



Spectral inequality with sensor sets of decaying density for Schrödinger operators with power growth potentials

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Abstract

We prove a spectral inequality (a specific type of uncertainty relation) for Schrödinger operators with confinement potentials, in particular of Shubin-type. The sensor sets are allowed to decay exponentially, where the precise allowed decay rate depends on the potential. The proof uses an interpolation inequality derived by Carleman estimates, quantitative weighted L^2 -estimates and an H^1 -concentration estimate, all of them for functions in a spectral subspace of the operator.

Keywords Spectral inequalities · Uncertainty relation · Schroedinger operator · Shubin operator · Decay of eigenfunctions · Confinement potential · Observability

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1 Introduction, main results, and discussion

In the context of control theory, a *spectral inequality* for a nonnegative selfadjoint operator H in $L^2(\mathbb{R}^d)$ is an inequality of the form

$$\|f\|_{L^2(\mathbb{R}^d)} \leq d_0 e^{d_1 \lambda^s} \|f\|_{L^2(\omega)} \quad \text{for all } f \in \text{Ran } P_\lambda(H), \lambda \geq 1, \quad (1.1)$$

with a measurable set $\omega \subset \mathbb{R}^d$ and some $s \in (0, 1)$ and $d_0, d_1 > 0$. Having the application in control theory in mind, we will refer to ω as a sensor set. Here, $\lambda \mapsto P_\lambda(H) = \mathbf{1}_{(-\infty, \lambda]}(H)$ denotes the resolution of identity associated to H . Once Inequality (1.1) is at disposal, the famous Lebeau-Robbiano method [33], see also [6, 23, 43, 51], allows to conclude (final state)

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observability and null-controllability from ω of the abstract Cauchy problem associated to $-H$. Note that the spectral inequality is a manifestation of the uncertainty principle and has been used also in other contexts albeit with a different name, see, e.g., [17].

The new results in this paper concern Schrödinger operators $H = -\Delta + V$ with potentials V growing at infinity. A paradigmatic model in this context is the harmonic oscillator $H = -\Delta + |x|^2$, for which spectral inequalities were proven in [4, 6, 15, 18, 40]. See also [16] for a treatment of partial harmonic oscillators. In particular, [15] shows that (1.1) holds for $H = -\Delta + |x|^2$ with measurable sensor sets $\omega \subset \mathbb{R}^d$ satisfying

$$\frac{|\omega \cap (k + (-\rho/2, \rho/2)^d)|}{\rho^d} \geq \gamma^{1+|k|^\alpha} \quad \text{for all } k \in \mathbb{Z}^d, \quad (1.2)$$

with some parameters $\rho > 0$, $\gamma \in (0, 1]$, and $\alpha \in [0, 1)$.

Sets satisfying (1.2) with $\alpha = 0$ are called *thick sets*. They are crucial in two aspects: First, for the pure Laplacian $H = -\Delta$ an inequality like (1.1) holds *if and only if* ω is a thick set. This has been studied in [27, 35, 45, 46] culminating in sharp inequalities established in [29, 30]. These results are in the literature often referred under the term ‘Logvinenko–Sereda inequality’. Second, as shown in [19, 52] independently, the heat equation on \mathbb{R}^d is observable (hence final state controllable) from a set $\omega \subset \mathbb{R}^d$ *if and only if* ω is a thick set. This illustrates that this type of set is a benchmark for comparison for the topics at hand.

For a Schrödinger operator $H = -\Delta + V$ with a bounded, real-valued, suitably analytic potential V vanishing at infinity, a spectral inequality with thick sets ω was shown in [32]. Thick sets are also relevant for spectral and parabolic observability inequalities on bounded domains, mostly if one considers a sequence of them like $(-L, L)^d$ and aims for bounds uniform in $L \in \mathbb{N}$ cf. [20] and [49]. See also [5, 34] for related results for divergence type operators.

Although we conjecture that a condition similar to (1.2) would be appropriate in the context of this paper, see Sect. 2 for a more precise statement, due to the methods applied, we need to assume more regularity of the sensor set. More precisely, we have to work in the category of open rather than measurable sets.

In this setting the natural analog of thick sets is the following: For $0 < \delta < 1/2$, a measurable ω is called a δ -*equidistributed set*, if each intersection $\omega \cap (k + (-1/2, 1/2)^d)$, $k \in \mathbb{Z}^d$, contains a ball of radius δ . This term was introduced in [47] (but see also [21, 22] for related notions) in the context of random Schrödinger operators with bounded potentials and shown to be sufficient for a weaker form of uncertainty relation than (1.1), where f needs to be an individual eigenfunction of H , rather than an element of a spectral subspace $\text{Ran } P_\lambda(H)$. This condition on f was subsequently relaxed in [28] to allow for $f \in \text{Ran } \mathbf{1}_{(\lambda-\varepsilon, \lambda]}(H)$, for specified, but small $\varepsilon > 0$. In [31] the argument was extended to potentials with uniformly controlled local singularities. Finally, a proper spectral inequality for Schrödinger operators with bounded potentials of the form (1.1) was obtained in [42, 44], the difference being that the first paper treats domains of the type $(-L, L)^d$, whereas the second allows also certain ‘rectangular’ unbounded domains, as well, including \mathbb{R}^d itself. Again, this result was extended in [12] to potentials with certain local singularities, giving an improvement of [31], as well. A common feature of these papers is that they trace the dependence of uncertainty relation on the geometry and model parameters, in contrast to a many earlier papers with qualitative results. In particular, [43] extends the techniques of the ingenious paper [25] (itself based on the seminal [11]) from compact domains to \mathbb{R}^d using a geometric covering construction and quantifying the above mentioned parameter dependence.

Let us come back to the harmonic oscillator. For the harmonic oscillator on the domain \mathbb{R}^d thick set have been shown in [4] to be sufficient for a spectral inequality. In [18] this

was generalized to certain unbounded domains exhibiting sufficient symmetry. However, in contrast to the pure Laplacian, this condition is not optimal here. In fact, the fast growth of the quadratic potential yields fast decay of eigenfunctions of the harmonic oscillator, i.e., Hermite functions. This explains why one can weaken the geometric condition on the sensor sets to a decay of the form (1.2). In particular, (1.2) allows the sensor set to have finite Lebesgue measure, see, e.g., the set ω considered in (1.6) below.

