Further results on A -numerical radius inequalities

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Abstract

Let A be a bounded linear positive operator on a complex Hilbert space $\mathcal{H}$. Furthermore, let $\mathcal{B}_A \Leftarrow \mathcal{H} \Rightarrow$ denote the set of all bounded linear operators on $\mathcal{H}$ whose A -adjoint exists, and $\mathbb{A}$ signify a diagonal operator matrix with diagonal entries are A . Very recently, several $\mathbb{A}$ -numerical radius inequalities of 2×2 operator matrices were established. In this paper, we prove a few new $\mathbb{A}$ -numerical radius inequalities for 2×2 and $n \times n$ operator matrices. We also provide a new proof of an existing result by relaxing a sufficient condition “ A is strictly positive”. Our proofs show the importance of the theory of the Moore–Penrose inverse of a bounded linear operator in this field of study.

Keywords $\mathbb{A}$ -numerical radius · Moore–Penrose inverse · Positive operator · Semi-inner product · Inequality · Operator matrix

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1 Introduction

Throughout, $\mathcal{H}$ denotes a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. By $\mathcal{B}(\mathcal{H})$, we mean the C^* -algebra of all bounded linear operators on $\mathcal{H}$. Let $\| \cdot \|$ be the norm induced from $\langle \cdot, \cdot \rangle$. For $A \in \mathcal{B}(\mathcal{H})$, $\mathcal{R}(A)$ stands for the range space of A and $\overline{\mathcal{R}(A)}$ for the norm closure of $\mathcal{R}(A)$ in $\mathcal{H}$. And A^* represents the adjoint operator of A . An operator $A \in \mathcal{B}(\mathcal{H})$ is called *selfadjoint* if $A = A^*$. A selfadjoint operator $A \in \mathcal{B}(\mathcal{H})$ is called *positive* if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$, and is called *strictly positive* if $\langle Ax, x \rangle > 0$ for all non-zero $x \in \mathcal{H}$. If A is a positive (strictly positive)

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operator, then we use the notation $A \geq 0$ ($A > 0$). Let $\mathbb{A}$ be an $n \times n$ diagonal operator matrix whose diagonal entries are positive operator A for $n = 1, 2, \dots$. Then, $\mathbb{A} \in \mathcal{B}(\bigoplus_{i=1}^n \mathcal{H})$ and $\mathbb{A} \geq 0$. If $A \geq 0$, then it induces a positive semidefinite sesquilinear form, $\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ defined by $\langle x, y \rangle_A = \langle Ax, y \rangle$, $x, y \in \mathcal{H}$. Let $\| \cdot \|_A$ denote the seminorm on $\mathcal{H}$ induced by $\langle \cdot, \cdot \rangle_A$, i.e., $\|x\|_A = \sqrt{\langle x, x \rangle_A}$ for all $x \in \mathcal{H}$. Then, $\|x\|_A$ is a norm if and only if $A > 0$. Also, $(\mathcal{H}, \| \cdot \|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}$. Henceforth, we use the symbol A and $\mathbb{A}$ for positive operators on $\mathcal{H}$ and $\bigoplus_{i=1}^n \mathcal{H}$, respectively. We retain the notations O and I for the null operator and the identity operator on $\mathcal{H}$, respectively. Given $T \in \mathcal{B} \Leftarrow \mathcal{H} \Rightarrow$, the A -operator seminorm $\|T\|_A$ is defined as follows:

$$\|T\|_A = \sup_{x \in \overline{\mathcal{R}(A)}, x \neq 0} \frac{\|Tx\|_A}{\|x\|_A} = \inf \left\{ c > 0 : \|Tx\|_A \leq c\|x\|_A, 0 \neq x \in \overline{\mathcal{R}(A)} \right\} < \infty.$$

We set $\mathcal{B}^A \Leftarrow \mathcal{H} \Rightarrow = \{T \in \mathcal{B} \Leftarrow \mathcal{H} \Rightarrow : \|T\|_A < \infty\}$. Then $\mathcal{B}^A \Leftarrow \mathcal{H} \Rightarrow$ is not a subalgebra of $\mathcal{B} \Leftarrow \mathcal{H} \Rightarrow$. It is pertinent to point out that $\|T\|_A = 0$ if and only if $ATA = O$. For $T \in \mathcal{B}^A \Leftarrow \mathcal{H} \Rightarrow$, we have

$$\|T\|_A = \sup \{ |\langle Tx, y \rangle_A| : x, y \in \overline{\mathcal{R}(A)}, \|x\|_A = \|y\|_A = 1 \}.$$

If $AT \geq 0$, then the operator T is called A -positive. Note that if T is A -positive, then

$$\|T\|_A = \sup \{ \langle Tx, x \rangle_A : x \in \mathcal{H}, \|x\|_A = 1 \}.$$

Before we proceed further, it is necessary to introduce the concept of A -adjoint operator. We say an operator $X \in \mathcal{B} \Leftarrow \mathcal{H} \Rightarrow$ to be A -adjoint operator of $T \in \mathcal{B} \Leftarrow \mathcal{H} \Rightarrow$ if $\langle Tx, y \rangle_A = \langle x, Xy \rangle_A$ for every $x, y \in \mathcal{H}$, i.e., $AX = T^*A$. By Douglas Theorem [5], the existence of an A -adjoint operator is not guaranteed. An operator $T \in \mathcal{B} \Leftarrow \mathcal{H} \Rightarrow$ may admit none, one or many A -adjoints. A rather well-known result states that A -adjoint of an operator $T \in \mathcal{B} \Leftarrow \mathcal{H} \Rightarrow$ exists if and only if $\mathcal{R}(T^*A) \subseteq \mathcal{R}(A)$. Let us now denote $\mathcal{B}_A \Leftarrow \mathcal{H} \Rightarrow = \{T \in \mathcal{B} \Leftarrow \mathcal{H} \Rightarrow : \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\}$. Note that $\mathcal{B}_A \Leftarrow \mathcal{H} \Rightarrow$ is a subalgebra of $\mathcal{B} \Leftarrow \mathcal{H} \Rightarrow$ which is neither closed nor dense in $\mathcal{B} \Leftarrow \mathcal{H} \Rightarrow$. Moreover, we have the following inclusion relations:

$$\mathcal{B}_A \Leftarrow \mathcal{H} \Rightarrow \subseteq \mathcal{B}^A \Leftarrow \mathcal{H} \Rightarrow \subseteq \mathcal{B} \Leftarrow \mathcal{H} \Rightarrow.$$

And the equality holds if A is injective and has a closed range.

