A more accurate half-discrete Hilbert-type inequality in the whole plane and the reverses

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Abstract

A more accurate half-discrete Hilbert-type inequality in the whole plane with multi-parameters is established by the use of Hermite–Hadamard’s inequality and weight functions. Furthermore, some equivalent forms and some special types of inequalities and operator representations as well as reverses are considered.

Keywords Hermite–Hadamard inequality · Half-discrete Hilbert-type inequality · Weight function · Equivalent form · Operator expression · Reverse

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1 Introduction

For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n > 0$,

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$$0 < \sum_{m=1}^{\infty} a_m^p < \infty \text{ and } 0 < \sum_{n=1}^{\infty} b_n^q < \infty,$$

the following discrete Hardy-Hilbert inequality (cf. [10], Theorem 315, and [4, 11, 36, 40]) holds true:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}. \quad (1)$$

The constant factor $\frac{\pi}{\sin(\pi/p)}$ is optimal.

Let $f(x), g(y) \geq 0$, such that

$$0 < \int_0^{\infty} f^p(x) dx < \infty \text{ and } 0 < \int_0^{\infty} g^q(y) dy < \infty.$$

Then the following Hardy-Hilbert integral inequality with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [11], Theorem 316) is valid:

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^{\infty} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^{\infty} g^q(y) dy \right)^{\frac{1}{q}}. \quad (2)$$

The following half-discrete Hardy-Hilbert inequality with the same best possible constant factor was recently formulated and proved (cf. [39]):

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{b_n f(x)}{x+n} dx < \frac{\pi}{\sin(\pi/p)} \left(\int_0^{\infty} f^p(x) dx \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}. \quad (3)$$

Several inequalities with homogenous kernels of degree 0 as well as with non-homogenous kernels have been proved in [7, 11, 19, 37, 42, 45]. For a large variety of integral inequalities of Hilbert-type the interested reader is referred to [1–6, 8, 12, 15, 16, 20, 22–24, 26, 41, 46, 47]. The above inequalities are constructed in the quarter plane of the first quadrant.

A Hilbert-type integral inequality in the whole plane was proved in [33] by Yang. Furthermore, a generalized form of a Hilbert-type integral inequality in the whole plane was considered in [34]:

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{|1+xy|^{\lambda}} f(x)g(y) dx dy \\ & < k_{\lambda} \left[\int_{-\infty}^{\infty} |x|^{p(1-\frac{\lambda}{2})-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\frac{\lambda}{2})-1} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \quad (4)$$

The constant factor

$$k_{\lambda} = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) + 2B\left(1-\lambda, \frac{\lambda}{2}\right) \quad (0 < \lambda < 1)$$

is optimal. Additionally, in [9, 13, 14, 25, 27, 29, 30, 43, 44] several integral and discrete Hilbert-type inequalities in the whole plane where formulated and proved.

The goal of the present paper is to study the following half-discrete Hilbert-type inequality in the whole plane with parameters and a best possible constant factor, by applying the Hermite–Hadamard inequality and weight functions:

$$\sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} \frac{\ln \left(\frac{|n|}{|x|} \right)}{|n|^{\lambda} - |x|^{\lambda}} f(x) b_n dx < 2 \left[\frac{\pi}{\lambda \sin \left(\frac{\pi \sigma}{\lambda} \right)} \right]^2 \quad (5)$$

$$\times \left[\int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{|n|=1}^{\infty} |n|^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}},$$

where $\sigma, \mu > 0, \sigma + \mu = \lambda \leq 1$. Moreover, a more accurate half-discrete Hilbert-type inequality with multiparameters is proved. Some equivalent forms, a few special types of inequalities as well as operator representations and reverses are studied.

2 Some lemmas

In what follows, we assume that $\delta \in \{-1, 1\}, a, b \in (-1, 1), \sigma, \mu > 0, \sigma + \mu = \lambda \leq 1, \xi \in (-\infty, \infty), \eta \in [0, \frac{1}{2}]$. We set

$$H(x, y) := \frac{\ln \left(\frac{|y-\eta|+b(y-\eta)}{[|x-\xi|+a(x-\xi)]^{\delta}} \right)}{\left(\frac{|y-\eta|+b(y-\eta)}{[|x-\xi|+a(x-\xi)]^{\delta}} \right)^{\lambda} - 1} \quad (x \neq \xi, y \neq \eta),$$

wherefrom

$$H(x, y) = \frac{\ln \left(\frac{(y-\eta)(1+b)}{[|x-\xi|+a(x-\xi)]^{\delta}} \right)}{\left(\frac{|y-\eta|+b(y-\eta)}{[|x-\xi|+a(x-\xi)]^{\delta}} \right)^{\lambda} - 1} \quad (y > \eta),$$

$$H(x, y) = \frac{\ln \left(\frac{|y-\eta|+b(y-\eta)}{[(x-\xi)(1+a)]^{\delta}} \right)}{\left(\frac{|y-\eta|+b(y-\eta)}{[(x-\xi)(1+a)]^{\delta}} \right)^{\lambda} - 1} \quad (x > \xi),$$

$$H(-x, y) = \frac{\ln \left(\frac{|y-\eta|+b(y-\eta)}{[(x+\xi)(1-a)]^{\delta}} \right)}{\left(\frac{|y-\eta|+b(y-\eta)}{[(x+\xi)(1-a)]^{\delta}} \right)^{\lambda} - 1} \quad (x > -\xi),$$

$$H(x, -y) = \frac{\ln \left(\frac{(y+\eta)(1-b)}{[|x-\xi|+a(x-\xi)]^{\delta}} \right)}{\left(\frac{(y+\eta)(1-b)}{[|x-\xi|+a(x-\xi)]^{\delta}} \right)^{\lambda} - 1} \quad (y > -\eta).$$

Lemma 2.1 We define two weight functions $\omega(\sigma, n)$ and $\varpi(\sigma, x)$ as follows:

$$\omega(\sigma, n) := \int_{-\infty}^{\infty} H(x, n) \frac{[|n - \eta| + b(n - \eta)]^\sigma dx}{[|x - \xi| + a(x - \xi)]^{1+\delta\sigma}} \quad (|n| \in \mathbf{N}), \quad (6)$$

$$\varpi(\sigma, x) := \sum_{|n|=1}^{\infty} H(x, n) \frac{[|x - \xi| + a(x - \xi)]^{-\delta\sigma}}{[|n - \eta| + b(n - \eta)]^{1-\sigma}} \quad (x \in \mathbf{R} \setminus \{\xi\}). \quad (7)$$

Then:

(i) we have

$$\omega(\sigma, n) = k_a(\sigma) := \frac{2}{1 - a^2} \left[\frac{\pi}{\lambda \sin\left(\frac{\pi\sigma}{\lambda}\right)} \right]^2 \in \mathbf{R}_+ \quad (|n| \in \mathbf{N}); \quad (8)$$

(ii) we also have

$$k_b(\sigma)(1 - \theta(\sigma, x)) < \varpi(\sigma, x) < k_b(\sigma) \quad (x \in \mathbf{R} \setminus \{\xi\}), \quad (9)$$

where

$$k_b(\sigma) = \frac{2}{1 - b^2} \left[\frac{\pi}{\lambda \sin\left(\frac{\pi\sigma}{\lambda}\right)} \right]^2,$$

and

$$\begin{aligned} \theta(\sigma, x) &:= \left[\frac{\sin\left(\frac{\pi\sigma}{\lambda}\right)}{\pi} \right]^2 \int_0^{\left\{ \frac{(1+\eta)(1+b)}{[|x-\xi|+a(x-\xi)]^\delta} \right\}^\lambda} \frac{\ln u}{u-1} u^{\frac{\sigma}{\lambda}-1} du \\ &= O\left(\frac{1}{[|x-\xi|+a(x-\xi)]^{\delta(\sigma-\lambda\kappa)}} \right) \in (0, 1) \quad \left(0 < \kappa < \frac{\sigma}{\lambda} \right). \end{aligned} \quad (10)$$

Proof It holds that

$$\begin{aligned}
\omega(\sigma, n) &= \int_{-\infty}^{\xi} H(x, n) \frac{[|n - \eta| + b(n - \eta)]^{\sigma}}{[(x - \xi)(a - 1)]^{1+\delta\sigma}} dx \\
&\quad + \int_{\xi}^{\infty} H(x, n) \frac{[|n - \eta| + b(n - \eta)]^{\sigma}}{[(x - \xi)(a + 1)]^{1+\delta\sigma}} dx \\
&= \int_{-\xi}^{\infty} H(-x, n) \frac{[|n - \eta| + b(n - \eta)]^{\sigma}}{[(x + \xi)(1 - a)]^{1+\delta\sigma}} dx \\
&\quad + \int_{\xi}^{\infty} H(x, n) \frac{[|n - \eta| + b(n - \eta)]^{\sigma}}{[(x - \xi)(1 + a)]^{1+\delta\sigma}} dx.
\end{aligned}$$

Setting

$$u = \left(\frac{[|n - \eta| + b(n - \eta)]}{[(x + \xi)(1 - a)]^{\delta}} \right)^{\lambda} \quad \left(\text{resp. } u = \left(\frac{[|n - \eta| + b(n - \eta)]}{[(x - \xi)(1 + a)]^{\delta}} \right)^{\lambda} \right)$$

in the above first (resp. second) integral, by simplifications, we deduce that

$$\begin{aligned}
\omega(\sigma, n) &= \frac{1}{\lambda^2(1 - a)} \int_0^{\infty} \frac{\ln u}{u - 1} u^{\frac{\sigma}{\lambda} - 1} du \\
&\quad + \frac{1}{\lambda^2(1 + a)} \int_0^{\infty} \frac{\ln u}{u - 1} u^{\frac{\sigma}{\lambda} - 1} du \\
&= \frac{2}{\lambda^2(1 - a^2)} \left[\frac{\pi}{\sin\left(\frac{\pi\sigma}{\lambda}\right)} \right]^2.
\end{aligned}$$

Hence, (8) follows.