One naturally expects similar phenomena for Schrödinger operators of the form $H = -\Delta + |x|^\tau$ with general $\tau > 0$, in particular that the unbounded potential enforces fast decay of the eigenfunctions. This should make it possible to allow a similar (or even faster) decay in the sensor set as in (1.2) above, while still obtaining a spectral inequality of the form (1.1) with explicit dependence on the geometry of ω .

An important step in this direction was achieved in [39], who proves a spectral inequality for $\tau \in 2\mathbb{N}$ with the sensor set ω being a cone of the form

$$\{x \in \mathbb{R}^d : |x| \geq r_0 \text{ and } x/|x| \in \Omega_0\} \text{ for some open } \Omega_0 \subset \mathbb{S}^{d-1}. \tag{1.3}$$

Note that the cone has infinite Lebesgue measure in contrast to typical sets satisfying (1.2) with $\alpha > 0$.

The operator considered in [39] is not only a Schrödinger operator, but at the same time a Shubin operator, see Sect. 2 for a definition and discussion of the latter class. In contrast, the methods of this paper do not require any integer condition on the power τ of the potential.

More precisely, our main new result is a spectral inequality for Schrödinger operators $H = -\Delta + V$ with potentials V of the following type.

Assumption A Suppose that $V \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^d)$ is such that

- (i) for some $c_1, c_2 > 0$ and some $0 < \tau_1 \leq \tau_2$ we have $c_1|x|^{\tau_1} \leq V(x) \leq c_2|x|^{\tau_2}$ for all $x \in \mathbb{R}^d$;
- (ii) for some $\nu > 0$ we have

$$M_\nu := \|e^{-\nu|x|}|\nabla V|\|_{L^\infty(\mathbb{R}^d \setminus B(0,1))} < \infty. \tag{1.4}$$

Theorem 1.1 (Spectral inequality) *Let $H = -\Delta + V$ with V as in Assumption A, and let $\omega \subset \mathbb{R}^d$ be measurable such that for some $\delta \in (0, 1/2)$ and $\alpha \geq 0$ each intersection $\omega \cap (k + (-1/2, 1/2)^d)$, $k \in \mathbb{Z}^d$, contains a ball of radius $\delta^{1+|k|^\alpha}$. Then there is a constant $C > 0$ depending only on $\tau_1, \tau_2, c_1, c_2, \nu, M_\nu$, and the dimension d such that*

$$\|f\|_{L^2(\mathbb{R}^d)} \leq \left(\frac{1}{\delta}\right)^{C^{1+\alpha} \lambda^{(\alpha+2\tau_2/3)/\tau_1}} \|f\|_{L^2(\omega)} \tag{1.5}$$

for all $f \in \text{Ran } P_\lambda(H)$, $\lambda \geq 1$.

In the case $\alpha = 0$, the sets ω considered in Theorem 1.1 correspond to δ^2 -equidistributed sets defined above. More interesting and enlightening is the set

$$\omega = \bigcup_{k \in \mathbb{Z}^d} B(k, 2^{-(1+|k|^\alpha)}), \tag{1.6}$$

which has finite measure if $\alpha > 0$. Moreover, if $\alpha < 1$, ω satisfies (1.2) with $\rho = 1$ and suitably chosen $\gamma \in (0, 1)$.

Note that inequality (1.5) is of the form (1.1), with the exponent s satisfying $s = (\alpha + 2\tau_2/3)/\tau_1 < 1$ if and only if $0 \leq \alpha < \tau_1 - 2\tau_2/3$. Hence, if Assumption A holds and additionally $2\tau_2 < 3\tau_1$, Theorem 1.1 directly leads to observability of the abstract Cauchy problem associated to $-H$. More precisely, we get the following result using the variant of the Lebeau-Robbiano method spelled out in [43, Theorem 2.8].

Corollary 1.2 (Final state observability inequality) *Let V be as in Assumption A, let $(e^{t(\Delta-V)})_{t \geq 0}$ be the C_0 -semigroup with generator $\Delta - V$, and let ω be as in Theorem 1.1 with $0 \leq \alpha < \tau_1 - 2\tau_2/3$.*

Then, for all $T > 0$ and all $g \in L^2(\mathbb{R}^d)$ we have

$$\|e^{T(\Delta-V)}g\|_{L^2(\mathbb{R}^d)}^2 \leq C_{\text{obs}}^2 \int_0^T \|e^{t(\Delta-V)}g\|_{L^2(\omega)}^2 dt. \tag{1.7}$$

The constant satisfies the asymptotics $C_{\text{obs}} = C_{\text{obs}}(T) = \mathcal{O}(T^{-1/2})$ as $T \rightarrow \infty$ and the bound

$$C_{\text{obs}}^2 \leq \frac{K}{T} (2d_0 + 1)^K \exp\left[K \left(\frac{d_1}{T^s}\right)^{1/(1-s)}\right] \tag{1.8}$$

with $s = \frac{\alpha+2\tau_2/3}{\tau_1}$, $d_1 = -C^{1+\alpha} \ln \delta$, $d_0 = \delta^{-C^{1+\alpha}}$, $C = C(\tau_1, \tau_2, c_1, c_2, \nu, M_\nu, d) > 0$ depending only on $\tau_1, \tau_2, c_1, c_2, \nu, M_\nu$, and the dimension d , and $K = K(s) \geq 1$ depending only on s .

Remark 1.3 (Bound on the observability constant) For general potentials satisfying Assumption A the bound (1.8) seems rather complicated. Let us concentrate on the question, how C_{obs} depends on the time parameter T and specialize to potentials of the form $V(x) = |x|^\tau$ (for some $\tau > 3\alpha$). The bound (1.8) on the observability constant C_{obs} in (1.7) can then be rewritten as

$$C_{\text{obs}}^2 \leq \frac{D}{T} \exp\left(\frac{D}{T^{1+\frac{2\alpha+\tau/3}{\tau/3-\alpha}}}\right) \text{ for all } T > 0$$

where $D > 0$ is a constant that depends only on τ, α, δ , and the dimension.