For $T \in \mathcal{B} \Leftarrow \mathcal{H} \Rightarrow$, $w_A(T)$, the A -numerical radius of T was proposed by Saddi [13]. And is defined as follows:

$$w_A(T) = \sup \{ |\langle Tx, x \rangle_A| : x \in \mathcal{H}, \|x\|_A = 1 \}. \quad (1)$$

When $T = (T_{ij})$ is an $n \times n$ operator matrix with $T_{ij} \in \mathcal{B} \Leftarrow \mathcal{H} \Rightarrow$, then (1) can be written as

$$w_{\mathbb{A}}(T) = \sup \left\{ |\langle Tx, x \rangle_{\mathbb{A}}| : x \in \bigoplus_{n=1}^n \mathcal{H}, \|x\|_{\mathbb{A}} = 1 \right\}.$$

Very recently, Zamani [16] obtained the following A -numerical radius inequality for $T \in \mathcal{B}_A(\mathcal{H})$:

$$\frac{1}{2}\|T\|_A \leq w_A(T) \leq \|T\|_A. \quad (2)$$

The first inequality in (2) becomes an equality if $T^2 = O$ and the second inequality becomes an equality if T is A -selfadjoint. The A -Crawford number of $T \in \mathcal{B}_A(\mathcal{H})$ is defined as

$$c_A(T) = \inf\{|\langle Tx, x \rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\}.$$

This terminology was introduced by Zamani [16]. Furthermore, if T is A -selfadjoint, then $w_A(T) = \|T\|_A$. Moslehian et al. [8] continued the study of A -numerical radius and obtained some new A -numerical radius inequalities. In this year, Bhunia et al. [3] presented several $\mathbb{A}$ -numerical radius inequalities for a strictly positive operator A . We refer the interested reader to [12, 14–16] and the references cited therein for further generalizations and refinements of A -numerical radius inequalities.

The objective of this paper is to present a few new $\mathbb{A}$ -numerical radius inequalities for 2×2 and $n \times n$ operator matrices. Besides these, we aim to establish some existing $\mathbb{A}$ -numerical radius inequalities by relaxing sufficient condition $A > 0$. To this end, the paper is sectioned as follows. In Sect. 2, we define additional mathematical constructs including the definition of the Moore–Penrose inverse of an operator, A -adjoint, A -selfadjoint and A -unitary operator, that are required to state and prove the results in the subsequent sections. Section 3 contains several new A -numerical radius inequalities. More interestingly, it also provides some recent existing results in the literature on A -numerical radius inequalities by dropping a sufficient condition.

2 Preliminaries

This section gathers a few more definitions and results that are useful in proving our main results. It starts with the definition of the Moore–Penrose inverse of a bounded operator A in $\mathcal{H}$. The *Moore–Penrose inverse* of $A \in \mathcal{B} \Leftarrow \mathcal{H} \Rightarrow$ [9] is the operator $X : \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \longrightarrow \mathcal{H}$ which satisfies the following four equations:

$$(1)AXA = A, (2)XAX = X, (3)XA = P_{\mathcal{N}(A)^\perp}, (4)$$

$$AX = P_{\overline{\mathcal{R}(A)}}|_{\mathcal{R}(A) \oplus \mathcal{R}(A)^\perp}.$$

Here, $\mathcal{N}(A)$ and P_L denote the null space of A and the orthogonal projection onto L , respectively. The Moore–Penrose inverse is unique, and is denoted by $A^\dagger$. In general, $A^\dagger \notin \mathcal{B} \Leftarrow \mathcal{H} \Rightarrow$. It is bounded if and only if $\mathcal{R}(A)$ is closed. If $A \in \mathcal{B} \Leftarrow \mathcal{H} \Rightarrow$ is invertible, then $A^\dagger = A^{-1}$. If $T \in \mathcal{B}_A \Leftarrow \mathcal{H} \Rightarrow$, the reduced solution of the equation $AX = T^*A$ is a distinguished A -adjoint operator of T , which is denoted by $T^{\#_A}$ (see [2, 7]). Note that $T^{\#_A} = A^\dagger T^*A$. If $T \in \mathcal{B}_A(\mathcal{H})$, then $AT^{\#_A} = T^*A$, $\mathcal{R}(T^{\#_A}) \subseteq \mathcal{R}(A)$ and $\mathcal{N}(T^{\#_A}) = \mathcal{N}(T^*A)$ (see [5]). One can observe that

$$I^{\#_A} = A^\dagger I^* A = A^\dagger A = P_{\overline{\mathcal{R}(A)}} \quad (\because \mathcal{N}(A)^\perp = \mathcal{R}(A^*)). \quad (3)$$

Besides, we derive below two new properties of A -adjoint of an operator $T \in \mathcal{B}_A(\mathcal{H})$, which are crucial in providing some new proofs of the existing results and in proving new results on A -numerical radius inequalities.

$$T^{\#_A} P_{\overline{\mathcal{R}(A)}} = A^\dagger T^* A A^\dagger A = A^\dagger T^* A = T^{\#_A}, \quad (4)$$

and

$$P_{\overline{\mathcal{R}(A)}} T^{\#_A} = A^\dagger A A^\dagger T^* A = A^\dagger T^* A = T^{\#_A}. \quad (5)$$

An operator $T \in \mathcal{B} \Leftarrow \mathcal{H} \Rightarrow$ is said to be A -selfadjoint if AT is selfadjoint, i.e., $AT = T^*A$. Observe that if T is A -selfadjoint, then $T \in \mathcal{B}_A(\mathcal{H})$. However, in general, $T \neq T^{\#_A}$. But, $T = T^{\#_A}$ if and only if T is A -selfadjoint and $\mathcal{R}(T) \subseteq \mathcal{R}(A)$. If $T \in \mathcal{B}_A(\mathcal{H})$, then $T^{\#_A} \in \mathcal{B}_A(\mathcal{H})$, $(T^{\#_A})^{\#_A} = P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}}$, and $((T^{\#_A})^{\#_A})^{\#_A} = T^{\#_A}$. Also, $T^{\#_A} T$ and $TT^{\#_A}$ are A -positive operators, and

$$\|T^{\#_A} T\|_A = \|TT^{\#_A}\|_A = \|T\|_A^2 = \|T^{\#_A}\|_A^2. \quad (6)$$

For any $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$, we have

$$\begin{aligned} \|T_1^{\#_A} T_2\|_A &= \sup\{|\langle T_1^{\#_A} T_2 x, y \rangle| : x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1\} \\ &= \sup\{|\langle T_2 x, T_1 y \rangle| : x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1\} \\ &= \sup\{|\langle x, T_2^{\#_A} T_1 y \rangle| : x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1\} \\ &= \sup\{|\langle T_2^{\#_A} T_1 y, x \rangle| : x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1\} \\ &= \|T_2^{\#_A} T_1\|_A. \end{aligned} \quad (7)$$

