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Phillips symmetric operators and functional calculus of maximal symmetric operators

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Abstract

The aim of this work is to develop a functional calculus for simple maximal symmetric operators. The proposed approach is based on the properties of self-adjoint extensions of Phillips symmetric operators. The obtained results are applied to the description of non-cyclic vectors of backward shift operators.

Keywords Maximal symmetric operator · Unilateral shift · Functional calculus · Non-cyclic vectors · Inner function

Mathematics Subject Classification 47B25 · 47A10

1 Introduction

Let S be a symmetric operator acting in a Hilbert space $\mathfrak{H}$ and let $\mathfrak{U}$ be a family of unitary operators in $\mathfrak{H}$ such that the inclusion $U \in \mathfrak{U}$ implies $U^* \in \mathfrak{U}$. The operator S is called $\mathfrak{U}$-invariant if S commutes with all $U \in \mathfrak{U}$. Does there exist at least one $\mathfrak{U}$-invariant self-adjoint extension of S? The answer is affirmative if S is a semibounded operator and the Friedrichs extension of S gives the required example.

In the general case of non-semibounded operators, Phillips constructed a symmetric operator S and a family $\mathfrak{U}$ of unitary operators commuting with S such that S has no $\mathfrak{U}$-invariant self-adjoint extensions [14, p. 382]. It was discovered by Kochubei [7]

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that the characteristic function of the symmetric operator[1] constructed in the Phillips work is a constant in the upper half-plane $\mathbb{C}_+$. This observation was used in [9] for the general definition of Phillips symmetric operators. Namely, we say that a closed densely defined symmetric operator with equal defect numbers is *a Phillips symmetric operator* (PSO) if its characteristic function is an operator-constant on $\mathbb{C}_+$.

Self-adjoint and, more generally, proper extensions of PSO possess a lot of curious properties [3, 7, 9, 10]. Part of them is used for the development of the functional calculus of simple maximal symmetric operators[2] in the present paper.

The functional calculus for simple maximal symmetric operators was announced by Plesner in two short papers [15, 16] without proofs. To the best of our knowledge, these papers have not been translated.

In the present paper, we propose an approach to the functional calculus that is based on the properties of self-adjoint extensions of PSO (Theorems 2.2, 2.4) and it gives rise to the all-around development of Plesner's ideas. Such kind of functional calculus turns out to be useful in the Lax–Phillips scattering theory [5, 6].

Let B be a simple maximal symmetric operator in $\mathfrak{H}$. In Sect. 3 we define an operator $\psi(B)$ for each Lebesgue measurable function ψ and investigate its properties (Proposition 3.2, Theorem 3.3). The results are simplified if $\psi \in H^\infty$. In this case, Plesner's functional calculus is reduced to functional calculus for the special class of self-adjoint operators (Theorem 3.4, Corollaries 3.5, 3.6). Further, we discuss a relationship of our results with functional calculus for unilateral shifts which is a special case of functional calculus for completely nonunitary contractions [18, Chapter III].

Special attention is paid to the case of inner functions ψ (Proposition 3.7, Corollary 3.8). An application of the functional calculus to the description of non-cyclic vectors of the backward shift operator is considered (Propositions 3.10, 3.11).

Throughout the paper $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\ker A$, $\rho(A)$, and $\sigma(A)$ denote *the domain*, *the range*, *the null-space*, *the resolvent set*, and *the spectrum* of a linear operator A, respectively, while $A|_{\mathcal{D}}$ stands for *the restriction* of A to the set $\mathcal{D}$. *The continuous spectrum* $\sigma_c(A)$ of a linear operator A consists of $\lambda \in \sigma(A)$ for which there exists a non-compact sequence $\{f_n\}$ such that $\|f_n\| = 1$ and $\lim_{n\to\infty} \|(A - \lambda I)f_n\| = 0$.

A subspace $\mathcal{K}$ of a Hilbert space $\mathfrak{H}$ is said to be *an invariant subspace* of the operator A in $\mathfrak{H}$ if for $f \in \mathcal{D}(A) \cap \mathcal{K}$ we have $Af \in \mathcal{K}$. A subspace $\mathcal{K}$ of $\mathfrak{H}$ is said to be *a reducing subspace* for A if $P_{\mathcal{K}}\mathcal{D}(A) \subset \mathcal{D}(A)$, where $P_{\mathcal{K}}$ is a orthogonal projection onto $\mathcal{K}$ and the subspaces $\mathcal{K}$ and $\mathcal{K}^\perp = \mathcal{H} \ominus \mathcal{K}$ are invariant for A.

The symbols $H^p(\mathbb{D})$ and $H^p(\mathbb{C}_+)$ are used for the Hardy spaces in $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, respectively. The Sobolev space is denoted as $W_2^p(I)$ ($I \in \{\mathbb{R}, \mathbb{R}_+ = [0, \infty)\}$, $p \in \{1, 2\}$). The notations $H^p(\mathbb{D}, N)$ and $H^p(\mathbb{C}_+, N)$, and $W_2^p(I, N)$ are used for the Hardy and Sobolev spaces of vector functions with values in an auxiliary Hilbert space N. The symbol $< n >$ means linear span of an element $n \in N$, while $\bigvee_\alpha X_\alpha$ means the closure of the linear span of the sets (vectors) X_α.

[1] The concept of characteristic functions of a symmetric operator was introduced by Livšic [11] and, further, substantially developed by Shtraus [17] and Kochubei [8].

[2] See the corresponding definitions in Sect. 2.1

2 Phillips symmetric operators

2.1 Simple maximal symmetric operator

Let B be a densely defined symmetric operator in a Hilbert space $\mathfrak{H}$ with inner product $(\cdot,\cdot)$ linear in the first argument. The *defect numbers* of B in $\mathbb{C}_+$ and $\mathbb{C}_-$ are defined as

$$m_+(B) = \dim\ker(B^* - iI) \quad \text{and} \quad m_-(B) = \dim\ \ker(B^* + iI), \tag{2.1}$$

respectively, where B^* is the adjoint of B.

A closed symmetric operator B is called *maximal symmetric* if one of its defect numbers is equal to zero. If B is a maximal symmetric operator, then $\mathbb{C}_- \subset \rho(B)$ for $m_+(B) = 0$ and $\mathbb{C}_+ \subset \rho(B)$ for $m_-(B) = 0$.

A closed symmetric operator B is *simple* if there is no subspace that reduces it and on which it induces a self-adjoint operator.

