\mathbb{N}}\in\ell_1(\mathbb{N},\mathscr{M}\hat{\otimes}_\pi\mathscr{N})$ and $\alpha=\sum_{n\in\mathbb{N}}v_n\otimes w_n$, then

$$|\alpha|_\pi=\left|\sum_{n\in\mathbb{N}}v_n\otimes w_n\right|_\pi\overset{(3.6)}{\leq}\sum_{n\in\mathbb{N}}|v_n||w_n|,$$

whence it follows that $|\alpha|_\pi\leq q(\alpha)$ for every $\alpha\in\mathscr{M}\hat{\otimes}_\pi\mathscr{N}$. On the other hand, let us denote by

$$\begin{aligned}\varphi&\colon\ell_1(\mathbb{S}_{\mathscr{M}},L^0(\mathbb{X}))\to\mathscr{M},\\ \phi&\colon L^0(\mathbb{X};\ell_1(\mathbb{S}_{\mathscr{M}}))\to\ell_1(\mathbb{S}_{\mathscr{M}},L^0(\mathbb{X})),\\ \mathtt{i}&\colon L^0(\mathbb{X};\ell_1(\mathbb{S}_{\mathscr{M}}))\hat{\otimes}_\pi\mathscr{N}\to\ell_1(\mathbb{S}_{\mathscr{M}},\mathscr{N})\end{aligned}$$

the operators given by Lemma 4.8, Corollary 3.12, and Theorem 4.6, respectively. Recall that φ is a quotient operator, while ϕ and i are isomorphisms of Banach $L^0(\mathbb{X})$-modules. In particular, $\tilde{\varphi}:=\varphi \circ \phi \colon L^0(\mathbb{X};\ell_1(\mathbb{S}_{\mathscr{M}})) \to \mathscr{M}$ is a quotient operator, so that accordingly

$$\psi:=(\tilde{\varphi} \otimes_\pi \mathrm{id}_{\mathscr{N}}) \circ \mathrm{i}^{-1} \colon \ell_1(\mathbb{S}_{\mathscr{M}},\mathscr{N}) \to \mathscr{M}\hat{\otimes}_\pi\mathscr{N} \quad \text{is a quotient operator}$$

by Lemma 4.4. Hence, for any $\alpha \in \mathscr{M}\hat{\otimes}_\pi\mathscr{N}$ and $\varepsilon > 0$ we can find an element $w = (w_v)_{v\in\mathbb{S}_{\mathscr{M}}} \in \ell_1(\mathbb{S}_{\mathscr{M}},\mathscr{N})$ such that $\psi(w) = \alpha$ and $|w|_1 \leq |\alpha|_\pi + \varepsilon$. Since

$$\bigvee_{F\in\mathscr{P}_f(\mathbb{S}_{\mathscr{M}})} \sum_{v\in F} |v \otimes w_v|_\pi \leq \bigvee_{F\in\mathscr{P}_f(\mathbb{S}_{\mathscr{M}})} \sum_{v\in F} |w_v| \in L^0(\mathbb{X})^+,$$

we see that $(v \otimes w_v)_{v\in\mathbb{S}_{\mathscr{M}}} \in \ell_1(\mathbb{S}_{\mathscr{M}},\mathscr{M}\hat{\otimes}_\pi\mathscr{N})$. By unwrapping the various definitions, we obtain

$$\begin{aligned}
\alpha &= \psi\big((w_v)_{v\in\mathbb{S}_{\mathscr{M}}}\big) \overset{(3.5)}{=} \psi\left(\sum_{v\in\mathbb{S}_{\mathscr{M}}} (\delta_{vu} w_v)_{u\in\mathbb{S}_{\mathscr{M}}}\right) \overset{(3.7)}{=} \sum_{v\in\mathbb{S}_{\mathscr{M}}} \psi\big((\delta_{vu} w_v)_{u\in\mathbb{S}_{\mathscr{M}}}\big) \\
&= \sum_{v\in\mathbb{S}_{\mathscr{M}}} \psi\big((\underline{\mathrm{e}}_v(\cdot)_u \cdot w_v)_{u\in\mathbb{S}_{\mathscr{M}}}\big) \overset{(4.3)}{=} \sum_{v\in\mathbb{S}_{\mathscr{M}}} (\tilde{\varphi} \otimes_\pi \mathrm{id}_{\mathscr{N}})(\underline{\mathrm{e}}_v \otimes w_v) \\
&= \sum_{v\in\mathbb{S}_{\mathscr{M}}} \varphi\big(\phi(\underline{\mathrm{e}}_v)\big) \otimes w_v \overset{(3.11)}{=} \sum_{v\in\mathbb{S}_{\mathscr{M}}} \varphi\big((\delta_{vu}\mathbb{1}_{\mathrm{X}})_{u\in\mathbb{S}_{\mathscr{M}}}\big) \otimes w_v \\
&\overset{(4.6)}{=} \sum_{v\in\mathbb{S}_{\mathscr{M}}} \left(\sum_{u\in\mathbb{S}_{\mathscr{M}}} \delta_{vu} u\right) \otimes w_v = \sum_{v\in\mathbb{S}_{\mathscr{M}}} v \otimes w_v.
\end{aligned}$$

It follows that there exists $(v_n)_{n\in\mathbb{N}} \subseteq \mathbb{S}_{\mathscr{M}}$ such that, letting $w_n:=w_{v_n}$ for every $n \in \mathbb{N}$, we have $(v_n \otimes w_n)_{n\in\mathbb{N}} \in \ell_1(\mathbb{N},\mathscr{M}\hat{\otimes}_\pi\mathscr{N})$, $\alpha = \sum_{n\in\mathbb{N}} v_n \otimes w_n$, and $\sum_{n\in\mathbb{N}} |v_n||w_n| = |w|_1 \leq |\alpha|_\pi + \varepsilon$. Therefore, we proved that $q(\alpha) \leq |\alpha|_\pi + \varepsilon$. By letting $\varepsilon \searrow 0$, we conclude that $|\alpha|_\pi = q(\alpha)$. $\square$

4.2 Relation with duals and pullbacks

In order to provide a characterisation of the dual of the projective tensor product in Theorem 4.11, we need to apply the following universal property:

Theorem 4.10 (Universal property of the projective tensor product) *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$, $\mathscr{N}$, $\mathscr{Q}$ be Banach $L^0(\mathbb{X})$-modules. Then for any $b \in$* $\mathrm{B}(\mathscr{M},\mathscr{N};\mathscr{Q})$ *there exists a unique $\tilde{b}_\pi \in$* $\mathrm{H{\small OM}}(\mathscr{M}\hat{\otimes}_\pi\mathscr{N};\mathscr{Q})$ *for which the following* *diagram commutes:*

$$\begin{tikzcd} \mathscr{M} \times \mathscr{N} \arrow[r, "b"] \arrow[d, "\otimes"'] & \mathscr{Q} \\ \mathscr{M} \hat{\otimes}_\pi \mathscr{N} \arrow[ur, "\tilde{b}_\pi"'] & \end{tikzcd}$$

Also, $\mathrm{B}(\mathscr{M}, \mathscr{N}; \mathscr{Q}) \ni b \mapsto \tilde{b}_\pi \in \mathrm{H{\small OM}}(\mathscr{M} \hat{\otimes}_\pi \mathscr{N}; \mathscr{Q})$ *is an isomorphism of Banach* $L^0(\mathbb{X})$*-modules.*

Proof Let $b \in \mathrm{B}(\mathscr{M}, \mathscr{N}; \mathscr{Q})$ be fixed. Denote by $\tilde{b} \colon \mathscr{M} \otimes \mathscr{N} \to \mathscr{Q}$ the $L^0(\mathbb{X})$-linearisation of b given by Lemma 2.2. For any $\alpha = \sum_{i=1}^n v_i \otimes w_i \in \mathscr{M} \otimes \mathscr{N}$, we can estimate

$$|\tilde{b}(\alpha)| \le \sum_{i=1}^n \big|\tilde{b}(v_i \otimes w_i)\big| = \sum_{i=1}^n |b(v_i, w_i)| \le |b| \sum_{i=1}^n |v_i||w_i|.$$

By taking the infimum over all representations of α, we get $|\tilde{b}(\alpha)| \le |b||\alpha|_\pi$, whence it follows that $\tilde{b} \in \mathrm{H{\small OM}}(\mathscr{M} \otimes_\pi \mathscr{N}; \mathscr{Q})$ and $|\tilde{b}| \le |b|$. Letting $\tilde{b}_\pi$ be the unique element of $\mathrm{H{\small OM}}(\mathscr{M} \hat{\otimes}_\pi \mathscr{N}; \mathscr{Q})$ extending $\tilde{b}$, we have $|\tilde{b}_\pi| = |\tilde{b}| \le |b|$. On the other hand, we have that

$$|b(v, w)| = \big|\tilde{b}_\pi(v \otimes w)\big| \le |\tilde{b}_\pi||v \otimes w|_\pi = |\tilde{b}_\pi||v||w| \quad \forall (v, w) \in \mathscr{M} \times \mathscr{N},$$

which implies that $|b| \le |\tilde{b}_\pi|$. All in all, we have shown that $|\tilde{b}_\pi| = |b|$. Moreover, the resulting map $\mathrm{B}(\mathscr{M}, \mathscr{N}; \mathscr{Q}) \ni b \mapsto \tilde{b}_\pi \in \mathrm{H{\small OM}}(\mathscr{M} \hat{\otimes}_\pi \mathscr{N}; \mathscr{Q})$ is a homomorphism of Banach $L^0(\mathbb{X})$-modules. In order to conclude, it remains to check that such map is surjective. To this aim, let $T \in \mathrm{H{\small OM}}(\mathscr{M} \hat{\otimes}_\pi \mathscr{N}; \mathscr{Q})$ be fixed. Now define $b^T : \mathscr{M} \times \mathscr{N} \to \mathscr{Q}$ as $b^T(v, w) := T(v \otimes w)$ for every $(v, w) \in \mathscr{M} \times \mathscr{N}$. Then $b^T \in \mathrm{B}(\mathscr{M}, \mathscr{N}; \mathscr{Q})$ by construction and $\tilde{b}^T_\pi = T$ by the uniqueness part of the statement. Therefore, the proof is complete. □

Choosing $\mathscr{Q} := L^0(\mathbb{X})$ in Theorem 4.10, we obtain the following characterisation of $\mathscr{M} \hat{\otimes}_\pi \mathscr{N}$:

Theorem 4.11 (Dual of $\mathscr{M} \hat{\otimes}_\pi \mathscr{N}$) *Let* $\mathbb{X}$ *be a* σ*-finite measure space. Let* $\mathscr{M}$ *and* $\mathscr{N}$ *be Banach* $L^0(\mathbb{X})$*-modules. Then it holds that*

$$(\mathscr{M} \hat{\otimes}_\pi \mathscr{N})^* \cong \mathrm{B}(\mathscr{M}, \mathscr{N}),$$

an isomorphism of Banach $L^0(\mathbb{X})$*-modules being given by the map*

$$\mathrm{B}(\mathscr{M}, \mathscr{N}) \ni b \mapsto \tilde{b}_\pi \in (\mathscr{M} \hat{\otimes}_\pi \mathscr{N})^*.$$

As a consequence of Theorem 4.11, we obtain a useful 'dual representation formula' for $|\cdot|_\pi$:

Corollary 4.12 *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$, $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Then*

$$|\alpha|_\pi = \bigvee \left\{ \tilde{b}_\pi(\alpha) \;\middle|\; b \in \mathrm{B}(\mathscr{M}, \mathscr{N}),\ |b| \le 1 \right\} \quad \textit{for every } \alpha \in \mathscr{M} \hat{\otimes}_\pi \mathscr{N}. \tag{4.8}$$

Proof The statement follows from Theorem 4.11 and the Hahn–Banach theorem for normed L^0-modules. □

We conclude the section by proving that 'pullbacks and projective tensor products commute':

