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ORIGINAL PAPER

On symmetry of strong Birkhoff orthogonality in $B(\mathcal{H}, \mathcal{K})$ and $K(\mathcal{H}, \mathcal{K})$

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Abstract

In this paper, complete characterizations of left (or right) symmetric points for strong Birkhoff orthogonality in $B(\mathcal{H}, \mathcal{K})$ and $K(\mathcal{H}, \mathcal{K})$ are given, where $\mathcal{H}, \mathcal{K}$ are complex Hilbert spaces and $B(\mathcal{H}, \mathcal{K})$ ($K(\mathcal{H}, \mathcal{K})$) is the space of all bounded linear (compact) operators from $\mathcal{H}$ into $\mathcal{K}$.

Keywords (Strong) Birkhoff orthogonality · Symmetric point · Bounded linear operator · Compact operator

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1 Introduction

This paper is concerned with a strengthened version of Birkhoff orthogonality in Hilbert C^*-modules $B(\mathcal{H}, \mathcal{K})$ and $K(\mathcal{H}, \mathcal{K})$, where $\mathcal{H}, \mathcal{K}$ are complex Hilbert spaces and $B(\mathcal{H}, \mathcal{K})$ ($K(\mathcal{H}, \mathcal{K})$) is the space of all bounded linear (compact) operators from $\mathcal{H}$ into $\mathcal{K}$. A *Hilbert C^*-module* $\mathfrak{M}$ over a C^*-algebra $\mathfrak{A}$ is the norm completion of an *inner product $\mathfrak{A}$-module* which is a complex vector space endowed with the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$ given by an $\mathfrak{A}$-valued inner product $\langle \cdot, \cdot\rangle$ which is compatible with a right $\mathfrak{A}$-module structure $\mathfrak{M} \times \mathfrak{A} \ni (x, a) \mapsto xa \in \mathfrak{M}$, where $\langle \cdot, \cdot\rangle$ satisfies the following properties:

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(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ for each $x, y, z \in \mathfrak{M}$ and each $\alpha, \beta \in \mathbb{C}$.
(ii) $\langle x, ya\rangle = \langle x, y\rangle a$ for each $x, y \in \mathfrak{M}$ and each $a \in \mathfrak{A}$.
(iii) $\langle x, y\rangle = \langle y, x\rangle^*$ for each $x, y \in \mathfrak{M}$.
(iv) $\langle x, x\rangle \geq 0$ for each $x \in \mathfrak{M}$; and $\langle x, x\rangle = 0$ if and only if $x = 0$.

The spaces $B(\mathcal{H}, \mathcal{K})$ and $K(\mathcal{H}, \mathcal{K})$ are Hilbert C^*-modules over, respectively, $B(\mathcal{H})$ and $K(\mathcal{H})$, under the right module structure $B(\mathcal{H}, \mathcal{K}) \times B(\mathcal{H}) \ni (S, T) \mapsto ST \in B(\mathcal{H}, \mathcal{K})$ and inner product defined by $\langle S, T\rangle = S^*T$ for each $S, T \in B(\mathcal{H}, \mathcal{K})$ (or $S, T \in K(\mathcal{H}, \mathcal{K})$).

The main object of this paper is Birkhoff orthogonality which was first introduced by Birkhoff [8] and given many important properties by James [14,15].

Definition 1 Let X be a Banach space over the scalar field $\mathbb{K}$. Then $x \in X$ is said to be *Birkhoff orthogonal* to $y \in X$, denoted by $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for each $\lambda \in \mathbb{K}$.

This is viewed as a generalization of the usual orthogonality relation in Hilbert spaces defined by using inner products since they are equivalent to each other in Hilbert spaces; and is known as one of the most important generalized orthogonality relations in Banach spaces because of its geometric meaning which is closely related to the best approximation in norms and support hyperplanes of unit balls. Indeed, if x is a unit vector of a Banach space X and $y \in X$, then $x \perp_B y$ means that the straight line $\{x + \lambda y : \lambda \in \mathbb{K}\}$ is tangent to the unit ball of X at x.

From the definition, Birkhoff orthogonality is obviously *homogeneous*; that is, if $x \perp_B y$ then $\alpha x \perp_B \beta y$ for each scalars α, β. However, the relation $\perp_B$ is not *symmetric* in general. Indeed, James [15] proved that if $x \perp_B y$ implies $y \perp_B x$ in a Banach space X with $\dim X \geq 3$, then X is a Hilbert space. See also the book of Amir [2] for related topics. The survey by Alonso–Martini–Wu [1] provides a good exposition of basic and important results on Birkhoff orthogonality.

Although Birkhoff orthogonality is rarely symmetric in the global sense, it can have some interesting properties for local symmetry. The following definitions (originally for Birkhoff orthogonality) were given by Sain [25] and Sain–Ghosh–Paul [26].

Definition 2 Let X be a Banach space, and let $\perp$ be a generalized orthogonality relation in X.

(L) $x \in X$ is said to be *left symmetric* for $\perp$ in X if $y \in X$ and $x \perp y$ imply that $y \perp x$.
(R) $x \in X$ is said to be *right symmetric* for $\perp$ in X if $y \in X$ and $y \perp x$ imply that $x \perp y$.