Let us briefly comment on the hypotheses in Assumption A: The lower bound in part (i) allows to bound the eigenvalue counting function for H , cf. (3.9) below. It also implies a suitable L^2 -decay for eigenfunctions of H , see Proposition 3.3 below. The bound in part (ii) allows to obtain a similar decay for partial derivatives of eigenfunctions by differentiating the eigenvalue equation $Hf = \lambda f$, which introduces partial derivatives of the potential to the equation, see Proposition 3.4 below. Together with the bound on the eigenvalue counting function, this amounts to the fact that the H^1 -mass of $f \in \text{Ran } P_\lambda(H)$ is strongly localized. This is made precise in the following theorem.

Theorem 1.4 *Let $H = -\Delta + V$ with V as in Assumption A. Then there is a constant $C' > 0$, depending only on τ_1, c_1, ν, M_ν , and the dimension d , such that for all $\lambda \geq 1$ and all $f \in \text{Ran } P_\lambda(H)$ we have*

$$\|f\|_{H^1(\mathbb{R}^d \setminus B(0, C'\lambda^{1/\tau_1}))}^2 \leq \frac{1}{2} \|f\|_{L^2(\mathbb{R}^d)}^2. \tag{1.9}$$

This result enables us essentially to reduce the considerations to a sufficiently large cube $\Lambda = (-L, L)^d$ in \mathbb{R}^d by a ‘cut-off procedure’ (cf. Remark 4.3 below). At this point the upper bound in part (i) of Assumption A comes into play and ensures an effective bound on V . More precisely, $\|V\|_{\infty, \Lambda} := \sup_{x \in \Lambda} V(x) \leq c_2 d^{\tau_2/2} L^{\tau_2}$.

We now discuss under which conditions on the parameter α (encoding the decay rate) the spectral inequality provided by Theorem 1.1 is of the type (1.1) and hence usable as input for the observability inequality in Corollary 1.2.

To this end, let us focus on the harmonic oscillator, that is, $H = -\Delta + V$ with $V(x) = |x|^2$. This operator meets the hypotheses of Theorem 1.1 (with $\tau_1 = \tau_2 = 2, c_1 = c_2 = 1$). Furthermore, condition $0 \leq \alpha < \tau_1 - 2\tau_2/3$ required for $s \in (0, 1)$ in (1.1), simplifies to $0 \leq \alpha < 2 - 4/3 = 2/3$. By contrast, [15] derives Inequality (1.1) merely under the condition $\alpha < 1$. Let us further restrict our considerations to the case $\alpha = 0$, i.e. that ω is a thick set. Then the exponent on the right-hand side of (1.5) reads $C\lambda^{2/3}$, while the corresponding term in [15] is $C\lambda^{1/2}$ which is the expected behaviour, see for instance the discussion on page 14 in [29] or in Section 3 of [20]. The slightly worse behaviour in our Theorem 1.1 above is due to the application of a Carleman estimate after the mentioned ‘cut-off procedure’ to the cube $\Lambda = (-L, L)^d$. While according to a conjecture associated with the name of Landis, the sup norm of the potential should enter the unique continuation estimate in the scaling $\|V\|_{\infty, \Lambda}^{1/2}$ the current technology based on Carleman estimates yields the scaling $\|V\|_{\infty, \Lambda}^{2/3}$. Tracing this suboptimality further leads to the suboptimal condition on the exponent α . See also the discussion in the next section.

Let us finally mention that our results can also be extended to potentials $V: \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \rightarrow [0, \infty)$ which grow only in certain coordinate directions. For simplicity, we here restrict ourselves to potentials $V(x) = |x_1|^\tau$ with $x = (x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $\tau > 0$. The more general analogue to Theorem 1.1 is presented in Theorem 5.1 below.

Theorem 1.5 *Let $H = -\Delta + |x_1|^\tau$, and let $\omega \in \mathbb{R}^d$ be measurable such that for some $\delta \in (0, 1/2)$ and $\alpha \geq 0$ each intersection $\omega \cap (k + (-1/2, 1/2)^d)$, $k = (k_1, k_2) \in \mathbb{Z}^d$, contains a ball of radius $\delta^{1+|k_1|^\alpha}$. Then there is a constant $C > 0$ depending only on the dimension d and the parameter τ such that for all $\lambda \geq 1$ and all $f \in \text{Ran } P_\lambda(H)$ we have*

$$\|f\|_{L^2(\mathbb{R}^d)} \leq \left(\frac{1}{\delta}\right)^{C^{1+\alpha} \cdot \lambda^{\alpha/\tau+2/3}} \|f\|_{L^2(\omega)}.$$

The rest of this note is organized as follows. The purpose of Sect. 2 is threefold: We consider some related models, in particular Shubin operators, formulate conjectures, that have been spelled out in the first version of this manuscript, and finally discuss papers of other authors, which have been obtained during the refereeing procedure of this manuscript, actually confirming the conjectures. Based on [24], Sect. 3 discusses decay properties of eigenfunctions and establishes the H^1 -localization for elements $f \in \text{Ran } P_\lambda(H)$, i.e. Theorem 1.4. Section 4 then revisits the proofs of the spectral inequalities developed in [42] and improved in [12, 44] and adapts them towards a proof of Theorem 1.1. Finally, in Sect. 5 we discuss how to extend our results to partial confinement potentials based on a tensorization technique.

In all these considerations, we frequently write $A \lesssim B$ with quantities A and B if there is a constant $c > 0$ depending on the model parameters such that $A \leq cB$. If the constant depends only on a subset of the model parameters, we occasionally write these parameters in the subscript of \lesssim , for instance, $A \lesssim_d B$ if the constant only depends on the dimension d .

2 Conjectures, generalizations and further developments

In this section, we discuss some related models, in particular Shubin operators, and formulate some conjectures that were spelled out in our original preprint. In fact, recent papers obtain more general results than ours and mostly confirm our original conjectures, in some cases quoting them as motivation. Let us discuss this properly.

We expected that a (partial) improvement of Theorem 1.1 of the following form is valid.