Theorem 4.13 (Pullbacks vs. projective tensor products) *Let $\mathbb{X}$, $\mathbb{Y}$ be separable, σ-finite measure spaces. Let $\varphi \colon \mathrm{X} \to \mathrm{Y}$ be a measurable map such that $\varphi_\# \mathfrak{m}_\mathrm{X} \ll \mathfrak{m}_\mathrm{Y}$. Let $\mathscr{M}$ and $\mathscr{N}$ be Banach $L^0(\mathbb{Y})$-modules. Then it holds that*

$$\varphi^*(\mathscr{M} \hat{\otimes}_\pi \mathscr{N}) \cong (\varphi^* \mathscr{M}) \hat{\otimes}_\pi (\varphi^* \mathscr{N}),$$

the pullback map $\varphi^* \colon \mathscr{M} \hat{\otimes}_\pi \mathscr{N} \to (\varphi^* \mathscr{M}) \hat{\otimes}_\pi (\varphi^* \mathscr{N})$ being the unique homomorphism such that

$$\varphi^*(v \otimes w) = (\varphi^* v) \otimes (\varphi^* w) \quad \textit{for every } v \in \mathscr{M} \textit{ and } w \in \mathscr{N}.$$

Proof First, we define the map $T \colon \mathscr{M} \otimes_\pi \mathscr{N} \to (\varphi^* \mathscr{M}) \otimes_\pi (\varphi^* \mathscr{N})$ as

$$T\left(\sum_{i=1}^n v_i \otimes w_i \right) := \sum_{i=1}^n (\varphi^* v_i) \otimes (\varphi^* w_i) \quad \forall \sum_{i=1}^n v_i \otimes w_i \in \mathscr{M} \otimes_\pi \mathscr{N}.$$

In order to prove that the map T is well-posed, it is sufficient to show that

$$(v_i)_{i=1}^n \subseteq \mathscr{M},\ (w_i)_{i=1}^n \subseteq \mathscr{N},\ \sum_{i=1}^n v_i \otimes w_i = 0 \quad \Longrightarrow \quad \sum_{i=1}^n (\varphi^* v_i) \otimes (\varphi^* w_i) = 0. \tag{4.9}$$

Let $\mathsf{I}_\varphi \colon \varphi^* \mathscr{N}^* \to (\varphi^* \mathscr{N})^*$ be the isometric embedding defined in (2.6). Corollary 3.21 yields

$$\sum_{i=1}^n \mathsf{I}_\varphi(\varphi^* \eta)(\varphi^* w_i) \cdot (\varphi^* v_i) = \sum_{i=1}^n (\eta(w_i) \circ \varphi) \cdot (\varphi^* v_i) = \varphi^* \left(\sum_{i=1}^n \eta(w_i) \cdot v_i \right) = 0$$

for every $\eta \in \mathscr{N}^*$, whence it follows that $\sum_{i=1}^n \mathsf{I}_\varphi(\theta)(\varphi^* w_i) \cdot (\varphi^* v_i) = 0$ for every $\theta \in \mathscr{G}(\varphi^*[\mathscr{N}^*])$. Using Theorem 2.13 and the density of $\mathscr{G}(\varphi^*[\mathscr{N}^*])$ in $\varphi^* \mathscr{N}^*$, we obtain $\sum_{i=1}^n \Theta(\varphi^* w_i) \cdot (\varphi^* v_i) = 0$ for every $\Theta \in (\varphi^* \mathscr{N})^*$, so that $\sum_{i=1}^n (\varphi^* v_i) \otimes (\varphi^* w_i) = 0$ by Corollary 3.21. This proves (4.9).

Observe that T is linear by construction. Moreover, for any $\alpha \in \mathscr{M} \otimes_\pi \mathscr{N}$ we can estimate

$$
\begin{aligned}
|T(\alpha)|_\pi &\leq \bigwedge \left\{ \sum_{i=1}^n |\varphi^* v_i||\varphi^* w_i| \;\middle|\; (v_i)_{i=1}^n \subseteq \mathscr{M},\ (w_i)_{i=1}^n \subseteq \mathscr{N},\ \alpha = \sum_{i=1}^n v_i \otimes w_i \right\} \\
&= \bigwedge \left\{ \left(\sum_{i=1}^n |v_i||w_i| \right) \circ \varphi \;\middle|\; (v_i)_{i=1}^n \subseteq \mathscr{M},\ (w_i)_{i=1}^n \subseteq \mathscr{N},\ \alpha = \sum_{i=1}^n v_i \otimes w_i \right\} \\
&= |\alpha|_\pi \circ \varphi.
\end{aligned}
$$

Now let us pass to the verification of the converse inequality. Given any $b \in \mathrm{B}(\mathscr{M}, \mathscr{N})$, we define

$$
b^\varphi \left(\sum_{i=1}^n \mathbb{1}_{E_i} \cdot \varphi^* v_i, \sum_{j=1}^m \mathbb{1}_{F_j} \cdot \varphi^* w_j \right) := \sum_{i=1}^n \sum_{j=1}^m \mathbb{1}_{E_i \cap F_j}\, b(v_i, w_j) \circ \varphi \in L^0(\mathbb{X})
$$

for every $\sum_{i=1}^n \mathbb{1}_{E_i} \cdot \varphi^* v_i \in \mathscr{G}(\varphi^*[\mathscr{M}])$ and $\sum_{j=1}^m \mathbb{1}_{F_j} \cdot \varphi^* w_j \in \mathscr{G}(\varphi^*[\mathscr{N}])$. Notice that

$$
\begin{aligned}
\left| \sum_{i=1}^n \sum_{j=1}^m \mathbb{1}_{E_i \cap F_j}\, b(v_i, w_j) \circ \varphi \right| &= \sum_{i=1}^n \sum_{j=1}^m \mathbb{1}_{E_i \cap F_j} |b(v_i, w_j)| \circ \varphi \\
&\leq |b| \circ \varphi \left| \sum_{i=1}^n \mathbb{1}_{E_i} \cdot \varphi^* v_i \right| \left| \sum_{j=1}^m \mathbb{1}_{F_j} \cdot \varphi^* w_j \right|.
\end{aligned}
$$

Therefore, $b^\varphi : \mathscr{G}(\varphi^*[\mathscr{M}]) \times \mathscr{G}(\varphi^*[\mathscr{N}]) \to L^0(\mathbb{X})$ can be uniquely extended to an $L^0(\mathbb{X})$-bilinear operator $b^\varphi \in \mathrm{B}(\varphi^* \mathscr{M}, \varphi^* \mathscr{N})$ satisfying $|b^\varphi| \leq |b| \circ \varphi$. Thanks to Corollary 4.12, we deduce that

$$
\begin{aligned}
|\alpha|_\pi \circ \varphi &= \bigvee \left\{ \tilde{b}_\pi(\alpha) \circ \varphi \;\middle|\; b \in \mathrm{B}(\mathscr{M}, \mathscr{N}),\ |b| \leq 1 \right\} \\
&= \bigvee \left\{ \sum_{i=1}^n b^\varphi(\varphi^* v_i, \varphi^* w_i) \;\middle|\; b \in \mathrm{B}(\mathscr{M}, \mathscr{N}),\ |b| \leq 1 \right\} \\
&\leq \bigvee \left\{ \tilde{B}_\pi(T(\alpha)) \;\middle|\; B \in \mathrm{B}(\varphi^* \mathscr{M}, \varphi^* \mathscr{N}),\ |B| \leq 1 \right\} = |T(\alpha)|_\pi
\end{aligned}
$$

for every tensor $\alpha = \sum_{i=1}^n v_i \otimes w_i \in \mathscr{M} \otimes_\pi \mathscr{N}$. All in all, we have shown that $|T(\alpha)|_\pi = |\alpha|_\pi \circ \varphi$ for every $\alpha \in \mathscr{M} \otimes_\pi \mathscr{N}$. It follows that T can be uniquely extended to a linear operator

$$
\varphi^* : \mathscr{M} \hat{\otimes}_\pi \mathscr{N} \to (\varphi^* \mathscr{M}) \hat{\otimes}_\pi (\varphi^* \mathscr{N})
$$

satisfying $|\varphi^* \alpha|_\pi = |\alpha|_\pi \circ \varphi$ for every $\alpha \in \mathscr{M} \hat{\otimes}_\pi \mathscr{N}$. Finally, it remains to check that $\varphi^*[\mathscr{M} \hat{\otimes}_\pi \mathscr{N}]$ generates $(\varphi^* \mathscr{M}) \hat{\otimes}_\pi (\varphi^* \mathscr{N})$. Given any $v \in \mathscr{M}$ and $w \in \mathscr{N}$, we have $(\varphi^* v) \otimes (\varphi^* w) = \varphi^*(v \otimes w)$. This shows that

$$S := \left\{ (\varphi^* v) \otimes (\varphi^* w) \;\middle|\; v \in \mathscr{M},\ w \in \mathscr{N} \right\} \subseteq \varphi^*[\mathscr{M} \hat{\otimes}_\pi \mathscr{N}].$$

Given that $\varphi^*[\mathscr{M}]$ and $\varphi^*[\mathscr{N}]$ generate $\varphi^*\mathscr{M}$ and $\varphi^*\mathscr{N}$, respectively, we know from Lemma 4.5 that S generates the space $(\varphi^*\mathscr{M}) \hat{\otimes}_\pi (\varphi^*\mathscr{N})$, thus a fortiori $\varphi^*[\mathscr{M} \hat{\otimes}_\pi \mathscr{N}]$ generates $(\varphi^*\mathscr{M}) \hat{\otimes}_\pi (\varphi^*\mathscr{N})$. □

4.3 Other consequences of the universal property

Let us now discuss other consequences of Theorem 4.10 and Corollary 4.12. Our first goal is to rewrite (4.8) in a different fashion.

Proposition 4.14 *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$, $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Then the map sending b to $v \to b(v, \cdot)$ is an isomorphism of Banach $L^0(\mathbb{X})$-modules from $\mathrm{B}(\mathscr{M}, \mathscr{N})$ to $\mathrm{H{\scriptsize OM}}(\mathscr{M}; \mathscr{N}^*)$. In particular, it holds that*

$$\mathrm{B}(\mathscr{M}, \mathscr{N}) \cong \mathrm{H{\scriptsize OM}}(\mathscr{M}; \mathscr{N}^*).$$

Proof One can readily check that the map

$$\varphi \colon b \mapsto \big(\mathscr{M} \ni v \to b(v, \cdot) \in \mathscr{N}^* \big)$$

is a homomorphism of Banach $L^0(\mathbb{X})$-modules between the spaces $\mathrm{B}(\mathscr{M}, \mathscr{N})$ and $\mathrm{H{\scriptsize OM}}(\mathscr{M}; \mathscr{N}^*)$. For any $b \in \mathrm{B}(\mathscr{M}, \mathscr{N})$, we have that

$$|\varphi(b)| = \bigvee_{v \in \mathscr{M}} \frac{\mathbb{1}_{\{|v|>0\}} |b(v, \cdot)|}{|v|} = \bigvee_{v \in \mathscr{M}} \bigvee_{w \in \mathscr{N}} \frac{\mathbb{1}_{\{|v|>0\}} \mathbb{1}_{\{|w|>0\}} |b(v, w)|}{|v||w|} = |b|.$$

Finally, we check that φ is surjective. For any $T \in \mathrm{H{\scriptsize OM}}(\mathscr{M}; \mathscr{N}^*)$, we define $b^T : \mathscr{M} \times \mathscr{N} \to L^0(\mathbb{X})$ as $b^T(v, w) := T(v)(w)$ for every $v \in \mathscr{M}$ and $w \in \mathscr{N}$. Then $b^T \in \mathrm{B}(\mathscr{M}, \mathscr{N})$ and $\varphi(b^T) = T$. □

Similarly, we have that $\mathrm{B}(\mathscr{M}, \mathscr{N}) \cong \mathrm{H{\scriptsize OM}}(\mathscr{N}; \mathscr{M}^*)$, an isomorphism of Banach $L^0(\mathbb{X})$-modules being given by the operator

$$\mathrm{B}(\mathscr{M}, \mathscr{N}) \ni b \mapsto \big(\mathscr{N} \ni w \mapsto b(\cdot, w) \in \mathscr{M}^* \big) \in \mathrm{H{\scriptsize OM}}(\mathscr{N}; \mathscr{M}^*).$$

