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ORIGINAL PAPER

On the smoothness of normed spaces

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Abstract
The aim of the paper is to discuss and clarify some concepts of the geometric theory of normed spaces. We mainly intend to present recent results concerning the concept of smoothness of normed spaces in connection with the concepts of the strict and uniform convexity of those spaces.

Keywords Normed space · Strict convexity · Uniform convexity · Smoothnes · Uniform smoothness · Modulus of convexity · Modulus of smoothness

Mathematics Subject Classification 46B20

1 Introduction

The present paper has mostly the survey character and is dedicated to discuss some results concerning geometry of normed spaces. Let us pay attention to the fact that geometry of normed spaces deals mainly with geometric properties of balls in those spaces (cf. [15, 18, 21, 22, 28, 31]). Indeed, the geometry of the unit ball of a normed space seems to thoroughly reflect the most important geometric properties of the space in question.

It turns out that the basic concepts considered in the geometry of normed spaces are the concepts associated with the convexity and the smoothness of the unit ball of a normed space (see [15, 22, 31], for example). Those concepts are investigated,

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first of all, through the functions characterizing the mentioned properties of convexity and smoothness of normed spaces, known as moduli of convexity and moduli of smoothness [15, 22].

In the paper, we focus mainly on considerations connected with some moduli of smoothness.

Since the concept of uniform convexity introduced firstly by Clarkson [10] is investigated very broadly in several papers and monographs (see [21, 22] and references therein), in this paper we pay attention to the study of some moduli of smooothness which were introduced and investigated quite recently (cf. [3, 5]).

However, in our considerations we start with the discussion of the concepts of convexity and the modulus of convexity in normed spaces. Next, we are going to describe the concepts of smoothness and uniform smoothness of normed spaces and the classical modulus of smoothness associated with this concept of uniform smoothness. The classical modulus of smoothness was introduced by Day [13] and its properties were investigated also by Lindenstrauss [27]. In this paper we will investigate another modulus of smooothness defined in paper [3] (cf. also [5]) which seems to be connected naturally with the modulus of convexity. Apart from this, the modulus of smoothness defined in [3] has very useful properties and is very closely related to the classical Day's modulus of convexity. Based on modulus of smoothness defined in [3] and the classical Day's modulus of convexity we can introduce very useful and natural concept of the deformation of normed spaces.

The results of this survey paper are an outcome of results presented in papers [3, 5, 16, 19, 24, 30, 36]. It is worthwhile mentioning that the concept of the modulus of smoothness discussed in this paper was studied and applied in several papers and monographs (cf. [1–7, 11, 30, 37, 42, 43]).

2 Preliminaries

Throughout this paper, we consider a real normed space E with the norm $||\cdot||_E$ denoted also by $||\cdot||$ if it does not lead to misunderstanding. Further, the symbol E^* will stand for the dual space of E. The closed unit ball centered at θ in the space E will be denoted by B_E while S_E denotes the unit sphere with the center θ.

We give now the definitions of basic concepts considered in the paper (cf. [22, 34]).

Definition 2.1 We say that a norm $||\cdot||$ in the linear space E is *strictly convex*, if for arbitrary vectors $x, y \in E$ the following implication holds:

$$||x|| \leq 1, \quad ||y|| \leq 1, \quad x \neq y \quad \Longrightarrow \quad ||x+y|| < 2.$$

Let us quote the following theorem from [34] containing conditions equivalent to strict convexity.

Theorem 2.2 *Let $||\cdot||$ be a norm in the linear space E. The following conditions are equivalent:*

1. *The norm $||\cdot||$ is strictly convex.*

2. *The unit sphere S_E does not contain a segment distinct from a point.*
3. *If $x, y \in E$ are such that $||x + y|| = ||x|| + ||y||$, then $x = \theta$ or $y = \theta$ or $y = tx$ for some $t > 0$.*

Definition 2.3 We say that a norm $|| \cdot ||$ in the linear space E is *uniformly convex*, if the following condition is satisfied:

$$\underset{0<\varepsilon\leq 2}{\forall} \quad \underset{\delta>0}{\exists} \quad \underset{x,y\in B_E}{\forall} \left[||x - y|| \geq \varepsilon \implies \frac{1}{2}||x + y|| \leq 1 - \delta \right].$$

It can be shown that every uniformly convex norm is strictly convex (see [21]). To prove this fact let us take $x, y \in E$ such that $||x|| \leq 1$, $||y|| \leq 1$ and $x \neq y$. Put $\varepsilon = ||x - y||$. Obviously $\varepsilon \in (0, 2]$. In view of the assumption there exists $\delta > 0$ such that $\frac{1}{2}||x + y|| \leq 1 - \delta < 1$. Hence $||x + y|| < 2$. Thus the norm $|| \cdot ||$ is strictly convex.