Confirmed conjecture 2.1 Let $\tau > 0$, and suppose that $\omega \subset \mathbb{R}^d$ is measurable such that for some $\delta \in (0, 1/2)$ and $\alpha \geq 0$ each intersection $\omega \cap (k + (-1/2, 1/2)^d)$, $k \in \mathbb{Z}^d$, contains a ball of radius $\delta^{1+|k|^\alpha}$. Then there is a constant $C > 0$ depending only on τ, δ, α , and the dimension d such that for all $\lambda \geq 1$ and all $f \in \text{Ran } P_\lambda(-\Delta + |x|^\tau)$ we have

$$\|f\|_{L^2(\mathbb{R}^d)} \leq \left(\frac{1}{\delta}\right)^{C \cdot \lambda^{\frac{\alpha}{\tau} + \frac{1}{2}}} \|f\|_{L^2(\omega)}.$$

In fact, recently [53] confirmed the conjecture considering even more general potentials of the following type:

Assumption B Suppose that $V = V_1 + V_2$ with $V_1 \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^d)$ and $V_2 \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ satisfies

(i) for some $\tau_1, c_1 > 0$ we have

$$c_1(|x| - 1)_+^{\tau_1} \leq V(x) \quad \text{for all } x \in \mathbb{R}^d \text{ and}$$

(ii) for some $\tau_2, c_2 > 0$ we have

$$|V_1(x)| + |\nabla V_1(x)| + |V_2(x)|^{4/3} \leq c_2(|x| + 1)^{\tau_2} \quad \text{for all } x \in \mathbb{R}^d.$$

Theorem 2.2 (Theorem 1 in [53]) Let $H = -\Delta + V$ with V as in Assumption B, and let $\omega \subset \mathbb{R}^d$ be measurable such that for some $\delta \in (0, 1/2)$ and $\alpha \geq 0$ each intersection $\omega \cap (k + (-1/2, 1/2)^d)$, $k \in \mathbb{Z}^d$, contains a ball of radius $\delta^{1+|k|^\alpha}$. Then there is a constant $C > 0$ depending only on $\tau_1, \tau_2, c_1, c_2, \alpha$, and the dimension d such that

$$\|f\|_{L^2(\mathbb{R}^d)} \leq \left(\frac{1}{\delta}\right)^{C \cdot \lambda^{(\alpha/\tau_1) + (\tau_2/2\tau_1)}} \|f\|_{L^2(\omega)}$$

for all $f \in \text{Ran } P_\lambda(H)$.

The proof of [53] follows the same line of argument as ours but develops several crucial technical improvements, among them a Carleman estimate using a bound on the gradient of the potential, not just the sup norm of the potential itself, and Cacciopoli inequalities. We recommend the reader interested in the currently best obtainable estimates in our context to consult their proofs.

[53] in turn motivated the study of null-controllability of the heat equation associated to fractional Baouendi-Grushin operators in [26]. The paper contains several related results. We quote a particular case of one of them as an example that considers potentials of the following class:

Assumption C Suppose that $V = V_1 + V_2$ with $V_1 \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^d)$ and $V_2 \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ satisfies for some $\tau, c_1, c_2 > 0$

(i)

$$c_1|x|^\tau \leq V(x) \quad \text{for all } x \in \mathbb{R}^d \text{ and}$$

(ii)

$$|V_1(x)| + |\nabla V_1(x)| + |V_2(x)|^{4/3} \leq c_2(|x| + 1)^\tau \quad \text{for all } x \in \mathbb{R}^d.$$

Theorem 2.3 (Cf. Theorem 1.7 in [26]) Let $\delta \in (0, 1/2)$ and V be as in Assumption C. Set $\theta_* := \frac{\tau+2}{3}$. Let $\omega \subset \mathbb{R}^d$ be a δ -equidistributed set. Then for every initial condition $\phi_0 \in L^2(\mathbb{R}^d \times \mathbb{R}^m)$ the problem

$$\begin{aligned} \partial_t \phi(t, x, y) + (-\Delta_x - V(x)\Delta_y)^{\theta_*} \phi(t, x, y) &= (\mathbf{1}_{\omega \times \mathbb{R}^m} \cdot u)(t, x, y), & t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^m, \\ \phi(0, x, y) &= \phi_0(x, y), & x \in \mathbb{R}^d, y \in \mathbb{R}^m. \end{aligned}$$

is final state null-controllable in every time $T > 0$, if $\theta > \theta_*$. Furthermore, there exists a time $T_* > 0$ such that for $\theta = \theta_*$ the above problem is final state null-controllable provided $T > T_*$. For $\delta \searrow 0$, the time T_* diverges like $\ln\left(\frac{1}{\delta}\right)$.

Due to the use of Carleman inequalities, we can prove the spectral inequality in Theorem 1.1 only for sensor sets containing suitable open balls, but not for measurable sets of the form (1.2). This prompts the question, how and under what conditions this restriction can be removed. Natural candidates are polynomial or analytic potentials, in particular in view of [32].

In this vein, let us discuss next results on Shubin operators, that is, Hamiltonians of the form $H = (-\Delta)^m + |x|^{2l}$ with $m, l \in \mathbb{N}$. Note that the particular case of $m = l = 1$ here agrees with the harmonic oscillator. For $l > m = 1$, a first proof of a spectral inequality of the form (1.1), and thus observability of the system generated by $-H$, was established in [39], but only for sensor sets ω as in (1.3). Arbitrary sets ω with strictly positive measure were treated recently in [37]. There, the author verifies that for such sets and general $m, l \in \mathbb{N}$ we have the inequality

$$\|f\|_{L^2(\mathbb{R}^d)} \leq d_0 e^{d_1 \lambda^{\frac{1}{2m} + \frac{1}{2l}} \log \lambda} \|f\|_{L^2(\omega)} \quad \text{for all } f \in \text{Ran } P_\lambda(H), \lambda \geq 1,$$

which in the case of $\max\{m, l\} > 1$ is indeed of the form (1.1) with the choice $s = 1/(2m) + 1/(2l) + \varepsilon < 1$ for $\varepsilon \in (0, 1/4)$. The constants d_0 and d_1 are, however, not explicit in this case.

A similar result in [37] treated the case of sensor sets ω that are thick with respect to certain unbounded scales, which allows for holes in ω of growing size. This takes into account more precise information on ω and results in an improved dependence on λ , but the sensor sets are required to have infinite Lebesgue measure and the constants d_0 and d_1 are still not explicit.