Corollary 4.15 *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$, $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Then it holds that*

$$\big(T(v_n)(w_n) \big)_{n \in \mathbb{N}} \in \ell_1(\mathbb{N}, L^0(\mathbb{X})) \quad \begin{array}{l} \text{for every } T \in \mathrm{H{\scriptsize OM}}(\mathscr{M}; \mathscr{N}^*) \\ \text{and } (v_n \otimes w_n)_{n \in \mathbb{N}} \in \ell_1(\mathbb{N}, \mathscr{M} \hat{\otimes}_\pi \mathscr{N}). \end{array}$$

Moreover, for every $\alpha \in \mathscr{M}\hat{\otimes}_\pi \mathscr{N}$ *we have that*

$$|\alpha|_\pi = \bigvee \left\{ \left| \sum_{n \in \mathbb{N}} T(v_n)(w_n) \right| \,\middle|\, \begin{array}{l} (v_n \otimes w_n)_n \in \ell_1(\mathbb{N}, \mathscr{M}\hat{\otimes}_\pi \mathscr{N}), \\ \sum_{n \in \mathbb{N}} v_n \otimes w_n = \alpha,\ T \in \mathbb{D}_{\mathrm{HOM}(\mathscr{M};\mathscr{N}^*)} \end{array} \right\}.$$

Proof Let us fix any sequence $(v_n \otimes w_n)_{n \in \mathbb{N}} \in \ell_1(\mathbb{N}, \mathscr{M}\hat{\otimes}_\pi \mathscr{N})$. We denote $\alpha := \sum_{n \in \mathbb{N}} v_n \otimes w_n \in \mathscr{M}\hat{\otimes}_\pi \mathscr{N}$. Then

$$\left| \sum_{n \in F} T(v_n)(w_n) \right| \leq |T| \sum_{n \in F} |v_n||w_n| = |T| \sum_{n \in F} |v_n \otimes w_n|_\pi$$

for every $T \in \mathrm{HOM}(\mathscr{M}; \mathscr{N}^*)$ and $F \in \mathscr{P}_f(\mathbb{N})$. By passing to the supremum over all $F \in \mathscr{P}_f(\mathbb{N})$, we thus obtain that

$$\left| \big(T(v_n)(w_n)\big)_{n \in \mathbb{N}} \right|_1 \leq |T| \left| (v_n \otimes w_n)_{n \in \mathbb{N}} \right|_1 \in L^0(\mathbb{X})^+, \tag{4.10}$$

which ensures that $\big(T(v_n)(w_n)\big)_{n \in \mathbb{N}} \in \ell_1(\mathbb{N}, L^0(\mathbb{X}))$. Now, let us introduce the shorthand notation

$$Q(\alpha) := \bigvee \left\{ \left| \sum_{n \in \mathbb{N}} T(v_n)(w_n) \right| \,\middle|\, \begin{array}{l} (v_n \otimes w_n)_n \in \ell_1(\mathbb{N}, \mathscr{M}\hat{\otimes}_\pi \mathscr{N}), \\ \sum_{n \in \mathbb{N}} v_n \otimes w_n = \alpha,\ T \in \\ \mathbb{D}_{\mathrm{HOM}(\mathscr{M};\mathscr{N}^*)} \end{array} \right\}$$

for every $\alpha \in \mathscr{M}\hat{\otimes}_\pi \mathscr{N}$. On the one hand, whenever $(v_n \otimes w_n)_n$ and T are competitors for $Q(\alpha)$, we have that $\left| \sum_{n \in \mathbb{N}} T(v_n)(w_n) \right| \leq \sum_{n \in \mathbb{N}} |v_n||w_n|$ by (4.10), so that $Q(\alpha) \leq |\alpha|_\pi$ by Theorem 4.9. On the other hand, take any $b \in \mathrm{B}(\mathscr{M}, \mathscr{N})$ such that $|b| \leq 1$. Proposition 4.14 tells that the element $T_b \in \mathrm{HOM}(\mathscr{M}; \mathscr{N}^*)$, which we define as $T_b(v) := b(v, \cdot)$ for all $v \in \mathscr{M}$, satisfies $|T_b| \leq 1$. Hence, Lemma 3.8 yields

$$\tilde{b}_\pi(\alpha) = \sum_{n \in \mathbb{N}} \tilde{b}_\pi(v_n \otimes w_n) = \sum_{n \in \mathbb{N}} T_b(v_n)(w_n).$$

It follows that

$$|\alpha|_\pi = \bigvee_{b \in \mathbb{D}_{\mathrm{B}(\mathscr{M},\mathscr{N})}} \tilde{b}_\pi(\alpha) = \bigvee_{b \in \mathbb{D}_{\mathrm{B}(\mathscr{M},\mathscr{N})}} \sum_{n \in \mathbb{N}} T_b(v_n)(w_n) \leq Q(\alpha)$$

thanks to Corollary 4.12. Consequently, the statement is finally achieved. □

The following symmetric statement is verified as well:

$$\big(S(w_n)(v_n)\big)_n \in \ell_1(\mathbb{N}, L^0(\mathbb{X}))$$

for every $(v_n \otimes w_n)_n \in \ell_1(\mathbb{N}, \mathscr{M}\hat{\otimes}_\pi \mathscr{N})$ and $S \in \mathrm{H\small{OM}}(\mathscr{N}; \mathscr{M}^*)$, and we have

$$|\alpha|_\pi = \bigvee \left\{ \left| \sum_{n\in\mathbb{N}} S(w_n)(v_n) \right| \,\middle|\, \begin{array}{l} (v_n \otimes w_n)_n \in \ell_1(\mathbb{N}, \mathscr{M}\hat{\otimes}_\pi \mathscr{N}), \\ \sum_{n\in\mathbb{N}} v_n \otimes w_n = \alpha,\ S \in \\ \mathbb{D}_{\mathrm{H\small{OM}}(\mathscr{N};\mathscr{M}^*)} \end{array} \right\}.$$

These claims can be proved by arguing exactly as we did in the proof of Corollary 4.15.

Next, we use Corollary 4.12 to characterise the 'tensor diagonal' in the space $L^0(\mathbb{X}; \ell_2(I))\hat{\otimes}_\pi L^0(\mathbb{X}; \ell_2(I))$:

Proposition 4.16 *Let $\mathbb{X}$ be a σ-finite measure space. Then the Banach $L^0(\mathbb{X})$-submodule of $L^0(\mathbb{X}; \ell_2(I))\hat{\otimes}_\pi L^0(\mathbb{X}; \ell_2(I))$ that is generated by $\{\underline{\mathsf{e}}_i \otimes \underline{\mathsf{e}}_i : i \in I\}$ is isomorphic to $L^0(\mathbb{X}; \ell_1(I))$.*

Proof Let $\mathscr{M}$ be the Banach $L^0(\mathbb{X})$-submodule of $L^0(\mathbb{X}; \ell_2(I))\hat{\otimes}_\pi L^0(\mathbb{X}; \ell_2(I))$ that is generated by $\{\underline{\mathsf{e}}_i \otimes \underline{\mathsf{e}}_i : i \in I\}$. Observe that $\mathscr{M}$ can be described as $\mathscr{M} = \mathrm{cl}_{L^0(\mathbb{X};\ell_2(I))\hat{\otimes}_\pi L^0(\mathbb{X};\ell_2(I))}(M)$, where

$$M := \left\{ \sum_{i\in F} f_i \cdot (\underline{\mathsf{e}}_i \otimes \underline{\mathsf{e}}_i) \,\middle|\, F \in \mathscr{P}_f(I),\ \{f_i\}_{i\in F} \subseteq L^0(\mathbb{X}) \right\}.$$

Given any $F \in \mathscr{P}_f(I)$ and $f = \{f_i\}_{i\in F} \subseteq L^0(\mathbb{X})$, let us define the operator $b^f : L^0(\mathbb{X}; \ell_2(I)) \times L^0(\mathbb{X}; \ell_2(I)) \to L^0(\mathbb{X})$ as

$$b^f(g, h) := \sum_{i\in F} \mathrm{sgn}(f_i) g(\cdot)_i\, h(\cdot)_i \quad \text{for every } g, h \in L^0(\mathbb{X}; \ell_2(I)).$$

The map b^f is $L^0(\mathbb{X})$-bilinear by construction. Also, for any $g, h \in L^0(\mathbb{X}; \ell_2(I))$ we can estimate

$$|b^f(g, h)| \le \sum_{i\in F} |g(\cdot)_i||h(\cdot)_i| \le \left(\sum_{i\in F} |g(\cdot)_i|^2 \right)^{1/2} \left(\sum_{i\in F} |h(\cdot)_i|^2 \right)^{1/2} \le |g||h|,$$

which yields $b^f \in \mathrm{B}\big(L^0(\mathbb{X}; \ell_2(I)), L^0(\mathbb{X}; \ell_2(I))\big)$ and $|b^f| \le 1$. Now define the map $\psi : M \to \ell_1(I, L^0(\mathbb{X}))$ as

$$\psi\left(\sum_{i\in F} f_i \cdot (\underline{\mathsf{e}}_i \otimes \underline{\mathsf{e}}_i) \right) := \sum_{i\in F} (\delta_{ij} f_i)_{j\in I} \quad \forall F \in \mathscr{P}_f(I),\ \{f_i\}_{i\in F} \subseteq L^0(\mathbb{X}).$$

By virtue of Corollary 4.12, for any $F \in \mathscr{P}_f(I)$ and $f = \{f_i\}_{i\in F} \subseteq L^0(\mathbb{X})$ we can estimate

$$\begin{aligned}\Big|\sum_{i\in F}(\delta_{ij} f_i)_{j\in I}\Big|_1 &= \Big|\Big(\sum_{i\in F}\delta_{ij} f_i\Big)_{j\in I}\Big|_1 = \sum_{i\in F}|f_i| = \sum_{i\in F} f_i \operatorname{sgn}(f_i)\\ &= \sum_{i\in F} f_i \sum_{j\in F}\operatorname{sgn}(f_j)\delta_{ij}^2 = \sum_{i\in F} f_i \sum_{j\in F}\operatorname{sgn}(f_j)\underline{\mathsf{e}}_i(\cdot)_j^2\\ &= \sum_{i\in F} f_i\, b^f(\underline{\mathsf{e}}_i, \underline{\mathsf{e}}_i) = \sum_{i\in F} f_i\, \tilde{b}^f_\pi(\underline{\mathsf{e}}_i \otimes \underline{\mathsf{e}}_i)\\ &= \tilde{b}^f_\pi\Bigg(\sum_{i\in F} f_i \cdot (\underline{\mathsf{e}}_i \otimes \underline{\mathsf{e}}_i)\Bigg) \le \Big|\sum_{i\in F} f_i \cdot (\underline{\mathsf{e}}_i \otimes \underline{\mathsf{e}}_i)\Big|_\pi .\end{aligned}$$

This shows that the operator ψ is well-defined (thus also $L^0(\mathbb{X})$-linear by construction) and that it satisfies $|\psi(\alpha)|_1 \le |\alpha|_\pi$ for all $\alpha \in M$. Conversely, for any $\alpha = \sum_{i\in F} f_i \cdot (\underline{\mathsf{e}}_i \otimes \underline{\mathsf{e}}_i) \in M$ we have

$$|\alpha|_\pi \le \sum_{i\in F}|f_i||\underline{\mathsf{e}}_i \otimes \underline{\mathsf{e}}_i|_\pi = \sum_{i\in F}|f_i||\underline{\mathsf{e}}_i|^2 = \sum_{i\in F}|f_i| = |\psi(\alpha)|_1.$$