One of the simplest examples of a simple maximal symmetric operator is the first derivative operator considered in $L_2(\mathbb{R}_+, N)$

$$\mathcal{B} = i\frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathcal{D}(\mathcal{B}) = \{u \in W_2^1(\mathbb{R}_+, N) : u(0) = 0\}. \tag{2.2}$$

In this case, $m_+(\mathcal{B}) = 0$ and $m_-(\mathcal{B}) = \dim N$. Each simple maximal symmetric operator B with $m_+(B) = 0$ is unitarily equivalent to the operator $\mathcal{B}$ defined by (2.2), where $m_-(B) = \dim N$ [2, § 104]. Namely, there exists a unitary operator $\Xi_+ : \mathfrak{H} \to L_2(\mathbb{R}_+, N)$ such that

$$B = \Xi_+^{-1}\mathcal{B}\Xi_+, \quad \mathcal{D}(B) = \Xi_+^{-1}\mathcal{D}(\mathcal{B}). \tag{2.3}$$

Similarly, if B' is a simple maximal symmetric operator with $m_-(B) = 0$ acting in a Hilbert space $\mathfrak{H}'$ then there exists a unitary operator $\Xi_- : \mathfrak{H}' \to L_2(\mathbb{R}_-, N)$ such that

$$B' = \Xi_-^{-1}\mathcal{B}'\Xi_-, \quad \mathcal{D}(B') = \Xi_-^{-1}\mathcal{D}(\mathcal{B}'), \tag{2.4}$$

where

$$\mathcal{B}' = i\frac{\mathrm{d}}{\mathrm{d}x}, \qquad \mathcal{D}(\mathcal{B}') = \{u \in W_2^1(\mathbb{R}_-, N) : u(0) = 0\}, \quad \dim N = m_+(B').$$

2.2 Phillips symmetric operators

The original definition of PSO deals with the concept of the characteristic function. For our considerations, it is convenient to use an equivalent definition established in [9].

Definition 2.1 A symmetric operator S in a Hilbert space $\widehat{\mathfrak{H}}$ with equal defect numbers $m = m_-(S) = m_+(S)$ is called a Phillips symmetric operator (PSO) if $\widehat{\mathfrak{H}}$ can be decomposed into the orthogonal sum $\widehat{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}' \oplus \mathfrak{H}''$ of reducing subspaces of S and such that

$$S = B \oplus B' \oplus B'',$$

where $B = S|_{\mathfrak{H}}$ and $B' = S|_{\mathfrak{H}'}$ are simple maximal symmetric operators in $\mathfrak{H}$ and $\mathfrak{H}'$, respectively with non-zero defect numbers $m_-(B) = m_+(B') = m$ and $B'' = S|_{\mathfrak{H}''}$ is a self-adjoint operator in $\mathfrak{H}''$.

It follows from the definition above and the relations (2.3), (2.4) that every simple PSO S is unitarily equivalent to the symmetric momentum operator with one-point interaction

$$\mathcal{S} = i\frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathcal{D}(\mathcal{S}) = \{u \in W_2^1(\mathbb{R}, N) : u(0) = 0\} \tag{2.5}$$

acting in the Hilbert space $L_2(\mathbb{R}, N)$, where $\dim N = m$.

The following two theorems are principal for our presentation.

Theorem 2.2 [3, 9] *Self-adjoint extensions of a PSO S are unitarily equivalent to each other. If S is a simple PSO, then its self-adjoint extensions are unitarily equivalent to the self-adjoint momentum operator in $L_2(\mathbb{R}, N)$*

$$\mathcal{A} = i\frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathcal{D}(\mathcal{A}) = W_2^1(\mathbb{R}, N). \tag{2.6}$$

Definition 2.3 [2, § 111] Let B be a maximal symmetric operator in $\mathfrak{H}$. A self-adjoint extension A of B that acts in a wider Hilbert space $\widehat{\mathfrak{H}} \supset \mathfrak{H}$ is called *minimal* if it has no reducing subspaces of $\widehat{\mathfrak{H}} \ominus \mathfrak{H}$ except trivial one.

Theorem 2.4 *Assume that an operator A acting in a Hilbert space $\widehat{\mathfrak{H}}$ is a minimal self-adjoint extension of a simple maximal symmetric operator B acting in a subspace $\mathfrak{H} \subset \widehat{\mathfrak{H}}$. Then A is unitarily equivalent to the multiplication operator*

$$(\mathcal{M}f)(\delta) = \delta f(\delta), \quad \mathcal{D}(\mathcal{M}) = \{f \in L_2(\mathbb{R}, N) : \delta f(\delta) \in L_2(\mathbb{R}, N)\} \tag{2.7}$$

by independent variable in the space $L_2(\mathbb{R}, N)$, where $\dim N$ *coincides with the nonzero defect number of B.*

Proof Denote $\mathfrak{H}' = \widehat{\mathfrak{H}} \ominus \mathfrak{H}$ and consider the operator

$$B' = A|_{\mathcal{D}(B')}, \quad \mathcal{D}(B') = (A - iI)^{-1}\mathfrak{H}'. \tag{2.8}$$

Assume that $m_+(B) = 0$ and verify that the operator B' defined by (2.8) turns out to be simple maximal symmetric in $\mathfrak{H}'$ and $m_-(B') = 0$. Indeed, assume that $u \in \mathcal{D}(B')$ and $g \in \mathfrak{H}$. In view of (2.8), there exists $h \in \mathfrak{H}'$ such that $u = (A - iI)^{-1}h$. On the other hand, $\mathcal{R}(B+iI)=\mathfrak{H}$ since $m_+(B)=0$ and, hence, there exists $v\in\mathcal{D}(B)$ such that $g=(B+iI)v$. Therefore,

$$(u,g)=((A-iI)^{-1}h,(B+iI)v)=(h,v)=0$$

that means that $\mathcal{D}(B')\subset\mathfrak{H}'$. Moreover,

$$B'u=A(A-iI)^{-1}h=h+i(A-iI)^{-1}h=h+iu. \tag{2.9}$$

That gives $B'u\in\mathfrak{H}'$. Therefore, B' is a symmetric operator in $\mathcal{H}'$.

In view of (2.9), $(B'-iI)u=h$, where h is an arbitrary vector from $\mathfrak{H}'$. Hence, $\mathcal{R}(B'-iI)=\mathfrak{H}'$ and B' is a maximal symmetric operator in $\mathfrak{H}'$ with $m_-(B')=0$. The operator B' is simple because A is minimal in the sense of Definition 2.3.

Consider a simple symmetric operator

$$S=B\oplus B',\quad \mathcal{D}(S)=\mathcal{D}(B)\oplus\mathcal{D}(B') \tag{2.10}$$

in the Hilbert space $\widehat{\mathfrak{H}}=\mathfrak{H}\oplus\mathfrak{H}'$. Its defect numbers are $m_-(S)=m_-(B)$ and $m_+(S)=m_+(B')$ by the construction. Taking into account that A is a self-adjoint extension of S in $\widehat{\mathfrak{H}}$ we arrive at the conclusion that $m_-(B)=m_+(B')$. In view of Definition 2.1, S is a simple PSO. By means Theorem 2.2, A is unitarily equivalent to the momentum operator $\mathcal{A}$ in $L_2(\mathbb{R},N)$, where $\dim N=m_-(B)$. Taking into account that the operators $\mathcal{A}$ and $\mathcal{M}$ are unitarily equivalent we complete the proof for the case $m_+(B)=0$.