Let us pay attention to the fact that the inverse implication is not valid i.e., a strictly convex norm in a linear space E is not necessarily uniformly convex.

Indeed (cf. [34]) consider the space c_0 consisting of all real sequences converging to zero. Then the norm in the space c_0 defined by the formula

$$||x|| = ||(x_n)|| = \sup\{|x_n| : n = 1, 2, \ldots\} + \left(\sum_{n=1}^{\infty} \frac{1}{2^n} x_n^2\right)^{\frac{1}{2}}$$

is strictly convex in c_0 but is not uniformly convex. An example of a norm which is similar to the norm given above can be found in [14] (see also [35]).

Now, we proceed to the concept of the smoothness. To this end assume that $E \neq \{\theta\}$ is a normed space with a norm $|| \cdot ||$. Let us recall that the Hahn–Banach theorem asserts that for every $x \in E$ there exists a functional $x^* \in E^*$ such that $||x^*|| = 1$ and $x^*(x) = ||x||$. This implies that the set

$$J(x) = \left\{x^* \in E^* : ||x^*|| = 1, \quad x^*(x) = ||x||\right\}$$

is nonempty. Obviously if $x \neq \theta$ then the set $J(x)$ is closed and convex. The above observation leads us to the following definition.

Definition 2.4 We say that a norm $|| \cdot ||$ in the linear space E is *smooth* if for every $x \in E$, $x \neq \theta$, there exists a unique functional $x^* \in E^*$ such that $||x^*|| = 1$ and $x^*(x) = ||x||$.

If the norm $|| \cdot ||$ is smooth then in the light of above definition the set $J(x)$ defined above is a singleton for any $x \neq \theta$.

Observe that if the norm $|| \cdot ||$ in E is not smooth then there exists $x \in S_E$ such that the set $J(x)$ is not a singleton. Hence, taking into account that $J(x)$ is a convex subset of the unit sphere S_{E^*}, we deduce that S_{E^*} contains a segment. This means that E^* is not strictly convex.

The above remarks allow us to infer the following theorem [17].

Theorem 2.5 *Let E be a Banach space with the norm $||\cdot||_E$. Then the following assertions hold.*

1. *If the norm $||\cdot||_{E^*}$ in E^* is strictly convex then the norm $||\cdot||_E$ is smooth.*
2. *If the norm $||\cdot||_{E^*}$ in E^* is smooth then the norm $||\cdot||_E$ is strictly convex.*

We will not develop here the theory of the smoothness and further facts associated with this concept (cf. [21, 22, 34] for other facts). However, we provide now the basic facts connected with the concept of the uniform smoothness.

We will start with the following definitions [13] (cf. also [26]).

Definition 2.6 We say that a normed space E (with the norm $||\cdot||$) is *uniformly smooth* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for $x, y \in E$ such that $||x|| \geq 1$, $||y|| \geq 1$ and $||x-y|| \leq \delta$ we have that $||x|| + ||y|| - ||x+y|| \leq \varepsilon||x-y||$.

Definition 2.7 A normed space E is referred to as uniformly smooth if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $||x|| = 1$, $||y|| \leq \delta$ implies that $||x+y|| + ||x-y|| \leq 2 + \varepsilon||y||$.

It can be shown that the above definitions are equivalent [26] and that every uniformly smooth space is smooth (see [8]).

For our further purposes let us notice that the assumptions $||x|| \geq 1$, $||y|| \geq 1$ in Definition 2.6 have no significance. Indeed, we may accept the following definition (cf. [3]).

Definition 2.8 A normed space E is called uniformly smooth if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for $x, y \in S_E$, $||x-y|| \leq \delta$, the inequality

$$1 - \frac{||x+y||}{2} \leq \varepsilon||x-y|| \tag{2.1}$$

holds true.

We show that Definition 2.8 is equivalent to those given previously.

To this end notice that if E is uniformly smooth in the sense of Definition 2.6 then it is also uniformly smooth in the sense of Definition 2.8. Coversely, let us assume that E is uniformly smooth in the sense of Definition 2.8 but is not uniformly smooth in the sense of Definition 2.6. Then there exist a number $\varepsilon_0 > 0$ and two sequences (x_n), (y_n) such that $||x_n|| \geq 1$, $||y_n|| \geq 1$, $||x_n - y_n|| \to 0$ as n tends to infinity. Moreover, we have

$$||x_n|| + ||y_n|| - ||x_n + y_n|| > \varepsilon_0||x_n - y_n|| \tag{2.2}$$

for $n = 1, 2, \ldots$. Without loss of generality we may assume that $||y_n|| \leq ||x_n||$ for $n \in \mathbb{N}$.