All in all, we have shown that ψ preserves the pointwise norm, thus it can be uniquely extended to a homomorphism $\bar{\psi} \in \textsc{Hom}\big(\mathscr{M}; \ell_1(I, L^0(\mathbb{X}))\big)$ that satisfies $|\bar{\psi}(\alpha)|_1 = |\alpha|_\pi$ for all $\alpha \in \mathscr{M}$. Letting $\phi\colon L^0(\mathbb{X}; \ell_1(I)) \to \ell_1(I, L^0(\mathbb{X}))$ be the isomorphism given by Corollary 3.12, we deduce that

$$\varphi := \phi^{-1} \circ \bar{\psi} \in \textsc{Hom}\big(\mathscr{M}; L^0(\mathbb{X}; \ell_1(I))\big)$$

satisfies $|\varphi(\alpha)| = |\alpha|_\pi$ for every $\alpha \in \mathscr{M}$. Finally, we verify that φ is surjective. Fix any $a \in L^0(\mathbb{X}; \ell_1(I))$. Thanks to Lemma 3.11, we can find an increasing sequence $(F_n)_{n\in\mathbb{N}} \subseteq \mathscr{P}_f(I)$ with $a_n := \sum_{i\in F_n} a(\cdot)_i \cdot \underline{\mathsf{e}}_i \to a$ as $n \to \infty$. Since

$$\varphi\Bigg(\sum_{i\in F_n} a(\cdot)_i \cdot (\underline{\mathsf{e}}_i \otimes \underline{\mathsf{e}}_i)\Bigg) = \sum_{i\in F_n}\phi^{-1}\Big(\big(\delta_{ij} a(\cdot)_i\big)_{j\in I}\Big) = \sum_{i\in F_n} a(\cdot)_i \cdot \underline{\mathsf{e}}_i = a_n,$$

we deduce that $(a_n)_{n\in\mathbb{N}} \subseteq \varphi[M]$, whence it follows that $a \in \varphi[\mathscr{M}]$. The proof is complete. □

Finally, we discuss a categorical consequence of Theorem 4.11. First, we need to introduce the two functors $\mathscr{M}\hat{\otimes}_\pi - \colon \mathbf{BanMod}_{\mathbb{X}} \to \mathbf{BanMod}_{\mathbb{X}}$ and $\textsc{Hom}(\mathscr{M}; -)\colon \mathbf{BanMod}_{\mathbb{X}} \to \mathbf{BanMod}_{\mathbb{X}}$, where $\mathscr{M}$ is a Banach $L^0(\mathbb{X})$-module. The functors $\mathscr{M}\hat{\otimes}_\pi -$ and $\textsc{Hom}(\mathscr{M}; -)$ are given as follows:

(i) For any object $\mathscr{N}$ of $\mathbf{BanMod}_{\mathbb{X}}$, we define $(\mathscr{M}\hat{\otimes}_\pi -)(\mathscr{N}) := \mathscr{M}\hat{\otimes}_\pi \mathscr{N}$. For any morphism $T\colon \mathscr{N} \to \tilde{\mathscr{N}}$ in $\mathbf{BanMod}_{\mathbb{X}}$, we define the morphism $(\mathscr{M}\hat{\otimes}_\pi -)(T)\colon \mathscr{M}\hat{\otimes}_\pi \mathscr{N} \to \mathscr{M}\hat{\otimes}_\pi \tilde{\mathscr{N}}$ as $(\mathscr{M}\hat{\otimes}_\pi -)(T) := \mathrm{id}_{\mathscr{M}} \otimes_\pi T$.

(ii) For any object $\mathscr{Q}$ of $\mathbf{BanMod}_{\mathbb{X}}$, we set

$$\textsc{Hom}(\mathscr{M};-)(\mathscr{Q}):=\textsc{Hom}(\mathscr{M};\mathscr{Q}).$$

For any morphism $T\colon\mathscr{Q}\to\tilde{\mathscr{Q}}$ in $\mathbf{BanMod}_{\mathbb{X}}$, we define the morphism

$$\textsc{Hom}(\mathscr{M};-)(T)\colon\textsc{Hom}(\mathscr{M};\mathscr{Q})\to\textsc{Hom}(\mathscr{M};\tilde{\mathscr{Q}})$$

as $\textsc{Hom}(\mathscr{M};-)(T)(S):=T\circ S$ for every $S\in\textsc{Hom}(\mathscr{M};\mathscr{Q})$.

We can now pass to the ensuing result, which states that $\mathscr{M}\hat{\otimes}_\pi-$ is the left adjoint of $\textsc{Hom}(\mathscr{M};-)$:

Proposition 4.17 *Let $\mathbb{X}$ be a σ-finite measure space and let $\mathscr{M}$ be a Banach $L^0(\mathbb{X})$-module. Then*

$$(\mathscr{M}\hat{\otimes}_\pi-)\dashv\textsc{Hom}(\mathscr{M};-).$$

Proof Our goal is to find a natural isomorphism

$$\Phi\colon\textsc{Hom}(\mathscr{M}\hat{\otimes}_\pi-;-)\to\textsc{Hom}(-;\textsc{Hom}(\mathscr{M};-)),$$

which means that $\big(\mathscr{M}\hat{\otimes}_\pi-,\textsc{Hom}(\mathscr{M};-),\Phi\big)$ is a hom-set adjunction. To this aim, fix two Banach $L^0(\mathbb{X})$-modules $\mathscr{N}$ and $\mathscr{Q}$. We define

$$\Phi_{\mathscr{N},\mathscr{Q}}\colon\textsc{Hom}(\mathscr{M}\hat{\otimes}_\pi\mathscr{N};\mathscr{Q})\to\textsc{Hom}(\mathscr{N};\textsc{Hom}(\mathscr{M};\mathscr{Q}))$$

as follows:

$$\Phi_{\mathscr{N},\mathscr{Q}}(T)(w)(v):=T(v\otimes w)\quad\forall T\in\textsc{Hom}(\mathscr{M}\hat{\otimes}_\pi\mathscr{N};\mathscr{Q}),\ (w,v)\in\mathscr{N}\times\mathscr{M}.$$

One can readily check that $\Phi_{\mathscr{N},\mathscr{Q}}$ is a morphism and $|\Phi_{\mathscr{N},\mathscr{Q}}|\leq 1$. On the other hand, let L be a given element of $\textsc{Hom}(\mathscr{N};\textsc{Hom}(\mathscr{M};\mathscr{Q}))$. Define $b^L\colon\mathscr{M}\times\mathscr{N}\to\mathscr{Q}$ as $b^L(v,w):=L(w)(v)$ for every $(v,w)\in\mathscr{M}\times\mathscr{N}$. Since $b^L\in\mathrm{B}(\mathscr{M},\mathscr{N};\mathscr{Q})$ and $|b^L|\leq|L|$, we know from Theorem 4.11 that the element $\tilde{b}^L_\pi\in\textsc{Hom}(\mathscr{M}\hat{\otimes}_\pi\mathscr{N};\mathscr{Q})$ satisfies $|\tilde{b}^L_\pi|\leq|L|$. Since

$$\tilde{b}^L_\pi(v\otimes w)=b^L(v,w)=L(w)(v)\quad\text{for every }(v,w)\in\mathscr{M}\times\mathscr{N},$$

we deduce that $\Phi_{\mathscr{N},\mathscr{Q}}(\tilde{b}^L_\pi)=L$ and $|\Phi_{\mathscr{N},\mathscr{Q}}(\tilde{b}^L_\pi)|=|L|\geq|\tilde{b}^L_\pi|$. All in all, we have shown that $\Phi_{\mathscr{N},\mathscr{Q}}$ is an isomorphism. Let us finally check the naturality of Φ. Given any two morphisms $T\colon\tilde{\mathscr{N}}\to\mathscr{N}$ and $S\colon\mathscr{Q}\to\tilde{\mathscr{Q}}$ in $\mathbf{BanMod}_{\mathbb{X}}$, we consider the morphisms

$$\begin{aligned}\textsc{Hom}(\mathscr{M}\hat{\otimes}_\pi T;S)&\colon\textsc{Hom}(\mathscr{M}\hat{\otimes}_\pi\mathscr{N};\mathscr{Q})\to\textsc{Hom}(\mathscr{M}\hat{\otimes}_\pi\tilde{\mathscr{N}};\tilde{\mathscr{Q}}),\\ \textsc{Hom}(T;\textsc{Hom}(\mathscr{M};S))&\colon\textsc{Hom}(\mathscr{N};\textsc{Hom}(\mathscr{M};\mathscr{Q}))\to\textsc{Hom}(\tilde{\mathscr{N}};\textsc{Hom}(\mathscr{M};\tilde{\mathscr{Q}})),\end{aligned}$$

which are given by $\mathrm{HOM}(\mathscr{M}\hat{\otimes}_\pi T;S)(\varphi):=S\circ\varphi\circ(\mathrm{id}_{\mathscr{M}}\otimes_\pi T)$ for every $\varphi\in\mathrm{HOM}(\mathscr{M}\hat{\otimes}_\pi\mathscr{N};\mathscr{Q})$ and $\mathrm{HOM}(T;\mathrm{HOM}(\mathscr{M};S))(\psi)(\tilde{w}):=S\circ(\psi\circ T)(\tilde{w})$ for every $\psi\in\mathrm{HOM}(\mathscr{N};\mathrm{HOM}(\mathscr{M};\mathscr{Q}))$ and $\tilde{w}\in\tilde{\mathscr{N}}$. Unwrapping the various definitions, one can see that the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{HOM}(\mathscr{M}\hat{\otimes}_\pi\mathscr{N};\mathscr{Q}) \arrow[r, "\Phi_{\mathscr{N},\mathscr{Q}}"] \arrow[d, "\mathrm{HOM}(\mathscr{M}\hat{\otimes}_\pi T;S)"'] & \mathrm{HOM}(\mathscr{N};\mathrm{HOM}(\mathscr{M};\mathscr{Q})) \arrow[d, "\mathrm{HOM}(T;\mathrm{HOM}(\mathscr{M};S))"] \\
\mathrm{HOM}(\mathscr{M}\hat{\otimes}_\pi\tilde{\mathscr{N}};\tilde{\mathscr{Q}}) \arrow[r, "\Phi_{\tilde{\mathscr{N}},\tilde{\mathscr{Q}}}"'] & \mathrm{HOM}(\tilde{\mathscr{N}};\mathrm{HOM}(\mathscr{M};\tilde{\mathscr{Q}}))
\end{tikzcd}$$

whence it follows that Φ is a natural isomorphism. Consequently, the proof is complete. □

The previous result implies that the functor $\mathscr{M}\hat{\otimes}_\pi-$ is cocontinuous, i.e. it preserves colimits. Notice however that we are considering $\mathscr{M}\hat{\otimes}_\pi-$ as an endofunctor on $\mathbf{BanMod}_{\mathbb{X}}$, which is only finitely cocomplete, and not on the cocomplete category $\mathbf{BanMod}^1_{\mathbb{X}}$.

5 Injective tensor products of Banach L^0-modules

5.1 Definition and main properties

We begin by introducing the injective pointwise norm:

Theorem 5.1 *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$, $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Define*

$$|\alpha|_\varepsilon:=\bigvee\left\{\left|\sum_{i=1}^n\omega(v_i)\eta(w_i)\right|\;\middle|\;\sum_{i=1}^n v_i\otimes w_i=\alpha,\ \omega\in\mathbb{D}_{\mathscr{M}^*},\ \eta\in\mathbb{D}_{\mathscr{N}^*}\right\}\tag{5.1}$$

for every $\alpha\in\mathscr{M}\otimes\mathscr{N}$. Then $|\cdot|_\varepsilon\colon\mathscr{M}\otimes\mathscr{N}\to L^0(\mathbb{X})^+$ is a pointwise norm on $\mathscr{M}\otimes\mathscr{N}$. Moreover,

$$|v\otimes w|_\varepsilon=|v||w|\quad\textit{for every } v\in\mathscr{M}\textit{ and } w\in\mathscr{N}.\tag{5.2}$$

Proof One can readily check that $|\cdot|_\varepsilon$ verifies the pointwise norm axioms; the fact that $|\alpha|_\varepsilon=0$ implies $\alpha=0$ is a consequence of Lemma 3.19. To prove (5.2), notice first that Lemma 3.19 yields

$$\left|\sum_{i=1}^n\omega(v_i)\eta(w_i)\right|=\big|\omega(v)\eta(w)\big|\leq|\omega||v||\eta||w|\leq|v||w|$$

whenever $\sum_{i=1}^n v_i\otimes w_i$ is a representation of the tensor $v\otimes w$ and for all $(\omega,\eta)\in\mathbb{D}_{\mathscr{M}^*}\times\mathbb{D}_{\mathscr{N}^*}$. Hence, we obtain that $|v\otimes w|_\varepsilon\leq|v||w|$. Conversely, an application of the Hahn–Banach theorem gives two elements $\omega_v \in \mathbb{S}_{\mathscr{M}^*} \cup \{0\}$ and $\eta_w \in \mathbb{S}_{\mathscr{N}^*} \cup \{0\}$ such that $\omega_v(v) = |v|$ and $\eta_w(w) = |w|$. Therefore, we have that $|v||w| = \omega_v(v)\eta_w(w) \leq |v \otimes w|_\varepsilon$. All in all, (5.2) is proved. □

Remark 5.2 Observe that $|\alpha|_\varepsilon \leq |\alpha|_\pi$ for every $\alpha \in \mathscr{M} \otimes \mathscr{N}$. Indeed, if we write $\alpha = \sum_{i=1}^n v_i \otimes w_i$, then for any $\omega \in \mathbb{D}_{\mathscr{M}^*}$ and $\eta \in \mathbb{D}_{\mathscr{N}^*}$ we have that

$$\left|\sum_{i=1}^n \omega(v_i)\eta(w_i)\right| \leq \sum_{i=1}^n |\omega(v_i)||\eta(w_i)| \leq \sum_{i=1}^n |v_i||w_i|,$$

whence the claim follows.