In terms of the preceding definition, Turnšek [29,30] showed that the set of all right symmetric points for $\perp_B$ in $B(\mathcal{H})$ coincides with that of scalar multiples of isometries or coisometries on $\mathcal{H}$, while there is no nonzero left symmetric point for $\perp_B$ in $B(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space. Since then, many results on local symmetry of Birkhoff orthogonality have been published, especially, in the fields of Banach (or Hilbert) space operators; see [11,13,21,26] for results on spaces of bounded linear operators between Banach spaces, and [3,7,12,22–25,27,28] for important techniques about Birkhoff orthogonality in such spaces. See also [18–20] for developments in the setting of operator algebras. Similar investigations have been carried out for approximate symmetry of Birkhoff orthogonality in [10].

The aim of this paper is to advance the study of local symmetry for a strengthened version of Birkhoff orthogonality. The following definition was given by Arambašić and Rajić [4].

Definition 3 Let $\mathfrak{M}$ be a Hilbert C^*-module over a C^*-algebra $\mathfrak{A}$, and let $x, y \in \mathfrak{M}$. Then x is said to be $\mathfrak{A}$-*strongly Birkhoff orthogonal* to y, denoted by $x \perp_{\mathfrak{A}} y$, if $x \perp_B ya$ for each $a \in \mathfrak{A}$.

Basics for this generalized orthogonality relation are found in [4–6]. In particular, we have $\langle x, y\rangle = 0 \Rightarrow x \perp_{\mathfrak{A}} y \Rightarrow x \perp_B y$ in Hilbert C^*-modules over $\mathfrak{A}$; see [4, page 112]. For local symmetry of strong Birkhoff orthogonality, in the setting of von Neumann algebras $\mathcal{R}$ (a very special case of Hilbert C^*-modules), it is known that an element $A \in \mathcal{R}$ is left symmetric for $\perp_{\mathcal{R}}$ in $\mathcal{R}$ if and only if $|A|$ is a scalar multiple of a minimal projection in $\mathcal{R}$; while A is right symmetric for $\perp_{\mathcal{R}}$ in $\mathcal{R}$ if and only if it is right invertible in $\mathcal{R}$; see [18]. However, we have not known complete characterizations of left (or right) symmetric points for strong Birkhoff orthogonality in general Hilbert C^*-modules.

In this paper, taking a step forward, we present complete characterizations of left (or right) symmetric points for strong Birkhoff orthogonality in $B(\mathcal{H}, \mathcal{K})$ and $K(\mathcal{H}, \mathcal{K})$, which may provide some hints for the general case. It is shown that an element S in $B(\mathcal{H}, \mathcal{K})$ ($K(\mathcal{H}, \mathcal{K})$) is left symmetric for $\perp_{B(\mathcal{H})}$ ($\perp_{K(\mathcal{H})}$) in $B(\mathcal{H}, \mathcal{K})$ ($K(\mathcal{H}, \mathcal{K})$) if and only S is rank one (Theorem 2). For the right symmetry, it turns out that $S \in B(\mathcal{H}, \mathcal{K})$ is right symmetric for $\perp_{B(\mathcal{H})}$ in $B(\mathcal{H}, \mathcal{K})$ if and only if

(I) S is right invertible in the cases that $\mathcal{H}$ is infinite dimensional or $\dim \mathcal{H} \geq \dim \mathcal{K}$ (Theorem 3);

(II) S is a scalar multiple of an isometry in the case that $\mathcal{H}$ is finite dimensional and $\dim \mathcal{H} < \dim \mathcal{K}$ (Theorem 4);

while $S \in K(\mathcal{H}, \mathcal{K})$ is right symmetric for $\perp_{K(\mathcal{H})}$ in $K(\mathcal{H}, \mathcal{K})$ if and only if S has the dense range (Theorem 5).

2 Preliminaries

Throughout this paper, the term "Hilbert space" always means a nontrivial complex Hilbert space. If $\mathcal{H}$ is a Hilbert space, its inner product is denoted by $\langle \cdot, \cdot \rangle$. An element $x \in \mathcal{H}$ is *orthogonal* to $y \in \mathcal{H}$, denoted by $x \perp y$, if $\langle x, y\rangle = 0$ holds. If A is a subset of $\mathcal{H}$, then $[A]$ denotes the closed linear span of A.

For basics about Hilbert space operators, the readers are referred to, for example, the books of Blackadar [9] and Kadison and Ringrose [16,17]. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. Then the symbols $B(\mathcal{H}, \mathcal{K})$ and $K(\mathcal{H}, \mathcal{K})$ stand for the Banach space of all bounded linear and compact operators from $\mathcal{H}$ into $\mathcal{K}$, respectively. The spaces $B(\mathcal{H}, \mathcal{H})$ and $K(\mathcal{H}, \mathcal{H})$ are simply denoted by $B(\mathcal{H})$ and $K(\mathcal{H})$. Let $T \in B(\mathcal{H}, \mathcal{K})$. Then there exists a unique $T^* \in B(\mathcal{K}, \mathcal{H})$ (called the *adjoint* of T) satisfying $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for each $x \in \mathcal{H}$ and each $y \in \mathcal{K}$. We note that $(T^*)^* = T$, $\|T^*T\| = \|T\|^2$, and that $T \mapsto T^*$ is a conjugate linear isometry from $B(\mathcal{H}, \mathcal{K})$ onto $B(\mathcal{K}, \mathcal{H})$; see [9, I.2.3.1]. Moreover, if $\mathcal{L}$ is another Hilbert space, $T \in B(\mathcal{H}, \mathcal{L})$ and $S \in B(\mathcal{L}, \mathcal{K})$, then $(ST)^* = T^*S^*$ holds.