However, in light of the Bernstein inequalities proved in [37, Proposition 4.1] we expected that the approach presented in [15] could be adapted for Shubin operators $H = (-\Delta)^m + |x|^{2l}$, $m, l \in \mathbb{N}$ in order to obtain a result of the following form:

Confirmed conjecture 2.4 *Let $m, l \in \mathbb{N}$, and suppose that $\omega \subset \mathbb{R}^d$ satisfies (1.2) with some $\rho > 0$, $\gamma \in (0, 1]$, and $0 \leq \alpha < l$. Then there are constants $K, C > 0$, with K depending only on m, l , and the dimension d , and C depending additionally also on ρ and α , such that for all $\lambda \geq 1$ and all $f \in \text{Ran } P_\lambda((-\Delta)^m + |x|^{2l})$ we have*

$$\|f\|_{L^2(\mathbb{R}^d)} \leq \left(\frac{K}{\gamma}\right)^{C \cdot \lambda^{\frac{\alpha}{2l} + \frac{1}{2m}}} \|f\|_{L^2(\omega)}. \tag{2.1}$$

This conjecture was indeed confirmed recently in [3] by the second author and P. Alphonse. In fact, while [37] relies on smoothing estimates for the semigroup generated by $-H$ established in [2], the paper [3] directly works with the underlying Agmon estimates from [2] in order to prove a variant of the Bernstein inequalities that allow for very explicit spectral inequalities under fairly general assumptions on the sensor set. In particular, the estimates of [3] confirm (2.1) with the explicit bound on the constant

$$C \leq \tilde{K}^{1+\alpha} (1 + \rho^{1+\frac{k}{m}} + \rho),$$

with $\tilde{K} > 0$ depending only on m, l , and the dimension d , see Theorem 2.3 and Remark 2.5 in [3]. Note that the general case of [3] actually treats sensor sets with a more general geometry than (1.2).

3 Decay of eigenfunctions

In this section we quantify decay properties of linear combinations of eigenfunctions for the operator $H = -\Delta + V$ with V as in Assumption A. Although there are several results available for eigenfunctions establishing a fast decay in L^2 -sense, see, e.g., [1, 7, 9, 13, 50], we need an explicit weighted L^2 -estimate also for the partial derivatives of first order. The approach in [24] seems to be the most convenient one for this task. However, since it is essential for us to have the dependence of the decay on the spectral parameter explicitly quantified, we have to revisit the reasoning from [24] and extract the statements we need.

3.1 Properties of the operator and main objective

We begin with a review of the construction of the operator $H = -\Delta + V$ with V as in Assumption A and a collection of its basic properties.

Consider the forms

$$\mathfrak{a}[f, g] := \int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla g(x) \, dx, \quad f, g \in \mathcal{D}[\mathfrak{a}] := H^1(\mathbb{R}^d),$$

as well as $\mathcal{D}[\mathfrak{v}] := \{f \in L^2(\mathbb{R}^d) : V^{1/2}f \in L^2(\mathbb{R}^d)\}$,

$$\mathfrak{v}[f, g] := \langle V^{1/2}f, V^{1/2}g \rangle_{L^2(\mathbb{R}^d)}, \quad f, g \in \mathcal{D}[\mathfrak{v}],$$

and

$$\mathfrak{h} := \mathfrak{a} + \mathfrak{v}, \quad \mathcal{D}[\mathfrak{h}] := \mathcal{D}[\mathfrak{a}] \cap \mathcal{D}[\mathfrak{v}].$$

The nonnegative form \mathfrak{a} is closed since $H^1(\mathbb{R}^d)$ is complete, and \mathfrak{v} is nonnegative and closed by [48, Proposition 10.5 (ii)]. Thus, the form \mathfrak{h} is densely defined, nonnegative, and closed by [48, Corollary 10.2], so that there is a unique (nonnegative) selfadjoint operator H on $L^2(\mathbb{R}^d)$ given by

$$\mathcal{D}(H) = \{f \in \mathcal{D}[\mathfrak{h}] : \exists h \in L^2(\mathbb{R}^d) \text{ s.t. } \mathfrak{h}[f, g] = \langle h, g \rangle_{L^2(\mathbb{R}^d)} \forall g \in \mathcal{D}[\mathfrak{h}]\}$$

and

$$\mathfrak{h}[f, g] = \langle Hf, g \rangle_{L^2(\mathbb{R}^d)}, \quad f \in \mathcal{D}(H), \quad g \in \mathcal{D}[\mathfrak{h}].$$

Since $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, it is well known that H has purely discrete spectrum, see, e.g., [48, Proposition 12.7]. Moreover, a form core for H is given by $C_c^\infty(\mathbb{R}^d)$, see, e.g., [8, Theorem 1.13], that is, every function in $\mathcal{D}[\mathfrak{h}]$ can be approximated in the form norm for H by functions in $C_c^\infty(\mathbb{R}^d)$; a simple proof of this fact for the current type of potential (in particular, $V \in L_{\text{loc}}^\infty(\mathbb{R}^d)$) can also be obtained from [24, Lemma 2.2].

Classic elliptic regularity results (see, e.g., [7, Theorem S2.2.1]) imply that $\mathcal{D}(H) \subset H_{\text{loc}}^2(\mathbb{R}^d)$ with $Hf = -\Delta f + Vf$ almost everywhere on \mathbb{R}^d for all $f \in \mathcal{D}(H)$. In addition, if $Hf \in H_{\text{loc}}^1(\mathbb{R}^d)$ for some $f \in \mathcal{D}(H)$, then $f \in H_{\text{loc}}^3(\mathbb{R}^d)$.

The main objective of the present section is now to prove Theorem 1.4. In this course, if desired, the dependence of C' in Theorem 1.4 on the parameters τ_1, c_1, ν, M_ν can be traced explicitly from the proof. We refrain from doing so here for simplicity and brevity.

3.2 Weighted inequalities

We prove Theorem 1.4 by establishing L^2 -estimates for f and $|\nabla f|$ with an exponential weight. As a preparation, we need the following technical lemma.