Definition 5.3 (*Injective tensor product*) Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$ and $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Then we denote by $\mathscr{M} \otimes_\varepsilon \mathscr{N}$ the normed $L^0(\mathbb{X})$-module $(\mathscr{M} \otimes \mathscr{N}, |\cdot|_\varepsilon)$, where the pointwise norm $|\cdot|_\varepsilon$ is defined as in (5.1). Moreover, the **injective tensor product** of $\mathscr{M}$ and $\mathscr{N}$ is the Banach $L^0(\mathbb{X})$-module $\mathscr{M} \hat{\otimes}_\varepsilon \mathscr{N}$ defined as the $L^0(\mathbb{X})$-completion of $\mathscr{M} \otimes_\varepsilon \mathscr{N}$.

The space $\mathscr{M} \hat{\otimes}_\varepsilon \mathscr{N}$ is a Banach $L^0(\mathbb{X})$-submodule of $\mathrm{B}(\mathscr{M}^*, \mathscr{N}^*)$:

Proposition 5.4 *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$ and $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Given any tensor $\alpha = \sum_{i=1}^n v_i \otimes w_i \in \mathscr{M} \otimes \mathscr{N}$, we define the map $B_\alpha \colon \mathscr{M}^* \times \mathscr{N}^* \to L^0(\mathbb{X})$ as*

$$B_\alpha(\omega, \eta) := \sum_{i=1}^n \omega(v_i)\eta(w_i) \quad \text{for every } \omega \in \mathscr{M}^* \text{ and } \eta \in \mathscr{N}^*. \tag{5.3}$$

Then B_α is well-defined and belongs to $\mathrm{B}(\mathscr{M}^*, \mathscr{N}^*)$. Moreover, the resulting operator

$$\mathscr{M} \otimes_\varepsilon \mathscr{N} \ni \alpha \mapsto B_\alpha \in \mathrm{B}(\mathscr{M}^*, \mathscr{N}^*)$$

can be uniquely extended to an isomorphism of Banach $L^0(\mathbb{X})$-modules from the injective tensor product $\mathscr{M} \hat{\otimes}_\varepsilon \mathscr{N}$ to the closure of $\{B_\alpha : \alpha \in \mathscr{M} \otimes \mathscr{N}\}$ in $\mathrm{B}(\mathscr{M}^*, \mathscr{N}^*)$.

Proof The well-posedness of (5.3) follows from Lemma 3.19, while the rest is straightforward. □

The following result provides other two representations of the injective tensor product $\mathscr{M} \hat{\otimes}_\varepsilon \mathscr{N}$:

Proposition 5.5 *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$ and $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Given any tensor $\alpha = \sum_{i=1}^n v_i \otimes w_i \in \mathscr{M} \otimes \mathscr{N}$, we define the maps $L_\alpha \colon \mathscr{M}^* \to \mathscr{N}$ and $R_\alpha \colon \mathscr{N}^* \to \mathscr{M}$ as*

$$L_\alpha(\omega) := \sum_{i=1}^n \omega(v_i) \cdot w_i, \quad R_\alpha(\eta) := \sum_{i=1}^n \eta(w_i) \cdot v_i \quad \forall \omega \in \mathscr{M}^*,\ \eta \in \mathscr{N}^*,$$

respectively. Then $L_\alpha \in \mathrm{HOM}(\mathscr{M}^*;\mathscr{N})$ *and* $R_\alpha \in \mathrm{HOM}(\mathscr{N}^*;\mathscr{M})$*. Moreover, the resulting maps*

$$\mathscr{M} \otimes_\varepsilon \mathscr{N} \ni \alpha \mapsto L_\alpha \in \mathrm{HOM}(\mathscr{M}^*;\mathscr{N}),$$
$$\mathscr{M} \otimes_\varepsilon \mathscr{N} \ni \alpha \mapsto R_\alpha \in \mathrm{HOM}(\mathscr{N}^*;\mathscr{M})$$

can be uniquely extended to pointwise norm preserving homomorphisms defined on $\mathscr{M}\hat{\otimes}_\varepsilon\mathscr{N}$.

Proof We consider only L_α, the proof for R_α being analogous. The well-posedness of the map L_α follows from Corollary 3.21. It is then easy to check that $L_\alpha \in \mathrm{HOM}(\mathscr{M}^*;\mathscr{N})$ holds for every $\alpha \in \mathscr{M} \otimes \mathscr{N}$ and that the mapping $\mathscr{M} \otimes_\varepsilon \mathscr{N} \ni \alpha \mapsto L_\alpha \in \mathrm{HOM}(\mathscr{M}^*;\mathscr{N})$ is a homomorphism of normed $L^0(\mathbb{X})$-modules. Also, we have that

$$\begin{aligned}|L_\alpha| &= \bigvee_{\omega\in\mathbb{D}_{\mathscr{M}^*}} |L_\alpha(\omega)| = \bigvee_{\omega\in\mathbb{D}_{\mathscr{M}^*}} \bigvee_{\eta\in\mathbb{D}_{\mathscr{N}^*}} \big|\eta\big(L_\alpha(\omega)\big)\big| \\ &= \bigvee_{\omega\in\mathbb{D}_{\mathscr{M}^*}} \bigvee_{\eta\in\mathbb{D}_{\mathscr{N}^*}} \left|\sum_{i=1}^n \omega(v_i)\eta(w_i)\right| = |\alpha|_\varepsilon\end{aligned}$$

for every $\alpha = \sum_{i=1}^n v_i \otimes w_i \in \mathscr{M} \otimes_\varepsilon \mathscr{N}$ by the Hahn–Banach theorem. The statement follows. □

Corollary 5.6 *Let* $\mathbb{X}$ *be a* σ*-finite measure space. Let* $\mathscr{M}$, $\mathscr{N}$ *be Banach* $L^0(\mathbb{X})$*-modules. Let* $\mathcal{T}$ *and* $\mathcal{S}$ *be norming subsets of* $\mathscr{M}^*$ *and* $\mathscr{N}^*$*, respectively. Then it holds that*

$$|\alpha|_\varepsilon = \bigvee_{\omega\in\mathcal{T}} \left|\sum_{i=1}^n \omega(v_i)\cdot w_i\right| = \bigvee_{\eta\in\mathcal{S}} \left|\sum_{i=1}^n \eta(w_i)\cdot v_i\right| \quad \forall \alpha = \sum_{i=1}^n v_i \otimes w_i \in \mathscr{M} \otimes_\varepsilon \mathscr{N}.$$

Proof Given that $|L_\alpha| = \bigvee_{\omega\in\mathcal{T}} |L_\alpha(\omega)|$ and $|R_\alpha| = \bigvee_{\eta\in\mathcal{S}} |R_\alpha(\eta)|$, the statement follows from Proposition 5.5. □

Let us now consider the injective tensor product of homomorphisms of Banach $L^0(\mathbb{X})$-modules:

Proposition 5.7 (Injective tensor products of homomorphisms) *Let* $\mathbb{X}$ *be a* σ*-finite measure space. Let* $T: \mathscr{M} \to \tilde{\mathscr{M}}$ *and* $S: \mathscr{N} \to \tilde{\mathscr{N}}$ *be homomorphisms of Banach* $L^0(\mathbb{X})$*-modules. Then there exists a unique homomorphism of Banach* $L^0(\mathbb{X})$*-modules* $T \otimes_\varepsilon S: \mathscr{M}\hat{\otimes}_\varepsilon\mathscr{N} \to \tilde{\mathscr{M}}\hat{\otimes}_\varepsilon\tilde{\mathscr{N}}$ *with*

$$(T \otimes_\varepsilon S)(v \otimes w) = T(v) \otimes S(w) \quad \textit{for every } v \in \mathscr{M} \textit{ and } w \in \mathscr{N}.$$

Moreover, it holds that $|T \otimes_\varepsilon S| = |T||S|$.

Proof Let $T \otimes S \colon \mathscr{M} \otimes \mathscr{N} \to \tilde{\mathscr{M}} \otimes \tilde{\mathscr{N}}$ be as in Lemma 2.2. Given any element $\alpha = \sum_{i=1}^n v_i \otimes w_i \in \mathscr{M} \otimes \mathscr{N}$, we have that

$$
\begin{aligned}
\big|(T \otimes S)(\alpha)\big|_\varepsilon &= \bigvee \left\{ \left| \sum_{i=1}^n \tilde{\omega}(T(v_i))\tilde{\eta}(S(w_i)) \right| \;\middle|\; \tilde{\omega} \in \mathbb{D}_{\tilde{\mathscr{M}}^*},\ \tilde{\eta} \in \mathbb{D}_{\tilde{\mathscr{N}}^*} \right\} \\
&\leq |T||S| \bigvee \left\{ \left| \sum_{i=1}^n \omega(v_i)\eta(w_i) \right| \;\middle|\; \omega \in \mathbb{D}_{\mathscr{M}^*},\ \eta \in \mathbb{D}_{\mathscr{N}^*} \right\} \\
&= |T||S||\alpha|_\varepsilon,
\end{aligned}
$$

where we used the fact that $\frac{\mathbb{1}_{\{|T|>0\}}}{|T|} \cdot (\tilde{\omega} \circ T) \in \mathbb{D}_{\mathscr{M}^*}$ and $\frac{\mathbb{1}_{\{|S|>0\}}}{|S|} \cdot (\tilde{\eta} \circ S) \in \mathbb{D}_{\mathscr{N}^*}$. It follows that the map $T \otimes S \colon \mathscr{M} \otimes \mathscr{N} \to \tilde{\mathscr{M}} \otimes \tilde{\mathscr{N}}$ can be uniquely extended to a homomorphism of Banach $L^0(\mathbb{X})$-modules $T \otimes_\varepsilon S \colon \mathscr{M} \hat{\otimes}_\varepsilon \mathscr{N} \to \tilde{\mathscr{M}} \hat{\otimes}_\varepsilon \tilde{\mathscr{N}}$ satisfying $|T \otimes_\varepsilon S| \leq |T||S|$. Finally, the validity of the converse inequality $|T \otimes_\varepsilon S| \geq |T||S|$ can be proved arguing as in Proposition 4.3. □

One can easily check that $L^0(\mathbb{X}) \hat{\otimes}_\varepsilon L^0(\mathbb{X}) = L^0(\mathbb{X}) \otimes_\varepsilon L^0(\mathbb{X}) \cong L^0(\mathbb{X})$ as Banach $L^0(\mathbb{X})$-modules via the isomorphism

$$
L^0(\mathbb{X}) \otimes_\varepsilon L^0(\mathbb{X}) \ni \sum_{i=1}^n f_i \otimes g_i \mapsto \sum_{i=1}^n f_i g_i \in L^0(\mathbb{X}).
$$

In particular, up to this identification, we have that

$$
\omega \otimes_\varepsilon \eta \in (\mathscr{M} \hat{\otimes}_\varepsilon \mathscr{N})^*, \quad |\omega \otimes_\varepsilon \eta| = |\omega||\eta| \quad \text{for every } \omega \in \mathscr{M}^* \text{ and } \eta \in \mathscr{N}^*.
$$

Lemma 5.8 *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$, $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Let $G \subseteq \mathscr{M}$ and $H \subseteq \mathscr{N}$ be generating subsets. Then it holds that the set $\{v \otimes w \mid v \in G,\ w \in H\}$ generates $\mathscr{M} \hat{\otimes}_\varepsilon \mathscr{N}$.*

Proof The statement follows from Lemma 4.5 and Remark 5.2. □

5.2 Relation with order-continuous maps

As we already mentioned in the first paragraph of Sect. 3.4, the Banach space $\mathrm{C}(K)$ (where K is a compact, Hausdorff topological space) has a special relevance in connection with injective tensor products. For instance, it holds that $\mathrm{C}(K) \hat{\otimes}_\varepsilon \mathbb{B} \cong \mathrm{C}(K; \mathbb{B})$ for every Banach space $\mathbb{B}$, whence it follows that any quotient operator $f \colon \mathbb{B}_1 \to \mathbb{B}_2$ between Banach spaces induces a quotient operator $\mathrm{id} \otimes_\varepsilon f \colon \mathrm{C}(K) \hat{\otimes}_\varepsilon \mathbb{B}_1 \to \mathrm{C}(K) \hat{\otimes}_\varepsilon \mathbb{B}_2$. The goal of the present section is to extend these results to the setting of Banach $L^0(\mathbb{X})$-modules, taking as K a compact, Hausdorff uniform space, and replacing $\mathrm{C}(K)$ with $\mathrm{UC}_{\mathrm{ord}}(K; L^0(\mathbb{X}))$.