Next, putting $u_n = \frac{x_n}{||x_n||}$, $v_n = \frac{y_n}{||y_n||}$ we have $||u_n|| = ||v_n|| = 1$ and

$$||u_n - v_n|| \leq \left|\left|\frac{x_n}{||x_n||} - \frac{y_n}{||x_n||} + \frac{y_n}{||x_n||} - \frac{y_n}{||y_n||}\right|\right| \leq \frac{2}{||x_n||}||x_n - y_n||.$$

Hence we infer that $||u_n - v_n|| \to 0$ as $n \to \infty$. Additionally, we obtain

$$||x_n - y_n|| \geq \frac{||x_n||}{2}||u_n - v_n|| \tag{2.3}$$

for $n = 1, 2, \ldots$. Now, let us observe that in view of (2.2) we get

$$\begin{aligned}
2 - ||u_n + v_n|| &= \left|\left|\frac{x_n}{||x_n||}\right|\right| + \left|\left|\frac{y_n}{||y_n||}\right|\right| - \left|\left|\frac{x_n}{||x_n||} + \frac{y_n}{||y_n||}\right|\right| \\
&= \frac{1}{||x_n||}||x_n|| + \frac{1}{||x_n||}||y_n|| - \frac{1}{||x_n||}||y_n|| + \frac{1}{||y_n||}||y_n|| \\
&\quad - \left|\left|\left(\frac{x_n}{||x_n||} + \frac{y_n}{||x_n||}\right) + \left(-\frac{y_n}{||x_n||} + \frac{y_n}{||y_n||}\right)\right|\right| \\
&\geq \frac{1}{||x_n||}(||x_n|| + ||y_n||) \\
&\quad + ||y_n||\left(\frac{1}{||y_n||} - \frac{1}{||x_n||}\right) \\
&\quad - \frac{1}{||x_n||}||x_n + y_n|| - \frac{1}{||x_n||}\,|||x_n|| - ||y_n||| \\
&= \frac{1}{||x_n||}\,(||x_n|| + ||y_n|| - ||x_n + y_n||) \\
&\quad + ||y_n||\left(\frac{1}{||y_n||} - \frac{1}{||x_n||}\right) \\
&\quad - \frac{1}{||x_n||}\,|||x_n|| - ||y_n||| > \frac{1}{||x_n||}\varepsilon_0||x_n - y_n|| \\
&\quad + 1 - \frac{||y_n||}{||x_n||} - 1 + \frac{||y_n||}{||x_n||} = \frac{\varepsilon_0}{||x_n||}||x_n - y_n||.
\end{aligned}$$

The above estimate in combination with (2.3) implies

$$2 - ||u_n + v_n|| > \frac{\varepsilon_0}{2}||u_n - v_n||$$

for $n = 1, 2, \ldots$. But this inequality contradicts to our assumption and the proof is complete. □

3 Modulus of convexity

In this section we provide in short the basic facts concerning the concept of the modulus of convexity. This concept is related to the concept of a uniformly convex norm presented in Definition 2.3 and was introduced by Day [13].

Definition 3.1 Let E be a normed space with the norm $||\cdot||$. The function $\delta_E : [0, 2] \to [0, 1]$ defined by the formula

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{||x + y||}{2} : x, y \in B_E, ||x - y|| \geq \varepsilon \right\}$$

is called *the modulus of convexity* of the space E (or of the norm $||\cdot||_E$).

It can be shown that this modulus can be defined equivalently in the form

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{||x + y||}{2} : x, y \in S_E, ||x - y|| = \varepsilon \right\}$$

(see [12, 18, 22]).

Let us recall some properties of the modulus δ_E [2, 22, 31]. First of all let us recall that the function δ_E is nondecreasing on the interval $[0, 2]$. Moreover, it can be shown that the function δ_E is continuous on the interval $[0, 2)$ and may be discontinuous at the point $\varepsilon = 2$ only.

In order to present further properties of the modulus δ_E let us remind that the number $\varepsilon_0(E)$ defined as

$$\varepsilon_0(E) = \sup \{\varepsilon \geq 0 : \delta_E(\varepsilon) = 0\}$$

is called *the characteristic of convexity* of a space E. We have that E is uniformly convex if and only if $\varepsilon_0(E) = 0$ [22]. Moreover, we can also prove that the function δ_E is increasing on the interval $[\varepsilon_0(E), 2]$ (cf. [22]).