However, the above proof is a very simple one and directly follows from the definition of A -norm. An operator $U \in \mathcal{B}_A(\mathcal{H})$ is said to be A -unitary if $\|Ux\|_A = \|U^{\#_A} x\|_A = \|x\|_A$ for all $x \in \mathcal{H}$. If $T \in \mathcal{B}_A(\mathcal{H})$ and U is A -unitary, then $w_A(U^{\#_A} T U) = w_A(T)$. For $T, S \in \mathcal{B}_A(\mathcal{H})$, we have $(TS)^{\#_A} = S^{\#_A} T^{\#_A}$, $(T + S)^{\#_A} = T^{\#_A} + S^{\#_A}$, $\|TS\|_A \leq \|T\|_A \|S\|_A$ and $\|Tx\|_A \leq \|T\|_A \|x\|_A$ for all $x \in \mathcal{H}$. The real and imaginary part of an operator $T \in \mathcal{B}_A(\mathcal{H})$ as $\operatorname{Re}_A(T) = \frac{T + T^{\#_A}}{2}$ and $\operatorname{Im}_A(T) = \frac{T - T^{\#_A}}{2i}$. An interested reader may refer [1, 2] for further properties of operators on semi-Hilbertian space. From the definition of A -numerical radius of an operator, it follows that

$$w_A(T) = w_A(T^{\#_A}) \text{ for any } T \in \mathcal{B}_A(\mathcal{H}). \quad (8)$$

Some interesting results are collected hereunder for further use.

The next result is a combination of [3, Lemma 2.4(i)] and [11, Lemma 2.2].

Lemma 2.1 *Let $T_1, T_2, T_3, T_4 \in \mathcal{B}_A(\mathcal{H})$. Then*

- (i) $\max\{w_A(T_1), w_A(T_4)\} = w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & O \\ O & T_4 \end{bmatrix}\right) \leq w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right).$
- (ii) $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right).$

The other parts of [3, Lemma 2.4] assume the condition A is strictly positive. Rout et al. [11] proved the same result for positive A , and the same is stated below.

Lemma 2.2 [11, Lemma 2.4]

Let $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$. Then

- (i) $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right) = w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_1 & O \end{bmatrix}\right).$
- (ii) $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_1 \\ e^{i\theta}T_2 & O \end{bmatrix}\right) = w_{\mathbb{A}}\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right)$ for any $\theta \in \mathbb{R}$.
- (iii) $w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix}\right) = \max\{w_A(T_1 + T_2), w_A(T_1 - T_2)\}.$ In particular,
 $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_2 & O \end{bmatrix}\right) = w_A(T_2).$

The next result establishes upper and lower bounds for the $\mathbb{A}$ -numerical radius of a particular type of 2×2 operator matrix that is a generalization of (2).

Lemma 2.3 [11, Theorem 2.6]

Let $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$. Then

$$\max\{w_A(T_1), w_A(T_2)\} \leq w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}\right) \leq w_A(T_1) + w_A(T_2). \quad (9)$$

Lemma 2.4 [11, Lemma 2.9]

Let $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$. Then,

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_2 & -T_1 \\ T_1 & T_2 \end{bmatrix}\right) = \max\{w_A(T_1 + iT_2), w_A(T_1 - iT_2)\}.$$

3 Main results

This section begins with the power inequality for semi-Hilbert space that has been proved by Moslehian et al. [8], which states that for $T \in \mathcal{B} \Leftarrow \mathcal{H} \Rightarrow$, $w_A(T^n) \leq w_A^n(T)$ for $n \in \mathbb{N}$. Using this, we prove the following theorem.

Theorem 3.1 Let $T_1, T_2, T_3, T_4 \in \mathcal{B}_A(\mathcal{H})$ and $T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$. Then

$$\max\{w_A^{1/2}(T_2T_3), w_A^{1/2}(T_3T_2)\} \leq w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right).$$

Proof Here,

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \text{ by Lemma 2.1 .}$$

Now,

$$\begin{aligned} \max\{w_A(T_2T_3), w_A(T_3T_2)\} &= w_{\mathbb{A}}\left(\begin{bmatrix} T_2T_3 & O \\ O & T_3T_2 \end{bmatrix}\right) \\ &= w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix} \begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \\ &= w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}^2\right) \\ &\leq w_{\mathbb{A}}^2\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \quad \because w_A(T^n) \leq w_A^n(T) \text{ (see [8]).} \end{aligned}$$

This implies

$$\max\{w_A^{1/2}(T_2T_3), w_A^{1/2}(T_3T_2)\} \leq w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right).$$

Remark 3.2 Using Lemma 2.1 and a known inequality $w_{\mathbb{A}}(T) \leq \frac{1}{2}(\|T\|_{\mathbb{A}} + \|T^2\|_{\mathbb{A}}^{1/2})$, Theorem 3.1 implies that

$$\max\{w_A^{1/2}(T_2T_3), w_A^{1/2}(T_3T_2)\} \leq w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq \frac{1}{2}(\|T\|_{\mathbb{A}} + \|T^2\|_{\mathbb{A}}^{1/2}).$$

□

Example Let $T_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $T_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then,

$$w_A(T_2T_3) = w_A(T_3T_2) = w_A\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = 1.$$

By Theorem 3.1, we thus have

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \geq 1.$$

We generalize some of the results of [6] now. Using Lemma 2.2, one can now state [3, Theorem 3.1] without assuming the condition $A > 0$. Its proof takes the

same steps as in the [3, proof of Theorem 3.1] after the use of Lemma 2.2, and hence the proof is omitted.

Lemma 3.3 *Let $T, S, X, Y \in \mathcal{B}_A(\mathcal{H})$. Then*

$$w_A(TXS^{\#_A} \pm SYT^{\#_A}) \leq 2\|T\|_A\|S\|_Aw_A \begin{bmatrix} O & X \\ Y & O \end{bmatrix}.$$

In particular, putting $Y = X$

$$w_A(TXS^{\#_A} \pm SXT^{\#_A}) \leq 2\|T\|_A\|S\|_Aw_A(X).$$

Considering $X = Y = Q$ and $T = I$ in Lemma 3.3, we get the following corollary.