Theorem 5.9 *Let (K,Φ) be a compact, Hausdorff uniform space. Let $\mathbb{X}$ be a σ-finite measure space and $\mathscr{M}$ a Banach $L^0(\mathbb{X})$-module. Then the unique linear and continuous operator*

$$\mathrm{j}\colon \mathrm{UC}_{\mathrm{ord}}(K;L^0(\mathbb{X}))\hat{\otimes}_\varepsilon\mathscr{M}\to\mathrm{UC}_{\mathrm{ord}}(K;\mathscr{M})$$

satisfying $\mathrm{j}(f\otimes v)(\cdot)=f(\cdot)\cdot v$ for every $f\in\mathrm{UC}_{\mathrm{ord}}(K;L^0(\mathbb{X}))$ and $v\in\mathscr{M}$ is an isomorphism.

Proof Notice that $|f(\cdot)\cdot v|=|f(\cdot)||v|$ and $\mathrm{Var}(f(\cdot)\cdot v;\mathcal{U})=|v|\mathrm{Var}(f;\mathcal{U})$ for every $f\in\mathrm{UC}_{\mathrm{ord}}(K;L^0(\mathbb{X}))$, $v\in\mathscr{M}$, and $\mathcal{U}\in\Phi$, which implies that

$$f(\cdot)\cdot v\in\mathrm{UC}_{\mathrm{ord}}(K;\mathscr{M}).$$

Therefore, it makes sense to define $\mathrm{j}\colon\mathrm{UC}_{\mathrm{ord}}(K;L^0(\mathbb{X}))\otimes_\varepsilon\mathscr{M}\to\mathrm{UC}_{\mathrm{ord}}(K;\mathscr{M})$ in the following way:

$$\mathrm{j}\left(\sum_{i=1}^n f_i\otimes v_i\right):=\sum_{i=1}^n f_i(\cdot)\cdot v_i\quad\forall\sum_{i=1}^n f_i\otimes v_i\in\mathrm{UC}_{\mathrm{ord}}(K;L^0(\mathbb{X}))\otimes_\varepsilon\mathscr{M}.$$

To prove that the definition of j is well-posed amounts to showing that

$$\begin{array}{l}(f_i)_{i=1}^n\subseteq\mathrm{UC}_{\mathrm{ord}}(K;L^0(\mathbb{X})),\\ (v_i)_{i=1}^n\subseteq\mathscr{M},\ \sum_{i=1}^n f_i\otimes v_i=0\end{array}\quad\Longrightarrow\quad\sum_{i=1}^n f_i(\cdot)\cdot v_i=0.\tag{5.4}$$

Assuming $\sum_{i=1}^n f_i\otimes v_i=0$, we have that $\sum_{i=1}^n f_i(p)\cdot v_i=\sum_{i=1}^n\delta_p(f_i)\cdot v_i=0$ for every $p\in K$ by Remark 3.15 and Corollary 3.21, thus showing that (5.4) holds. Moreover, if $\alpha=\sum_{i=1}^n f_i\otimes v_i$ is an element of $\mathrm{UC}_{\mathrm{ord}}(K;L^0(\mathbb{X}))\otimes_\varepsilon\mathscr{M}$, then by Corollary 5.6 and Remark 3.15 we can compute

$$|\mathrm{j}(\alpha)|=\bigvee_{p\in K}\left|\sum_{i=1}^n f_i(p)\cdot v_i\right|=\bigvee_{p\in K}\left|\sum_{i=1}^n\delta_p(f_i)\cdot v_i\right|=|\alpha|_\varepsilon.$$

Since j is also linear by construction, it can be uniquely extended to a homomorphism of Banach $L^0(\mathbb{X})$-modules

$$\mathrm{j}\colon \mathrm{UC}_{\mathrm{ord}}(K;L^0(\mathbb{X}))\hat{\otimes}_\varepsilon\mathscr{M}\to\mathrm{UC}_{\mathrm{ord}}(K;\mathscr{M})$$

that preserves the pointwise norm.

In order to conclude, it remains to check that the isometric embedding map j is also surjective. Let $v\in\mathrm{UC}_{\mathrm{ord}}(K;\mathscr{M})$ be given. Then we can find a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}\subseteq\Phi$ with $\mathrm{Var}(v;\mathcal{U}_n)\to 0$ in $L^0(\mathbb{X})$. Fix any $n\in\mathbb{N}$. Given that $\{\mathcal{U}_n[p]\}_{p\in K}$ is an open cover of the compact set K, there exist $k_n\in\mathbb{N}$ and $(p_i^n)_{i=1}^{k_n}\subseteq K$ such that

$K = \bigcup_{i=1}^{k_n} \mathcal{U}_n[p_i^n]$. Now, take a continuous partition of unity $(\eta_i^n)_{i=1}^{k_n}$ subordinated to $(\mathcal{U}_n[p_i^n])_{i=1}^{k_n}$ (see e.g. [20]), i.e. $\eta_i^n : K \to [0,1]$ is continuous, supported in $\mathcal{U}_n[p_i^n]$, and $\sum_{i=1}^{k_n} \eta_i^n = 1$ on K. With no loss of generality, we can also assume that for any $i = 1, \ldots, k_n$ there exists $q_i^n \in \mathcal{U}_n[p_i^n]$ such that $\eta_i^n(q_i^n) = 1$; this fact will be used in Remark 5.10. Let us define

$$\alpha_n := \sum_{i=1}^{k_n} (\eta_i^n(\cdot)\mathbb{1}_{\mathrm{X}}) \otimes v(p_i^n) \in \mathrm{UC}_{\mathrm{ord}}(K; L^0(\mathbb{X})) \otimes_\varepsilon \mathscr{M}.$$

Observe that for any given point $p \in K$ we can estimate

$$\begin{aligned}\big|\mathrm{j}(\alpha_n)(p) - v(p)\big| &= \left|\sum_{i=1}^{k_n} \eta_i^n(p)\big(v(p_i^n) - v(p)\big)\right| \leq \sum_{i=1}^{k_n} \eta_i^n(p)\big|v(p_i^n) - v(p)\big| \\ &\leq \mathrm{Var}(v; \mathcal{U}_n).\end{aligned}$$

By passing to the supremum over all $p \in K$, we get $|\mathrm{j}(\alpha_n) - v| \leq \mathrm{Var}(v; \mathcal{U}_n)$, whence it follows that $\mathrm{j}(\alpha_n) \to v$ in $\mathrm{UC}_{\mathrm{ord}}(K; \mathscr{M})$. This shows that j is surjective, as desired. □

Remark 5.10 We isolate a useful byproduct of the proof of Theorem 5.9. For any $n \in \mathbb{N}$, we denote

$$\mathscr{F}_n := \left\{ (\eta_i)_{i=1}^n \subseteq \mathrm{C}(K; [0,1]) \;\middle|\; \{\eta_i = 1\} \neq \varnothing \;\forall i = 1, \ldots, n, \; \sum_{i=1}^n \eta_i = 1 \right\},$$

where $\mathrm{C}(K; [0,1])$ stands for the set of continuous functions from K to $[0,1]$. Then we have that

$$\mathscr{D} := \bigcup_{n \in \mathbb{N}} \left\{ \sum_{i=1}^n (\eta_i(\cdot)\mathbb{1}_{\mathrm{X}}) \otimes v_i \;\middle|\; (\eta_i)_{i=1}^n \in \mathscr{F}_n, \; (v_i)_{i=1}^n \subseteq \mathscr{M} \right\}$$

is dense in $\mathrm{UC}_{\mathrm{ord}}(K; L^0(\mathbb{X})) \hat{\otimes}_\varepsilon \mathscr{M}$, or equivalently

$$\bigcup_{n \in \mathbb{N}} \left\{ \sum_{i=1}^n \eta_i(\cdot) v_i \;\middle|\; (\eta_i)_{i=1}^n \in \mathscr{F}_n, \; (v_i)_{i=1}^n \subseteq \mathscr{M} \right\}$$

is dense in $\mathrm{UC}_{\mathrm{ord}}(K; \mathscr{M})$. Moreover, it holds that

$$|\alpha|_\varepsilon = \bigvee_{i=1}^n |v_i| \quad \text{for every } \alpha = \sum_{i=1}^n (\eta_i(\cdot)\mathbb{1}_{\mathrm{X}}) \otimes v_i \in \mathscr{D}. \tag{5.5}$$

To prove it, take $(q_i)_{i=1}^n \subseteq K$ such that $\eta_i(q_i) = 1$ for every $i = 1, \dots, n$. Therefore, we can estimate

$$
\begin{aligned}
|\alpha|_\varepsilon = |\mathfrak{j}(\alpha)| &= \bigvee_{p\in K} \left| \sum_{i=1}^n \eta_i(p) v_i \right| \le \bigvee_{p\in K} \sum_{i=1}^n \eta_i(p)|v_i| \le \bigvee_{i=1}^n |v_i| \\
&= \bigvee_{i=1}^n |\mathfrak{j}(\alpha)(q_i)| \le |\alpha|_\varepsilon,
\end{aligned}
$$

which shows the validity of (5.5).

Proposition 5.11 *Let (K, Φ) be a compact, Hausdorff uniform space. Let $\mathbb{X}$ be a σ-finite measure space and let $\mathscr{M}$, $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Let $T : \mathscr{M} \to \mathscr{N}$ be a quotient operator. Then the map*

$$
\mathrm{id} \otimes_\varepsilon T : \mathrm{UC}_{\mathrm{ord}}(K; L^0(\mathbb{X}))\hat{\otimes}_\varepsilon \mathscr{M} \to \mathrm{UC}_{\mathrm{ord}}(K; L^0(\mathbb{X}))\hat{\otimes}_\varepsilon \mathscr{N}
$$

is a quotient operator.