If $m_-(B)=0$, then one should put $\mathcal{D}(B')=(A+iI)^{-1}\mathfrak{H}'$ in (2.8) and, repeating the argumentation above, to show that B' is a simple maximal symmetric operator in $\mathfrak{H}'$ with $m_+(B')=0$. □

The self-adjoint momentum operator $\mathcal{A}$ acting in $L_2(\mathbb{R},N)$ and defined by (2.6) is an example of a minimal self-adjoint extension of a simple maximal symmetric operator $\mathcal{B}$ defined by (2.2). Applying the Fourier transformation[3]

$$Ff(\delta)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{i\delta x}f(x)\mathrm{d}x \tag{2.11}$$

we obtain $F\mathcal{A}=\mathcal{M}F$ i.e., $\mathcal{A}$ is unitarily equivalent to the multiplication operator $\mathcal{M}$ defined by (2.7).

[3] The Fourier transformation is considered as the unitary mapping of $L_2(\mathbb{R},N)$ itself. The integral has the usual meaning of L_2-limits of the partial transformations $(2\pi)^{-\frac{1}{2}}\int_{-a}^{a}\mathrm{e}^{\pm i\delta x}f(x)\mathrm{d}x$, $a\to\infty$.

3 Functional calculus of a simple maximal symmetric operator

3.1 Definition of $\psi(B)$

Let B be a symmetric operator in a Hilbert space $\mathfrak{H}$. A family of bounded operators $\{E_\delta\}$ $(\delta \in \mathbb{R})$ in $\mathfrak{H}$ is called *a spectral function of* B if the following conditions hold [2, p. 396]:

1. $E_{\delta_2} - E_{\delta_1}$ is nonnegative for $\delta_2 > \delta_1$;
2. $E_{\delta+0} = E_\delta$;
3. $\lim_{\delta\to-\infty} E_\delta f = 0,\ \lim_{\delta\to\infty} E_\delta f = f,\ f \in \mathfrak{H}$;
4. $(Bf, g) = \int_{-\infty}^{\infty} \delta d(E_\delta f, g),\ f \in \mathcal{D}(B),\ g \in \mathfrak{H}$;
5. $\|Bf\|^2 = \int_{-\infty}^{\infty} \delta^2 d(E_\delta f, f),\ f \in \mathcal{D}(B)$.

According to Naimark's results [2, Section 9], each symmetric operator possesses at least one spectral function and every spectral function $\{E_\delta\}$ of a symmetric operator B has the form

$$E_\delta f = P E_\delta^A f, \quad f \in \mathfrak{H}, \tag{3.1}$$

where $\{E_\delta^A\}$ is a spectral function of a self-adjoint extension A of B, which is obtained by emerging from the space $\mathfrak{H}$ into a space $\widehat{\mathfrak{H}} \supset \mathfrak{H}$ and P is the orthogonal projection operator in $\widehat{\mathfrak{H}}$ onto $\mathfrak{H}$.

It is important that a spectral function of a *maximal* symmetric operator is determined uniquely[4] [2, p. 402]). For this reason, similarly to the self-adjoint case, one can try to define operators

$$\psi(B) = \int_{-\infty}^{\infty} \psi(\delta) \mathrm{d}E_\delta \tag{3.2}$$

for some class of functions ψ. To do that we consider an operator of multiplication by a Lebesgue measurable function ψ in the space $L_2(\mathbb{R}, N)$. It can be presented as a function $\psi(\mathcal{M})$ of the multiplication operator $\mathcal{M}$ by an independent variable in $L_2(\mathbb{R}, N)$, i.e.,

$$\begin{aligned}
&\psi(\mathcal{M}) = \int_{-\infty}^{\infty} \psi(\delta) \mathrm{d}E_\delta^{\mathcal{M}}, \\
&\mathcal{D}(\psi(\mathcal{M})) = \{f \in L_2(\mathbb{R}, N) : \\
&\int_{-\infty}^{\infty} |\psi(\delta)|^2 \mathrm{d}(E_\delta^{\mathcal{M}} f, f) = \int_{-\infty}^{\infty} |\psi(\delta) f(\delta)|^2 \mathrm{d}\delta < \infty\},
\end{aligned}$$

where the operators $E_\delta^{\mathcal{M}}$ of the spectral function of $\mathcal{M}$ act as the multiplication by characteristic function $\chi_{(-\infty,\delta]}$ of the intervals $(-\infty, \delta]$.

[4] The difference with the self-adjoint case consists in the fact that E_δ are not orthogonal projections and the relation $E_r E_s = E_{\min\{r,s\}}$ does not hold.

Let an operator A in $\widehat{\mathfrak{H}}$ be a minimal self-adjoint extension of a simple maximal symmetric operator B acting in $\mathfrak{H} \subset \widehat{\mathfrak{H}}$. In view of Theorem 2.4, A is unitarily equivalent to $\mathcal{M}$, i.e., there exists a unitary mapping $G : \widehat{\mathfrak{H}} \to L_2(\mathbb{R}, N)$ such that

$$GA = \mathcal{M}G \quad \text{and} \quad GE^A_\delta = E^{\mathcal{M}}_\delta G.$$

This means that the operator

$$\psi(A) = \int_{-\infty}^{\infty} \psi(\delta)\mathrm{d}E^A_\delta, \quad \mathcal{D}(\psi(A)) = G^{-1}\mathcal{D}(\psi(\mathcal{M}))$$

is well defined and

$$\psi(\mathcal{M})G = \int_{-\infty}^{\infty} \psi(\delta)\mathrm{d}E^{\mathcal{M}}_\delta G = \int_{-\infty}^{\infty} \psi(\delta)\mathrm{d}GE^A_\delta = G\int_{-\infty}^{\infty} \psi(\delta)\mathrm{d}E^A_\delta = G\psi(A). \tag{3.3}$$

By virtue of (3.1) and (3.2),

$$(\psi(A)f, g) = \int_{-\infty}^{\infty} \psi(\delta)\mathrm{d}(E^A_\delta f, g) = \int_{-\infty}^{\infty} \psi(\delta)\mathrm{d}(E_\delta f, g) = (\psi(B)f, g) \tag{3.4}$$

where $f \in \mathcal{D}(\psi(A)) \cap \mathfrak{H}$ and $g \in \mathfrak{H}$. Therefore, for a simple maximal symmetric operator B, the operator $\psi(B)$ in (3.2) can be defined as follows

$$\psi(B)f = P\psi(A)f, \quad f \in \mathcal{D}(\psi(B)) = \mathcal{D}(\psi(A)) \cap \mathfrak{H}, \tag{3.5}$$

where A is a minimal self-adjoint extension of B in $\widehat{\mathfrak{H}}$, P is an orthoprojector in $\widehat{\mathfrak{H}}$ onto $\mathfrak{H}$, and ψ is a Lebesgue measurable function.

The operator $\psi(B)$ in (3.5) does not depend on the choice of minimal self-adjoint extension A since the operators $E_\delta = PE^A_\delta$ in (3.4) do not depend on the choice of[5] A.

Remark 3.1 The condition of the simplicity of a maximal symmetric operator in (3.5) is important. If there exists a reducing subspace $\mathfrak{H}_0 \subset \mathfrak{H}$ of B, then its minimal self-adjoint extension A involves a self-adjoint part $B_0 = B|_{\mathfrak{H}_0}$ acting in $\mathfrak{H}_0$. In this case, one cannot guarantee the existence of $\psi(A)$ for arbitrary Lebesgue measurable function.