Let $\mathfrak{A}$ be a C^*-algebra. Then $\mathfrak{A}$ is said to be *unital* if it has the multiplicative unit. If $\mathfrak{A}$ is non-unital, there exists a unital C^*-algebra $\mathfrak{A}_I$, called the *unitization* of $\mathfrak{A}$, such that $\mathfrak{A}$ is an ideal of $\mathfrak{A}_I$ with $\dim(\mathfrak{A}_I/\mathfrak{A}) = 1$. Let $a \in \mathfrak{A}$. Then a is *self-adjoint* if $a^* = a$; and is *positive* if it is self-adjoint and has the nonnegative spectrum in $\mathfrak{A}$ (or $\mathfrak{A}_I$ if $\mathfrak{A}$ is non-unital). Let $\mathfrak{B}$ be another C^*-algebra. A mapping $\varphi : \mathfrak{A} \to \mathfrak{B}$ is called a $*$-*isomorphism* if it is a linear bijection satisfying ($\varphi(I_{\mathfrak{A}}) = I_{\mathfrak{B}}$ if $\mathfrak{A}$ and $\mathfrak{B}$ are unital), $\varphi(a^*) = \varphi(a)^*$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for each $a, b \in \mathfrak{A}$; in which case, $\mathfrak{A}$ and $\mathfrak{B}$ are $*$-*isomorphic*. It is well-known that each C^*-algebra is $*$-isomorphic to a C^*-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (by the Gelfand-Naimark-Segal theorem); see [16, Theorem 4.5.6]. Another important fact is that if $\mathfrak{A}$ is an *abelian* C^*-algebra (that is, $ab = ba$ holds for each $a, b \in \mathfrak{A}$) then it is $*$-isomorphic to the C^*-algebra $C_0(K)$ of all continuous functions on a locally compact Hausdorff space K vanishing at infinity. If, additionally, $\mathfrak{A}$ is unital, then K can be chosen as compact. See [9, Theorem II.2.2.4] and [16, Theorem 4.4.3] for these results.

A *von Neumann algebra* $\mathcal{R}$ acting on a Hilbert space $\mathcal{H}$ is a C^*-subalgebra of $B(\mathcal{H})$ that is closed with respect to the *weak-operator topology* defined as the weak topology induced by the family of functionals on $B(\mathcal{H})$ of the form $A \mapsto \langle Ax, y\rangle$. A bit stronger one, the strong-operator topology, is also important. A net $(A_a) \subset \mathcal{R}$ converges to A in the strong-operator topology if $\|(A_a - A)x\| \to 0$ for each $x \in \mathcal{H}$. The weak-operator and strong-operator closures coincide for convex subsets; see [16, Sect. 5.1]. If $\mathcal{A}$ is an abelian von Neumann algebra and it is $*$-isomorphic to $C(K)$, then K is Stonean (by [16, Theorem 5.2.1]), where K is said to be *Stonean* if it is compact, Hausdorff and extremally disconnected (that is, the closure of an open set is also open).

If $T \in B(\mathcal{H})$, then T is self-adjoint if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for each $x \in \mathcal{H}$; and is positive if and only if $\langle Tx, x\rangle \geq 0$ for each $x \in \mathcal{H}$. An element $E \in B(\mathcal{H})$ is called a *projection* if $E^2 = E = E^*$. An *isometry* is an element $U \in B(\mathcal{H}, \mathcal{K})$ with $U^*U = I_{\mathcal{H}}$; and a *coisometry* is the adjoint of an isometry, that is, an operator $U \in B(\mathcal{H}, \mathcal{K})$ with $UU^* = I_{\mathcal{K}}$. These notions are special cases of partial isometries. An element $V \in B(\mathcal{H}, \mathcal{K})$ is called a *partial isometry* if V^*V is a projection in $B(\mathcal{H})$ (and VV^* is a projection in $B(\mathcal{K})$); in which case, V^*V is the *initial projection* and VV^* is the *final projection*. We note that if V is a partial isometry then it is an isometry on $V^*V(\mathcal{H})$ and $V(V^*V) = V$ holds. Moreover, V^* is also a partial isometry with the initial projection VV^* and the final projection V^*V.

We recall that each $T \in B(\mathcal{H}, \mathcal{K})$ has the polar decomposition $T = U|T| = |T^*|U$, where $|T| = (T^*T)^{1/2}$ and U is a partial isometry from the range projection of $|T|$ onto the range projection of T, where the *range projection* of T is defined to be the projection from H onto $\overline{T(\mathcal{H})}$; see [9, I.5.2.2] or [17, Theorem 6.1.11].

3 Left symmetric points

In this section, we provide characterizations of left symmetric points for $\perp_{B(\mathcal{H})}$ ($\perp_{K(\mathcal{H})}$) in $B(\mathcal{H},\mathcal{K})$ ($K(\mathcal{H},\mathcal{K})$), where $\mathcal{H},\mathcal{K}$ are Hilbert spaces. We begin with an important result due to Arambašić and Rajić [6, Theorem 2.2].