We obtain

$$\begin{aligned}
\varpi(\sigma, x) &= \sum_{n=-1}^{-\infty} H(x, n) \frac{[|x - \xi| + a(x - \xi)]^{-\delta\sigma}}{[|n - \eta| + b(n - \eta)]^{1-\sigma}} \\
&\quad + \sum_{n=1}^{\infty} H(x, n) \frac{[|x - \xi| + a(x - \xi)]^{-\delta\sigma}}{[|n - \eta| + b(n - \eta)]^{1-\sigma}} \\
&= \frac{[|x - \xi| + a(x - \xi)]^{-\delta\sigma}}{(1 - b)^{1-\sigma}} \sum_{n=1}^{\infty} \frac{H(x, -n)}{(n + \eta)^{1-\sigma}} \\
&\quad + \frac{[|x - \xi| + a(x - \xi)]^{-\delta\sigma}}{(1 + b)^{1-\sigma}} \sum_{n=1}^{\infty} \frac{H(x, n)}{(n - \eta)^{1-\sigma}}.
\end{aligned} \tag{11}$$

For $0 < \sigma < \lambda \leq 1$, since $(-1)^i \frac{d^{(i)}}{du^{(i)}} \frac{\ln u}{u^{\lambda} - 1} > 0$ ($i = 1, 2$) (cf. [36]), we find that both $\frac{H(x, -y)}{(y + \eta)^{1-\sigma}}$ and $\frac{H(x, y)}{(y - \eta)^{1-\sigma}}$ are strictly decreasing and strictly convex in $y \in (\frac{1}{2}, \infty)$, satisfying

$$(-1)^i \frac{d^{(i)}}{dy^{(i)}} \frac{H(x, -y)}{(y + \eta)^{1-\sigma}} > 0, (-1)^i \frac{d^{(i)}}{dy^{(i)}} \frac{H(x, y)}{(y - \eta)^{1-\sigma}} > 0 \quad (i = 1, 2).$$

By (11) and Hermite–Hadamard's inequality (cf. [17]), in view of $\eta \in [0, \frac{1}{2}]$, we have that

$$\begin{aligned} \varpi(\sigma, x) &< \frac{[|x - \xi| + a(x - \xi)]^{-\delta\sigma}}{(1 - b)^{1-\sigma}} \int_{\frac{1}{2}}^{\infty} \frac{H(x, -y)}{(y + \eta)^{1-\sigma}} dy \\ &\quad + \frac{[|x - \xi| + a(x - \xi)]^{-\delta\sigma}}{(1 + b)^{1-\sigma}} \int_{\frac{1}{2}}^{\infty} \frac{H(x, y)}{(y - \eta)^{1-\sigma}} dy \\ &\leq \frac{[|x - \xi| + a(x - \xi)]^{-\delta\sigma}}{(1 - b)^{1-\sigma}} \int_{-\eta}^{\infty} \frac{H(x, -y) dy}{(y + \eta)^{1-\sigma}} \\ &\quad + \frac{[|x - \xi| + a(x - \xi)]^{-\delta\sigma}}{(1 + b)^{1-\sigma}} \int_{\eta}^{\infty} \frac{H(x, y) dy}{(y - \eta)^{1-\sigma}}. \end{aligned}$$

Setting

$$u = \left\{ \frac{(y + \eta)(1 - b)}{[|x - \xi| + a(x - \xi)]^{\delta}} \right\}^{\lambda} \quad \left(\text{resp. } u = \left\{ \frac{(y - \eta)(1 + b)}{[|x - \xi| + a(x - \xi)]^{\delta}} \right\}^{\lambda} \right)$$

in the above first (resp. second) integral, by simplifications, we deduce that

$$\varpi(\sigma, x) < \frac{1}{1 - b^2} \left[\frac{\pi}{\lambda \sin\left(\frac{\pi\sigma}{\lambda}\right)} \right]^2 = k_b(\sigma).$$

By (11) and the decreasing property of series, we also have that

$$\begin{aligned} \varpi(\sigma, x) &> \frac{[|x - \xi| + a(x - \xi)]^{-\delta\sigma}}{(1 - b)^{1-\sigma}} \int_1^{\infty} \frac{H(x, -y)}{(y + \eta)^{1-\sigma}} dy \\ &\quad + \frac{[|x - \xi| + a(x - \xi)]^{-\delta\sigma}}{(1 + b)^{1-\sigma}} \int_1^{\infty} \frac{H(x, y)}{(y - \eta)^{1-\sigma}} dy. \end{aligned}$$

Setting

$$u = \left\{ \frac{(y + \eta)(1 - b)}{[|x - \xi| + a(x - \xi)]^{\delta}} \right\}^{\lambda} \quad \left(\text{resp. } u = \left\{ \frac{(y - \eta)(1 + b)}{[|x - \xi| + a(x - \xi)]^{\delta}} \right\}^{\lambda} \right)$$

in the above first (resp. second) integral, by simplifications, we derive that

$$\begin{aligned}
\varpi(\sigma, x) &> \frac{1}{\lambda^2(1-b)} \int_{\left\{ \frac{(1+\eta)(1-b)}{[|x-\xi|+a(x-\xi)]^\delta} \right\}^\lambda}^\infty \frac{\ln u}{u-1} u^{\frac{\sigma}{\lambda}-1} du \\
&\quad + \frac{1}{\lambda^2(1+b)} \int_{\left\{ \frac{(1-\eta)(1+b)}{[|x-\xi|+a(x-\xi)]^\delta} \right\}^\lambda}^\infty \frac{\ln u}{u-1} u^{\frac{\sigma}{\lambda}-1} du \\
&\geq \frac{2}{\lambda^2(1-b^2)} \int_{\left\{ \frac{(1+\eta)(1+b)}{[|x-\xi|+a(x-\xi)]^\delta} \right\}^\lambda}^\infty \frac{\ln u}{u-1} u^{\frac{\sigma}{\lambda}-1} du \\
&= k_b(\sigma)(1-\theta(\sigma, x)) > 0.
\end{aligned}$$

For some $\kappa \in (0, \frac{\sigma}{\lambda})$, we obtain that

$$\lim_{u \rightarrow 0^+} u^\kappa \frac{\ln u}{u-1} = \lim_{u \rightarrow \infty} u^\kappa \frac{\ln u}{u-1} = 0$$

and thus there exists a constant $L > 0$, such that $0 < u^\kappa \frac{\ln u}{u-1} \leq L$ ($u \in (0, \infty)$). Hence, we have

$$\begin{aligned}
0 < \theta(\sigma, x) &= \left[\frac{\sin\left(\frac{\pi\sigma}{\lambda}\right)}{\pi} \right]^2 \int_0^{\left\{ \frac{(1+\eta)(1+b)}{[|x-\xi|+a(x-\xi)]^\delta} \right\}^\lambda} u^{\frac{\sigma}{\lambda}-\kappa-1} \left(u^\kappa \frac{\ln u}{u-1} \right) du \\
&\leq \left[\frac{\sin\left(\frac{\pi\sigma}{\lambda}\right)}{\pi} \right]^2 L \int_0^{\left\{ \frac{(1+\eta)(1+b)}{[|x-\xi|+a(x-\xi)]^\delta} \right\}^\lambda} u^{\frac{\sigma}{\lambda}-\kappa-1} du \\
&= \frac{L}{\left(\frac{\sigma}{\lambda} - \kappa\right)} \left[\frac{\sin\left(\frac{\pi\sigma}{\lambda}\right)}{\pi} \right]^2 \left\{ \frac{(1+\eta)(1+b)}{[|x-\xi|+a(x-\xi)]^\delta} \right\}^{\sigma-\kappa\lambda},
\end{aligned}$$

and therefore (9) and (10) follow.