The next statement makes more precise the domain of $\psi(B)$.

Proposition 3.2 *Let B be a simple maximal symmetric operator in $\mathfrak{H}$ and let $\psi \in L_p(\mathbb{R})$, $p \geq 2$. Then, for every $\mu \in \mathbb{C}\backslash\mathbb{R}$,*

$$\ker(B^* - \mu I) \subset \mathcal{D}(\psi(B)). \tag{3.6}$$

[5] Since the spectral function $\{E_\delta\}$ of a maximal symmetric operator is determined uniquely.

Proof Without loss of generality, one may assume $m_+(B) = 0$. Then (see (2.1)) the subspace $\ker(B^* - \mu I)$ is non-trivial for $\mu \in \mathbb{C}_-$.

Let A be a minimal self-adjoint extension of B. Due to the proof of Theorem 2.4, A is a self-adjoint extension of a simple PSO S (see (2.10)) in the space $\widehat{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}'$. By virtue of (2.3) and (2.4), the operator

$$\Xi = \Xi_+ \oplus \Xi_- \tag{3.7}$$

is a unitary mapping of $\widehat{\mathfrak{H}}$ onto $L_2(\mathbb{R}, N)$ ($\dim N = m_-(B)$) such that a simple PSO S and the symmetric momentum operator $\mathcal{S}$ (see (2.5)) are related as follows:

$$\Xi S = \mathcal{S}\Xi.$$

This means that self-adjoint extensions of S in $\widehat{\mathfrak{H}}$ are unitarily equivalent to self-adjoint extension of $\mathcal{S}$ in $L_2(\mathbb{R}, N)$. Since the operator $\psi(B)$ in (3.5) does not depend on the choice of self-adjoint extensions A of S, one can choose it in such a manner that

$$\Xi A = \mathcal{A}\Xi,$$

where $\mathcal{A}$ is the momentum operator in $L_2(\mathbb{R}, N)$ (see (2.6)). Applying the Fourier transformation (2.11) to the last relation and taking into account that $F\mathcal{A} = \mathcal{M}F$, where $\mathcal{M}$ is defined by (2.7), we obtain $GA = \mathcal{M}G$, where $G = F\Xi$ is a unitary mapping of $\widehat{\mathfrak{H}}$ onto $L_2(\mathbb{R}, N)$. By virtue of (3.3)

$$\psi(A) = G^{-1}\psi(\mathcal{M})G = \Xi^{-1}F^{-1}\psi(\mathcal{M})F\Xi. \tag{3.8}$$

Relations (3.5) and (3.8) mean that (3.6) is equivalent to the inclusion

$$F\Xi\ker(B^* - \mu I) \subset \mathcal{D}(\psi(\mathcal{M})) \cap F\Xi\mathfrak{H}. \tag{3.9}$$

It follows from (2.2), (2.3), and (3.7) that

$$\Xi\ker(B^* - \mu I) = \ker(\mathcal{B}^* - \mu I) = \{\gamma = \mathrm{e}^{-i\mu x}n : n \in N,\ x \in \mathbb{R}_+\}. \tag{3.10}$$

Since

$$F\gamma = F\begin{cases} \mathrm{e}^{-i\mu x}n, & x \geq 0 \\ 0, & x < 0 \end{cases} = \frac{i}{\sqrt{2\pi}}\frac{1}{\delta - \mu}n \in F\Xi\mathfrak{H}, \tag{3.11}$$

the inclusion (3.9) holds if $\frac{1}{\delta-\mu}n \in \mathcal{D}(\psi(\mathcal{M}))$. The last relation is equivalent to

$$\int_{\infty}^{\infty} \frac{|\psi(\delta)|^2}{(\delta - \operatorname{Re}\mu)^2 + (\operatorname{Im}\mu)^2}\mathrm{d}\delta < \infty. \tag{3.12}$$

It is easy to see that (3.12) holds when $\psi \in L_p(\mathbb{R})$ and $p \geq 2$. □

3.2 Spectral properties of $\psi(B)$

The formula (3.5) defines $\psi(B)$ as the projection of $\psi(A)$. The functional calculus of self-adjoint operators states that the spectrum of $\psi(A)$ is determined by the range of the function ψ considered on the spectrum of A. In our case, the operator A is a self-adjoint extension of a simple PSO (2.10). For this reason, its spectrum coincides with $\mathbb{R}$ (it follows from Theorem 2.4). Hence, one may expect that the spectrum of $\psi(B)$ will be determined by the range of ψ. To prove the corresponding result (Theorem 3.3) we start with the auxiliary notations and results from [13].

Let $\phi \in L_\infty(\mathbb{T})$, where $\mathbb{T} = \{\lambda \in \mathbb{C} \ \ |\lambda| = 1\}$. The operator

$$T_\phi f = P_+ \phi f, \quad f \in H^2(\mathbb{D}),$$

where P_+ is an orthogonal projection operator in $L_2(\mathbb{T})$ onto the subspace $H^2(\mathbb{D})$ is called a *Toeplitz operator with the symbol* ϕ. The operator T_ϕ is bounded in $H^2(\mathbb{D})$. The essential range of a symbol ϕ is defined as follows:

$$\mathsf{Range}(\phi) = \left\{\lambda \in \mathbb{C} \ \ \operatorname{ess}\inf_{\xi \in \mathbb{T}} |\phi(\xi) - \lambda| = 0\right\}.$$

A Toeplitz operator in $H^2(\mathbb{C}_+)$ with the symbol $\psi \in L_\infty(\mathbb{R})$ has the form

$$T_\psi f = P^+ \psi f, \quad f \in H^2(\mathbb{C}_+),$$

where P^+ is an orthogonal projection operator in $L_2(\mathbb{R})$ onto the subspace $H^2(\mathbb{C}_+)$. The essential range of ψ is:

$$\mathsf{Range}(\psi) = \left\{\lambda \in \mathbb{C} \ \ \operatorname{ess}\inf_{\delta \in \mathbb{R}} |\psi(\delta) - \lambda| = 0\right\}.$$

If the symbols $\phi \in L_\infty(\mathbb{T})$ and $\psi \in L_\infty(\mathbb{R})$ of the Toeplitz operators T_ϕ and T_ψ satisfy the relation

$$\psi(\delta) = \phi\left(\frac{\delta - i}{\delta + i}\right), \quad \text{a.e. } \delta \in \mathbb{R} \tag{3.13}$$

then T_ϕ and T_ψ are unitarily equivalent [13, p. 261].