In what follows let us observe that if F is a subspace of E then $\delta_E(\varepsilon) \leq \delta_F(\varepsilon)$ for $\varepsilon \in [0, 2]$. This observation leads to the following equality

$$\delta_E(\varepsilon) = \inf \{\delta_F(\varepsilon) : F \text{ is a subspace of } E,\ \dim F = 2\}.$$

This shows that the modulus of convexity has a two-dimensional character.

Further, let us notice that in the case of an unitary space H with the norm generated by a scalar product, using the parallelogram law it is easily seen that

$$\delta_H(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \tag{3.1}$$

for $\varepsilon \in [0, 2]$. It is an interesting fact (cf. [33]) that the modulus of convexity of an unitary space majorizes the modulus of convexity of an arbitrary normed space i.e., for any normed space E we have

$$\delta_E(\varepsilon) \leq \delta_H(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \tag{3.2}$$

for $\varepsilon \in [0, 2]$.

Finally, we quote a theorem showing the relation between the characteristic of convexity of a normed space E and the limit $\lim_{\varepsilon\to 2^-}\delta_E(\varepsilon)$. Indeed, we have the following theorem [22].

Theorem 3.2 *For any normed space E we have*

$$\delta_E(2^-)=\lim_{\varepsilon\to 2^-}\delta_E(\varepsilon)=1-\frac{\varepsilon_0(E)}{2}.$$

Proof The case $\varepsilon_0(E)=2$ is obvious. Thus, assume that $\varepsilon_0(E)<2$. Fix $\varepsilon\in[\varepsilon_0(E),2)$. Then, for any $t\in(0,1-\delta_E(\varepsilon))$ we can choose $x,y\in B_E$ such that $||x-y||=\varepsilon$ and

$$\frac{||x+y||}{2}\geq 1-\delta_E(\varepsilon)-t.$$

Then we have

$$\begin{aligned}\frac{\varepsilon}{2}=\frac{||x-y||}{2}&\leq 1-\delta_E(||x-(-y)||)=1-\delta_E(||x+y||)\\&\leq 1-\delta_E(2(1-\delta_E(\varepsilon)-t)).\end{aligned}$$

Using the continuity of δ_E passing with $t\to 0$ and then $\varepsilon\to 2^-$ we have

$$\delta_E(2(1-\delta_E(2^-)))=0.$$

Hence we get that

$$\delta_E(2^-)\geq 1-\frac{\varepsilon_0(E)}{2}.$$

The reverse inequality follows when we take $t\to 0$ and $\varepsilon\to\varepsilon_0(E)$. The proof is complete. □

Some other properties of the modulus of convexity and some related constants as well as formulas expressing the mentioned modulus and constants in some normed spaces can be found in the papers ([2, 25, 32, 39–41], for example).

4 Moduli of smoothness

Through this section we will assume that E is a real linear space with the norm $||\cdot||_E=||\cdot||$. For simplicity we will say that E is a normed space with the norm $||\cdot||$.

Let us notice that based on Definition 2.7 Day [13] defined the so-called *modulus of smoothness* of a normed space E in the following way

$$\rho(\varepsilon)=\sup\left\{\frac{||x+y||+||x-y||-2}{2}:x\in S_E,\,||y||\leq\varepsilon\right\},$$

for $\varepsilon \in [0, 1]$. The function ρ has several properties [27]. For example, it is continuous and convex on the interval $[0, 1]$ and $\sqrt{1+\varepsilon^2}-1 \leq \rho(\varepsilon) \leq \varepsilon$ for $\varepsilon \in [0, 1]$. Moreover, it is easily seen that the space E is uniformly smooth if and only if $\lim_{\varepsilon \to 0} \frac{\rho(\varepsilon)}{\varepsilon} = 0$.

In what follows, in view of Definition 2.8, we may define other modulus of smoothness of a space E with help of the following formula (cf. [3])

$$\overline{\rho}_E(\varepsilon) = \sup\left\{1 - \frac{||x+y||}{2} : x, y \in S_E, ||x-y|| \leq \varepsilon\right\}, \tag{4.1}$$

for $\varepsilon \in [0, 2]$. Obviously we have (in the light of Definition 2.8) that a space E is uniformly smooth if and only if $\lim_{\varepsilon \to 0} \frac{\overline{\rho}_E(\varepsilon)}{\varepsilon} = 0$.

Further, let us observe that the function $\overline{\rho}_E$ is nondecreasing on the interval $[0, 2]$. Later we show that $\overline{\rho}_E$ is increasing on $[0, 2]$.

Moreover, we have the following result [3].