This completes the proof of the lemma. $\square$

Lemma 2.2 For $\varepsilon > 0$,

$$H_\varepsilon(b) := \sum_{|n|=1}^\infty \frac{1}{[|n-\eta|+b(n-\eta)]^{1+\varepsilon}},$$

we have

$$H_\varepsilon(b) = \frac{1}{\varepsilon} \left(\frac{2}{1-b^2} + o_1(1) \right) (1 + o_2(1)) \quad (\varepsilon \rightarrow 0^+). \quad (12)$$

Proof It holds that

$$\begin{aligned}
H_\varepsilon(b) &= \sum_{n=-1}^{-\infty} \frac{1}{[(n-\eta)(b-1)]^{1+\varepsilon}} + \sum_{n=1}^{\infty} \frac{1}{[(n-\eta)(b+1)]^{1+\varepsilon}} \\
&= \frac{1}{(1-b)^{1+\varepsilon}} \sum_{n=1}^{\infty} \frac{1}{(n+\eta)^{1+\varepsilon}} + \frac{1}{(1+b)^{1+\varepsilon}} \sum_{n=1}^{\infty} \frac{1}{(n-\eta)^{1+\varepsilon}}.
\end{aligned} \tag{13}$$

By (13) and the decreasing property of series, we derive that

$$\begin{aligned}
H_\varepsilon(b) &\leq \left[\frac{1}{(1+b)^{1+\varepsilon}} + \frac{1}{(1-b)^{1+\varepsilon}} \right] \sum_{n=1}^{\infty} \frac{1}{(n-\eta)^{1+\varepsilon}} \\
&= \left[\frac{1}{(1+b)^{1+\varepsilon}} + \frac{1}{(1-b)^{1+\varepsilon}} \right] \left[\frac{1}{(1-\eta)^{1+\varepsilon}} + \sum_{n=2}^{\infty} \frac{1}{(n-\eta)^{1+\varepsilon}} \right] \\
&< \left[\frac{1}{(1+b)^{1+\varepsilon}} + \frac{1}{(1-b)^{1+\varepsilon}} \right] \left[\frac{1}{(1-\eta)^{1+\varepsilon}} + \int_1^{\infty} \frac{dy}{(y-\eta)^{1+\varepsilon}} \right] \\
&= \frac{1}{\varepsilon} \left(\frac{2}{1-b^2} + o_1(1) \right) \left\{ 1 + \left[\frac{\varepsilon}{(1-\eta)^{1+\varepsilon}} + \frac{1}{(1-\eta)^\varepsilon} - 1 \right] \right\}, \\
H_\varepsilon(b) &\geq \left[\frac{1}{(1+b)^{1+\varepsilon}} + \frac{1}{(1-b)^{1+\varepsilon}} \right] \sum_{n=1}^{\infty} \frac{1}{(n+\eta)^{1+\varepsilon}} \\
&> \left[\frac{1}{(1+b)^{1+\varepsilon}} + \frac{1}{(1-b)^{1+\varepsilon}} \right] \int_1^{\infty} \frac{dy}{(y+\eta)^{1+\varepsilon}} \\
&= \frac{1}{\varepsilon} \left(\frac{2}{1-b^2} + o_1(1) \right) \left\{ 1 + \left[\frac{1}{(1+\eta)^\varepsilon} - 1 \right] \right\}.
\end{aligned}$$

Hence, we obtain (12) and the lemma is proved. $\square$

Lemma 2.3 For $\varepsilon > 0$, setting

$$E_\delta := \left\{ x \in \mathbf{R} \setminus \{\xi\}; \frac{1}{[|x-\xi| + a(x-\xi)]^\delta} \geq 1 \right\},$$

we have

$$H_\delta := \int_{E_\delta} \frac{1}{[|x-\xi| + a(x-\xi)]^{1+\delta\varepsilon}} dx = \frac{1}{\varepsilon} \frac{2}{1-a^2}. \tag{14}$$

Proof Setting

$$\begin{aligned}
E_\delta^+ &:= \left\{ x > \xi; \frac{1}{[(x-\xi)(1+a)]^\delta} \geq 1 \right\}, \\
E_\delta^- &:= \left\{ x < \xi; \frac{1}{[(\xi-x)(1-a)]^\delta} \geq 1 \right\},
\end{aligned}$$

it follows that $E_\delta = E_\delta^+ \cup E_\delta^-$. We have that

$$H_\delta = \frac{1}{(1+a)^{1+\delta\epsilon}} \int_{E_\delta^+} \frac{1}{(x-\xi)^{1+\delta\epsilon}} dx \\ + \frac{1}{(1-a)^{1+\delta\epsilon}} \int_{E_\delta^-} \frac{1}{(\xi-x)^{1+\delta\epsilon}} dx.$$

Setting $u = [(x-\xi)(1+a)]^\delta$ (resp. $u = [(\xi-x)(1-a)]^\delta$) in the above first (resp. second) integral, we obtain

$$H_\delta = \left(\frac{1}{1+a} + \frac{1}{1-a} \right) \int_1^\infty \frac{du}{u^{1+\epsilon}} = \frac{1}{\epsilon} \frac{2}{1-a^2}.$$

Hence, we get (14) and thus the lemma is proved. $\square$

3 Main results

Theorem 3.1 Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1$,

$$K_{a,b}(\sigma) := k_a^q(\sigma) k_b^p(\sigma) = \frac{2 \left[\frac{\pi}{\lambda \sin\left(\frac{\pi\sigma}{\lambda}\right)} \right]^2}{(1-a^2)^{1/q} (1-b^2)^{1/p}}. \quad (15)$$

If $f(x), b_n \geq 0$, satisfying

$$0 < \int_{-\infty}^{\infty} [|x-\xi| + a(x-\xi)]^{p(1+\delta\sigma)-1} f^p(x) dx < \infty, \text{ and} \\ 0 < \sum_{|n|=1}^{\infty} [|n-\eta| + b(n-\eta)]^{q(1-\sigma)-1} b_n^q < \infty,$$

then we have the following equivalent inequalities:

$$I := \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} H(x,n) f(x) b_n dx \\ < K_{\alpha,\beta}(\sigma) \left\{ \int_{-\infty}^{\infty} [|x-\xi| + a(x-\xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} \\ \times \left\{ \sum_{|n|=1}^{\infty} [|n-\eta| + b(n-\eta)]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}}, \quad (16)$$

$$J_1 := \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{p\sigma-1} \left(\int_{-\infty}^{\infty} H(x, n) f(x) dx \right)^p \right\}^{\frac{1}{p}} \quad (17)$$

$$< K_{\alpha, \beta}(\sigma) \left\{ \int_{-\infty}^{\infty} [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}},$$

$$J_2 := \left\{ \int_{-\infty}^{\infty} [|x - \xi| + a(x - \xi)]^{-q\delta\sigma-1} \left(\sum_{|n|=1}^{\infty} H(x, n) b_n \right)^q dx \right\}^{\frac{1}{q}} \quad (18)$$

$$< K_{\alpha, \beta}(\sigma) \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}}.$$

In particular, for $a = b = 0$ we have the following equivalent inequalities:

$$\sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} \frac{\ln \left(\frac{|n-\eta|}{|x-\xi|^{\delta}} \right)}{\left(\frac{|n-\eta|}{|x-\xi|^{\delta}} \right)^{\lambda} - 1} f(x) b_n dx \quad (19)$$

$$< \left[\frac{\pi}{\lambda \sin(\frac{\pi\sigma}{\lambda})} \right]^2 \left[\int_{-\infty}^{\infty} |x - \xi|^{p(1+\delta\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}$$

$$\times \left[\sum_{|n|=1}^{\infty} |n - \eta|^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}},$$

$$\left\{ \sum_{|n|=1}^{\infty} |n - \eta|^{p\sigma-1} \left[\int_{-\infty}^{\infty} \frac{\ln \left(\frac{|n-\eta|}{|x-\xi|^{\delta}} \right)}{\left(\frac{|n-\eta|}{|x-\xi|^{\delta}} \right)^{\lambda} - 1} f(x) dx \right]^p \right\}^{\frac{1}{p}} \quad (20)$$

$$< \left[\frac{\pi}{\lambda \sin(\frac{\pi\sigma}{\lambda})} \right]^2 \left[\int_{-\infty}^{\infty} |x - \xi|^{p(1+\delta\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}},$$

$$\left\{ \int_{-\infty}^{\infty} |x - \xi|^{-q\delta\sigma-1} \left[\sum_{|n|=1}^{\infty} \frac{\ln \left(\frac{|n-\eta|}{|x-\xi|^{\delta}} \right)}{\left(\frac{|n-\eta|}{|x-\xi|^{\delta}} \right)^{\lambda} - 1} b_n \right]^q dx \right\}^{\frac{1}{q}} \quad (21)$$

$$< \left[\frac{\pi}{\lambda \sin(\frac{\pi\sigma}{\lambda})} \right]^2 \left[\sum_{|n|=1}^{\infty} |n - \eta|^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}}.$$