Theorem 3.3 *Let B be a simple maximal symmetric operator in $\mathfrak{H}$ and let the operator $\psi(B)$ be defined by (3.5), where $\psi \in L_\infty(\mathbb{R})$. Then $\psi(B)$ is a bounded operator in $\mathfrak{H}$ and the following relations hold:*

1. *the spectral radius of $\psi(B)$ coincides with $\|\psi\|_\infty$;*
2. *the continuous spectrum $\sigma_c(\psi(B))$ of $\psi(B))$ involves* $\mathsf{Range}(\psi)$, *while the spectrum $\sigma(\psi(B))$ is a subset of the closed linear span of* $\mathsf{Range}(\psi)$;
3. *if $\psi(B)$ is invertible, then* $\operatorname{ess}\inf_{\delta \in \mathbb{R}} |\psi(\delta)| > 0$;

4. *if ψ is real valued, then $\psi(B)$ is a self-adjoint operator and*

$$\sigma(\psi(B)) = \left[\operatorname*{ess\,inf}_{\delta\in\mathbb{R}} \psi(\delta), \quad \operatorname*{ess\,sup}_{\delta\in\mathbb{R}} \psi(\delta)\right];$$

5. *if ψ is a non-constant real valued function, then $\sigma_p(\psi(B)) = \emptyset$.*

Proof If $\psi \in L_\infty(\mathbb{R})$, then $\psi(A)$ has to be a bounded operator in $\widehat{\mathfrak{H}}$. In view of (3.5), $\psi(B)$ is also a bounded operator acting in $\mathfrak{H} \subset \widehat{\mathfrak{H}}$.

In the first place, we consider a simple maximal symmetric operator B with $m_+(B) = 0$. Repeating the proof of Proposition 3.2 we choose a minimal self-adjoint extension A of B for which the relation (3.8) holds. Using now the elementary modification of the Paley–Wiener theorem [13, p. 146] for the spaces $L_2(\mathbb{R}_+, N)$ and $H^2(\mathbb{C}_+, N)$:

$$FL_2(\mathbb{R}_+, N) = H^2(\mathbb{C}_+, N) \tag{3.14}$$

and relations (2.3) and (3.7) we arrive at the conclusion that $G = F\Xi$ maps unitarily the subspace $\mathfrak{H}$ of $\widehat{\mathfrak{H}}$ onto the subspace $H^2(\mathbb{C}_+, N)$ of $L_2(\mathbb{R}, N)$ and the orthogonal projection operator P in $\widehat{\mathfrak{H}}$ onto $\mathfrak{H}$ has the form $P = G^{-1}P^+G$, where P^+ is an orthogonal projection in $L_2(\mathbb{R}, N)$ onto $H^2(\mathbb{C}_+, N)$. Summing up, the formula (3.5) can be rewritten as follows:

$$\psi(B)f = P\psi(A)f = G^{-1}P^+GG^{-1}\psi(\mathcal{M})Gf = G^{-1}P^+\psi(\mathcal{M})Gf, \quad f \in \mathfrak{H}, \tag{3.15}$$

where $\psi(\mathcal{M})$ is the operator of multiplication by ψ in $L_2(\mathbb{R}, N)$. Therefore, the operator $\psi(B)$ is unitarily equivalent to the operator $P^+\psi(\mathcal{M})$ acting in $H^2(\mathbb{C}_+, N)$.

Let us consider an orthonormal basis $\{n_j\}_{j\in\mathbb{J}}$ of N, where the index set $\mathbb{J}$ is countable (or finite) and its cardinality coincides with $\dim N = m_-(B)$. Then one can present $H^2(\mathbb{C}_+, N)$ as the orthogonal sum of the Hardy spaces

$$H^2(\mathbb{C}_+, N) = \bigoplus_{j\in\mathbb{J}} H^2(\mathbb{C}_+, \langle n_j\rangle),$$

where $H^2(\mathbb{C}_+, \langle n\rangle)$ denotes a Hardy subspace of functions with values from $\langle n\rangle$. By the construction, each subspace $H^2(\mathbb{C}_+, \langle n_j\rangle)$ reduces the operator $P^+\psi(\mathcal{M})$ and its restriction onto $H^2(\mathbb{C}_+, \langle n_j\rangle)$ is unitarily equivalent to the Toeplitz operator T_ψ in $H^2(\mathbb{C}_+)$. Therefore, the operator $P^+\psi(\mathcal{M})$ in $H^2(\mathbb{C}_+, N)$ is unitarily equivalent to the orthogonal sum of operators

$$\underbrace{T_\psi \oplus T_\psi \oplus \cdots \oplus T_\psi}_{\dim N \text{ terms}}$$

acting in the space $H^2(\mathbb{C}_+) \oplus H^2(\mathbb{C}_+) \oplus \cdots \oplus H^2(\mathbb{C}_+)$. As a consequence, $\psi(B)$ is unitarily equivalent to the orthogonal sum of Toeplitz operators T_ϕ acting in $H^2(\mathbb{D})$

$$\underbrace{T_\phi \oplus T_\phi \oplus \cdots \oplus T_\phi,}_{\dim N \text{ terms}}$$

where functions ψ and ϕ satisfy (3.13). Now, taking into account the above consideration, we can justify statements of the theorem using the corresponding results for T_ϕ given in [13]. Precisely, in view of [13, Corollary 4.1.7, p. 246], the spectral radius of $\psi(B)$ coincides with the norm of ϕ in $L_\infty(\mathbb{T})$. Using (3.13) we complete the proof of statement 1. Further, [13, Theorem 4.2.4, p. 247] and the fact that $\mathsf{Range}(\phi) = \mathsf{Range}(\psi)$ justifies statement 2. Similarly, [13, Corollary 4.2.6, p. 248] and the relation

$$\operatorname{ess}\inf_{\xi\in\mathbb{T}} |\phi(\xi)| = \operatorname{ess}\inf_{\delta\in\mathbb{R}} |\psi(\delta)|, \quad \xi = \frac{\delta - i}{\delta + i}$$

establishes statement 3.

If ψ is real-valued, then the function ϕ in (3.13) is also real-valued. By virtue of [13, Corollary 4.2.5, p. 248], the operator T_ϕ is self-adjoint in $H^2(\mathbb{D})$. This means that T_ψ is self-adjoint in $H^2(\mathbb{C}_+)$ and hence, $\psi(B)$ is a self-adjoint operator in $\mathfrak{H}$ that proves the first part of statement 4. Its second part follows from [13, Theorem 4.2.7, p. 248]. Now, taking into account that statement 5. is a consequence of [13, Lemma 4.2.9, p. 249] we complete the proof for the case of a simple maximal symmetric operator B with $m_+(B) = 0$.

The case where $m_-(B) = 0$ is reduced to the previous one if we consider a simple maximal symmetric operator $-B$. □

3.3 The class H^∞

3.3.1 Functional calculus

Actually, (3.5) allows one to define $\psi(B)$ for an arbitrary Lebesgue measurable function ψ. This formula can be essentially simplified if $\psi \in H^\infty$. The next result was proved in [5]. For the reader's convenience, the principal stages of the proof are presented.

Theorem 3.4 *Let B be a simple maximal symmetric operator with $m_+(B) = 0$ acting in a Hilbert space $\mathfrak{H}$ and let A be a minimal self-adjoint extension of B in $\widehat{\mathfrak{H}} \supset \mathfrak{H}$. Then*

$$\psi(B)f = \psi(A)f, \quad f \in \mathfrak{H}$$

for every $\psi \in H^\infty(\mathbb{C}_+)$.