Lemma 3.1 (see [24, Lemma 2.1]) *Suppose that for some $\phi \in L^2(\mathbb{R}^d)$ and $\lambda \geq 0$ the function $f \in H^2_{loc}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ satisfies $-\Delta f + Vf - \lambda f = \phi$ almost everywhere. Then, $f \in \mathcal{D}[\mathfrak{h}]$ and for all $g \in \mathcal{D}[\mathfrak{h}]$ we have*

$$\mathfrak{h}[f, g] - \lambda \langle f, g \rangle_{L^2(\mathbb{R}^d)} = \langle \phi, g \rangle_{L^2(\mathbb{R}^d)}.$$

Proof The fact that $f \in \mathcal{D}[\mathfrak{h}]$ follows from

$$\int_{\mathbb{R}^d} (|\nabla f|^2 + V|f|^2) \leq \|\phi\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} + \lambda \|f\|_{L^2(\mathbb{R}^d)}^2,$$

which is proved verbatim as in [24, Lemma 2.1]; the smoothness of the potential V assumed there is actually not used and not needed.

For $g \in C_c^\infty(\mathbb{R}^d)$ we then obtain by integration by parts that

$$\begin{aligned} \mathfrak{h}[f, g] &= \mathfrak{a}[f, g] + \mathfrak{v}[f, g] = \int_{\mathbb{R}^d} (-\Delta f + Vf)\bar{g} = \int_{\mathbb{R}^d} (\phi + \lambda f)\bar{g} \\ &= \langle \phi + \lambda f, g \rangle_{L^2(\mathbb{R}^d)}, \end{aligned}$$

and the latter extends to all $g \in \mathcal{D}[\mathfrak{h}]$ by approximation since $C_c^\infty(\mathbb{R}^d)$ is a form core for H ; cf. the discussion after Lemma 2.2 in [24]. \square

The next result is now at the core of our proof of Theorem 1.4 and is a quantitative version of the statement in [24, Lemma 2.3]. Its proof is also extracted from that reference.

Lemma 3.2 *Let $\lambda \geq 0$, $\mu > 0$, and $R \geq 1$ be such that $V(x) \geq \mu^2 + \lambda + 1$ whenever $|x| \geq R$. Moreover, suppose that $f \in H^2_{loc}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ satisfies $-\Delta f + Vf - \lambda f = \phi$ almost everywhere with some $\phi \in L^2(\mathbb{R}^d)$. Then, if $e^{2\mu|x|}\phi \in L^2(\mathbb{R}^d)$, we have*

$$\|e^{\mu|x|}f\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{1}{2} \|e^{2\mu|x|}\phi\|_{L^2(\mathbb{R}^d \setminus B(0,R))}^2 + (4\mu + 6)e^{2\mu(R+1)} \|f\|_{L^2(\mathbb{R}^d)}^2. \tag{3.1}$$

Proof According to Lemma 3.1, f belongs to $\mathcal{D}[\mathfrak{h}]$. We first suppose that f is real-valued. Choose an infinitely differentiable function $\chi : \mathbb{R}^d \rightarrow [0, 1]$ with $\chi(x) = 0$ for $|x| \leq R$ and $\chi(x) = 1$ for $|x| \geq R + 1$ such that $\|\nabla \chi\|_{L^\infty(\mathbb{R}^d)} \leq 2$. For $\varepsilon > 0$ let $w(x) = w_\varepsilon(x) = \mu|x|/(1 + \varepsilon|x|)$. Then w is bounded and infinitely differentiable on $\mathbb{R}^d \setminus \{0\}$. Accordingly, the same is true for χe^w and χe^{2w} . Therefore, $\chi e^{2w}f$, $\chi^2 e^{2w}f$, and $g := \chi e^w f$ are all real-valued, belong to $\mathcal{D}[\mathfrak{h}]$, and vanish in the ball $B(0, R)$. In particular, the choice of R implies that $\mathfrak{v}[g, g] \geq (\mu^2 + \lambda + 1)\|g\|_{L^2(\mathbb{R}^d)}^2$. Moreover, with the relation $\nabla(e^{\pm w}g) = e^{\pm w}\nabla g \pm g e^{\pm w}\nabla w$ and the identity $\|\nabla w\|_{L^\infty(\mathbb{R}^d \setminus \{0\})} = \mu$ we obtain

$$\nabla(e^{-w}g) \cdot \nabla(e^w g) = |\nabla g|^2 - |g|^2 |\nabla w|^2 \geq -\mu^2 |g|^2,$$

so that

$$\mathfrak{h}[\chi f, \chi e^{2w}f] = \mathfrak{h}[e^{-w}g, e^w g] = \mathfrak{a}[e^{-w}g, e^w g] + \mathfrak{v}[g, g] \geq (\lambda + 1)\|g\|_{L^2(\mathbb{R}^d)}^2.$$

The latter can be rewritten as

$$\|\chi e^w f\|_{L^2(\mathbb{R}^d)}^2 \leq \mathfrak{h}[\chi f, \chi e^{2w}f] - \lambda \langle f, \chi^2 e^{2w}f \rangle_{L^2(\mathbb{R}^d)}. \tag{3.2}$$

Clearly, $v[\chi f, \chi e^{2w} f] = v[f, \chi^2 e^{2w} f]$. Moreover, a straightforward computation shows that $\nabla(\chi f) \cdot \nabla(\chi e^{2w} f) = \nabla f \cdot \nabla(\chi^2 e^{2w} f) + \eta e^{2w} |f|^2$ with

$$\eta := 2\chi \nabla \chi \cdot \nabla w + |\nabla \chi|^2. \tag{3.3}$$

Taking into account Lemma 3.1, we therefore have

$$\begin{aligned} \mathfrak{h}[\chi f, \chi e^{2w} f] &= \mathfrak{h}[f, \chi^2 e^{2w} f] + \langle f, \eta e^{2w} f \rangle_{L^2(\mathbb{R}^d)} \\ &= \langle \phi + \lambda f, \chi^2 e^{2w} f \rangle_{L^2(\mathbb{R}^d)} + \langle f, \eta e^{2w} f \rangle_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Plugging the latter into (3.2) gives

$$\begin{aligned} \|\chi e^w f\|_{L^2(\mathbb{R}^d)}^2 &\leq \langle \phi, \chi^2 e^{2w} f \rangle_{L^2(\mathbb{R}^d)} + \langle f, \eta e^{2w} f \rangle_{L^2(\mathbb{R}^d)} \\ &= \langle \chi^2 e^{2w} \phi, f \rangle_{L^2(\mathbb{R}^d)} + \langle f, \eta e^{2w} f \rangle_{L^2(\mathbb{R}^d)} \\ &\leq \|\chi^2 e^{2w} \phi\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} + \|\eta e^{2w}\|_{L^\infty(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)}^2. \end{aligned} \tag{3.4}$$