Corollary 3.4 *Let $Q, S \in \mathcal{B}_A(\mathcal{H})$. Then,*

$$w_A(QS^{\#_A} \pm SQ) \leq 2\|S\|_Aw_A(Q).$$

It is well known that $P_{\overline{\mathcal{R}(A)}}T \neq TP_{\overline{\mathcal{R}(A)}}$ for $T \in \mathcal{B}_A(\mathcal{H})$ (even if A and T are finite matrices). And the equality holds if $\mathcal{N}(A)^\perp$ is invariant under T . The following result shows that $w_A(P_{\overline{\mathcal{R}(A)}}T)$ and $w_A(TP_{\overline{\mathcal{R}(A)}})$ are same for all $T \in \mathcal{B}_A(\mathcal{H})$ even though $\mathcal{N}(A)^\perp$ is not invariant under T .

Theorem 3.5 $w_A(P_{\overline{\mathcal{R}(A)}}T) = w_A(TP_{\overline{\mathcal{R}(A)}}) = w_A(T)$ for any $T \in \mathcal{B}_A(\mathcal{H})$.

$$\begin{aligned} w_A(P_{\overline{\mathcal{R}(A)}}T) &= w_A((P_{\overline{\mathcal{R}(A)}}T)^{\#_A}) \quad (\because w_A(T) = w_A(T^{\#_A})) \\ &= w_A(T^{\#_A}P_{\overline{\mathcal{R}(A)}}) \quad (\because (TS)^{\#_A} = S^{\#_A}T^{\#_A} \text{ \& } (P_{\overline{\mathcal{R}(A)}})^{\#_A} = P_{\overline{\mathcal{R}(A)}}) \\ &= w_A(T^{\#_A}) \quad \text{by (4)} \\ &= w_A(T). \end{aligned} \quad (10)$$

Proof

Again,

$$\begin{aligned} w_A(TP_{\overline{\mathcal{R}(A)}}) &= w_A((TP_{\overline{\mathcal{R}(A)}})^{\#_A}) \quad (\because w_A(T) = w_A(T^{\#_A})) \\ &= w_A(P_{\overline{\mathcal{R}(A)}}T^{\#_A}) \quad (\because (TS)^{\#_A} = S^{\#_A}T^{\#_A} \text{ \& } (P_{\overline{\mathcal{R}(A)}})^{\#_A} = P_{\overline{\mathcal{R}(A)}}) \\ &= w_A(T^{\#_A}) \quad \text{by (5)} \\ &= w_A(T). \end{aligned} \quad (11)$$

We, therefore, have

$$w_A(P_{\overline{\mathcal{R} \Leftarrow A \Rightarrow}}T) = w_A(TP_{\overline{\mathcal{R} \Leftarrow A \Rightarrow}}) = w_A(T).$$

□

The next result provides an estimate for lower bound of $\mathbb{A}$ -numerical radius of a 2×2 operator matrix.

Theorem 3.6 Let $T_1, T_2, T_3, T_4 \in \mathcal{B}_A(\mathcal{H})$. Then $w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \geq \frac{1}{2} \max\{\alpha, \beta\}$, where $\alpha = \max\{w_A(T_1 + T_2 + T_3 + T_4), w_A(T_1 + T_4 - T_2 - T_3)\}$ and

$$\beta = \max\{w_A(T_1 + T_4 + i(T_2 - T_3)), w_A(T_1 + T_4 - i(T_2 - T_3))\}.$$

Proof Let $T = \begin{bmatrix} T_1^{\#_A} & T_3^{\#_A} \\ T_2^{\#_A} & T_4^{\#_A} \end{bmatrix}$ and $Q = \begin{bmatrix} O & I \\ I & O \end{bmatrix}$. To show that Q is $\mathbb{A}$ -unitary, we need to prove that $\|x\|_{\mathbb{A}} = \|Qx\|_{\mathbb{A}} = \|Q^{\#_A}x\|_{\mathbb{A}}$. So,

$$\begin{aligned} Q^{\#_A} &= \begin{bmatrix} O & I^{\#_A} \\ I^{\#_A} & O \end{bmatrix} \left(\because \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}^{\#_A} = \begin{bmatrix} T_1^{\#_A} & T_3^{\#_A} \\ T_2^{\#_A} & T_4^{\#_A} \end{bmatrix} \right) \\ &= \begin{bmatrix} O & P_{\overline{\mathcal{R}(A)}} \\ P_{\overline{\mathcal{R}(A)}} & O \end{bmatrix} \quad (\because \mathcal{N}(A)^{\perp} = \overline{\mathcal{R}(A^*)} \text{ \& } \mathcal{R}(A^*) = \mathcal{R}(A)). \end{aligned}$$

This in turn implies $QQ^{\#_A} = \begin{bmatrix} P_{\overline{\mathcal{R}(A)}} & O \\ O & P_{\overline{\mathcal{R}(A)}} \end{bmatrix} = Q^{\#_A}Q$. Now, for $x = (x_1, x_2) \in \mathcal{H} \oplus \mathcal{H}$, we have

$$\begin{aligned} \|Qx\|_{\mathbb{A}}^2 &= \langle Qx, Qx \rangle_{\mathbb{A}} = \langle Q^{\#_A}Qx, x \rangle_{\mathbb{A}} = \left\langle \begin{bmatrix} P_{\overline{\mathcal{R}(A)}} & O \\ O & P_{\overline{\mathcal{R}(A)}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle_{\mathbb{A}} \\ &= \left\langle \begin{bmatrix} AP_{\overline{\mathcal{R}(A)}} & O \\ O & AP_{\overline{\mathcal{R}(A)}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle_{\mathbb{A}} \\ &= \left\langle \begin{bmatrix} AA^{\dagger}A & O \\ O & AA^{\dagger}A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle_{\mathbb{A}} \\ &= \left\langle \begin{bmatrix} A & O \\ O & A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle_{\mathbb{A}} \\ &= \|x\|_{\mathbb{A}}^2. \end{aligned}$$