Theorem 1 (Arambašić and Rajić [6]) *Let $\mathcal{M}$ be a Hilbert C^*-module over a C^*-algebra $\mathfrak{A}$. Then $S \in \mathfrak{M}$ is left symmetric for $\perp_{\mathfrak{A}}$ if and only if $\langle S, S\rangle$ is a minimal projection.*

Recall that $B(\mathcal{H},\mathcal{K})$ ($K(\mathcal{H},\mathcal{K})$) is a Hilbert C^*-module over $B(\mathcal{H})$ ($K(\mathcal{H})$) under the $B(\mathcal{H})$- ($K(\mathcal{H})$-)valued inner product $\langle\cdot,\cdot\rangle$ given by $\langle S, T\rangle = S^*T$ for each $S, T \in B(\mathcal{H},\mathcal{K})$ ($S, T \in K(\mathcal{H},\mathcal{K})$). Moreover, a minimal projection in $B(\mathcal{H})$ is just a rank one projection in the usual sense. Hence, by the preceding theorem, we have a necessary condition for left symmetry for strong Birkhoff orthogonality in $B(\mathcal{H},\mathcal{K})$ and $K(\mathcal{H},\mathcal{K})$.

Lemma 1 *Let $\mathcal{H},\mathcal{K}$ be Hilbert spaces, and let $(\mathfrak{M},\mathfrak{A})$ be*

$$(B(\mathcal{H},\mathcal{K}), B(\mathcal{H})) \text{ or } (K(\mathcal{H},\mathcal{K}), K(\mathcal{H})).$$

If $S \in \mathfrak{M}$ is left symmetric for $\perp_{\mathfrak{A}}$ in $\mathfrak{M}$, then S is rank one.

We are now ready to provide characterizations of left symmetric points for strong Birkhoff orthogonality in $B(\mathcal{H},\mathcal{K})$ and $K(\mathcal{H},\mathcal{K})$. It will turn out that the converse of the preceding lemma holds true. The proof to be given here is essentially due to Arambašić and Rajić [5, Proposition 2.3]. Recall that $S \in B(\mathcal{H},\mathcal{K})$ is said to be *rank one* if $\dim S(\mathcal{H}) = 1$.

Theorem 2 *Let $\mathcal{H},\mathcal{K}$ be Hilbert spaces, and let $(\mathfrak{M},\mathfrak{A})$ be*

$$(B(\mathcal{H},\mathcal{K}), B(\mathcal{H})) \text{ or } (K(\mathcal{H},\mathcal{K}), K(\mathcal{H})).$$

Suppose that $S \in \mathfrak{M}$ is nonzero. Then S is left symmetric for $\perp_{\mathfrak{A}}$ in $\mathfrak{M}$ if and only if S is rank one.

Proof The "only if" part is completed by Lemma 1. We shall show the "if" part. Suppose that S is rank one. Let T be an element of $\mathfrak{M}$ such that $S \perp_{\mathfrak{A}} T$. Then, by [6, Lemma 2.1 (6)], one has $S^*S \perp_{\mathfrak{A}} S^*T$ in $\mathfrak{A}$; and then $S^*S \perp_B S^*T(T^*S)$ in $\mathfrak{A}$. On the other hand, the rank one operator S is represented by $Sx = \langle x, x_0\rangle y_0$ for some $x_0, y_0 \in \mathcal{H}$. It is easy to check that $S^*x = \langle x, y_0\rangle x_0$ for each $x \in \mathcal{H}$, and that $S^*TT^*S = aS^*S$, where $a = \|y_0\|^{-2}\|T^*y_0\|^2$. Hence it follows from $S^*S \perp_B aS^*S$ and $S^*S \neq 0$ that $a = 0$; which implies that $S^*TT^*S = 0$ and $(\langle S, T\rangle =)S^*T = 0$. Thus $T \perp_{\mathfrak{A}} S$; and, S is left symmetric for $\perp_{\mathfrak{A}}$ in $\mathfrak{A}$. ◻

The following are immediate consequences of Theorem 2.

Corollary 1 *Let $\mathcal{H},\mathcal{K}$ be Hilbert spaces. Suppose that $S \in B(\mathcal{H},\mathcal{K})$ is nonzero. Then the following are equivalent:*

(i) *S is rank one.*
(ii) *S is left symmetric for $\perp_{B(\mathcal{H})}$ in $B(\mathcal{H},\mathcal{K})$.*
(iii) *$S \in K(\mathcal{H},\mathcal{K})$, and is left symmetric for $\perp_{K(\mathcal{H})}$ in $K(\mathcal{H},\mathcal{K})$.*

Corollary 2 *Let $\mathcal{H},\mathcal{K}$ be Hilbert spaces, and let $S \in B(\mathcal{H},\mathcal{K})$ be nonzero. Suppose that* $\dim \mathcal{H} = 1$. *Then $\perp_{B(\mathcal{H})}$ is symmetric in $B(\mathcal{H},\mathcal{K})$.*

Proof Since all $S, T \in B(\mathcal{H},\mathcal{K})$ are at most rank one, $S \perp_{B(\mathcal{H})} T$ always implies that $T \perp_{B(\mathcal{H})} S$ by Theorem 2. □

It should be mentioned that Corollary 2 is a special case of [6, Theorem 2.6].