Proof If $\psi \in H^\infty(\mathbb{C}_+)$ then $H^2(\mathbb{C}_+, N)$ is an invariant subspace for the operator of multiplication $\psi(\mathcal{M})$ acting in $L_2(\mathbb{R}, N)$. Recalling that $G = F\Xi$ maps $\mathfrak{H}$ onto

$H^2(\mathbb{C}_+, N)$ (see the proof of Theorem 3.3) one conclude that $\mathfrak{H}$ is an invariant subspace for the operator $G^{-1}\psi(\mathcal{M})G$ in $\widehat{\mathfrak{H}}$. Now, one can rewrite (3.15) as follows:

$$\psi(B)f = P\psi(A)f = PG^{-1}\psi(\mathcal{M})Gf = G^{-1}\psi(\mathcal{M})Gf = \psi(A)f, \quad f \in \mathfrak{H}$$

that completes the proof. □

Corollary 3.5 *Let B be a simple maximal symmetric operator with $m_+(B) = 0$ and let $\psi \in H^\infty(\mathbb{C}_+)$. Then the following statements are true*:

1. $B\psi(B)u = \psi(B)Bu, \ u \in \mathcal{D}(B)$;
2. *if* $|\psi(\delta)| = 1$ *a.e., then* $\psi(B)$ *is an isometric operator in* $\mathfrak{H}$;
3. *if* $\psi(\delta) = \frac{1}{\delta+i}$, *then* $\psi(B) = (B + iI)^{-1}$;
4. *if* $\psi_i \in H^\infty(\mathbb{C}_+)$, *then* $\psi_1(B)\psi_2(B) = \psi_2(B)\psi_1(B)$.

Proof Follows from Theorem 3.4 and functional calculus of self-adjoint operators. □

Corollary 3.6 *Let B be a simple maximal symmetric operator with $m_+(B) = 0$ and let $\psi \in H^\infty(\mathbb{C}_+)$. Then, for all $f \in \ker(B^* - \mu I)$ where $\mu \in \mathbb{C}_-$, the following relation is true*

$$\psi(B)^* f = \overline{\psi(\overline{\mu})} f.$$

Proof It follows from (3.7), (3.8) and Theorem 3.4 that

$$\Xi_+\psi(B)f = \Xi\psi(A)f = F^{-1}\psi(\mathcal{M})F\,\Xi f = \psi(\mathcal{B})\Xi_+ f, \quad f \in \mathfrak{H}, \tag{3.16}$$

where $\mathcal{B}$ is defined by (2.2) and $\psi(\mathcal{B}) = F^{-1}\psi(\mathcal{M})F$. Therefore, $\psi(B)$ and $\psi(\mathcal{B})$ are unitarily equivalent and, without loss of generality, one can consider the case, where $\mathfrak{H} = L_2(\mathbb{R}_+, N)$ and $B = \mathcal{B}$.

By virtue of Corollary 3.5, for arbitrary $f \in \ker(\mathcal{B}^* - \mu I)$ and $u \in \mathcal{D}(\mathcal{B})$,

$$\begin{aligned}((\mathcal{B} - \overline{\mu}I)u, \psi(\mathcal{B})^* f) &= (\psi(\mathcal{B})(\mathcal{B} - \overline{\mu}I)u, f) = ((\mathcal{B} - \overline{\mu}I)\psi(\mathcal{B})u, f) \\ &= (\psi(\mathcal{B})u, (\mathcal{B}^* - \mu I)f) = 0.\end{aligned}$$

This means that $\psi(\mathcal{B})^* f$ is orthogonal to $\mathcal{R}(\mathcal{B} - \overline{\mu}I)$ and hence, $\psi(\mathcal{B})^* f$ belongs to $\ker(\mathcal{B}^* - \mu I)$. Recalling now (3.10), we arrive at the conclusion that

$$\psi(\mathcal{B})^* f = \psi(\mathcal{B})^* \mathrm{e}^{-i\mu x} n = \mathrm{e}^{-i\mu x} n', \quad n, n' \in N, \ x \in \mathbb{R}_+, \tag{3.17}$$

where $n' \in N$ is determined uniquely by the vector $f = \mathrm{e}^{-i\mu x} n$. Relation (3.17) and the fact that the subspace $L_2(\mathbb{R}_+, \langle n \rangle)$ is a reducing subspace for the operator $\psi(\mathcal{B}) = F^{-1}\psi(M)F$ acting in $L_2(\mathbb{R}_+, N)$ lead to the conclusion that

$$\psi(\mathcal{B})^* f \in L_2(\mathbb{R}_+, \langle n \rangle) \cap \ker(\mathcal{B}^* - \mu I).$$

Hence, the vector n' in (3.17) has the form $n' = cn$, where a constant $c \in \mathbb{C}$ should be specified. To do that, one first calculates

$$(\psi(\mathcal{B})^* f, f) = c(n, n)_N \int_0^\infty e^{-i(\mu-\overline{\mu})x} dx = -\frac{c\|n\|_N^2}{2\mathrm{Im}\,\mu}, \tag{3.18}$$

where $(\cdot, \cdot)_N$ means a scalar product in N.

On the other hand, $(\psi(\mathcal{B})^* f, f) = (f, \psi(\mathcal{B}) f) = (Ff, \psi Ff)$, where (see (3.11))

$$(Ff, \psi Ff) = \frac{\|n\|_N^2}{2\pi} \int_{-\infty}^{\infty} \frac{\overline{\psi(\delta)}}{(\mathrm{Re}\,\mu - \delta)^2 + (\mathrm{Im}\,\mu)^2} d\delta. \tag{3.19}$$

After the comparison of relations (3.18) and (3.19) we get

$$-\frac{c\|n\|_N^2}{2\mathrm{Im}\,\mu} = \frac{\|n\|_N^2}{2\pi} \int_{-\infty}^{\infty} \frac{\overline{\psi(\delta)}}{(\mathrm{Re}\,\mu - \delta)^2 + (\mathrm{Im}\,\mu)^2} d\delta.$$

The application of the Poisson formula [13, p. 147]

$$\psi(z) = \psi(\mathrm{Re}\,z + i\,\mathrm{Im}\,z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{Im}\,z}{(\mathrm{Re}\,z - \delta)^2 + (\mathrm{Im}\,z)^2} \psi(\delta)\, d\delta, \quad z \in \mathbb{C}_+,$$

leads to the equality

$$c = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{-(\mathrm{Im}\,\mu)\overline{\psi(\delta)}}{(\mathrm{Re}\,\mu - \delta)^2 + (\mathrm{Im}\,\mu)^2} d\delta = \overline{\psi(\overline{\mu})}$$

that completes the proof. □

3.3.2 Relationship with unilateral shifts

Let B be a simple maximal symmetric operator in $\mathfrak{H}$. Without loss of generality, we assume that $m_+(B) = 0$. In this case there exists a bounded operator $(B + iI)^{-1}$ and the Cayley transform of B:

$$T = (B - iI)(B + iI)^{-1} \tag{3.20}$$

turns out to be a unilateral shift in $\mathfrak{H}$. In other words, T is an isometric operator on $\mathfrak{H}$ and there exists a subspace $\mathcal{L} = \ker T^* = \ker(B^* + iI) \subset \mathfrak{H}$, which is *wandering* for T i.e.,

$$T^n\mathcal{L} \perp \mathcal{L},\ n = 1, 2, 3, \ldots, \quad \bigoplus_{n=0}^{\infty} T^n\mathcal{L} = \mathfrak{H}.$$

The dimension of the wandering subspace is called *a multiplicity* of the unilateral shift T and, by virtue of (2.1), it is equal to $m_-(B)$.