Lemma 4.1 *For any* $\varepsilon \in [0, 2]$ *the following inequality holds*

$$\overline{\rho}_E(\varepsilon) \leq \frac{\varepsilon}{2}.$$

Proof We have the following estimates

$$\begin{aligned}
\overline{\rho}_E(\varepsilon) &= \sup\left\{\frac{2-||x+y||}{2} : x, y \in S_E, ||x-y|| \leq \varepsilon\right\}\\
&= \sup\left\{\frac{2||x||-||x+y||}{2} : x, y \in S_E, ||x-y|| \leq \varepsilon\right\}\\
&= \sup\left\{\frac{||x+x||-||x+y||}{2} : x, y \in S_E, ||x-y|| \leq \varepsilon\right\}\\
&= \sup\left\{\frac{||(x+y)+(x-y)||-||x+y||}{2} : x, y \in S_E, ||x-y|| \leq \varepsilon\right\}\\
&\leq \sup\left\{\frac{||x+y||+||x-y||-||x+y||}{2} : x, y \in S_E, ||x-y|| \leq \varepsilon\right\}\\
&= \sup\left\{\frac{||x-y||}{2} : x, y \in S_E, ||x-y|| \leq \varepsilon\right\} = \frac{\varepsilon}{2}.
\end{aligned}$$

The proof is complete. □

Let us notice that in the case of the Banach space $C([a, b])$ with the standard maximum norm it is easily seen that $\overline{\rho}(\varepsilon) = \frac{\varepsilon}{2}$. This shows that the estimate from Lemma 4.1 is exact.

Further, let us observe that in the case of an unitary space H with the norm $||\cdot||$ generated by the scalar product of H, using the parallelogram identity it is easy to prove that

$$\overline{\rho}_H(\varepsilon) = \delta_H(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}, \quad \varepsilon \in [0, 2], \tag{4.2}$$

where $\delta_H(\varepsilon)$ is the modulus of convexity of the unitary space H (cf. formula (3.1)).

On the other hand, using the same argumentation as in paper [33] we can show that for any normed space E the following estimate holds

$$\overline{\rho}_H(\varepsilon) \leq \overline{\rho}_E(\varepsilon) \tag{4.3}$$

for any $\varepsilon \in [0, 2]$.

Thus, linking (4.2), (4.3) and Lemma 4.1 we obtain

$$1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \leq \overline{\rho}_E(\varepsilon) \leq \frac{\varepsilon}{2}, \quad \varepsilon \in [0, 2] \tag{4.4}$$

for every normed space E.

In what follows we recall a few facts associated with the geometry of two-dimensional normed space [23]. To this end assume that x, y are linearly independent elements of the normed space E. The set $L = L(x, y)$ defined as

$$L(x, y) = \{\alpha x + \beta y : \alpha \in \mathbb{R}, \beta \geq 0\}$$

is called two-dimensional half-plane (in the space E). In this situation the vector x is said to be *diametral point* of the half-plane L.

Theorem 4.2 *Let $\mathcal{L}$ denote the family of all two-dimensional half-planes in E. Then*

$$\overline{\rho}_E(\varepsilon) = \sup_{L \in \mathcal{L}} \overline{\rho}_L(\varepsilon), \quad \varepsilon \in [0, 2].$$

Next, we recall also the following lemma [4] (cf. also [23]).

Lemma 4.3 *Let $\varepsilon_1, \varepsilon_2$ be fixed positive numbers such that $\varepsilon_1 < \varepsilon_2 < 2$. Next, assume that y_1, y_2 are linearly independent vectors of S_E such that $||y_1 - y_2|| = \varepsilon_2$. Then, there exist vectors $z_1, z_2 \in S_E$ belonging to the half-plane $L(y_1, y_2)$ such that $||z_1 - z_2|| = \varepsilon_1$ and*

$$\left(1 - \frac{||y_1 + y_2||}{2}\right) - \left(1 - \frac{||z_1 + z_2||}{2}\right) \leq \frac{2\sqrt{5} + 1}{2} \cdot \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}.$$

Now, we are prepared to formulate the result on the continuity of the modulus of smoothness $\overline{\rho}_E$.

Theorem 4.4 [4] *The modulus of smoothness $\overline{\rho}_E = \overline{\rho}_E(\varepsilon)$ is continuous on the interval* $[0, 2]$.