The function η in (3.3) vanishes outside of the annulus $R < |x| < R + 1$ and satisfies $|\eta| \leq 2|\nabla \chi| |\nabla w| + |\nabla \chi|^2 \leq 4(\mu + 1)$. Hence,

$$\|\eta e^{2w}\|_{L^\infty(\mathbb{R}^d)} \leq 4(\mu + 1)e^{2\mu(R+1)}.$$

We thus conclude from (3.4) that

$$\begin{aligned} \|e^w f\|_{L^2(\mathbb{R}^d)}^2 &= \|e^w f\|_{L^2(B(0, R+1))}^2 + \|e^w f\|_{L^2(\mathbb{R}^d \setminus B(0, R+1))}^2 \\ &\leq e^{2\mu(R+1)} \|f\|_{L^2(\mathbb{R}^d)}^2 + \|\chi e^w f\|_{L^2(\mathbb{R}^d)}^2 \\ &\leq \|\chi^2 e^{2w} \phi\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} + (4\mu + 5)e^{2\mu(R+1)} \|f\|_{L^2(\mathbb{R}^d)}^2 \\ &\leq \|e^{2w} \phi\|_{L^2(\mathbb{R}^d \setminus B(0, R))} \|f\|_{L^2(\mathbb{R}^d)} + (4\mu + 5)e^{2\mu(R+1)} \|f\|_{L^2(\mathbb{R}^d)}^2 \\ &\leq \frac{1}{2} \|e^{2w} \phi\|_{L^2(\mathbb{R}^d \setminus B(0, R))}^2 + (4\mu + 6)e^{2\mu(R+1)} \|f\|_{L^2(\mathbb{R}^d)}^2, \end{aligned}$$

where we used Young’s inequality for the last estimate. Since $w(x) = w_\varepsilon(x) \rightarrow \mu|x|$ as $\varepsilon \rightarrow 0$ pointwise and monotonically, (3.1) now follows by monotone convergence theorem.

If f is not real-valued, we proceed analogously for $\text{Re } f$ and $\text{Im } f$ separately and combine the obtained inequalities to arrive again at (3.1). \square

Applying Lemma 3.2 with $\phi = 0$ allows us to obtain the desired weighted L^2 -estimates for eigenfunctions of H , where R can be computed from λ and the constants in part (i) of Assumption A.

Proposition 3.3 *Suppose that $f \in \mathcal{D}(H)$ with $Hf = \lambda f$ for some $\lambda \geq 0$, and choose $R \geq 1$ such that $R^{\tau_1} \geq (\lambda + 2)/c_1$. Then, we have*

$$\|e^{|x|/2} f\|_{L^2(\mathbb{R}^d)}^2 \leq 8e^{R+1} \|f\|_{L^2(\mathbb{R}^d)}^2.$$

Proof According to the discussion in Subsection 3.1, the function f belongs to $H_{\text{loc}}^2(\mathbb{R}^d)$ and satisfies $-\Delta f + Vf - \lambda f = 0$ almost everywhere. Applying Lemma 3.2 with $\mu = 1/2$ and $\phi = 0$ therefore proves the claim. \square

In order to obtain by means of Lemma 3.2 an analogous result for the partial derivatives of an eigenfunction, we follow the approach of [24] and differentiate the eigenvalue equation

$Hf = \lambda f$. Indeed, since $Hf \in H_{loc}^2(\mathbb{R}^d)$, we know that, in fact, f belongs to $H_{loc}^3(\mathbb{R}^d)$, and it follows that each $\partial_j f \in H_{loc}^2(\mathbb{R}^d)$, $j = 1, \dots, d$, satisfies

$$-\Delta \partial_j f + V \partial_j f - \lambda \partial_j f = -f \partial_j V \tag{3.5}$$

almost everywhere. This allows to apply Lemma 3.2 to $\partial_j f$ with a corresponding right-hand side and, thus, leads to the following result.

Proposition 3.4 *Let $f \in \mathcal{D}(H)$ with $Hf = \lambda f$ for some $\lambda \geq 0$, and choose $R \geq 1$ such that $R^{\tau_1} \geq ((\nu + 1)^2 + \lambda + 1)/c_1$. Then, we have*

$$\|e^{|\cdot|/2} |\nabla f|\|_{L^2(\mathbb{R}^d)}^2 \leq (8\lambda + (2\nu + 5)M_\nu^2) e^{2(1+\nu)(R+1)} \|f\|_{L^2(\mathbb{R}^d)}^2.$$

Proof Denote by $\phi_j := -f \partial_j V$ the right-hand side of (3.5).

In light of the hypothesis on R , we may first apply Lemma 3.2 to f with $\mu = \nu + 1$ and $\phi = 0$ to obtain

$$\|e^{(1+\nu)|\cdot|} f\|_{L^2(\mathbb{R}^d)}^2 \leq (4\nu + 10) e^{2(1+\nu)(R+1)} \|f\|_{L^2(\mathbb{R}^d)}^2. \tag{3.6}$$

Since $|\phi_j(x)| \leq M_\nu e^{\nu|\cdot|} |f|$ on $\mathbb{R}^d \setminus B(0, 1)$, we conclude that $e^{|\cdot|} \phi_j \in L^2(\mathbb{R}^d \setminus B(0, 1))$. In view of (3.5), we may then again apply Lemma 3.2, this time to $\partial_j f$ with $\mu = 1/2$ and $\phi = \phi_j = -f \partial_j V$, which gives

$$\|e^{|\cdot|/2} \partial_j f\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{1}{2} \|e^{|\cdot|} \phi_j\|_{L^2(\mathbb{R}^d \setminus B(0, 1))}^2 + 8e^{R+1} \|\partial_j f\|_{L^2(\mathbb{R}^d)}^2. \tag{3.7}$$