4 Right symmetric points

Our next aim is to characterize right symmetric points for strong Birkhoff orthogonality in $B(\mathcal{H},\mathcal{K})$ and $K(\mathcal{H},\mathcal{K})$. The following elementary facts are needed in the sequel. The proof can be essentially found, for example, in [16, Proposition 2.5.13].

Lemma 2 *Let $\mathcal{H},\mathcal{K}$ be Hilbert spaces. Suppose that $S \in B(\mathcal{H},\mathcal{K})$. Then the following hold:*

(i) $\overline{S(\mathcal{H})} = \overline{SS^*(\mathcal{K})} = \overline{|S^*|(\mathcal{K})}$.
(ii) *$\overline{S(\mathcal{H})} = \mathcal{K}$ if and only if $|S^*|$ is injective.*

One of the most important arguments in our study of right symmetry is the following.

Lemma 3 *Let $\mathcal{H},\mathcal{K}$ be Hilbert spaces, and let $(\mathfrak{M},\mathfrak{A})$ be*

$$(B(\mathcal{H},\mathcal{K}), B(\mathcal{H})) \text{ or } (K(\mathcal{H},\mathcal{K}), K(\mathcal{H})).$$

Suppose that $S \in \mathfrak{M}$ is nonzero. If $|S| \neq \|S\| I_{\mathcal{H}}$ and $\overline{S(\mathcal{H})} \neq \mathcal{K}$, then S is not right symmetric for $\perp_{\mathfrak{A}}$ in $\mathfrak{M}$.

Proof We may assume that $\|S\| = 1$. Let $\mathcal{A}$ be the von Neumann algebra generated by $|S|$ and $I_{\mathcal{H}}$. Then there exists a Stonean space K such that $\mathcal{A}$ is $*$-isomorphic to $C(K)$ under a $*$-isomorphism $\varphi : \mathcal{A} \to C(K)$. Since $(f =)\varphi(|S|) \neq \mathbf{1}$, we have $O = \{t \in K : f(t) > \alpha\} \neq K$ for some $\alpha \in (0, 1)$. Let e be the characteristic function of the clopen set $\overline{O}$, and let $E = \varphi^{-1}(e)$. Next, pick an arbitrary unit vector $x_0 \in (I_{\mathcal{H}} - E)(\mathcal{H})$. Since $\overline{S(\mathcal{H})} \neq \mathcal{K}$, the projection F from $\mathcal{K}$ onto $\overline{S(\mathcal{H})}$ satisfies $I_{\mathcal{K}} - F \neq 0$. Take a unit vector $y_0 \in (I_{\mathcal{K}} - F)(\mathcal{H})$. Let $Vx = \langle x, x_0\rangle y_0$ for each $x \in \mathcal{H}$, and let $T = V + SE$. Then we have $T \in \mathfrak{M}$ since V is compact. Moreover, it follows from $\|V\| = \|S\| = 1$, $SE = FSE$ and $V = (I_{\mathcal{K}} - F)V(I_{\mathcal{H}} - E)$ that

$$\|Tx\|^2 = \|(I_{\mathcal{K}} - F)V(I_{\mathcal{H}} - E)x\|^2 + \|FSEx\|^2 \leq \|x\|^2$$

for each $x \in \mathcal{H}$; which implies that $\|T\| = \|Tx_0\| = 1$. From this, we obtain $T \perp_{\mathfrak{A}} S$ since

$$\|T + \lambda SR\| \geq \|(I_{\mathcal{K}} - F)(T + \lambda SR)\| = \|V\| = 1 = \|T\|$$

for each $R \in \mathfrak{A}$ and each λ. On the other hand, since

$$\|S - 2^{-1}TE\| = \|S(I_{\mathcal{H}} - 2^{-1}E)\| = \max\{2^{-1}, \alpha\} < 1$$

it follows that $S \not\perp_B T$; and then $S \not\perp_{\mathfrak{A}} T$. Thus S is not right symmetric for $\perp_{\mathfrak{A}}$ in $\mathfrak{M}$. $\square$

Recall that an element S of $B(\mathcal{H}, \mathcal{K})$ is said to be *right invertible* if $ST = I_{\mathcal{K}}$ for some $T \in B(\mathcal{K}, \mathcal{H})$. Typical examples are given by coisometries in $B(\mathcal{H}, \mathcal{K})$.

Theorem 3 *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, and let $S \in B(\mathcal{H}, \mathcal{K})$ be nonzero. Suppose that either of the following statements holds:*

(i) *$\mathcal{H}$ is infinite dimensional.*
(ii) $\dim \mathcal{H} \geq \dim \mathcal{K}$.

Then S is right symmetric for $\perp_{B(\mathcal{H})}$ in $B(\mathcal{H}, \mathcal{K})$ if and only if S is right invertible.