So, $\|Qx\|_{\mathbb{A}} = \|x\|_{\mathbb{A}}$. Similarly, it can be proved that $\|Q^{\#_A}x\|_{\mathbb{A}} = \|x\|_{\mathbb{A}}$. Thus, Q is an $\mathbb{A}$ -unitary operator. By Corollary 3.4, we obtain

$$w_A(TQ^{\#_A} \pm QT) \leq 2w_A(T). \quad (12)$$

So,

$$\begin{aligned}
2w_{\mathbb{A}}(T) &\geq w_{\mathbb{A}}\left(\begin{bmatrix} T_1^{\#A} & T_3^{\#A} \\ T_2^{\#A} & T_4^{\#A} \end{bmatrix} \begin{bmatrix} O & P_{\overline{\mathcal{R}(A)}} \\ P_{\overline{\mathcal{R}(A)}} & O \end{bmatrix} + \begin{bmatrix} O & I \\ I & O \end{bmatrix} \begin{bmatrix} T_1^{\#A} & T_3^{\#A} \\ T_2^{\#A} & T_4^{\#A} \end{bmatrix}\right) \\
&= w_{\mathbb{A}}\left(\begin{bmatrix} T_3^{\#A} P_{\overline{\mathcal{R}(A)}} & T_1^{\#A} P_{\overline{\mathcal{R}(A)}} \\ T_4^{\#A} P_{\overline{\mathcal{R}(A)}} & T_2^{\#A} P_{\overline{\mathcal{R}(A)}} \end{bmatrix} + \begin{bmatrix} T_2^{\#A} & T_4^{\#A} \\ T_1^{\#A} & T_3^{\#A} \end{bmatrix}\right) \\
&= w_{\mathbb{A}}\left(\begin{bmatrix} T_3^{\#A} & T_1^{\#A} \\ T_4^{\#A} & T_2^{\#A} \end{bmatrix} + \begin{bmatrix} T_2^{\#A} & T_4^{\#A} \\ T_1^{\#A} & T_3^{\#A} \end{bmatrix}\right) \quad \text{by (4)} \\
&= w_{\mathbb{A}}\left(\begin{bmatrix} T_3^{\#A} + T_2^{\#A} & T_1^{\#A} + T_4^{\#A} \\ T_4^{\#A} + T_1^{\#A} & T_2^{\#A} + T_3^{\#A} \end{bmatrix}\right) \\
&= w_{\mathbb{A}}\left(\begin{bmatrix} T_2 + T_3 & T_4 + T_1 \\ T_4 + T_1 & T_2 + T_3 \end{bmatrix}^{\#A}\right) \\
&= w_{\mathbb{A}}\left(\begin{bmatrix} T_2 + T_3 & T_4 + T_1 \\ T_4 + T_1 & T_2 + T_3 \end{bmatrix}\right).
\end{aligned}$$

Hence, we have

$$2w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) = 2w_{\mathbb{A}}\left(\begin{bmatrix} T_1^{\#A} & T_3^{\#A} \\ T_2^{\#A} & T_4^{\#A} \end{bmatrix}\right) \geq w_{\mathbb{A}}\left(\begin{bmatrix} T_2 + T_3 & T_4 + T_1 \\ T_4 + T_1 & T_2 + T_3 \end{bmatrix}\right). \quad (13)$$

By (13) and Lemma 2.2, we obtain

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \geq \frac{1}{2} \max\{w_A(T_1 + T_2 + T_3 + T_4), w_A(T_2 + T_3 - T_4 - T_1)\}. \quad (14)$$

Again, applying Corollary 3.4 and taking $T = \begin{bmatrix} T_1^{\#A} & T_3^{\#A} \\ T_2^{\#A} & T_4^{\#A} \end{bmatrix}$ and $Q = \begin{bmatrix} O & I \\ -I & O \end{bmatrix}$, it can be shown that Q is $\mathbb{A}$ -unitary and $w_{\mathbb{A}}(TQ^{\#A} \pm QT) \leq 2w_{\mathbb{A}}(T)$. Therefore,

$$\begin{aligned}
2w_{\mathbb{A}}(T) &\geq w_{\mathbb{A}}\left(\begin{bmatrix} T_1^{\#A} & T_3^{\#A} \\ T_2^{\#A} & T_4^{\#A} \end{bmatrix} \begin{bmatrix} O & -P_{\overline{\mathcal{R}(A)}} \\ P_{\overline{\mathcal{R}(A)}} & O \end{bmatrix} - \begin{bmatrix} O & I \\ -I & O \end{bmatrix} \begin{bmatrix} T_1^{\#A} & T_3^{\#A} \\ T_2^{\#A} & T_4^{\#A} \end{bmatrix}\right) \\
&= w_{\mathbb{A}}\left(\begin{bmatrix} T_3^{\#A} P_{\overline{\mathcal{R}(A)}} & -T_1^{\#A} P_{\overline{\mathcal{R}(A)}} \\ T_4^{\#A} P_{\overline{\mathcal{R}(A)}} & -T_2^{\#A} P_{\overline{\mathcal{R}(A)}} \end{bmatrix} - \begin{bmatrix} T_2^{\#A} & T_4^{\#A} \\ -T_1^{\#A} & -T_3^{\#A} \end{bmatrix}\right) \\
&= w_{\mathbb{A}}\left(\begin{bmatrix} -T_2^{\#A} + T_3^{\#A} & -T_4^{\#A} - T_1^{\#A} \\ T_4^{\#A} + T_1^{\#A} & -T_2^{\#A} + T_3^{\#A} \end{bmatrix}\right) \quad \text{by (4)} \\
&= w_{\mathbb{A}}\left(\begin{bmatrix} -T_2 + T_3 & T_4 + T_1 \\ -T_4 - T_1 & -T_2 + T_3 \end{bmatrix}\right).
\end{aligned}$$

By Lemma 2.4, we, therefore, achieve the following:

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \geq \frac{1}{2} \max\{w_A(T_4 + T_1 - i(T_2 - T_3)), w_A(T_4 + T_1 + i(T_2 - T_3))\}. \quad (15)$$

From (14) and (15), we get the desired result. $\square$

Example Let $T_1 = T_2 = T_3 = T_4 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $w_A\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = 1$. By Theorem 3.6, we have $\alpha = 4, \beta = 2$ and

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \geq 2.$$

Remark 3.7 Using [11, Theorem 2.1] in the above example, it is easy to find that

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \leq 2.$$

So, we have

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) = 2.$$

Remark 3.8 Using [11, Theorem 2.15] in the above example, it is easy to find

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \geq 1.$$

So, Theorem 3.6 is a refinement of Theorem 2.15 of [11].

As a special case of the above result, we obtain a corollary in Krein space setting. Let J be a non-scalar Hermitian involution operator in $\mathcal{B}(\mathbb{H})$, where $\mathbb{H}$ is a separable Hilbert space. Then, space $(\mathbb{H}, \langle \cdot, \cdot \rangle_J)$ is called *Krein space* [4]. Note that the J -adjoint operator of $T \in \mathcal{B}(\mathbb{H})$ is the unique operator in $\mathcal{B}(\mathbb{H})$ such that $\langle Tx, y \rangle_J = \langle x, T^{\#_J} y \rangle_J$, for all $x, y \in \mathbb{H}$. Therefore, we have $T^{\#_J} = JT^*J$ and $(T^{\#_J})^{\#_J} = T$. One can observe that Lemmas 2.1 and 2.2 hold in the setting of Krein spaces for $T_1, T_2 \in \mathcal{B}(\mathbb{H})$. We provide below a lemma for the J -adjoint operator of an $n \times n$ operator matrix.