Proof First, notice that $|\mathrm{id} \otimes_\varepsilon T| = |T| \le 1$. Our goal is to apply Lemma 3.4. To this aim, we shorten $\mathrm{UC}:=\mathrm{UC}_{\mathrm{ord}}(K; L^0(\mathbb{X}))$, and we fix $\beta \in \mathrm{UC}\hat{\otimes}_\varepsilon \mathscr{N}$ and $\varepsilon > 0$. By Remark 5.10, we can find $n \in \mathbb{N}$, $(\eta_i)_{i=1}^n \in \mathscr{F}_n$, and $(w_i)_{i=1}^n \subseteq \mathscr{N}$ such that $\tilde{\beta}:=\sum_{i=1}^n (\eta_i(\cdot)\mathbb{1}_{\mathrm{X}}) \otimes w_i$ satisfies $\mathsf{d}_{\mathrm{UC}\hat{\otimes}_\varepsilon \mathscr{N}}(\tilde{\beta}, \beta) < \varepsilon/2$. Since T is a quotient operator, for any $i = 1, \dots, n$ we can find $v_i \in \mathscr{M}$ such that $T(v_i) = w_i$ and $|v_i| \le |w_i| + \delta$, where we have chosen some $\delta > 0$ for which $\mathsf{d}_{L^0(\mathbb{X})}(\delta\mathbb{1}_{\mathrm{X}}, 0) < \varepsilon/2$. Now define $\alpha:=\sum_{i=1}^n (\eta_i(\cdot)\mathbb{1}_{\mathrm{X}}) \otimes v_i \in \mathrm{UC}\hat{\otimes}_\varepsilon \mathscr{M}$. Since $(\mathrm{id} \otimes_\varepsilon T)(\alpha) = \tilde{\beta}$, we have $\mathsf{d}_{\mathrm{UC}\hat{\otimes}_\varepsilon \mathscr{N}}((\mathrm{id} \otimes_\varepsilon T)(\alpha), \beta) < \varepsilon$. Moreover, recalling (5.5) we see that $|\alpha|_\varepsilon = \bigvee_{i=1}^n |v_i| \le \delta + \bigvee_{i=1}^n |w_i| = |\tilde{\beta}|_\varepsilon + \delta \le |\alpha|_\varepsilon + \delta$, which yields $\mathsf{d}_{L^0(\mathbb{X})}(|\alpha|_\varepsilon, |\beta|_\varepsilon) \le \mathsf{d}_{L^0(\mathbb{X})}(\delta\mathbb{1}_{\mathrm{X}}, 0) + \mathsf{d}_{L^0(\mathbb{X})}(|\tilde{\beta}|_\varepsilon, |\beta|_\varepsilon) < \varepsilon$. Therefore, we can apply Lemma 3.4, which gives that $\mathrm{id} \otimes_\varepsilon T : \mathrm{UC}\hat{\otimes}_\varepsilon \mathscr{M} \to \mathrm{UC}\hat{\otimes}_\varepsilon \mathscr{N}$ is a quotient operator. □

5.3 Relation with duals and pullbacks

First of all, we provide a characterisation of the dual of $\mathscr{M}\hat{\otimes}_\varepsilon \mathscr{N}$. By $\mathbb{D}^{w^*}_{\mathscr{M}^*}$ we will mean the unit disc of $\mathscr{M}^*$ endowed with the restriction of the weak* topology. Moreover, the space $\mathbb{D}^{w^*}_{\mathscr{M}^*} \times \mathbb{D}^{w^*}_{\mathscr{N}^*}$ will be tacitly equipped with the product topology.

Theorem 5.12 (Dual of $\mathscr{M}\hat{\otimes}_\varepsilon \mathscr{N}$) *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$ and $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Then there exists a unique homomorphism of Banach $L^0(\mathbb{X})$-modules*

$$
\iota : \mathscr{M}\hat{\otimes}_\varepsilon \mathscr{N} \to \mathrm{C}_{\mathrm{pb}}(\mathbb{D}^{w^*}_{\mathscr{M}^*} \times \mathbb{D}^{w^*}_{\mathscr{N}^*}; L^0(\mathbb{X}))
$$

such that

$$
\iota(v \otimes w)(\omega, \eta) = \omega(v)\eta(w) \quad \forall (v, w, \omega, \eta) \in \mathscr{M} \times \mathscr{N} \times \mathbb{D}_{\mathscr{M}^*} \times \mathbb{D}_{\mathscr{N}^*}.
$$

Moreover, the homomorphism ι preserves the pointwise norm. In particular, it holds that

$$(\mathscr{M}\hat{\otimes}_\varepsilon\mathscr{N})^* \cong \mathrm{C}_{\mathrm{pb}}(\mathbb{D}^{w^*}_{\mathscr{M}^*} \times \mathbb{D}^{w^*}_{\mathscr{N}^*}; L^0(\mathbb{X}))^*/(\mathscr{M}\hat{\otimes}_\varepsilon\mathscr{N})^\perp.$$

Proof First of all, let us define $\iota\colon \mathscr{M} \otimes_\varepsilon \mathscr{N} \to \mathrm{C}_{\mathrm{pb}}(\mathbb{D}^{w^*}_{\mathscr{M}^*} \times \mathbb{D}^{w^*}_{\mathscr{N}^*}; L^0(\mathbb{X}))$ as

$$\iota(\alpha)(\omega,\eta) := \sum_{i=1}^n \omega(v_i)\eta(w_i)$$

for every $\alpha = \sum_{i=1}^n v_i \otimes w_i \in \mathscr{M} \otimes_\varepsilon \mathscr{N}$ and $(\omega,\eta) \in \mathbb{D}_{\mathscr{M}^*} \times \mathbb{D}_{\mathscr{N}^*}$. It can be easily checked that ι is well-posed and $L^0(\mathbb{X})$-linear. Moreover, (3.12) and (5.1) yield

$$|\iota(\alpha)| = \bigvee_{(\omega,\eta)\in\mathbb{D}_{\mathscr{M}^*}\times\mathbb{D}_{\mathscr{N}^*}} \big|\iota(\alpha)(\omega,\eta)\big| = \bigvee_{(\omega,\eta)\in\mathbb{D}_{\mathscr{M}^*}\times\mathbb{D}_{\mathscr{N}^*}} \left|\sum_{i=1}^n \omega(v_i)\eta(w_i)\right| = |\alpha|_\varepsilon$$

for every $\alpha = \sum_{i=1}^n v_i \otimes w_i \in \mathscr{M} \otimes_\varepsilon \mathscr{N}$, thus ι can be uniquely extended to a pointwise norm preserving homomorphism

$$\iota\colon \mathscr{M}\hat{\otimes}_\varepsilon\mathscr{N} \to \mathrm{C}_{\mathrm{pb}}(\mathbb{D}^{w^*}_{\mathscr{M}^*} \times \mathbb{D}^{w^*}_{\mathscr{N}^*}; L^0(\mathbb{X})).$$

For the last claim, see Lemma 3.3. □

We stress that in Theorem 5.12 we consider the space C_{pb}, differently from Sect. 5.2. It seems that in Theorem 5.12 the space C_{pb} cannot be replaced by the smaller space $\mathrm{UC}_{\mathrm{ord}}$. Furthermore, we point out that the description of $(\mathscr{M}\hat{\otimes}_\varepsilon\mathscr{N})^*$ provided by Theorem 5.12 is rather implicit if compared with the corresponding one for Banach spaces (see [21, Proposition 3.14]). Indeed, it is not clear whether the space $\mathrm{C}_{\mathrm{pb}}(\mathbb{D}^{w^*}_{\mathscr{M}^*} \times \mathbb{D}^{w^*}_{\mathscr{N}^*}; L^0(\mathbb{X}))^*$ can be described as a space of measures.

We conclude this section by proving that 'pullbacks and injective tensor products commute':

Theorem 5.13 (Pullbacks vs. injective tensor products) *Let $\mathbb{X} = (\mathrm{X}, \Sigma_{\mathrm{X}}, \mathfrak{m}_{\mathrm{X}})$, $\mathbb{Y} = (\mathrm{Y}, \Sigma_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}})$ be separable, σ-finite measure spaces. Let $\varphi\colon \mathrm{X} \to \mathrm{Y}$ be a measurable map with $\varphi_\#\mathfrak{m}_{\mathrm{X}} \ll \mathfrak{m}_{\mathrm{Y}}$. Let $\mathscr{M}$, $\mathscr{N}$ be Banach $L^0(\mathbb{Y})$-modules. Then it holds that*

$$\varphi^*(\mathscr{M}\hat{\otimes}_\varepsilon\mathscr{N}) \cong (\varphi^*\mathscr{M})\hat{\otimes}_\varepsilon(\varphi^*\mathscr{N}),$$

the pullback map $\varphi^*\colon \mathscr{M}\hat{\otimes}_\varepsilon\mathscr{N} \to (\varphi^*\mathscr{M})\hat{\otimes}_\varepsilon(\varphi^*\mathscr{N})$ being the unique homomorphism such that

$$\varphi^*(v \otimes w) = (\varphi^* v) \otimes (\varphi^* w) \quad \text{for every } v \in \mathscr{M} \text{ and } w \in \mathscr{N}.$$

Proof Let us define $T\colon \mathscr{M} \otimes_\varepsilon \mathscr{N} \to (\varphi^*\mathscr{M}) \otimes_\varepsilon (\varphi^*\mathscr{N})$ as

$$T\Big(\sum_{i=1}^n v_i \otimes w_i\Big) := \sum_{i=1}^n (\varphi^* v_i) \otimes (\varphi^* w_i) \quad \text{for every } \sum_{i=1}^n v_i \otimes w_i \in \mathscr{M} \otimes_\varepsilon \mathscr{N}.$$

The well-posedness of T can be proved exactly as in Theorem 4.13, while its linearity is clear. Moreover, for any $\alpha = \sum_{i=1}^n v_i \otimes w_i \in \mathscr{M} \otimes_\varepsilon \mathscr{N}$ we have

$$\begin{aligned}
|\alpha|_\varepsilon \circ \varphi &= \bigvee \left\{ \left| \sum_{i=1}^n \mathsf{l}_\varphi(\varphi^*\omega)(\varphi^* v_i)\, \mathsf{l}_\varphi(\varphi^*\eta)(\varphi^* w_i) \right| \;\middle|\; \omega \in \mathbb{D}_{\mathscr{M}^*},\ \eta \in \mathbb{D}_{\mathscr{N}^*} \right\} \\
&\leq \bigvee \left\{ \left| \sum_{i=1}^n \Xi(\varphi^* v_i)\Theta(\varphi^* w_i) \right| \;\middle|\; \Xi \in \mathbb{D}_{(\varphi^*\mathscr{M})^*},\ \Theta \in \mathbb{D}_{(\varphi^*\mathscr{N})^*} \right\} \\
&= |T(\alpha)|_\varepsilon.
\end{aligned}$$

Conversely, if $\xi = \sum_{j=1}^m \mathbb{1}_{E_j} \cdot \varphi^*\omega_j$ and $\theta = \sum_{k=1}^\ell \mathbb{1}_{F_k} \cdot \varphi^*\eta_k$ are given elements of $\mathscr{G}(\varphi^*[\mathbb{D}_{\mathscr{M}^*}])$ and $\mathscr{G}(\varphi^*[\mathbb{D}_{\mathscr{N}^*}])$, respectively, then

$$\begin{aligned}
\left| \sum_{i=1}^n \mathsf{l}_\varphi(\xi)(\varphi^* v_i)\, \mathsf{l}_\varphi(\theta)(\varphi^* w_i) \right| &= \sum_{j=1}^m \sum_{k=1}^\ell \mathbb{1}_{E_j \cap F_k} \left| \sum_{i=1}^n \omega_j(v_i)\eta_k(w_i) \right| \circ \varphi \\
&\leq |\alpha|_\varepsilon \circ \varphi.
\end{aligned}$$

Using Theorem 2.13, as well as the density of $\mathscr{G}(\varphi^*[\mathbb{D}_{\mathscr{M}^*}])$ and $\mathscr{G}(\varphi^*[\mathbb{D}_{\mathscr{N}^*}])$ in $\mathbb{D}_{\varphi^*\mathscr{M}^*}$ and $\mathbb{D}_{\varphi^*\mathscr{N}^*}$, respectively, we get $\left|\sum_{i=1}^n \Xi(\varphi^* v_i)\Theta(\varphi^* w_i)\right| \leq |\alpha|_\varepsilon \circ \varphi$ for all $\Xi \in \mathbb{D}_{\varphi^*\mathscr{M}^*}$ and $\Theta \in \mathbb{D}_{\varphi^*\mathscr{N}^*}$. It follows that $|T(\alpha)|_\varepsilon \leq |\alpha|_\varepsilon \circ \varphi$ holds for every $\alpha \in \mathscr{M} \otimes_\varepsilon \mathscr{N}$, thus accordingly T can be uniquely extended to a linear map $\varphi^*\colon \mathscr{M} \hat{\otimes}_\varepsilon \mathscr{N} \to (\varphi^*\mathscr{M}) \hat{\otimes}_\varepsilon (\varphi^*\mathscr{N})$ satisfying $|\varphi^*\alpha|_\varepsilon = |\alpha|_\varepsilon \circ \varphi$ for all $\alpha \in \mathscr{M} \hat{\otimes}_\varepsilon \mathscr{N}$. Finally, the fact that $\varphi^*[\mathscr{M} \hat{\otimes}_\varepsilon \mathscr{N}]$ generates $(\varphi^*\mathscr{M}) \hat{\otimes}_\varepsilon (\varphi^*\mathscr{N})$ can be proved as in Theorem 4.13, using Lemma 5.8 in place of Lemma 4.5. The proof is then complete. □

5.4 Pointwise crossnorms

Let us now introduce a class of 'tensor product pointwise norms':

Definition 5.14 (*Reasonable pointwise crossnorm*) Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$ and $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Then a pointwise norm $|\cdot|_c$ on $\mathscr{M} \otimes \mathscr{N}$ is said to be a **reasonable pointwise crossnorm** provided the following properties are verified:

(i) $|v \otimes w|_c \leq |v||w|$ for every $v \in \mathscr{M}$ and $w \in \mathscr{N}$.
(ii) $\omega \otimes \eta \in (\mathscr{M} \otimes \mathscr{N})^*_c$ and $|\omega \otimes \eta|_{c^*} \leq |\omega||\eta|$ for every $\omega \in \mathscr{M}^*$ and $\eta \in \mathscr{N}^*$, where we denote by $((\mathscr{M} \otimes \mathscr{N})^*_c, |\cdot|_{c^*})$ the dual of the normed $L^0(\mathbb{X})$-module $(\mathscr{M} \otimes \mathscr{N}, |\cdot|_c)$.