Proof Assume that $\varepsilon_1, \varepsilon_2$ are arbitrarily fixed and $0 < \varepsilon_1 < \varepsilon_2 < 2$. Further, choose $\eta > 0$ sufficiently small. Based on Theorem 4.2 we can find $x, y \in S_E$ such that $||x - y|| = \varepsilon_2$ and

$$\overline{\rho}_E(\varepsilon_2) - \eta \leq 1 - \frac{||x + y||}{2} \leq \overline{\rho}_E(\varepsilon_2).$$

Next, in view of Lemma 4.3, we can select two points x_1, y_1 in the half-plane $L(x, y)$ such that $x_1, y_1 \in S_E$, $||x_1 - y_1|| = \varepsilon_1$ and

$$\left(1 - \frac{||x + y||}{2}\right) - \left(1 - \frac{||x_1 + y_1||}{2}\right) \leq \frac{2\sqrt{5} + 1}{2} \cdot \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}.$$

Hence, we obtain

$$\overline{\rho}_E(\varepsilon_2) - \eta - \left(1 - \frac{||x_1 + y_1||}{2}\right) \leq \frac{2\sqrt{5} + 1}{2} \cdot \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}$$

or, equivalently

$$\overline{\rho}_E(\varepsilon_2) - \left(1 - \frac{||x_1 + y_1||}{2}\right) \leq \frac{2\sqrt{5} + 1}{2} \cdot \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} + \eta.$$

Now, taking into account Theorem 4.2 we derive the following estimate

$$\overline{\rho}_E(\varepsilon_2) - \overline{\rho}_E(\varepsilon_1) \leq \frac{2\sqrt{5} + 1}{2} \cdot \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} + \eta.$$

The arbitrariness of the number η implies

$$\overline{\rho}_E(\varepsilon_2) - \overline{\rho}_E(\varepsilon_1) \leq \frac{2\sqrt{5} + 1}{2} \cdot \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}.$$

Hence we conclude that the function $\overline{\rho}_E(\varepsilon)$ is continuous on the interval $(0, 2)$. The continuity of the function $\overline{\rho}_E(\varepsilon)$ at the points $\varepsilon = 0$ and $\varepsilon = 2$ is a simple consequence of inequalities (4.4). The proof is complete. □

Now, we are going to prove that the modulus of smoothness $\overline{\rho}_E$ is a convex function on the interval $[0, 2]$. The proof of this fact given in [7] is not complete.

At the beginning let us fix $u, v \in E$ such that $u \neq \theta$, $v \neq \theta$ and denote by $N(u, v)$ the set of all pairs (x, y) such that $x, y \in B_E$ and $x - y = au$, $x + y = bv$ for some $a, b \geq 0$. Next, consider the function

$$\overline{\rho}_E(u, v, \varepsilon) = \sup\left\{1 - \frac{||x + y||}{2} : x, y \in N(u, v), ||x - y|| \leq \varepsilon\right\}. \tag{4.5}$$

Similarly as in the case of two-dimensional half-planes we can prove the following theorem.

Theorem 4.5 *Let $\mathcal{N}$ denote the family of all sets $N(u, v)$ in the space E. Then*

$$\overline{\rho}_E(\varepsilon) = \sup\left\{\overline{\rho}_E(u, v, \varepsilon) : N(u, v) \in \mathcal{N}\right\}.$$

In what follows assume that u, v are two fixed nonzero vectors in the space E and consider the function $\overline{\rho}_E(u, v, \varepsilon)$ defined by (4.5) for $\varepsilon \in [0, 2]$. Let $(x_1, y_1), (x_2, y_2) \in N(u, v)$ i. e., $x_1 - y_1 = a_1 u$, $x_1 + y_1 = b_1 v$, $x_2 - y_2 = a_2 u$, $x_2 + y_2 = b_2 v$ for some nonnegative numbers a_1, a_2, b_1, b_2. Moreover, we assume that $||x_1 - y_1|| \le \varepsilon_1$, $||x_2 - y_2|| \le \varepsilon_2$, where $\varepsilon_1 > 0$, $\varepsilon_2 > 0$. Put $x_3 = \frac{1}{2}(x_1 + x_2)$, $y_3 = \frac{1}{2}(y_1 + y_2)$. Then we have

$$x_3 - y_3 = \frac{x_1 + x_2}{2} - \frac{y_1 + y_2}{2} = \frac{x_1 - y_1}{2} + \frac{x_2 - y_2}{2} = \frac{a_1 u + a_2 u}{2} = \frac{a_1 + a_2}{2} u,$$
$$x_3 + y_3 = \frac{x_1 + x_2}{2} + \frac{y_1 + y_2}{2} = \frac{x_1 + y_1}{2} + \frac{x_2 + y_2}{2} = \frac{b_1 v + b_2 v}{2} = \frac{b_1 + b_2}{2} v.$$

This shows that $(x_3, y_3) \in N(u, v)$.