Proof By Hölder's inequality (cf. [17]) and (6), we obtain that

$$\begin{aligned}
 & \left[\int_{-\infty}^{\infty} H(x, n) f(x) dx \right]^p \\
 &= \left\{ \int_{-\infty}^{\infty} H(x, n) \frac{[|x - \xi| + a(x - \xi)]^{(1+\delta\sigma)/q}}{[|n - \eta| + b(n - \eta)]^{(1-\sigma)/p}} f(x) \right. \\
 &\quad \times \left. \frac{[|n - \eta| + b(n - \eta)]^{(1-\sigma)/p}}{[|x - \xi| + a(x - \xi)]^{(1+\delta\sigma)/q}} dx \right\}^p \\
 &\leq \int_{-\infty}^{\infty} H(x, n) \frac{[|x - \xi| + a(x - \xi)]^{(1+\delta\sigma)(p-1)}}{[|n - \eta| + b(n - \eta)]^{1-\sigma}} f^p(x) dx \\
 &\quad \times \left\{ \int_{-\infty}^{\infty} H(x, n) \frac{[|n - \eta| + b(n - \eta)]^{(1-\sigma)(q-1)}}{[|x - \xi| + a(x - \xi)]^{1+\delta\sigma}} dx \right\}^{p-1} \\
 &= \frac{\omega^{p-1}(\sigma, n)}{[|n - \eta| + b(n - \eta)]^{p\sigma-1}} \\
 &\quad \times \int_{-\infty}^{\infty} H(x, n) \frac{[|x - \xi| + a(x - \xi)]^{(1+\delta\sigma)(p-1)}}{[|n - \eta| + b(n - \eta)]^{1-\sigma}} f^p(x) dx.
 \end{aligned}$$

Then by (8) and the Lebesgue term by term integration theorem (cf. [18]), in view of (7), we derive that

$$\begin{aligned}
 J_1 &\leq k_a^{\frac{1}{q}}(\sigma) \left\{ \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} g(x, n) \frac{[|x - \xi| + a(x - \xi)]^{(1+\delta\sigma)(p-1)}}{[|n - \eta| + b(n - \eta)]^{1-\sigma}} f^p(x) dx \right\}^{\frac{1}{p}} \\
 &= k_a^{\frac{1}{q}}(\sigma) \left\{ \int_{-\infty}^{\infty} \sum_{|n|=1}^{\infty} g(x, n) \frac{[|x - \xi| + a(x - \xi)]^{(1+\delta\sigma)(p-1)}}{[|n - \eta| + b(n - \eta)]^{1-\sigma}} f^p(x) dx \right\}^{\frac{1}{p}} \quad (22) \\
 &= k_a^{\frac{1}{q}}(\sigma) \left\{ \int_{-\infty}^{\infty} \varpi(\sigma, x) [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}}.
 \end{aligned}$$

Hence, by (9), we deduce (17).

By Hölder's inequality (cf. [17]), we have

$$\begin{aligned}
 I &= \sum_{|n|=1}^{\infty} \left\{ [|n - \eta| + b(n - \eta)]^{\frac{-1}{p} + \sigma} \int_{-\infty}^{\infty} H(x, n) f(x) dx \right\} \\
 &\quad \times \left\{ [|n - \eta| + b(n - \eta)]^{\frac{1}{p} - \sigma} b_n \right\} \quad (23) \\
 &\leq J_1 \left[\sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}}.
 \end{aligned}$$

Then by (17), we get (16). On the other hand, assuming that (16) is valid, we set

$$b_n := [|n - \eta| + b(n - \eta)]^{p\sigma-1} \left[\int_{-\infty}^{\infty} H(x, n) f(x) dx \right]^{p-1} \quad (|n| \in \mathbb{N}).$$

Then we obtain that

$$J_1 = \left[\sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{p}}.$$

In view of (22), it follows that $J_1 < \infty$. If $J_1 = 0$, then (17) is trivially valid; if $J_1 > 0$, then by (16), we have

$$\begin{aligned} & \sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} b_n^q \\ &= J_1^p = I < K_{a,b}(\sigma) \left\{ \int_{-\infty}^{\infty} [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} \\ & \quad \times \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}}, \\ & \quad \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{p}} \\ &= J_1 < K_{a,b}(\sigma) \left\{ \int_{-\infty}^{\infty} [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}}, \end{aligned}$$

namely (17) holds, which is equivalent to (16).

Similarly to as we obtained (22), by Hölder's inequality, we have

$$\begin{aligned}
 & \left[\sum_{|n|=1}^{\infty} H(x, n) b_n \right]^q \\
 &= \left\{ \sum_{|n|=1}^{\infty} H(x, n) \frac{[|x - \xi| + a(x - \xi)]^{(1+\delta\sigma)/q}}{[|n - \eta| + b(n - \eta)]^{(1-\sigma)/p}} \right. \\
 &\quad \times \left. \frac{[|n - \eta| + b(n - \eta)]^{(1-\sigma)/p}}{[|x - \xi| + a(x - \xi)]^{(1+\delta\sigma)/q}} b_n \right\}^q \\
 &\leq \left\{ \sum_{|n|=1}^{\infty} H(x, n) \frac{[|x - \xi| + a(x - \xi)]^{(1+\delta\sigma)(p-1)}}{[|n - \eta| + b(n - \eta)]^{1-\sigma}} \right\}^{q-1} \\
 &\quad \times \sum_{|n|=1}^{\infty} H(x, n) \frac{[|n - \eta| + b(n - \eta)]^{(1-\sigma)(q-1)}}{[|x - \xi| + a(x - \xi)]^{1+\delta\sigma}} b_n^q \\
 &= \frac{(\varpi(\sigma, x))^{q-1}}{[|x - \xi| + a(x - \xi)]^{-q\delta\sigma-1}} \\
 &\quad \times \sum_{|n|=1}^{\infty} H(x, n) \frac{[|n - \eta| + b(n - \eta)]^{(1-\sigma)(q-1)}}{[|x - \xi| + a(x - \xi)]^{1+\delta\sigma}} b_n^q.
 \end{aligned}$$

By (9) and the Lebesgue term by term theorem, we have

$$\begin{aligned}
 J_2 &< k_a^{\frac{1}{p}}(\sigma) \left\{ \int_{-\infty}^{\infty} \sum_{|n|=1}^{\infty} H(x, n) \frac{[|n - \eta| + b(n - \eta)]^{(1-\sigma)(q-1)}}{[|x - \xi| + a(x - \xi)]^{1+\delta\sigma}} b_n^q dx \right\}^{\frac{1}{q}} \\
 &= k_a^{\frac{1}{p}}(\sigma) \left\{ \sum_{|n|=1}^{\infty} \omega(\sigma, n) [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}}.
 \end{aligned} \tag{24}$$

Hence, by (8), we deduce (18).

We have proved that (16) is satisfied. Setting

$$f(x) := [|x - \xi| + a(x - \xi)]^{-q\delta\sigma-1} \left[\sum_{|n|=1}^{\infty} H(x, n) b_n \right]^{q-1} \quad (x \in \mathbf{R} \setminus \{\xi\}),$$

it then follows that

$$J_2 = \left\{ \int_{-\infty}^{\infty} [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{q}},$$

and in view of (24), we obtain that $J_2 < \infty$. If $J_2 = 0$, then (18) is trivially valid; if $J_2 > 0$, then by (16) we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \\
&= J_2^q = I < K_{a,b}(\sigma) \left\{ \int_{-\infty}^{\infty} [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}}, \\
&\quad \left\{ \int_{-\infty}^{\infty} [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{q}} \\
&= J_2 < K_{a,b}(\sigma) \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}},
\end{aligned}$$

namely, (18) follows.

On the other hand, assuming that (18) is valid, by Hölder's inequality (cf. [17]) and similarly to as we obtained (23), we have

$$I \leq \left\{ \int_{-\infty}^{\infty} [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} J_2. \quad (25)$$

Then by (18) we derive (16), which is equivalent to (18).

Therefore, inequalities (16), (17) and (18) are equivalent.