The inverse statement is also true: each unilateral shift T with multiplicity m determines a simple maximal symmetric operator B with $m_+(B) = 0$ and $m_-(B) = m$ by the formula

$$B = i(I + T)(I - T)^{-1}. \tag{3.21}$$

The above-mentioned one-to-one correspondence between B and T can be easily deduced from [18, Chapter III, §9].

A unilateral shift is an example of a completely nonunitary contraction. For such operators, the functional calculus is well-developed [18]. This gives rise to an idea to define the operator $\psi(B)$ with the use of the Cayley transform T of B.

Let A be a minimal self-adjoint extension of B acting in $\widehat{\mathfrak{H}}$. Its Cayley transform

$$W = (A - iI)(A + iI)^{-1}$$

is a unitary operator in $\widehat{\mathfrak{H}}$ and, according to [18, p. 117],

$$\phi(T) = P\phi(W), \quad \phi \in H^\infty(\mathbb{D}),$$

where P is the orthogonal projection operator in $\widehat{\mathfrak{H}}$ on $\mathfrak{H}$.

The spectral function $\{E_\delta^A\}$ of A and the spectral function $\{E_\theta^W\}$ of W are closely related [1, § 79]:

$$E_\delta^A = E_\theta^W, \quad \text{where} \quad \delta = i\frac{1 + \mathrm{e}^{i\theta}}{1 - \mathrm{e}^{i\theta}} = i\frac{\mathrm{e}^{-i\theta/2} + \mathrm{e}^{i\theta/2}}{\mathrm{e}^{-i\theta/2} - \mathrm{e}^{i\theta/2}} = -\cot\frac{\theta}{2}, \quad \theta \in [0, 2\pi].$$

Moreover, by virtue of [4, p. 138]

$$\phi(W) = \int_0^{2\pi} \phi(\mathrm{e}^{i\theta})\mathrm{d}E_\theta^W = \int_{-\infty}^{\infty} \phi\left(\frac{\delta - i}{\delta + i}\right)\mathrm{d}E_\delta^A = \psi(A), \tag{3.22}$$

where $\psi(\delta) = \phi(\frac{\delta-i}{\delta+i})$ belongs to $H^\infty(\mathbb{C}_+)$.

Using (3.5) and (3.22) we arrive at the conclusion that

$$\phi(T)f = P\phi(W)f = P\psi(A)f = \psi(B)f, \quad f \in \mathfrak{H}, \tag{3.23}$$

where $\psi(\delta) = \phi(\frac{\delta-i}{\delta+i}) \in H^\infty(\mathbb{C}_+)$. The obtained relationship allows one to reduce the investigation of $\psi(B)$ to the investigation of $\phi(T)$.

3.3.3 The case of inner functions

Proposition 3.7 *Let B be a simple maximal symmetric operator in $\mathfrak{H}$ with $m_+(B) = 0$ and let $\psi \in H^\infty(\mathbb{C}_+)$ be a non-constant inner function. Then $\psi(B)$ is a unilateral shift with the wandering subspace $\mathcal{L} = \ker\psi(B)^*$.*

Proof In view of (3.23), it is sufficient to prove that $\phi(T)$ is a unilateral shift in $\mathfrak{H}$ with the wandering subspace $\mathcal{L} = \ker \phi(T)^*$.

Due to the relation $\psi(\delta) = \phi(\frac{\delta - i}{\delta + i})$, the function ϕ belongs to $H^\infty(\mathbb{D})$ and it is a non-constant inner function. Therefore, $|\phi(\lambda)| < 1$ for $\lambda \in \mathbb{D}$ [12, p. 49]. Taking into account that the unitary shift T is an example of completely nonunitary contraction and using [18, Chapter III, Theorem 2.1 (e)] we arrive at the conclusion that $\phi(T)$ is also a completely nonunitary contraction in $\mathfrak{H}$. Simultaneously, $\phi(T)$ is an isometric operator (it follows from (3.23) and Corollary 3.5). In view of the Wold decomposition [18, Chapter I, Theorem 1.1], $\phi(T)$ turns out to be a unilateral shift with the wandering subspace $\mathcal{L} = \mathfrak{H} \ominus \phi(T)\mathfrak{H} = \ker \phi(T)^*$. □

Corollary 3.8 *If assumptions of Proposition 3.7 hold, then $\psi(B)$ and B are strongly commuting operators, i.e., $\psi(B)B = B\psi(B)$.*

Proof The inclusion $\psi(B)B \subseteq B\psi(B)$ follows from Corollary 3.5.

Assume now that $f \in \mathcal{D}(B\psi(B))$ and consider the operator

$$C = Y^* B Y, \quad \mathcal{D}(C) = \mathcal{D}(B) \dot{+} \langle f \rangle,$$

where $Y = \psi(B)$ is a unilateral shift in $\mathfrak{H}$.

Elementary analysis with the use of Corollary 3.5 and the relation $Y^*Y = I$ gives rise to the following conclusions:

(a) $Cu = Y^*BYu = Y^*YBu = Bu$, $u \in \mathcal{D}(B)$;
(b) $(Cu, v) = (u, Cv)$, $u, v \in \mathcal{D}(B)$;
(c) $(Cf, f) = (BYf, Yf) = (Yf, BYf) = (f, Cf)$;
(d) $(Cu, f) = (BYu, Yf) = (Yu, BYf) = (u, Cf)$;
(e) $(Cf, u) = (BYf, Yu) = (Yf, BYu) = (f, Cu)$.

It follows from (a)–(e) that C is a symmetric extension of the maximal symmetric operator B in $\mathfrak{H}$. This means that $f \in \mathcal{D}(B)$ and the inverse inclusion $B\psi(B) \subseteq \psi(B)B$ is proved. □

Remark 3.9 Results of Sects. 3.3.1–3.3.3 hold true for a simple maximal symmetric operator B with $m_-(B) = 0$. In this case, one should consider functions ψ from $H^\infty(\mathbb{C}_-)$ and choose $\mu \in \mathbb{C}_+$ in Corollary 3.6.

3.3.4 Non-cyclic vectors

Let T be a unilateral shift in $\mathfrak{H}$. It's adjoint operator T^* is called a *backward shift operator*. A vector $f \in \mathfrak{H}$ is called *non-cyclic* for the backward shift operator T^* if

$$E_f = \bigvee_{n=0}^{\infty} T^{*n} f$$

is a proper subspace of $\mathfrak{H}$.

A non-cyclic vector f is called *simple* if $\dim E_f = 1$. Denote by $\mathfrak{M}_{T^*}$ and $\mathfrak{M}^s_{T^*}$ the sets of non-cyclic vectors and simple non-cyclic vectors, respectively. It is clear that $\mathfrak{M}^s_{T^*} \subset \mathfrak{M}_{T^*}$.