Proof Suppose that S is right symmetric for $\perp_{B(\mathcal{H})}$ in $B(\mathcal{H}, \mathcal{K})$. We first show that the existence of such an S assures that (i) $\Rightarrow$ (ii). To see this, it is enough to show that $\overline{S(\mathcal{H})} = \mathcal{K}$ if $\mathcal{H}$ is infinite dimensional since

$$\dim \overline{S(\mathcal{H})} = \dim \overline{|S|(\mathcal{H})} \leq \dim \mathcal{H}.$$

Suppose to the contrary that $\overline{S(\mathcal{H})} \neq \mathcal{K}$. Then S is an isometry by Lemma 3; which implies that $\mathcal{K}$ is also infinite dimensional. Let (e_a) be an orthonormal basis for $\mathcal{H}$. Then (Se_a) is an orthonormal basis for $S(\mathcal{H})$. If y_0 is a unit vector in $\mathcal{K}$ that is orthogonal to $\overline{S(\mathcal{H})}$, then there exists a bijection from $\{e_a\}$ onto $\{Se_a\} \cup \{y_0\}$. The linear extension $U \in B(\mathcal{H}, \mathcal{K})$ of such a bijection gives rise to an isometry satisfying $S(\mathcal{H}) \subset U(\mathcal{H})$. Now, letting F be the projection from $\mathcal{K}$ onto $[\{y_0\}]$, we have $U \perp_{B(\mathcal{H})} S$ since

$$\|U + \lambda SR\| \geq \|F(U + \lambda SR)\| = \|FU\| = 1 = \|U\|$$

for each $R \in B(\mathcal{H})$ and each λ. Hence $S \perp_{B(\mathcal{H})} U$ by the right symmetry of S for $\perp_{B(\mathcal{H})}$ in $B(\mathcal{H}, \mathcal{K})$. However, then, one has that $S \perp_B U(U^*S)$. Since $UU^*(\mathcal{K}) = \overline{U(\mathcal{H})}$, we derive that $UU^*S = S$; which implies that $S = 0$, a contradiction. Therefore $\overline{S(\mathcal{H})} = \mathcal{K}$ must hold.

From what we have shown in above, in either case, $\dim \mathcal{H} \geq \dim \mathcal{K}$ holds. By considering an injection from an orthonormal basis for $\mathcal{K}$ into that of $\mathcal{H}$, we can construct an isometry $V \in B(\mathcal{K}, \mathcal{H})$. We shall show that $k_0 = \inf\{\|S^*y\| : y \in K, \ \|y\| = 1\} > 0$. For this purpose, suppose to the contrary that $k_0 = 0$. Take a sequence (y_n) of unit vectors in $\mathcal{K}$ satisfying $\|S^*y_n\| \to 0$. Since V is an isometry, it follows that

$$\|V^* + \lambda SR\| = \|V + \overline{\lambda}R^*S^*\| \geq \|(V + \overline{\lambda}R^*S^*)y_n\| \to 1 = \|V^*\|$$

for each $R \in B(\mathcal{H})$ and each λ. Therefore $V^* \perp_{B(\mathcal{H})} S$. However, then, $S \perp_{B(\mathcal{H})} V^*$ since S is right symmetric for $\perp_{B(\mathcal{H})}$ in $B(\mathcal{H}, \mathcal{K})$. In particular, $S \perp_B V^*(VS)$ with $V^*S \in B(\mathcal{H})$; which implies that $S = 0$, a contradiction. Hence we have $k_0 > 0$.

Now we have $\|S^*y\| \geq k_0\|y\|$ for each $y \in \mathcal{K}$. From this, S^* is injective and has the closed range, that is, $S^*(\mathcal{K}) = \overline{S^*(\mathcal{K})}$. In particular, S^* can be viewed as an isomorphism between K and $S^*(\mathcal{K})$. Let $S_0 \in B(S^*(\mathcal{K}), \mathcal{K})$ be such that $S_0S^* = I_{\mathcal{K}}$, and let E be a projection from H onto $S^*(\mathcal{K})$. Putting $T = S_0E \in B(\mathcal{H}, \mathcal{K})$ yields that $TS^* = S_0ES^* = S_0S^* = I_{\mathcal{K}}$. Thus $ST^* = I_{\mathcal{K}}$ holds; and S is right invertible.

Conversely, if $ST = I_{\mathcal{K}}$ for some $T \in B(\mathcal{K}, \mathcal{H})$, then $R \in B(\mathcal{H}, \mathcal{K})$ and $R \perp_{B(\mathcal{H})} S$ imply that $R \perp_B S(TR)$. This shows that $R = 0$; and hence, $S \perp_{B(\mathcal{H})} R$. The proof is complete. $\square$

Hence the problem remains only in the case that $\mathcal{H}$ is finite dimensional and $\dim \mathcal{H} < \dim \mathcal{K}$. In this case, we have a consequence that is natural but completely different from that of Theorem 3. We note that the case $\dim \mathcal{H} = 1$ was already completed in Corollary 2.

Theorem 4 *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, and let $S \in B(\mathcal{H}, \mathcal{K})$ be nonzero. Suppose that $\mathcal{H}$ is finite dimensional with $2 \leq \dim \mathcal{H} < \dim \mathcal{K}$. Then S is right symmetric for $\perp_{B(\mathcal{H})}$ in $B(\mathcal{H}, \mathcal{K})$ if and only if S is a scalar multiple of an isometry.*

Proof It may be assumed that $\|S\| = 1$. Suppose that S is right symmetric for $\perp_{B(\mathcal{H})}$ in $B(\mathcal{H}, \mathcal{K})$. Since $\mathcal{H}$ is finite dimensional and $\dim S(\mathcal{H}) \leq \dim \mathcal{H} < \dim \mathcal{K}$, we have $\overline{S(\mathcal{H})} = S(\mathcal{H}) \neq \mathcal{K}$. By Lemma 3, it follows that $|S| = I_{\mathcal{H}}$. Therefore $S^*S = I_{\mathcal{H}}$ holds; that is, S is an isometry.