This completes the proof of the theorem. $\square$

Theorem 3.2 *With regards to the assumptions of Theorem 1, the constant factor $K_{a,b}(\sigma)$ in (16), (17) and (18) is the best possible.*

Proof For $0 < \varepsilon < q\sigma$, we set $\tilde{\sigma} = \sigma - \frac{\varepsilon}{q}$ ($\in (0, \lambda)$),

$$\tilde{f}(x) := \begin{cases} \frac{1}{[|x - \xi| + a(x - \xi)]^{\delta(\sigma + \frac{\varepsilon}{p}) + 1}}, & x \in E_{\delta}, \\ 0, & x \in \mathbf{R} \setminus E_{\delta}, \end{cases}$$

and

$$\tilde{b}_n := [|n - \eta| + b(n - \eta)]^{(\sigma - \frac{\varepsilon}{q}) - 1}, |n| \in \mathbf{N}.$$

Then by (12) and (14), we obtain that

$$\begin{aligned}
 \tilde{I}_1 &:= \left\{ \int_{-\infty}^{\infty} [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} \tilde{f}^p(x) dx \right\}^{\frac{1}{p}} \\
 &\quad \times \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} \tilde{b}_n^q \right\}^{\frac{1}{q}} \\
 &= \left\{ \int_{-\infty}^{\infty} \frac{dx}{[|x - \xi| + a(x - \xi)]^{\delta\epsilon+1}} \right\}^{\frac{1}{p}} \\
 &\quad \times \left\{ \sum_{|n|=1}^{\infty} \frac{1}{[|n - \eta| + b(n - \eta)]^{\epsilon+1}} \right\}^{\frac{1}{q}} \\
 &\leq \frac{1}{\epsilon} \left(\frac{2}{1-a^2} \right)^{\frac{1}{p}} \left[\left(\frac{2}{1-b^2} + o_1(1) \right) (1 + o_2(1)) \right]^{\frac{1}{q}}.
 \end{aligned}$$

By (9), we also have that

$$\begin{aligned}
 \tilde{I} &:= \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} H(x, n) \tilde{f}(x) \tilde{b}_n dx \\
 &= \int_{E_{\delta}} \sum_{|n|=1}^{\infty} H(x, n) \frac{[|x - \xi| + a(x - \xi)]^{-\delta(\tilde{\sigma}+\epsilon)-1}}{[|n - \eta| + b(n - \eta)]^{1-\tilde{\sigma}}} dx \\
 &= \int_{E_{\delta}} \frac{\varpi(\tilde{\sigma}, x)}{[|x - \xi| + a(x - \xi)]^{\delta\epsilon+1}} dx \\
 &\geq k_b(\tilde{\sigma}) \int_{E_{\delta}} \frac{1 - \theta(\tilde{\sigma}, x)}{[|x - \xi| + a(x - \xi)]^{\delta\epsilon+1}} dx \\
 &= k_b(\tilde{\sigma}) \left\{ \int_{E_{\delta}} \frac{1}{[|x - \xi| + a(x - \xi)]^{\delta\epsilon+1}} dx \right. \\
 &\quad \left. - \int_{E_{\delta}} \frac{1}{O([|x - \xi| + a(x - \xi)]^{\delta(\sigma+\frac{\epsilon}{p}-\lambda\kappa)+1})} dx \right\} \\
 &= \frac{1}{\epsilon} k_b \left(\sigma - \frac{\epsilon}{q} \right) \left(\frac{2}{1-a^2} - \epsilon O(1) \right) \left(0 < \kappa < \frac{\sigma - \epsilon/q}{\lambda} \right).
 \end{aligned}$$

If the constant factor $K_{a,b}(\sigma)$ in (16) is not the best possible, then there exists a positive number k , with $K_{a,b}(\sigma) > k$, such that (16) is valid when we replace $K_{a,b}(\sigma)$ by k . Then in particular, we have $\epsilon \tilde{I} < \epsilon k \tilde{I}_1$, namely,

$$\begin{aligned}
 &k_b \left(\sigma - \frac{\epsilon}{q} \right) \left(\frac{2}{1-a^2} - \epsilon O(1) \right) \\
 &< k \cdot \left(\frac{2}{1-a^2} \right)^{\frac{1}{p}} \left[\left(\frac{2}{1-b^2} + o_1(1) \right) (1 + o_2(1)) \right]^{\frac{1}{q}}.
 \end{aligned}$$

It follows that

$$k_b(\sigma) \frac{2}{1-a^2} \leq k \left(\frac{2}{1-a^2} \right)^{\frac{1}{p}} \left(\frac{2}{1-b^2} \right)^{\frac{1}{q}} (\epsilon \rightarrow 0^+),$$

that is,

$$K_{a,b}(\sigma) = \frac{2 \left[\frac{\pi}{\lambda \sin\left(\frac{\pi\sigma}{\lambda}\right)} \right]^2}{(1-a^2)^{1/q}(1-b^2)^{1/p}} \leq k.$$

This is a contradiction. Hence, the constant factor $K_{a,b}(\sigma)$ in (16) is the best possible.

The constant factor $K_{a,b}(\sigma)$ in (17) (resp. (18)) is still the best possible. Otherwise, we would reach a contradiction by (23) (resp. (25)) that the constant factor $K_{a,b}(\sigma)$ in (16) is not the best possible.

This completes the proof of the theorem. $\square$

4 Operator expressions

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. We set the following functions:

$$\Phi(x) := [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1},$$

$$\Psi(n) := [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1},$$

wherefrom,

$$\Phi^{1-q}(x) = [|x - \xi| + a(x - \xi)]^{-q\delta\sigma-1},$$

$$\Psi^{1-p}(n) = [|n - \eta| + b(n - \eta)]^{p\sigma-1} \quad (x \in \mathbf{R} \setminus \{\xi\}, |n| \in \mathbf{N}).$$

Define the following real weight normed linear spaces:

$$\begin{aligned} L_{p,\Phi}(\mathbf{R}) &:= \left\{ f; \|f\|_{p,\Phi} := \left(\int_{-\infty}^{\infty} \Phi(x) |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\}, \\ L_{q,\Phi^{1-q}}(\mathbf{R}) &:= \left\{ h; \|h\|_{q,\Phi^{1-q}} := \left(\int_{-\infty}^{\infty} \Phi^{1-q}(x) |h(x)|^q dx \right)^{\frac{1}{q}} < \infty \right\}, \\ l_{q,\Psi} &:= \left\{ b = \{b_n\}_{|n|=1}^{\infty}; \|b\|_{q,\Psi} := \left(\sum_{|n|=1}^{\infty} \Psi(n) |b_n|^q \right)^{\frac{1}{q}} < \infty \right\} \\ l_{p,\Psi^{1-p}} &:= \left\{ c = \{c_n\}_{|n|=1}^{\infty}; \|c\|_{p,\Psi^{1-p}} := \left(\sum_{|n|=1}^{\infty} \Psi^{1-p}(n) |c_n|^p \right)^{\frac{1}{p}} < \infty \right\}. \end{aligned}$$

(a) In view of Theorem 1, for $f \in L_{p,\Phi}(\mathbf{R})$, setting

$$H^{(1)}(n) := \int_{-\infty}^{\infty} H(x, n) |f(x)| dx \quad (|n| \in \mathbf{N}),$$

by (17), we have

$$\|H^{(1)}\|_{p,\Psi^{1-p}} = \left[\sum_{|n|=1}^{\infty} \Psi^{1-p}(n) (H^{(1)}(n))^p \right]^{\frac{1}{p}} < K_{a,b}(\sigma) \|f\|_{p,\Phi} < \infty, \quad (26)$$

namely, $H^{(1)} \in l_{p,\Psi^{1-p}}$.

Definition 4.1 Define a Hilbert-type operator in the whole plane

$$T^{(1)} : L_{p,\Phi}(\mathbf{R}) \rightarrow l_{p,\Psi^{1-p}}$$

as follows:

For any $f \in L_{p,\Phi}(\mathbf{R})$, there exists a unique representation

$$T^{(1)}f = H^{(1)} \in l_{p,\Psi^{1-p}},$$

satisfying

$$(T^{(1)}f)(n) = H^{(1)}(n),$$

for any $|n| \in \mathbf{N}$.