Further, we obtain

$$\begin{aligned}
||x_3 - y_3|| &= \left|\left|\frac{x_1 + x_2}{2} - \frac{y_1 + y_2}{2}\right|\right| = \left|\left|\frac{x_1 - y_1}{2} + \frac{x_2 - y_2}{2}\right|\right| \\
&= \left|\left|\frac{a_1 u}{2} + \frac{a_2 u}{2}\right|\right| = \frac{1}{2}(a_1 + a_2)\,||u|| = \frac{1}{2}(a_1 ||u|| + a_2 ||u||) \\
&= \frac{1}{2}(||x_1 - y_1|| + ||x_2 - y_2||) \le \frac{1}{2}(\varepsilon_1 + \varepsilon_2)
\end{aligned}$$

and

$$\begin{aligned}
1 - \frac{||x_3 + y_3||}{2} &= 1 - \frac{1}{2}\left|\left|\frac{x_1 + x_2}{2} + \frac{y_1 + y_2}{2}\right|\right| = 1 - \frac{1}{2}\left|\left|\frac{x_1 + y_1}{2} + \frac{x_2 + y_2}{2}\right|\right| \\
&= 1 - \frac{1}{2}\left|\left|\frac{b_1 v}{2} + \frac{b_2 v}{2}\right|\right| = 1 - \frac{1}{2} \cdot \frac{1}{2}(b_1 + b_2)||v|| = 1 \\
&\quad - \frac{1}{2}\left(\frac{1}{2} b_1 ||v|| + \frac{1}{2} b_2 ||v||\right) \\
&= 1 - \frac{1}{2}\left(\frac{1}{2}||x_1 + y_1|| + \frac{1}{2}||x_2 + y_2||\right) \\
&= \frac{1}{2}\left(2 - \frac{||x_1 + y_1||}{2} - \frac{||x_2 + y_2||}{2}\right) \\
&= \frac{1}{2}\left[\left(1 - \frac{||x_1 + y_1||}{2}\right) + \left(1 - \frac{||x_2 + y_2||}{2}\right)\right].
\end{aligned} \tag{4.6}$$

Now, taking the supremum over the right hand side and using the definition of the function $\overline{\rho}_E(u, v, \varepsilon)$, we get

$$\overline{\rho}_E\left(u, v, \frac{\varepsilon_1 + \varepsilon_2}{2}\right) \le \frac{1}{2}\left[\overline{\rho}_E(u, v, \varepsilon_1) + \overline{\rho}_E(u, v, \varepsilon_2)\right].$$

This shows that the function $\varepsilon \mapsto \overline{\rho}_E(u, v, \varepsilon)$ is J-convex. Hence, in view of Theorem 4.5 we conclude that the function $\overline{\rho}_E(\varepsilon)$ is J-convex, i.e., we have

$$\overline{\rho}_E\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right) \leq \frac{1}{2}\overline{\rho}_E(\varepsilon_1) + \frac{1}{2}\overline{\rho}_E(\varepsilon_2) \tag{4.7}$$

for $\varepsilon_1, \varepsilon_2 \in [0, 2]$.

Remark 4.6 Assume that I is an interval, $I \subset \mathbb{R}$. Let us recall that a function $f : I \to \mathbb{R}$ is called *J-convex* on the interval I, if for arbitrary $t, s \in I$ the following inequality holds

$$f\left(\frac{t+s}{2}\right) \leq \frac{f(t)+f(s)}{2}. \tag{4.8}$$

Moreover, the function f is said to be *convex* on the interval I if for $t, s \in I$ and for any $\alpha \in (0, 1)$ the following inequality is satisfied

$$f(\alpha t + (1-\alpha)s) \leq \alpha f(t) + (1-\alpha) f(s). \tag{4.9}$$

It is known that if f is J-convex on I then it satisfies inequality (4.9) for α belonging to a dense subset of the interval $(0, 1)$ and any convex function on an open interval I is continuous on I. Moreover, there exist functions being J-convex but not convex.

On the other hand we have the following results due to Bernstein and Doetsch [9] and Sierpiński [38].

Theorem 4.7 *Assume that I is an open interval and $f : I \to \mathbb{R}$ is a function J-convex on I. If f is bounded from above on the interval I then f is convex on I.*

Theorem 4.8 *Let I be an open interval and let $f : I \to \mathbb{R}$ be a J-convex and measurable function on I. Then f is convex on I.*

Corollary 4.9 *Assume that I is an open interval and $f : I \to \mathbb{R}$ is J-convex and nondecreasing on I. Then the function f is convex on I.*

Now, let us return to the considerations of the modulus of smoothness $\overline{\rho}_E$ defined by (4.1) on the interval $[0, 2]$. In view of the fact that that $\overline{\rho}_E$ is nondecreasing on the interval $[0, 2]$ and taking into account that $\overline{\rho}_E$ is J-convex on $[0, 2]$ (cf. (4.7)), on the basis of Corollary 4.9 we deduce that the modulus $\overline{\rho}_E$ is convex on the interval $(0, 2)$. Hence, in view of Remark 4.6 we conclude that the function $\overline{\rho}_E(\varepsilon)$ is continuous on the interval $(0, 2)$. Continuity of the function $\overline{\rho}_E(\varepsilon)$ at points $\varepsilon = 0$ and $\varepsilon = 2$ follows easily from inequalities (4.4).