Lemma 3.9 Let $T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}$, where $T_{ij} \in \mathcal{B}(\mathbb{H})$. Then

$$\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}^{\#_J} = \begin{bmatrix} T_{11}^{\#_J} & T_{21}^{\#_J} & \cdots & T_{n1}^{\#_J} \\ T_{12}^{\#_J} & T_{22}^{\#_J} & \cdots & T_{n2}^{\#_J} \\ \vdots & \vdots & \cdots & \vdots \\ T_{1n}^{\#_J} & T_{2n}^{\#_J} & \cdots & T_{nn}^{\#_J} \end{bmatrix}.$$

$$\begin{aligned}
& \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}^{\#_J} \\
&= \begin{bmatrix} J & O & \cdots & O \\ O & J & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ O & O & \cdots & J \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}^* \begin{bmatrix} J & O & \cdots & O \\ O & J & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ O & O & \cdots & J \end{bmatrix} \\
&= \begin{bmatrix} J & O & \cdots & O \\ O & J & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ O & O & \cdots & J \end{bmatrix} \begin{bmatrix} T_{11}^* & T_{21}^* & \cdots & T_{n1}^* \\ T_{12}^* & T_{22}^* & \cdots & T_{n2}^* \\ \vdots & \vdots & \cdots & \vdots \\ T_{1n}^* & T_{2n}^* & \cdots & T_{nn}^* \end{bmatrix} \begin{bmatrix} J & O & \cdots & O \\ O & J & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ O & O & \cdots & J \end{bmatrix} \\
&= \begin{bmatrix} JT_{11}^* J & JT_{21}^* J & \cdots & JT_{n1}^* J \\ JT_{12}^* J & JT_{22}^* J & \cdots & JT_{n2}^* J \\ \vdots & \vdots & \cdots & \vdots \\ JT_{1n}^* J & JT_{2n}^* J & \cdots & JT_{nn}^* J \end{bmatrix} \\
&= \begin{bmatrix} T_{11}^{\#_J} & T_{21}^{\#_J} & \cdots & T_{n1}^{\#_J} \\ T_{12}^{\#_J} & T_{22}^{\#_J} & \cdots & T_{n2}^{\#_J} \\ \vdots & \vdots & \cdots & \vdots \\ T_{1n}^{\#_J} & T_{2n}^{\#_J} & \cdots & T_{nn}^{\#_J} \end{bmatrix}.
\end{aligned}$$

Proof

□

We state below another lemma for Krein space operators whose proof follows from [3, Theorem 3.1].

Lemma 3.10 *Let $T, S, X, Y \in \mathcal{B}(\mathbb{H})$. Then,*

$$w_J(TXS^{\#_J} \pm SYT^{\#_J}) \leq 2\|T\|_J \|S\|_J w_J \left(\begin{bmatrix} O & X \\ Y & O \end{bmatrix} \right).$$

In particular, if $X = Y$, then

$$w_J(TXS^{\#_J} \pm SXT^{\#_J}) \leq 2\|T\|_J \|S\|_J w_J(X),$$

and if $X = Y = Q$, $T = I$, then

$$w_J(QS^{\#_J} \pm SQ) \leq 2\|S\|_J w_J(Q).$$

Based on these results, we have the following corollary of Theorem 3.6.

Corollary 3.11 *Let $T_1, T_2, T_3, T_4 \in \mathcal{B}(\mathbb{H})$. Then $w_J \left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) \geq \frac{1}{2} \max\{\alpha, \beta\}$, where*
 $\alpha = \max\{w_J(T_1 + T_2 + T_3 + T_4), w_J(T_1 + T_4 - T_2 - T_3)\}$ *and*
 $\beta = \max\{w_J(T_1 + T_4 + i(T_2 - T_3)), w_J(T_1 + T_4 - i(T_2 - T_3))\}.$

We provide below a lower bound for $\mathbb{A}$ -numerical radius inequality of an operator matrix.

Theorem 3.12 *Let $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$. Then,*

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ O & O \end{bmatrix}\right) \geq \frac{1}{2} \max\{w_A(T_1 + iT_2), w_A(T_1 - iT_2)\}.$$

Proof Suppose that $T = \begin{bmatrix} T_1^{\#_A} & O \\ T_2^{\#_A} & O \end{bmatrix}$ and $Q = \begin{bmatrix} O & -I \\ I & O \end{bmatrix}$. It then follows that Q is $\mathbb{A}$ -unitary. So, $\|Q\|_{\mathbb{A}} = 1$. Using Corollary 3.4, we get

$$2w_{\mathbb{A}}(T) \geq w_{\mathbb{A}}(TQ^{\#_A} - QT).$$

Now,

$$\begin{aligned} w_{\mathbb{A}}(T) &\geq \frac{1}{2} w_{\mathbb{A}}(TQ^{\#_A} - QT) \\ &= \frac{1}{2} w_{\mathbb{A}}\left(\begin{bmatrix} T_1^{\#_A} & O \\ T_2^{\#_A} & O \end{bmatrix} \begin{bmatrix} O & P_{\overline{\mathcal{R}(A)}} \\ -P_{\overline{\mathcal{R}(A)}} & O \end{bmatrix} - \begin{bmatrix} O & -I \\ I & O \end{bmatrix} \begin{bmatrix} T_1^{\#_A} & O \\ T_2^{\#_A} & O \end{bmatrix}\right) \\ &= \frac{1}{2} w_{\mathbb{A}}\left(\begin{bmatrix} O & T_1^{\#_A} P_{\overline{\mathcal{R}(A)}} \\ O & T_2^{\#_A} P_{\overline{\mathcal{R}(A)}} \end{bmatrix} - \begin{bmatrix} -T_2^{\#_A} & O \\ T_1^{\#_A} & O \end{bmatrix}\right) \\ &= \frac{1}{2} w_{\mathbb{A}}\left(\begin{bmatrix} T_2^{\#_A} & T_1^{\#_A} \\ -T_1^{\#_A} & T_2^{\#_A} \end{bmatrix}\right) \text{ by (4)} \\ &= \frac{1}{2} w_{\mathbb{A}}\left(\begin{bmatrix} T_2 & -T_1 \\ T_1 & T_2 \end{bmatrix}^{\#_{\mathbb{A}}}\right) \\ &= \frac{1}{2} w_{\mathbb{A}}\left(\begin{bmatrix} T_2 & -T_1 \\ T_1 & T_2 \end{bmatrix}\right). \end{aligned}$$

By Lemma 2.4, we thus have

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ O & O \end{bmatrix}\right) = w_{\mathbb{A}}\left(\begin{bmatrix} T_1^{\#_A} & O \\ T_2^{\#_A} & O \end{bmatrix}\right) \geq \frac{1}{2} \max\{w_{\mathbb{A}}(T_1 + iT_2), w_{\mathbb{A}}(T_1 - iT_2)\}.$$

□

Remark 3.13 One can observe that Theorem 3.12 can also be proved by taking $T_3 = T_4 = O$ in Theorem 3.6, and is shown below.