In view of (26), it follows that

$$\|T^{(1)}f\|_{p,\Psi^{1-p}} = \|H^{(1)}\|_{p,\Psi^{1-p}} \leq K_{a,b} \|f\|_{p,\Phi},$$

and then the operator $T^{(1)}$ is bounded satisfying

$$\|T^{(1)}\| = \sup_{f(\neq 0) \in L_{p,\Phi}(\mathbf{R})} \frac{\|T^{(1)}f\|_{p,\Psi^{1-p}}}{\|f\|_{p,\Phi}} \leq K_{a,b}(\sigma).$$

Since the constant factor $K_{a,b}(\sigma)$ in (26) is the best possible, we have

$$\|T^{(1)}\| = K_{a,b}(\sigma) = \frac{2 \left[\frac{\pi}{\lambda \sin(\frac{\pi a}{\lambda})} \right]^2}{(1-a^2)^{1/q} (1-b^2)^{1/p}}. \quad (27)$$

If we define the formal inner product of $T^{(1)}f$ and b ($\in l_{q,\Psi}$) as follows:

$$(T^{(1)}f, b) := \sum_{|n|=1}^{\infty} \left(\int_{-\infty}^{\infty} H(x, n) f(x) dx \right) b_n$$

we can then rewrite (16) and (17) as follows:

$$(T^{(1)}f, b) < \|T^{(1)}\| \cdot \|f\|_{p,\Psi} \|b\|_{q,\Phi}, \|T^{(1)}f\|_{p,\Psi^{1-p}} < \|T^{(1)}\| \cdot \|f\|_{p,\Phi}. \quad (28)$$

(b) In view of Theorem 1, for $b \in l_{q,\Psi}$, setting

$$H^{(2)}(x) := \sum_{|n|=1}^{\infty} H(x, n) b_n \quad (x \in \mathbf{R}),$$

then by (18) we have

$$\|H^{(2)}\|_{q,\Phi^{1-q}} = \left[\int_{-\infty}^{\infty} \Phi^{1-q}(x) (H^{(2)}(x))^q dx \right]^{\frac{1}{q}} < K_{a,b}(\sigma) \|b\|_{q,\Psi} < \infty, \quad (29)$$

namely $H^{(2)} \in L_{q,\Psi^{1-q}}(\mathbf{R})$.

Definition 4.2 Define a Hilbert-type operator in the whole plane

$$T^{(2)} : l_{q,\Psi} \rightarrow L_{q,\Psi^{1-q}}(\mathbf{R})$$

as follows:

For any $b \in l_{q,\Psi}$, there exists a unique representation

$$T^{(2)}b = H^{(2)} \in L_{q,\Psi^{1-q}}(\mathbf{R}),$$

satisfying

$$(T^{(2)}b)(x) = H^{(2)}(x),$$

for any $x \in \mathbf{R}$.

In view of (29), we have

$$\|T^{(2)}b\|_{q,\Phi^{1-q}} = \|H^{(2)}\|_{q,\Phi^{1-q}} \leq K_{a,b}(\sigma) \|b\|_{q,\Psi},$$

and then the operator $T^{(2)}$ is bounded satisfying

$$\|T^{(2)}\| = \sup_{b(\neq \theta) \in l_{q,\Psi}} \frac{\|T^{(2)}b\|_{q,\Phi^{1-q}}}{\|b\|_{q,\Psi}} \leq K_{a,b}(\sigma).$$

Since the constant factor $K_{a,b}(\sigma)$ in (29) is the best possible, we have

$$\|T^{(2)}\| = K_{a,b}(\sigma) = \|T^{(1)}\|. \quad (30)$$

If we define the formal inner product of $T^{(2)}b$ and $f \in L_{p,\Phi}(\mathbf{R})$ as follows:

$$(T^{(2)}b, f) := \int_{-\infty}^{\infty} \sum_{|n|=1}^{\infty} H(x, n) b_n f(x) dx,$$

then we can rewrite (16) and (18) in the following manner:

$$(T^{(2)}b, f) < \|T^{(2)}\| \cdot \|f\|_{p, \Psi} \|b\|_{q, \Phi}, \|T^{(2)}b\|_{q, \Phi^{1-q}} < \|T^{(2)}\| \cdot \|b\|_{q, \Psi}. \quad (31)$$

Remark 4.3 (i) For $\xi = \eta = 0$, $\delta = 1$, (19) reduces to (5). If $f(-x) = f(x)$ ($x > 0$), $b_{-n} = b_n$ ($n \in \mathbb{N}$), then (5) reduces to the following half-discrete Hilbert-type inequality (cf. [40]):

$$\begin{aligned} & \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\ln\left(\frac{n}{x}\right)}{\left(\frac{n}{x}\right)^{\lambda} - 1} f(x) b_n dx \\ & < \left[\frac{\pi}{\lambda \sin\left(\frac{\pi\sigma}{\lambda}\right)} \right]^2 \left[\int_0^{\infty} x^{p(1+\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}}. \end{aligned} \quad (32)$$

(ii) For $\delta = 1$, replacing $[|x - \xi| + a(x - \xi)]^{\lambda} f(x)$ to $f(x)$, (16) reduces to the following particular inequality with homogeneous kernel of degree $-\lambda$:

$$\begin{aligned} & \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} \frac{\ln\left(\frac{|n-\eta|+b(n-\eta)}{|x-\xi|+a(x-\xi)}\right) f(x) b_n}{[|n-\eta| + b(n-\eta)]^{\lambda} - [|x-\xi| + a(x-\xi)]^{\lambda}} dx \\ & < K_{a,b}(\sigma) \left[\int_{-\infty}^{\infty} [|x-\xi| + a(x-\xi)]^{p(1-\mu)-1} f^p(x) dx \right]^{\frac{1}{p}} \\ & \quad \times \left[\sum_{|n|=1}^{\infty} [|n-\eta| + b(n-\eta)]^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}}, \end{aligned} \quad (33)$$

(iii) For $\delta = -1$, (16) reduces to the following particular inequality with nonhomogeneous kernel:

$$\begin{aligned} & \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} \frac{\ln\{[x-\xi] + a(x-\xi)][|n-\eta| + b(n-\eta)]\} f(x)}{\{[|n-\eta| + b(n-\eta)][x-\xi] + a(x-\xi)\}^{\lambda} - 1} b_n dx \\ & < K_{a,b}(\sigma) \left\{ \int_{-\infty}^{\infty} [|x-\xi| + a(x-\xi)]^{p(1-\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} \\ & \quad \times \left\{ \sum_{|n|=1}^{\infty} [|n-\eta| + b(n-\eta)]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}}. \end{aligned} \quad (34)$$

The constant factors in the above inequalities are the best possible.

5 Some equivalent reverses

In the sequel, for the cases when $0 < p < 1$ and $p < 0$, we still use $\|b\|_{q,\Phi}$ and $\|f\|_{p,\Psi}$ as the formal symbols.

Theorem 5.1 *Suppose that $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$. If $f(x), b_n \geq 0$, satisfying $0 < \|f\|_{p,\Psi}, \|b\|_{q,\Phi} < \infty$, then we have the following equivalent inequalities:*

$$I = \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} H(x, n) f(x) b_n dx > K_{a,b}(\sigma) \times \left\{ \int_{-\infty}^{\infty} (1 - \theta(\sigma, x)) [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} \|b\|_{q,\Phi}, \quad (35)$$

$$J_1 = \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{p\sigma-1} \left(\int_{-\infty}^{\infty} H(x, n) f(x) dx \right)^p \right\}^{\frac{1}{p}} > K_{a,b}(\sigma) \left\{ \int_{-\infty}^{\infty} (1 - \theta(\sigma, x)) [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}}, \quad (36)$$

$$\tilde{J}_2 := \left\{ \int_{-\infty}^{\infty} \frac{(1 - \theta(\sigma, x))^{1-q}}{[|x - \xi| + a(x - \xi)]^{q\delta\sigma+1}} \left(\sum_{|n|=1}^{\infty} H(x, n) b_n \right)^q dx \right\}^{\frac{1}{q}} > K_{a,b}(\sigma) \|b\|_{q,\Phi}, \quad (37)$$

where the constant factor $K_{a,b}(\sigma)$ is the best possible.

Proof By the reverse Hölder inequality (cf. [17]) and (6), we obtain that

$$\left[\int_{-\infty}^{\infty} H(x, n) f(x) dx \right]^p \geq \frac{\omega^{p-1}(\sigma, n)}{[|n - \eta| + b(n - \eta)]^{p\sigma-1}} \times \int_{-\infty}^{\infty} H(x, n) \frac{[|x - \xi| + a(x - \xi)]^{(1+\delta\sigma)(p-1)}}{[|n - \eta| + b(n - \eta)]^{1-\sigma}} f^p(x) dx.$$

Then by (8) and the Lebesgue term by term integration theorem (cf. [18]), in view of (7), we derive that

$$J_1 \geq k_a^q(\sigma) \left\{ \int_{-\infty}^{\infty} \varpi(\sigma, x) [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}}. \quad (38)$$

Hence, by (9), we deduce (36).

By the reverse Hölder inequality (cf. [17]), we have

$$I \geq J_1 \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}}. \quad (39)$$

Then by (36), we deduce (35).