Conversely, we assume that S is an isometry. Let $T \in B(\mathcal{H}, \mathcal{K})$ be such that $T \perp_{B(\mathcal{H})} S$. Then it follows from $T \perp_B S(S^*T)$ that $\|(I_{\mathcal{K}} - SS^*)T\| = \|T\|$. In particular, we have $\|(I_{\mathcal{K}} - SS^*)Tx_0\| = \|T\|$ for some unit vector $x_0 \in \mathcal{H}$; in which case, $\|Tx_0\| = \|(I_{\mathcal{K}} - SS^*)Tx_0\|$. Hence $Tx_0 = (I_{\mathcal{K}} - SS^*)Tx_0$ and $Tx_0 \perp Sx_0$ holds. Now let F be the projection from $\mathcal{K}$ onto $S(\mathcal{H})$. Since $FTx_0 = 0$ and $\dim \mathcal{H} = \dim \ker FT + \dim FT(\mathcal{H})$, we have

$$\dim FT(\mathcal{H}) < \dim \mathcal{H} = \dim S(\mathcal{H}),$$

which implies that $FT(\mathcal{H})$ is a proper subspace of $S(\mathcal{H})$. Let $y_1(= Sx_1)$ be a unit vector in $S(\mathcal{H})$ that is orthogonal to $FT(\mathcal{H})$. Then one obtains that $y_1 \perp T(\mathcal{H})$ by $(I_{\mathcal{K}} - F)y_1 = 0$ and

$$\langle y_1, Tx \rangle = \langle y_1, (I_{\mathcal{K}} - F)Tx \rangle + \langle y_1, FTx \rangle = 0$$

for each $x \in \mathcal{H}$. Thus the inequality

$$\|S + \lambda TR\| \geq \|(S + \lambda TR)x_1\| \geq \|y_1\| = 1 = \|S\|$$

holds for each $R \in B(\mathcal{H})$ and each λ. Hence we have $S \perp_{B(\mathcal{H})} T$; that is, S is right symmetric for $\perp_{B(\mathcal{H})}$ in $B(\mathcal{H}, \mathcal{K})$. $\square$

Remark 1 We remark that if $\mathcal{H}$ is infinite dimensional or $\dim \mathcal{H} \geq \dim \mathcal{K}$, then each right symmetric point S for $\perp_{B(\mathcal{H})}$ in $B(\mathcal{H}, \mathcal{K})$ has no nonzero element $T \in B(\mathcal{H}, \mathcal{K})$ satisfying $T \perp_{B(\mathcal{H})} S$ by its right invertibility.

On the other hand, if $\mathcal{H}$ is nonzero finite dimensional and $\dim\mathcal{H} < \dim\mathcal{K}$, then each $S \in B(\mathcal{H},\mathcal{K})$ always has a nonzero element $T \in B(\mathcal{H},\mathcal{K})$ satisfying $T \perp_{B(\mathcal{H})} S$. Indeed, as in the first paragraph of the proof of Theorem 4, we have $\dim\overline{S(\mathcal{H})} < \dim\mathcal{K}$. Let F be the projection from $\mathcal{K}$ onto $\overline{S(\mathcal{H})}$, and let y_0 be an element of $(I-F)(\mathcal{K})$ with $\|y_0\| = 1$. Fix an arbitrary $x_0 \in \mathcal{H}$ with $\|x_0\| = 1$. Then $Tx = \langle x, x_0\rangle y_0$ defines a nonzero element of $B(\mathcal{H},\mathcal{K})$ satisfying $\langle Tx, Sx\rangle = 0$ for each $x \in \mathcal{H}$ (that is, $S^*T = 0$). Thus we obtain $T \perp_{B(\mathcal{H})} S$.

We finally consider right symmetric points for $\perp_{K(\mathcal{H})}$ in $K(\mathcal{H},\mathcal{K})$. Since $K(\mathcal{H},\mathcal{K}) = B(\mathcal{H},\mathcal{K})$ if $\mathcal{H}$ is finite dimensional, the remainder part is only the case that $\mathcal{H}$ is infinite dimensional.

Theorem 5 *Let $\mathcal{H},\mathcal{K}$ be Hilbert spaces, and let $S \in K(\mathcal{H},\mathcal{K})$ be nonzero. Suppose that $\mathcal{H}$ is infinite dimensional. Then S is right symmetric for $\perp_{K(\mathcal{H})}$ in $K(\mathcal{H},\mathcal{K})$ if and only if $\overline{S(\mathcal{H})} = \mathcal{K}$.*

Proof Throughout this proof, we may assume that $\|S\| = 1$. We note that $|S| \neq I_{\mathcal{H}}$ since $|S| \in K(\mathcal{H})$ and $\mathcal{H}$ is infinite dimensional. Hence, if S is right symmetric for $\perp_{K(\mathcal{H})}$ in $K(\mathcal{H},\mathcal{K})$, then, as in the first paragraph of the proof of Theorem 4, we have $\overline{S(\mathcal{H})} = \mathcal{K}$ by Lemma 3.