The projective pointwise norm and the injective pointwise norm are examples of reasonable pointwise crossnorms. In fact, they are the 'greatest' and the 'least' crossnorms, respectively:

Theorem 5.15 (Characterisation of reasonable pointwise crossnorms) *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$ and $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Let $|\cdot|_c$ be a given pointwise norm on $\mathscr{M} \otimes \mathscr{N}$. Then $|\cdot|_c$ is a reasonable pointwise crossnorm if and only if*

$$|\alpha|_\varepsilon \leq |\alpha|_c \leq |\alpha|_\pi \quad \textit{for every } \alpha \in \mathscr{M} \otimes \mathscr{N}. \tag{5.6}$$

Proof Suppose $|\cdot|_c$ is a reasonable pointwise crossnorm. Given any element $\alpha = \sum_{i=1}^n v_i \otimes w_i \in \mathscr{M} \otimes \mathscr{N}$, we can estimate

$$|\alpha|_c \leq \sum_{i=1}^n |v_i \otimes w_i|_c \leq \sum_{i=1}^n |v_i||w_i|,$$

thus by taking the infimum over all representations of α we get $|\alpha|_c \leq |\alpha|_\pi$. Moreover, we have that

$$\begin{aligned} |\alpha|_\varepsilon &= \bigvee \left\{ |(\omega \otimes \eta)(\alpha)| \;\middle|\; (\omega, \eta) \in \mathbb{D}_{\mathscr{M}^*} \times \mathbb{D}_{\mathscr{N}^*} \right\} \\ &\leq \bigvee \left\{ |\Theta(\alpha)| \;\middle|\; \Theta \in \mathbb{D}_{(\mathscr{M} \otimes \mathscr{N})_c^*} \right\} = |\alpha|_c. \end{aligned}$$

Conversely, if (5.6) is verified, then $|v \otimes w|_c \leq |v \otimes w|_\pi = |v||w|$ holds for every $(v, w) \in \mathscr{M} \times \mathscr{N}$. Moreover,

$$|(\omega \otimes \eta)(\alpha)| \leq |\omega \otimes \eta|_{\varepsilon^*} |\alpha|_\varepsilon \leq |\omega||\eta||\alpha|_c \quad \text{for every } (\omega, \eta) \in \mathscr{M}^* \times \mathscr{N}^*,$$

thus $\omega \otimes \eta \in (\mathscr{M} \otimes \mathscr{N})_c^*$ and $|\omega \otimes \eta|_{c^*} \leq |\omega||\eta|$. Hence, $|\cdot|_c$ is a reasonable pointwise crossnorm. □

Let $\mathbb{X}$ be a σ-finite measure space and $\mathscr{M}$, $\mathscr{N}$ Banach $L^0(\mathbb{X})$-modules. Let $|\cdot|_c$ be a reasonable pointwise crossnorm on $\mathscr{M} \otimes \mathscr{N}$. Let $G \subseteq \mathscr{M}$ and $H \subseteq \mathscr{N}$ be generating subsets. Then it follows from the second inequality in (5.6) and Lemma 4.5 that the set $\{v \otimes w \mid v \in G,\ w \in H\}$ generates the Banach $L^0(\mathbb{X})$-module obtained by taking the completion of $(\mathscr{M} \otimes \mathscr{N}, |\cdot|_c)$.

Proposition 5.16 *Let $\mathbb{X}$ be a σ-finite measure space. Let $\mathscr{M}$ and $\mathscr{N}$ be Banach $L^0(\mathbb{X})$-modules. Let $|\cdot|_c$ be a reasonable pointwise crossnorm on $\mathscr{M} \otimes \mathscr{N}$. Then it holds that*

$$|v \otimes w|_c = |v||w|, \quad |\omega \otimes \eta|_{c^*} = |\omega||\eta| \quad \forall (v, w, \omega, \eta) \in \mathscr{M} \times \mathscr{N} \times \mathscr{M}^* \times \mathscr{N}^*.$$

Proof First, (5.6) and (5.2) yield $|v \otimes w|_c \geq |v \otimes w|_\varepsilon = |v||w|$. Proposition 4.3 and (5.6) yield

$$|\omega \otimes \eta|_{c^*} = \bigvee \left\{ |(\omega \otimes \eta)(\alpha)| \;\middle|\; \alpha \in \mathscr{M} \otimes \mathscr{N},\ |\alpha|_c \leq 1 \right\}$$
$$\geq \bigvee \left\{ |(\omega \otimes_\pi \eta)(\alpha)| \;\middle|\; \alpha \in \mathbb{D}_{\mathscr{M} \otimes_\pi \mathscr{N}} \right\} = |\omega \otimes_\pi \eta| = |\omega||\eta|.$$

Therefore, the proof of the statement is complete. □

We conclude the paper with another important example of reasonable pointwise crossnorm. A Banach $L^0(\mathbb{X})$-module $\mathscr{H}$ is a **Hilbert $L^0(\mathbb{X})$-module** if $\langle \cdot, \cdot \rangle \in \mathrm{B}(\mathscr{H}, \mathscr{H})$, where we define

$$\langle v, w \rangle := \frac{|v + w|^2 - |v|^2 - |w|^2}{2} \in L^0(\mathbb{X}) \quad \text{for every } v, w \in \mathscr{H}.$$

The Riesz representation theorem for Hilbert $L^0(\mathbb{X})$-modules states that the space $\mathscr{H}$ is canonically isomorphic to its dual $\mathscr{H}^*$ via the operator

$$\mathscr{H} \ni v \mapsto \langle v, \cdot \rangle \in \mathscr{H}^*.$$

Now let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert $L^0(\mathbb{X})$-modules. Then we define the **Hilbert–Schmidt pointwise norm** on $\mathscr{H} \otimes \mathscr{K}$ as

$$|\alpha|_{\mathrm{HS}} := \left(\sum_{i,j=1}^n \langle v_i, v_j \rangle \langle w_i, w_j \rangle \right)^{1/2} \in L^0(\mathbb{X})^+ \tag{5.7}$$

for every $\alpha = \sum_{i=1}^n v_i \otimes w_i \in \mathscr{H} \otimes \mathscr{K}$. Following [11, Section 1.5], we define the **tensor product of Hilbert modules** $\mathscr{H} \otimes_{\mathrm{HS}} \mathscr{K}$ as the completion of the normed $L^0(\mathbb{X})$-module $(\mathscr{H} \otimes \mathscr{K}, |\cdot|_{\mathrm{HS}})$. It holds that $\mathscr{H} \otimes_{\mathrm{HS}} \mathscr{K}$ is a Hilbert $L^0(\mathbb{X})$-module. Also,

$$|\cdot|_{\mathrm{HS}} \quad \text{is a reasonable pointwise crossnorm on } \mathscr{H} \otimes \mathscr{K}.$$

Indeed, the identity $|v \otimes w|_{\mathrm{HS}} = |v||w|$ for all $(v, w) \in \mathscr{H} \times \mathscr{K}$ is a direct consequence of (5.7), thus Definition 5.14 i) holds. Definition 5.14 ii) then follows as well, by the Riesz representation theorem for Hilbert $L^0(\mathbb{X})$-modules. In particular, Theorem 5.15 ensures that

$$|\alpha|_\varepsilon \leq |\alpha|_{\mathrm{HS}} \leq |\alpha|_\pi \quad \text{for every } \alpha \in \mathscr{H} \otimes \mathscr{K}.$$

Acknowledgements I thank the reviewers for their comments and suggestions, which led to an improvement of the presentation.

Funding Open Access funding provided by University of Jyväskylä (JYU).

Data availability Not applicable

Declarations

Conflict of interest The author declares that he has no conflict of interest.

References

1. Aliprantis, C., Border, K.: Infinite Dimensional Analysis: A Hitchhiker's Guide, Studies in Economic Theory. Springer, Berlin (1999)
2. Benatti, L.: A review of differential calculus on metric measure spaces via L^0-normed modules. Master's thesis, University of Trieste (2018)
3. Bessaga, C., Pełczyński, A.: Selected Topics in Infinite-Dimensional Topology, Monografie Matematyczne. Polska Akademia Nauk, Instytut Matematyczny, Warszawa (1975)
4. Bogachev, V.I.: Measure Theory, vol. I, II. Springer, Berlin (2007)
5. Bourbaki, N.: Livre II algèbre chapitre III (état 4) algèbre multilinéaire (1948). http://archives-bourbaki.ahp-numerique.fr/files/original/97a9fed708bdde4dc55547ab5a8ff943.pdf. Accessed 06 August 2023
6. Bourbaki, N.: General Topology: Chapters 1–4. Elements of Mathematics. Springer, Berlin (1998)
7. Conrad, K.: Tensor products (2018). https://kconrad.math.uconn.edu/blurbs/linmultialg/tensorprod.pdf. Accessed 06 August 2023
8. Di Marino, S., Lučić, D., Pasqualetto, E.: Representation theorems for normed modules (2021). Preprint, arXiv:2109.03509
9. Fabian, M., Habala, P., Hájek, P., Montesinos, V., Zizler, V.: Banach Space Theory: The Basis for Linear and Nonlinear Analysis. CMS Books in Mathematics. Springer, New York (2010)
10. Gigli, N.: Lecture notes on differential calculus on RCD spaces. Publ. Res. Inst. Math. Sci. **54**, 855–918 (2018)
11. Gigli, N.: Nonsmooth differential geometry—an approach tailored for spaces with Ricci curvature bounded from below. Mem. Am. Math. Soc. **251**, 161 (2018)
12. Gigli, N., Pasqualetto, E.: Lectures on Nonsmooth Differential Geometry, SISSA Springer Series 2. Springer Cham, New York (2020)
13. Gigli, N., Lučić, D., Pasqualetto, E.: Duals and pullbacks of normed modules. To appear in: Isr. J. Math. (2022). arXiv:2207.04972
14. Guo, T.X.: Recent progress in random metric theory and its applications to conditional risk measures. Sci. China Math. **54**, 633–660 (2011)
15. Haydon, R., Levy, M., Raynaud, Y.: Randomly Normed Spaces. Travaux en Cours [Works in Progress], vol. 41. Hermann, Paris (1991)
16. Lučić, D., Pasqualetto, E.: The Serre–Swan theorem for normed modules. Rend. Circ. Mat. Palermo **2**(68), 385–404 (2019)
17. Lučić, D., Pasqualetto, E.: An axiomatic theory of normed modules via Riesz spaces (2023). Preprint, arXiv:2306.12238
18. Pabst, S.L.: On $\aleph_1$-free modules with trivial dual. Commun. Algebra **28**, 5053–5065 (2000)
19. Pasqualetto, E.: Limits and colimits in the category of Banach L^0-modules (2022). Preprint, arXiv:2211.13197
20. Rudin, W.: Real and Complex Analysis. Mathematics Series, McGraw-Hill, New York (1987)
21. Ryan, R.A.: Introduction to Tensor Products of Banach Spaces. Springer Monographs in Mathematics. Springer, London (2002)