On the other hand, assuming that (35) is satisfied, we set

$$b_n := [|n - \eta| + b(n - \eta)]^{p\sigma-1} \left[\int_{-\infty}^{\infty} H(x, n) f(x) dx \right]^{p-1} \quad (|n| \in \mathbb{N}).$$

Then we obtain that

$$J_1 = \left[\sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{p}}.$$

In view of (38), it follows that $J_1 > 0$. If $J_1 = \infty$, then (36) is trivially valid; if $J_1 < \infty$, then by (35), we have

$$\begin{aligned} \|b\|_{q,\Phi}^q &= J_1^p = I \\ &> K_{a,b}(\sigma) \left\{ \int_{-\infty}^{\infty} (1 - \theta(\sigma, x)) [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} \|b\|_{q,\Phi}, \\ \|b\|_{q,\Phi}^{q-1} &= J_1 \\ &> K_{a,b}(\sigma) \left\{ \int_{-\infty}^{\infty} (1 - \theta(\sigma, x)) [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}}, \end{aligned}$$

namely (36) holds true, which is equivalent to (35).

Similarly to as we obtained (38), we derive that

$$\tilde{J}_2 > k_a^{\frac{1}{p}}(\sigma) \left\{ \sum_{|n|=1}^{\infty} \omega(\sigma, n) [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}}. \quad (40)$$

Hence, by (8), we deduce (37). We have proved that (35) is valid. Setting

$$f(x) := \frac{(1 - \theta(\sigma, x))^{1-q}}{[|x - \xi| + a(x - \xi)]^{q\delta\sigma+1}} \left[\sum_{|n|=1}^{\infty} H(x, n) b_n \right]^{q-1} \quad (x \in \mathbb{R} \setminus \{\xi\}),$$

then it follows that

$$\tilde{J}_2 = \left\{ \int_{-\infty}^{\infty} (1 - \theta(\sigma, x)) [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{q}},$$

and in view of (40), we obtain that $\tilde{J}_2 > 0$. If $\tilde{J}_2 = \infty$, then (37) is trivially valid; if $\tilde{J}_2 < \infty$, then by (35), we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} (1 - \theta(\sigma, x)) [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx = \tilde{J}_2^q = I \\
& > K_{a,b}(\sigma) \left\{ \int_{-\infty}^{\infty} (1 - \theta(\sigma, x)) [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} \|b\|_{q,\Phi}, \\
& \left\{ \int_{-\infty}^{\infty} (1 - \theta(\sigma, x)) [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{q}} \\
& = \tilde{J}_2 > K_{a,b}(\sigma) \|b\|_{q,\Phi},
\end{aligned}$$

namely, (37) follows.

On the other hand, assuming that (37) is satisfied, by the reverse Hölder inequality (cf. [17]), we obtain

$$I \geq \left\{ \int_{-\infty}^{\infty} (1 - \theta(\sigma, x)) [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} \tilde{J}_2. \quad (41)$$

Then by (37), we derive (15), which is equivalent to (37).

Therefore, inequalities (35), (36) and (37) are equivalent.

For $0 < \varepsilon < p(\lambda - \sigma)$, we set $\tilde{\sigma} = \sigma + \frac{\varepsilon}{p} (< \lambda)$,

$$\tilde{f}(x) := \begin{cases} \frac{1}{[|x - \xi| + a(x - \xi)]^{\delta(\sigma + \frac{\varepsilon}{p}) + 1}}, & x \in E_{\delta}, \\ 0, & x \in \mathbf{R} \setminus E_{\delta}, \end{cases}$$

and

$$\tilde{b}_n := [|n - \eta| + b(n - \eta)]^{\left(\sigma - \frac{\varepsilon}{p}\right) - 1}, \quad |n| \in \mathbf{N}.$$

Then by (12) and (14), we find

$$\begin{aligned}
 \tilde{I}_1 &:= \left\{ \int_{-\infty}^{\infty} (1 - \theta(\sigma, x)) [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} \tilde{f}^p(x) dx \right\}^{\frac{1}{p}} \\
 &\quad \times \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} \tilde{b}_n^q \right\}^{\frac{1}{q}} \\
 &= \left\{ \int_{-\infty}^{\infty} \frac{(1 - \theta(\sigma, x)) dx}{[|x - \xi| + a(x - \xi)]^{\delta\epsilon+1}} \right\}^{\frac{1}{p}} \left\{ \sum_{|n|=1}^{\infty} \frac{1}{[|n - \eta| + b(n - \eta)]^{\epsilon+1}} \right\}^{\frac{1}{q}} \\
 &= \left\{ \frac{1}{\epsilon} \frac{2}{1 - a^2} - \int_{-\infty}^{\infty} \frac{dx}{[|x - \xi| + a(x - \xi)]^{\delta[\sigma - \lambda\kappa + \epsilon] + 1}} \right\}^{\frac{1}{p}} \\
 &\quad \times \left\{ \sum_{|n|=1}^{\infty} \frac{1}{[|n - \eta| + b(n - \eta)]^{\epsilon+1}} \right\}^{\frac{1}{q}} \\
 &= \frac{1}{\epsilon} \left(\frac{2}{1 - a^2} - \epsilon O(1) \right)^{\frac{1}{p}} \left[\left(\frac{2}{1 - b^2} + o_1(1) \right) (1 + o_2(1)) \right]^{\frac{1}{q}} \quad (0 < \kappa < \frac{\sigma}{\lambda}).
 \end{aligned}$$

By (9), we also have

$$\begin{aligned}
 \tilde{I} &:= \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} H(x, n) \tilde{f}(x) \tilde{b}_n dx \\
 &= \sum_{|n|=1}^{\infty} \int_{E_{\delta}} H(x, n) \frac{[|n - \eta| + b(n - \eta)]^{(\sigma - \frac{\epsilon}{q}) - 1}}{[|x - \xi| + a(x - \xi)]^{\delta(\sigma + \frac{\epsilon}{p}) + 1}} dx \\
 &\leq \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} H(x, n) \frac{[|n - \eta| + b(n - \eta)]^{(\tilde{\sigma} - \epsilon) - 1}}{[|x - \xi| + a(x - \xi)]^{\delta\tilde{\sigma} + 1}} dx \\
 &= \sum_{|n|=1}^{\infty} \frac{\omega(\tilde{\sigma}, n)}{[|n - \eta| + b(n - \eta)]^{\epsilon+1}} \\
 &= k_a(\tilde{\sigma}) \sum_{|n|=1}^{\infty} \frac{1}{[|n - \eta| + b(n - \eta)]^{\epsilon+1}} \\
 &= \frac{1}{\epsilon} k_a(\tilde{\sigma}) \left(\frac{2}{1 - b^2} + o_1(1) \right) (1 + o_2(1))
 \end{aligned}$$

If the constant factor $K_{a,b}(\sigma)$ in (36) is not the best possible, then there exists a positive number k , with $K_{a,b}(\sigma) < k$, such that (36) is satisfied when we replace $K_{a,b}(\sigma)$ by k . Then in particular, we have $\epsilon \tilde{I} > \epsilon k \tilde{I}_1$, namely

$$\begin{aligned}
 &k_a \left(\sigma + \frac{\epsilon}{p} \right) \left(\frac{2}{1 - b^2} + o_1(1) \right) (1 + o_2(1)) \\
 &> k \cdot \left(\frac{2}{1 - a^2} - \epsilon O(1) \right)^{\frac{1}{p}} \left[\left(\frac{2}{1 - b^2} + o_1(1) \right) (1 + o_2(1)) \right]^{\frac{1}{q}}.
 \end{aligned}$$

It follows that

$$k_a(\sigma) \frac{2}{1-b^2} \geq k \left(\frac{2}{1-a^2} \right)^{\frac{1}{p}} \left(\frac{2}{1-b^2} \right)^{\frac{1}{q}} (\varepsilon \rightarrow 0^+),$$

namely

$$K_{a,b}(\sigma) = \frac{2 \left[\frac{\pi}{\lambda \sin\left(\frac{\pi\sigma}{\lambda}\right)} \right]^2}{(1-a^2)^{1/q}(1-b^2)^{1/p}} \geq k.$$

This is a contradiction. Hence, the constant factor $K_{a,b}(\sigma)$ in (35) is the best possible.

The constant factor $K_{a,b}(\sigma)$ in (36) ((37)) is still the best possible. Otherwise, we would reach a contradiction by (39) ((41)) that the constant factor $K_{a,b}(\sigma)$ in (35) is not the best possible.

This completes the proof of the theorem. $\square$

Theorem 5.2 Suppose that $p < 0$, $\frac{1}{p} + \frac{1}{q} = 1$. If $f(x), b_n \geq 0$, $0 < \|f\|_{p,\Psi}, \|b\|_{q,\Phi} < \infty$. Then we have the following equivalent inequalities:

$$I = \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} H(x, n) f(x) b_n dx > K_{a,b}(\sigma) \|f\|_{p,\Psi} \|b\|_{q,\Phi}, \quad (42)$$

$$J_1 = \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{p\sigma-1} \left(\int_{-\infty}^{\infty} H(x, n) f(x) dx \right)^p \right\}^{\frac{1}{p}} > K_{a,b}(\sigma) \|f\|_{p,\Psi}, \quad (43)$$

$$J_2 = \left\{ \int_{-\infty}^{\infty} \frac{1}{[|x - \xi| + a(x - \xi)]^{q\delta\sigma+1}} \left(\sum_{|n|=1}^{\infty} H(x, n) b_n \right)^q dx \right\}^{\frac{1}{q}} > K_{a,b}(\sigma) \|b\|_{q,\Phi}, \quad (44)$$

where the constant factor $K_{a,b}(\sigma)$ is the best possible.