Proposition 3.10 *The set $\mathfrak{M}^s_{T^*}$ is complete in $\mathfrak{H}$ and*

$$\mathfrak{M}^s_{T^*} = \bigcup_{\mu\in\mathbb{C}_-} \ker(B^* - \mu I), \tag{3.24}$$

where a simple maximal symmetric operator B is defined by (3.21).

Proof The unilateral shift T defined by (3.20) coincides with $\psi(B)$, where the function $\psi(\delta) = \frac{\delta-i}{\delta+i}$ belongs to $H^\infty(\mathbb{C}_+)$ and a simple maximal symmetric operator B is defined by (3.21). By Corollary 3.6, for all $f \in \ker(B^* - \mu I)$,

$$T^* f = \psi(B)^* f = \overline{\psi(\overline{\mu})} f = \frac{\mu + i}{\mu - i} f, \quad \mu \in \mathbb{C}_-.$$

Therefore, $E_f = \langle f\rangle$ and $\mathfrak{M}^s_{T^*} \supseteq \bigcup_{\mu\in\mathbb{C}_-} \ker(B^* - \mu I)$. The last relation means that the set $\mathfrak{M}^s_{T^*}$ is complete in $\mathfrak{H}$ because $\bigvee_{\mu\in\mathbb{C}_-} \ker(B^* - \mu I) = \mathfrak{H}$.

Assume now that $f \in \mathfrak{M}^s_{T^*}$. Then there exists $\alpha_f \in \mathbb{C}$ such that $T^* f = \alpha_f f$. Here, $|\alpha_f| < 1$ since T^* is a backward shift operator.

Since T is a unilateral shift, each $f \in \mathfrak{H}$ admits the presentation

$$f = \sum_{n=0}^{\infty} T^n l_n, \quad l_n \in \mathcal{L}, \tag{3.25}$$

where $\mathcal{L} = \ker T^* = \ker(B^* + iI)$ is the wandering subspace of T. Comparing (3.25) with the relation $T^* f = \alpha_f f$, we arrive at the conclusion that $l_n = \alpha_f^n l_0$ and (3.25) takes the form

$$f = \sum_{n=0}^{\infty} (\alpha_f T)^n l_0 = (I - \alpha_f T)^{-1} l_0.$$

Since $|\alpha_f| < 1$ there exists $\mu \in \mathbb{C}_-$ such that $\alpha_f = \frac{\mu+i}{\mu-i}$. Then, recalling (3.20):

$$f = (I - \alpha_f T)^{-1} l_0 = \left(I - \frac{\mu + i}{\mu - i} T\right)^{-1} = \frac{1 + i\mu}{2}(B + iI)(B - \mu I)^{-1} l_0,$$

where $l_0 \in \ker(B^* + iI)$. We note that $(B + iI)(B - \mu I)^{-1}$ maps $\ker(B^* + iI)$ onto $\ker(B^* - \mu I)$. Hence, $f \in \ker(B^* - \mu I)$ and $\mathfrak{M}^s_{T^*} \subseteq \bigcup_{\mu\in\mathbb{C}_-} \ker(B^* - \mu I)$ that proves (3.24). □

It follows from (3.24) that the sum $f + g$ of vectors $f, g \in \mathfrak{M}^s_{T^*}$ remains in $\mathfrak{M}^s_{T^*}$ if and only if they are linearly dependent.

Proposition 3.11 *Let T^* be a backward shift operator in $\mathfrak{H}$ with multiplicity* 1. *Then $\mathfrak{M}_{T^*}$ is a linear manifold in $\mathfrak{H}$ and*

$$\mathfrak{M}_{T^*} = \bigcup_{\psi} \ker \psi(B)^*, \tag{3.26}$$

where $\psi \in H^\infty(\mathbb{C}_+)$ runs the set of inner functions and a simple maximal symmetric operator B is defined by (3.21).

Proof In view of Corollary 3.5, $T^n\psi(B) = \psi(B)T^n$, $n \in \mathbb{N} \cup \{0\}$. Hence,

$$T^{*n}\psi(B)^* = \psi(B)^*T^{*n}.$$

If $f \in \ker\psi(B)^*$, then $0 = T^{*n}\psi(B)^*f = \psi(B)^*T^{*n}f$. Therefore, $E_f \subset \ker\psi(B)^*$ and $f \in \mathfrak{M}_{T^*}$. The inclusion $\bigcup\ker\psi(B)^* \subseteq \mathfrak{M}_{T^*}$ is proved.

To prove an inverse inclusion we note that the operator $\psi(B)$ is unitarily equivalent to $\psi(\mathcal{B})$ (see (3.16)) and, without loss of generality, one can consider the case, where:

$$\mathfrak{H} = L_2(\mathbb{R}_+, N), \quad B = \mathcal{B}, \quad T = (\mathcal{B} - iI)(\mathcal{B} + iI)^{-1}.$$

Since the multiplicity of T is 1, the space N can be chosen as $\mathbb{C}$. In this case, $L_2(\mathbb{R}_+, \mathbb{C}) = L_2(\mathbb{R}_+)$ and for $f \in \mathfrak{M}_{T^*}$, the subspace

$$L_2(\mathbb{R}_+) \ominus E_f = L_2(\mathbb{R}_+) \ominus \bigvee_{n=0}^{\infty} T^{*n}f$$

turns out to be an invariant subspace for T in $L_2(\mathbb{R}_+)$. Applying the Fourier transformation (2.11) to the relation above and using (3.14) we conclude that the subspace $F[L_2(\mathbb{R}_+) \ominus E_f]$ of $H^2(\mathbb{C}_+)$ is invariant for the operator of multiplication by $\frac{\delta - i}{\delta + i}$ in $H^2(\mathbb{C}_+)$. By virtue of Beurling's theorem [12, p. 49] there exists an inner function $\psi \in H^\infty(\mathbb{C}_+)$ such that

$$F[L_2(\mathbb{R}_+) \ominus E_f] = \psi(\delta)\, H^2(\mathbb{C}_+).$$

Therefore,

$$L_2(\mathbb{R}_+) \ominus E_f = F^{-1}\psi(\delta)FL_2(\mathbb{R}_+) = F^{-1}\psi(\mathcal{M})FL_2(\mathbb{R}_+) = \psi(\mathcal{B})L_2(\mathbb{R}_+),$$

where $\mathcal{M}$ is defined by (2.7). The obtained relation means that f is orthogonal to $\psi(\mathcal{B})L_2(\mathbb{R}_+)$. Therefore, $f \in \ker\psi(B)^*$ and $\bigcup\ker\psi(B)^* \supseteq \mathfrak{M}_{T^*}$. The relation (3.26) is proved.

The set $\mathfrak{M}_{T^*} = \bigcup\ker\psi(B)^*$ is a linear manifold in $\mathfrak{H}$ because $f_j \in \ker\psi_j(B)^*$ yields that $f_1 + f_2 \in \ker\psi_3(B)^*$, where $\psi_3 = \psi_2\psi_1$. The proof is completed. □

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