By Theorem 3.6, we have

$$\begin{aligned} w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ O & O \end{bmatrix}\right) &\geq \frac{1}{2} \max\{w_A(T_1 + T_2), w_A(T_1 - T_2), w_A(T_1 + iT_2), w_A(T_1 - iT_2)\} \\ &\geq \frac{1}{2} \max\{w_A(T_1 + iT_2), w_A(T_1 - iT_2)\}. \end{aligned}$$

Corollary 3.14 *Let $T = P + iQ$ be the cartesian decomposition in $\mathcal{B}_A(\mathcal{H})$. Then*

$$\begin{aligned} \frac{1}{2}w_A(T) &\leq \min\left\{w_{\mathbb{A}}\left(\begin{bmatrix} P & Q \\ O & O \end{bmatrix}\right), w_{\mathbb{A}}\left(\begin{bmatrix} O & P \\ Q & O \end{bmatrix}\right)\right\}. \\ w_{\mathbb{A}}\left(\begin{bmatrix} P & Q \\ O & O \end{bmatrix}\right) &\geq \frac{1}{2} \max\{w_A(P + iQ), w_A(P - iQ)\} \\ &= \frac{1}{2} \max\{w_A(T), w_A(T^{\#_A})\} \\ &= \frac{1}{2}w_A(T). \end{aligned} \tag{16}$$

Proof

Using Lemma 2.2, we obtain

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & P \\ Q & O \end{bmatrix}\right) = w_{\mathbb{A}}\left(\begin{bmatrix} O & P \\ iQ & O \end{bmatrix}\right) \geq \frac{1}{2}w_A(P \pm iQ) = \frac{1}{2}w_A(T). \tag{17}$$

From (16) and (17), we have

$$\frac{1}{2}w_A(T) \leq \min\left\{w_{\mathbb{A}}\left(\begin{bmatrix} P & Q \\ O & O \end{bmatrix}\right), w_{\mathbb{A}}\left(\begin{bmatrix} O & P \\ Q & O \end{bmatrix}\right)\right\}.$$

□

One can easily prove the next lemma proceeding as in the proof of [11, Lemma 2.9], and is stated below.

Lemma 3.15 *Let $T_1, T_2 \in \mathcal{B}_J(\mathbb{H})$. Then*

$$w_{\mathbb{J}}\left(\begin{bmatrix} T_2 & -T_1 \\ T_1 & T_2 \end{bmatrix}\right) = \max\{w_J(T_1 + iT_2), w_J(T_1 - iT_2)\}.$$

The next corollary follows from the above lemma.

Corollary 3.16 *Let $T_1, T_2 \in \mathcal{B}_J(\mathbb{H})$. Then*

$$w_{\mathbb{J}}\left(\begin{bmatrix} T_1 & T_2 \\ O & O \end{bmatrix}\right) \geq \frac{1}{2} \max\{w_J(T_1 + iT_2), w_J(T_1 - iT_2)\}.$$

We conclude this section with the following two results for $n \times n$ operator matrices. First, we demonstrate an interesting property of $\mathbb{A}$ -numerical radius of an $n \times n$ operator matrix which is a generalization of [10, Lemma 2.1].

Theorem 3.17 Let $T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}$, where $T_{ij} \in \mathcal{B}_A(\mathcal{H})$ for $1 \leq i, j \leq n$. Then

$$w_{\mathbb{A}} \left(\begin{bmatrix} T_{11} & O & \cdots & O \\ O & T_{22} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & T_{nn} \end{bmatrix} \right) \leq w_{\mathbb{A}}(T).$$

Proof Let $z = e^{\frac{2\pi i}{n}}$ and $U = \begin{bmatrix} I & O & \cdots & O \\ O & zI & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & z^{n-1}I \end{bmatrix}$. It is easy to see that $\bar{z} = z^{-1} = z^{n-1}$ and

$|z| = 1$. To show that U is $\mathbb{A}$ -unitary, we need to prove that $\|x\|_{\mathbb{A}} = \|Ux\|_{\mathbb{A}} = \|U^{\#_{\mathbb{A}}}x\|_{\mathbb{A}}$, for $x = (x_1, x_2, \dots, x_n) \in \bigoplus_{i=1}^n \mathcal{H}$. Here,

$$\begin{aligned} U^{\#_{\mathbb{A}}} &= \begin{bmatrix} I & O & \cdots & O \\ O & zI & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & z^{n-1}I \end{bmatrix}^{\#_{\mathbb{A}}} \\ &= \begin{bmatrix} I^{\#_{\mathbb{A}}} & O & \cdots & O \\ O & \bar{z}I^{\#_{\mathbb{A}}} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \bar{z}^{n-1}I^{\#_{\mathbb{A}}} \end{bmatrix} \\ &= \begin{bmatrix} P_{\overline{\mathcal{R}(A)}} & O & \cdots & O \\ O & \bar{z}P_{\overline{\mathcal{R}(A)}} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \bar{z}^{n-1}P_{\overline{\mathcal{R}(A)}} \end{bmatrix}. \end{aligned}$$

$$\text{This in turn implies } UU^{\#_{\mathbb{A}}} = \begin{bmatrix} P_{\overline{\mathcal{R}(A)}} & O & \cdots & O \\ O & P_{\overline{\mathcal{R}(A)}} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & P_{\overline{\mathcal{R}(A)}} \end{bmatrix} = U^{\#_{\mathbb{A}}}U.$$

Now, for $x = (x_1, x_2, \dots, x_n) \in \bigoplus_{i=1}^n \mathcal{H}$, we have

$$\|Ux\|_{\mathbb{A}}^2 = \langle Ux, Ux \rangle_{\mathbb{A}} = \langle U^{\#_{\mathbb{A}}}Ux, x \rangle_{\mathbb{A}} = \|x\|_{\mathbb{A}}^2.$$