Proof By the reverse Hölder's inequality (cf. [17]) and (6), we obtain that

$$\begin{aligned} \left[\int_{-\infty}^{\infty} H(x, n) f(x) dx \right]^p &\leq \frac{\omega^{p-1}(\sigma, n)}{[|n - \eta| + b(n - \eta)]^{p\sigma-1}} \\ &\quad \times \int_{-\infty}^{\infty} H(x, n) \frac{[|x - \xi| + a(x - \xi)]^{(1+\delta\sigma)(p-1)}}{[|n - \eta| + b(n - \eta)]^{1-\sigma}} f^p(x) dx. \end{aligned}$$

Then by (8) and the Lebesgue term by term integration theorem (cf. [18]), in view of (7), we deduce that

$$J_1 \geq k_a^{\frac{1}{p}}(\sigma) \left\{ \int_{-\infty}^{\infty} \varpi(\sigma, x) [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}}. \quad (45)$$

Hence, by (9), we derive (43).

By the reverse Hölder inequality (cf. [17]), we have

$$I \geq J_1 \left[\sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}}. \quad (46)$$

Then by (43), we have (42).

On the other hand, assuming that (42) is valid, we set

$$b_n := [|n - \eta| + b(n - \eta)]^{p\sigma-1} \left[\int_{-\infty}^{\infty} H(x, n) f(x) dx \right]^{p-1} \quad (|n| \in \mathbb{N}).$$

Then we obtain that

$$J_1 = \left[\sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{p}}.$$

In view of (45), it follows that $J_1 > 0$. If $J_1 = \infty$, then (43) is trivially valid; if $J_1 < \infty$, then by (42), we have

$$\begin{aligned} \|b\|_{q,\Phi}^q &= J_1^p = I > K_{a,b}(\sigma) \|f\|_{p,\Psi} \|b\|_{q,\Phi}, \\ \|b\|_{q,\Phi}^{q-1} &= J_1 > K_{a,b}(\sigma) \|f\|_{p,\Psi}, \end{aligned}$$

namely, (43) holds true, which is equivalent to (42).

Similarly, we have

$$J_2 > k_b^{\frac{1}{p}}(\sigma) \left\{ \sum_{|n|=1}^{\infty} \omega(\sigma, n) [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}}. \quad (47)$$

Hence, by (8), we deduce (37). We have proved that (42) is satisfied. Setting

$$f(x) := \frac{1}{[|x - \xi| + a(x - \xi)]^{q\delta\sigma+1}} \left(\sum_{|n|=1}^{\infty} H(x, n) b_n \right)^{q-1} \quad (x \in \mathbb{R} \setminus \{\xi\}),$$

it follows that

$$J_2 = \left\{ \int_{-\infty}^{\infty} [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{q}},$$

and in view of (47), we get $J_2 > 0$. If $J_2 = \infty$, then (44) is trivially valid; if $J_2 < \infty$, then by (42), we have

$$\begin{aligned} \|f\|_{p,\Psi}^p &= J_2^q = I > K_{a,b}(\sigma) \|f\|_{p,\Psi} \|b\|_{q,\Phi}, \\ \|f\|_{p,\Psi}^{p-1} &= J_2 > K_{a,b}(\sigma) \|b\|_{q,\Phi}, \end{aligned}$$

namely, (44) follows.

On the other hand, assuming that (44) is valid, by the reverse Hölder inequality (cf. [17]), we obtain

$$I \geq \left\{ \int_{-\infty}^{\infty} [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} J_2. \quad (48)$$

Then by (44), we deduce (22), which is equivalent to (44). Therefore, inequalities (42), (43) and (44) are equivalent.

For $0 < \varepsilon < |p|\sigma$, we set $\tilde{\sigma} = \sigma + \frac{\varepsilon}{p}$ (> 0),

$$\tilde{f}(x) := \begin{cases} \frac{1}{[|x - \xi| + a(x - \xi)]^{\delta(\sigma + \frac{\varepsilon}{p}) + 1}}, & x \in E_{\delta}, \\ 0, & x \in \mathbf{R} \setminus E_{\delta}, \end{cases}$$

and

$$\tilde{b}_n := [|n - \eta| + b(n - \eta)]^{(\sigma - \frac{\varepsilon}{q}) - 1}, |n| \in \mathbf{N}.$$

Then by (12) and (14), we obtain that

$$\begin{aligned} \tilde{I}_1 &:= \left\{ \int_{-\infty}^{\infty} [|x - \xi| + a(x - \xi)]^{p(1+\delta\sigma)-1} \tilde{f}^p(x) dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + b(n - \eta)]^{q(1-\sigma)-1} \tilde{b}_n^q \right\}^{\frac{1}{q}} \\ &= \left\{ \int_{-\infty}^{\infty} \frac{dx}{[|x - \xi| + a(x - \xi)]^{\delta\varepsilon+1}} \right\}^{\frac{1}{p}} \left\{ \sum_{|n|=1}^{\infty} \frac{1}{[|n - \eta| + b(n - \eta)]^{\varepsilon+1}} \right\}^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} \left(\frac{2}{1 - a^2} \right)^{\frac{1}{p}} \left[\left(\frac{2}{1 - b^2} + o_1(1) \right) (1 + o_2(1)) \right]^{\frac{1}{q}}. \end{aligned}$$

By (9), we also have

$$\begin{aligned}
 \tilde{I} &:= \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} H(x, n) \tilde{f}(x) \tilde{b}_n dx \\
 &= \sum_{|n|=1}^{\infty} \int_{E_{\delta}} H(x, n) \frac{[|n - \eta| + b(n - \eta)]^{(\sigma - \frac{\varepsilon}{q}) - 1}}{[|x - \xi| + a(x - \xi)]^{\delta(\sigma + \frac{\varepsilon}{p}) + 1}} dx \\
 &\leq \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} H(x, n) \frac{[|n - \eta| + b(n - \eta)]^{(\tilde{\sigma} - \varepsilon) - 1}}{[|x - \xi| + a(x - \xi)]^{\delta\tilde{\sigma} + 1}} dx \\
 &= \sum_{|n|=1}^{\infty} \frac{\omega(\tilde{\sigma}, n)}{[|n - \eta| + b(n - \eta)]^{\varepsilon + 1}} \\
 &= k_a(\tilde{\sigma}) \sum_{|n|=1}^{\infty} \frac{1}{[|n - \eta| + b(n - \eta)]^{\varepsilon + 1}} \\
 &= \frac{1}{\varepsilon} k_a(\tilde{\sigma}) \left(\frac{2}{1 - b^2} + o_1(1) \right) (1 + o_2(1))
 \end{aligned}$$

If the constant factor $K_{a,b}(\sigma)$ in (42) is not the best possible, then there exists a positive number k , with $K_{a,b}(\sigma) < k$, such that (42) is satisfied when we replace $K_{a,b}(\sigma)$ by k . Then in particular, we have $\varepsilon \tilde{I} > \varepsilon k \tilde{I}_1$, namely

$$\begin{aligned}
 &k_a \left(\sigma + \frac{\varepsilon}{p} \right) \left(\frac{2}{1 - b^2} + o_1(1) \right) (1 + o_2(1)) \\
 &> k \cdot \left(\frac{2}{1 - a^2} \right)^{\frac{1}{p}} \left[\left(\frac{2}{1 - b^2} + o_1(1) \right) (1 + o_2(1)) \right]^{\frac{1}{q}}.
 \end{aligned}$$

It follows that

$$k_a(\sigma) \frac{2}{1 - b^2} \geq k \cdot \left(\frac{2}{1 - a^2} \right)^{1/p} \left(\frac{2}{1 - b^2} \right)^{1/q} \quad (\varepsilon \rightarrow 0^+),$$

that is

$$K_{a,b}(\sigma) = \frac{2 \left[\frac{\pi}{\lambda \sin \left(\frac{\pi \sigma}{\lambda} \right)} \right]^2}{(1 - a^2)^{1/q} (1 - b^2)^{1/p}} \geq k.$$

This is a contradiction. Hence, the constant factor $K_{a,b}(\sigma)$ in (42) is the best possible.

The constant factor $K_{a,b}(\sigma)$ in (43) ((44)) is still the best possible. Otherwise, we would reach a contradiction by (46) ((48)) that the constant factor $K_{a,b}(\sigma)$ in (42) is not the best possible.

This completes the proof of the theorem. $\square$

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