Moreover, keeping in mind that $\overline{\rho}_E$ is nondecreasing and convex on the interval $[0, 2]$, in view of (4.4) we infer that $\overline{\rho}_E$ is increasing on the interval $[0, 2]$.

Gathering all above established facts we obtain another proof of Theorem 4.4. More precisely, we obtain the proof of the following theorem.

Theorem 4.10 *The modulus of smoothness $\overline{\rho}_E = \overline{\rho}_E(\varepsilon)$ is convex, increasing and continuous on the interval* $[0, 2]$.

Remark 4.11 Analogous considerations to those leading to estimate (4.6) and inequality (4.7) but with inequalities $||x_1 - y_1|| \leq \varepsilon_1$, $||x_2 - y_2|| \leq \varepsilon_2$ replaced by $||x_1 - y_1|| \geq \varepsilon_1$, $||x_2 - y_2|| \geq \varepsilon_2$ were conducted also in paper [20] (cf. also [7]). Namely, in the mentioned paper the author used incorrectly the "inequality" asserting that the infimum of the sum of two functions is less or equal than the sum of infimum of those functions. From that "reasoning" the author of [20] concluded that the modulus of convexity is a convex function on the interval $[0, 2]$. But such a conclusion is not valid, in general (cf. [29]).

5 Deformation of normed spaces

This final and short section is devoted to introduce a concept joining the concepts of convexity and smoothness of a normed space. Thus, assume that E is a normed space with a norm $||\cdot|| = ||\cdot||_E$. Let $\delta_E, \overline{\rho}_E$ denote the modulus of convexity (cf. Definition 3.1) and the modulus of smoothness defined by (4.1) of the space E, respectively. Then, in view of estimates (3.2) and (4.4) we have that

$$\overline{\rho}_E(\varepsilon) - \delta_E(\varepsilon) \geq 0 \tag{5.1}$$

for each $\varepsilon \in [0, 2]$.

Based on inequality (5.1) we can define the concept of the modulus of deformation of the normed space E [4].

Definition 5.1 The function $d_E : [0, 2] \to [0, 1]$ defined in the following way

$$d_E(\varepsilon) = \overline{\rho}_E(\varepsilon) - \delta_E(\varepsilon), \qquad \varepsilon \in [0, 2], \tag{5.2}$$

will be called *the modulus of deformation* of the space E.

Let us notice that in view of inequalities (4.4), (3.2) and the obvious inequality $\delta_E(\varepsilon) \geq 0$ for $\varepsilon \in [0, 2]$, we deduce that

$$0 \leq d_E(\varepsilon) \leq \overline{\rho}(\varepsilon) \leq \frac{\varepsilon}{2} \leq 1 \tag{5.3}$$

for each $\varepsilon \in [0, 2]$. Thus, the values of the modulus of deformation d_E are really located in the interval $[0, 1]$.

Based on (5.2) we can introduce the concept of the *deformation of the space* E as a number D_E defined as

$$D_E = \sup\{d_E(\varepsilon) : \varepsilon \in [0, 2]\}$$

(cf. [7]).

For example, if we take $E = H$, where H denotes a Hilbert space, then in view of (3.1) and (4.2) we have that $d_H(\varepsilon) = 0$ for $\varepsilon \in [0, 2]$. Thus $D_H = 0$ and this fact can be interpreted that the Hilbert space has the smallest deformation among all normed spaces.

On the other hand taking the classical space $C = C([a,b])$ (with the maximum norm) we have that $\overline{\rho}_C(\varepsilon) = \frac{\varepsilon}{2}$ and $\delta_C(\varepsilon) = 0$ for $\varepsilon \in [0,2]$. Thus $d_C(\varepsilon) = \frac{\varepsilon}{2}$ for $\varepsilon \in [0,2]$ and $D_C = 1$. Thus the space C has the possible biggest deformation in the class of normed spaces.

We will not discuss other facts concerning both the concept of the modulus of deformation and the deformation of normed spaces. Our knowledge about mentioned concept is not large, unfortunately (cf. [4, 7], for example). At present stage the properties of $d_E(\varepsilon)$ and D_E create the challenge for investigators.

Data availability The authors declare that the data supporting the findings of this study are available within the paper and its supplementary information files.

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