For the converse, suppose that $\overline{S(\mathcal{H})} = \mathcal{K}$. Then $|S^*|$ is injective by Lemma 2; which implies that $\langle |S^*|y, y\rangle > 0$ for each nonzero $y \in \mathcal{K}$. Let H be a nonzero positive element of $K(\mathcal{H})$. Since $0 \leq |S^*| \leq I_{\mathcal{K}}$, at least, we have $\|H - |S^*|H\| \leq \|I_{\mathcal{K}} - |S^*|\|\|H\| \leq \|H\|$. Suppose that $\|H - |S^*|H\| = \|H\|$. Then there exists a sequence (y_n) of unit vectors in $\mathcal{K}$ such that $\|(H - |S^*|H)y_n\| \to \|H\|$. Since $H - |S^*|H \in K(\mathcal{K})$, we obtain a subsequence (y_{n_k}) of (y_n) with the property that $((H - |S^*|H)y_{n_k})$ converges to some $z_0 \in \mathcal{K}$. On the other hand, the reflexivity of $\mathcal{K}$ generates a subsequence $(y_{n_{k_l}})$ of (y_{n_k}) that converges weakly to some $y_0 \in \mathcal{K}$ with $\|y_0\| \leq 1$. Since bounded linear operators are weak to weak continuous, we drive that $((H - |S^*|H)y_{n_{k_l}})$ converges weakly to $(H - |S^*|H)y_0$. Hence one has that $(H - |S^*|H)y_0 = z_0$, and that

$$\|(H - |S^*|H)y_0\| = \|z_0\| = \lim_k \|(H - |S^*|H)y_{n_k}\| = \|H\|.$$

However, then, it turns out that

$$\begin{aligned}\|H\|^2 = \|(H - |S^*|H)y_0\|^2 &\leq (\|(I_{\mathcal{K}} - |S^*|)^{1/2}\|\|(I - |S^*|)^{1/2}Hy_0\|)^2\\ &\leq \langle H(I - |S^*|)Hy_0, y_0\rangle\\ &= \langle H^2y_0, y_0\rangle - \langle |S^*|Hy_0, Hy_0\rangle\\ &< \langle H^2y_0, y_0\rangle\\ &\leq \|H\|^2.\end{aligned}$$

This is a contradiction. Hence $\|H - |S^*|H\| < \|H\|$ must hold.

Now let $T \in K(\mathcal{H}, \mathcal{K})$ be nonzero, and let $S = |S^*|U$ and $T = |T^*|V$ be the polar decompositions. Since $UU^*|S^*| = |S^*|$, putting $R = U^*T \in K(\mathcal{H})$ yields that

$$\begin{aligned} \|T - SR\| &= \||T^*|V - (|S^*|U)(U^*|T^*|V)\| \\ &\leq \|(|T^*| - |S^*||T^*|)V\| \\ &\leq \||T^*| - |S^*||T^*|\| \\ &< \||T^*|\| = \|T\|. \end{aligned}$$

This proves that there is no nonzero element $T \in K(\mathcal{H}, \mathcal{K})$ satisfying $T \perp_{K(\mathcal{H})} S$; and thus, S is right symmetric for $\perp_{K(\mathcal{H})}$ in $K(\mathcal{H}, \mathcal{K})$. This completes the proof. □

Remark 2 As was mentioned in the last paragraph of the proof of Theorem 5, if $\mathcal{H}$ is infinite dimensional, then each right symmetric point S for $\perp_{K(\mathcal{H})}$ in $K(\mathcal{H}, \mathcal{K})$ has no nonzero element $T \in K(\mathcal{H}, \mathcal{K})$ satisfying $T \perp_{K(\mathcal{H})} S$.

As a consequence of Theorem 5, it turns out that, if $\mathcal{H}$ is infinite dimensional, the existence of nonzero right symmetric points for $\perp_{K(\mathcal{H})}$ in $K(\mathcal{H}, \mathcal{K})$ depends on the dimension of $\mathcal{K}$.

Corollary 3 *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. Suppose that $\mathcal{H}$ is infinite dimensional. Then there exists a nonzero right symmetric point for $\perp_{K(\mathcal{H})}$ in $K(\mathcal{H}, \mathcal{K})$ if and only if $\mathcal{K}$ is separable.*

Proof Suppose first that there exists a nonzero right symmetric point S for $\perp_{K(\mathcal{H})}$ in $K(\mathcal{H}, \mathcal{K})$. Then, by Theorem 5, we have $\mathcal{K} = \overline{S(\mathcal{H})}$; and $\overline{S(\mathcal{H})}$ is separable since S is compact.

Conversely, we assume that $\mathcal{K}$ is separable. Let (e_a) and $(f_n)_{n\in\mathbb{N}}$ be orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$, respectively, and let $(e_{a_n})_{n\in\mathbb{N}}$ be a countably infinite subset of (e_a). Define an operator $S \in K(\mathcal{H}, \mathcal{K})$ by letting $Sx = \sum_{n\in\mathbb{N}} 2^{-n}\langle x, e_{a_n}\rangle f_n$ for each $x \in \mathcal{H}$. Then we have $\overline{S(\mathcal{H})} = \mathcal{K}$ since $\{f_n : n \in \mathbb{N}\} \subset S(\mathcal{H})$. Therefore, by Theorem 5, S is a (nonzero) right symmetric point for $\perp_{K(\mathcal{H})}$ in $K(\mathcal{H}, \mathcal{K})$. □

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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