So, $\|Ux\|_{\mathbb{A}} = \|x\|_{\mathbb{A}}$. Similarly, $\|U^{\#_{\mathbb{A}}}x\|_{\mathbb{A}} = \|x\|_{\mathbb{A}}$. Thus, U is an $\mathbb{A}$ -unitary operator. Further, a simple calculation shows that

$$\begin{bmatrix} T_{11}^{\#_{\mathbb{A}}} & O & \cdots & O \\ O & T_{22}^{\#_{\mathbb{A}}} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & T_{nn}^{\#_{\mathbb{A}}} \end{bmatrix} = \frac{1}{n} \sum_{k=0}^{n-1} U^{\#_{\mathbb{A}}}{}^k T^{\#_{\mathbb{A}}} U^k.$$

Therefore,

$$\begin{aligned} w_{\mathbb{A}} \left(\begin{bmatrix} T_{11}^{\#_{\mathbb{A}}} & O & \cdots & O \\ O & T_{22}^{\#_{\mathbb{A}}} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & T_{nn}^{\#_{\mathbb{A}}} \end{bmatrix} \right) &\leq \frac{1}{n} \sum_{k=0}^{n-1} w_{\mathbb{A}}(U^{\#_{\mathbb{A}}}{}^k T^{\#_{\mathbb{A}}} U^k) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} w_{\mathbb{A}}(T^{\#_{\mathbb{A}}}) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} w_{\mathbb{A}}(T) \\ &= w_{\mathbb{A}}(T). \end{aligned}$$

This implies that

$$w_{\mathbb{A}} \left(\begin{bmatrix} T_{11} & O & \cdots & O \\ O & T_{22} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & T_{nn} \end{bmatrix}^{\#_{\mathbb{A}}} \right) = w_{\mathbb{A}} \left(\begin{bmatrix} T_{11} & O & \cdots & O \\ O & T_{22} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & T_{nn} \end{bmatrix} \right) \leq w_{\mathbb{A}}(T).$$

□

The last result provides a relation between $\mathbb{A}$ -numerical radius of two diagonal operator matrices, where $\text{diag}(T_1, \dots, T_n)$ means an $n \times n$ diagonal operator matrix with entries $T_1, \dots, T_n$.

Theorem 3.18 *Let $T_i \in \mathcal{B}_{\mathbb{A}}(\mathcal{H})$ for $1 \leq i \leq n$. Then*

$$w_{\mathbb{A}} \left(\text{diag} \left(\sum_{i=1}^n T_i, \dots, \sum_{i=1}^n T_i \right) \right) \leq n w_{\mathbb{A}}(\text{diag}(T_1, \dots, T_n)).$$

Proof Here,

$$\begin{aligned}
w_{\mathbb{A}}\left(\operatorname{diag}\left(\sum_{i=1}^n T_i, \dots, \sum_{i=1}^n T_i\right)\right) &= w_A\left(\sum_{i=1}^n T_i\right) \text{ by Lemma 2.1} \\
&\leq \sum_{i=1}^n w_A(T_i) \\
&\leq n \max\{w_A(T_i) : 1 \leq i \leq n\} \\
&= n w_{\mathbb{A}}(\operatorname{diag}(T_1, \dots, T_n)).
\end{aligned}$$

□

4 Concluding remarks

In this paper, we have further studied A -numerical radius inequalities for operators on semi-Hilbertian space and numerical radius inequalities for operators in Krein space. The important findings are summarized as follows:

- Several upper and lower bounds for a 2×2 operator matrices are demonstrated.
- Some existing $\mathbb{A}$ -numerical radius inequalities are provided by relaxing sufficient condition $A > 0$. This shows the importance of the theory of the Moore–Penrose inverse in this field of research.
- Some numerical radius inequalities for operator matrices in Krein space setting are shown as corollaries to our main results.

This paper ends with the note that further work on $\mathbb{A}$ -numerical radius for $n \times n$ operator matrices and numerical radius inequalities for operators in Krein space can be studied.

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References

1. Arias, M.L., Corach, G., Gonzalez, M.C.: Metric properties of projections in semi-Hilbertian spaces. *Integral Equ. Oper. Theory* **62**, 11–28 (2008)
2. Arias, M.L., Corach, G., Gonzalez, M.C.: Partial isometries in semi-Hilbertian spaces. *Linear Algebra Appl.* **428**, 1460–1475 (2008)
3. Bhunia, P., Paul, K., Nayak, R.K.: On inequalities for A -numerical radius of operators. *Electron. J. Linear Algebra* **36**, 143–157 (2020)
4. Bognar, J.: *Indefinite Inner Product Spaces*. Springer, Berlin (1974)
5. Douglas, R.G.: On majorization, factorization, and range inclusion of operators on Hilbert space. *Proc. Am. Math. Soc.* **17**, 413–415 (1966)
6. Hirzallah, O., Kittaneh, F., Shebrawi, K.: Numerical radius inequalities for commutators of Hilbert space operators. *Numer. Funct. Anal. Optim.* **32**, 739–749 (2011)

7. Moslehian, M.S., Kian, M., Xu, Q.: Positivity of 2×2 block matrices of operators. *Banach J. Math. Anal.* **13**, 726–743 (2019)
8. Moslehian, M.S., Xu, Q., Zamani, A.: Seminorm and numerical radius inequalities of operators in semi-Hilbertian spaces. *Linear Algebra Appl.* **591**, 299–321 (2020)
9. Nashed, M.Z.: *Generalized Inverses and Applications*. Academic Press, New York (1976)
10. Omidvar, M.E., Moslehian, M.S., Niknam, A.: Some numerical radius inequalities for Hilbert space operators. *Involve* **2**, 471–478 (2009)
11. Rout, N.C., Sahoo, S., Mishra, D.: On $\mathbb{A}$ -numerical radius inequalities for 2×2 operator matrices. *Linear Multilinear Algebra* (2020). <https://doi.org/10.1080/03081087.2020.1810201>
12. Rout, N.C., Sahoo, S., Mishra, D.: Some A -numerical radius inequalities for semi-Hilbertian space operators. *Linear Multilinear Algebra* **69**, 980–996 (2021)
13. Saddi, A.: A -normal operators in semi Hilbertian spaces. *Aust. J. Math. Anal. Appl.* **9**, 1–12 (2012)
14. Xu, Q., Ye, Z., Zamani, A.: Some upper bounds for the $\mathbb{A}$ -numerical radius of 2×2 block matrices. *Adv. Oper. Theory* (2021). <https://doi.org/10.1007/s43036-020-00102-5>
15. Zamani, A.: A -numerical radius and product of semi-Hilbertian operators. *Bull. Iran. Math. Soc.* **47**, 371–377 (2021)
16. Zamani, A.: A -Numerical radius inequalities for semi-Hilbertian space operators. *Linear Algebra Appl.* **578**, 159–183 (2019)