Taking into account (3.6) and that

$$\| |\nabla f| \|_{L^2(\mathbb{R}^d)}^2 = \mathfrak{a}[f, f] \leq \mathfrak{h}[f, f] = \langle Hf, f \rangle_{L^2(\mathbb{R}^d)} = \lambda \|f\|_{L^2(\mathbb{R}^d)}^2,$$

summing over j then yields

$$\begin{aligned} \|e^{|\cdot|/2} |\nabla f|\|_{L^2(\mathbb{R}^d)}^2 &\leq \frac{1}{2} \|e^{|\cdot|} f |\nabla V|\|_{L^2(\mathbb{R}^d \setminus B(0, 1))}^2 + 8e^{R+1} \| |\nabla f| \|_{L^2(\mathbb{R}^d)}^2 \\ &\leq \frac{M_\nu^2}{2} \|e^{(1+\nu)|\cdot|} f\|_{L^2(\mathbb{R}^d \setminus B(0, 1))}^2 + 8\lambda e^{R+1} \|f\|_{L^2(\mathbb{R}^d)}^2 \\ &\leq (8\lambda + (2\nu + 5)M_\nu^2) e^{2(1+\nu)(R+1)} \|f\|_{L^2(\mathbb{R}^d)}^2, \end{aligned}$$

which proves the claim. □

3.3 Proof of Theorem 1.4

Recall that H has purely discrete spectrum, and let $(\lambda_k)_{k \in \mathbb{N}}$ be an enumeration of its spectrum $\sigma(H)$ in nondecreasing order (without multiplicities). With

$$N(\lambda) := \#\{\sigma(H) \cap (-\infty, \lambda]\},$$

we may then expand every $f \in \text{Ran } P_\lambda(H)$ as

$$f = \sum_{k=1}^{N(\lambda)} f_k \tag{3.8}$$

where $f_k = \mathbf{1}_{\{\lambda_k\}}(H) f$ for $k \in \{1, \dots, N(\lambda)\}$. Note that we have the simple bound

$$N(\lambda) \leq \#\{k : \lambda_k \leq \lambda\} \leq \sum_{k : \lambda_k \leq \lambda} (\lambda + 1 - \lambda_k) \leq \sum_{k : \lambda_k \leq \lambda+1} (\lambda + 1 - \lambda_k)$$

and in light of the lower bound $V(x) \geq c_1|x|^{\tau_1}$ on the potential in part (i) of Assumption A, the right hand side can be estimated explicitly by means of the classic Lieb-Thirring bound from [36, Theorem 1]. More precisely, for $\lambda \geq 1$ we have

$$\begin{aligned} \sum_{k: \lambda_k \leq \lambda+1} (\lambda + 1 - \lambda_k) &\lesssim_d \int_{\mathbb{R}^d} \max\{\lambda + 1 - V(x), 0\}^{d/2+1} dx \\ &\leq \int_{B(0, ((\lambda+1)/c_1)^{1/\tau_1})} (\lambda + 1)^{d/2+1} dx \\ &\lesssim_{d, \tau_1, c_1} \lambda^{1+d(1/2+1/\tau_1)}, \end{aligned}$$

and, therefore,

$$N(\lambda) \lesssim_{d, \tau_1, c_1} \lambda^{1+d(1/2+1/\tau_1)}. \tag{3.9}$$

Remark 3.5 Note that the Lieb-Thirring bound actually also takes into account multiplicities. It is worth to mention that for $d \geq 3$ the classic Cwikel-Lieb-Rozenblum bound provides a sharper bound on $N(\lambda)$, but the above is more than sufficient for our purposes.

We are now in position to prove the main result of this section.

Proof of Theorem 1.4 For every $r > 0$, we have

$$\begin{aligned} \|f\|_{H^1(\mathbb{R}^d \setminus B(0,r))}^2 &= \|f\|_{L^2(\mathbb{R}^d \setminus B(0,r))}^2 + \|\nabla f\|_{L^2(\mathbb{R}^d \setminus B(0,r))}^2 \\ &\leq e^{-r} (\|e^{|\cdot|/2} f\|_{L^2(\mathbb{R}^d)}^2 + \|e^{|\cdot|/2} |\nabla f|\|_{L^2(\mathbb{R}^d)}^2). \end{aligned}$$

Moreover, using the expansion (3.8) and Hölder’s inequality, we may estimate

$$\|e^{|\cdot|/2} f\|_{L^2(\mathbb{R}^d)}^2 \leq \left(\sum_{k=1}^{N(\lambda)} \|e^{|\cdot|/2} f_k\|_{L^2(\mathbb{R}^d)} \right)^2 \leq N(\lambda) \sum_{k=1}^{N(\lambda)} \|e^{|\cdot|/2} f_k\|_{L^2(\mathbb{R}^d)}^2$$

and similarly, taking into account $|\nabla f| \leq \sum_{k=1}^{N(\lambda)} |\nabla f_k|$,

$$\|e^{|\cdot|/2} |\nabla f|\|_{L^2(\mathbb{R}^d)}^2 \leq N(\lambda) \sum_{k=1}^{N(\lambda)} \|e^{|\cdot|/2} |\nabla f_k|\|_{L^2(\mathbb{R}^d)}^2.$$

We choose $R := ((v + 1)^2 + \lambda + 1)^{1/\tau_1} / c_1 \lesssim_{v, \tau_1, c_1} \lambda^{1/\tau_1}$, which meets the requirement on R in both Propositions 3.3 and 3.4 for all eigenfunctions corresponding to eigenvalues not exceeding λ . In particular, this is the case for the functions f_k in the expansion (3.8).

Since $\sum_{k=1}^{N(\lambda)} \|f_k\|_{L^2(\mathbb{R}^d)}^2 = \|f\|_{L^2(\mathbb{R}^d)}^2$ and in light of (3.9), applying Propositions 3.3 and 3.4 for each f_k separately therefore implies that there is a constant $\tilde{C} > 0$, depending only on c_1, τ_1, v, M_v , and d , such that

$$\|f\|_{H^1(\mathbb{R}^d \setminus B(0,r))}^2 \leq e^{-r} e^{\tilde{C}\lambda^{1/\tau_1}} \|f\|_{L^2(\mathbb{R}^d)}^2.$$

Choosing $r := \log 2 + \tilde{C}\lambda^{1/\tau_1} \leq (\tilde{C} + \log 2)\lambda^{1/\tau_1}$ then proves the claim with the constant $C' = \tilde{C} + \log 2$. □

4 Proof of the spectral inequality

Due to Theorem 1.4 it is sufficient to derive an analog of the spectral inequality on a large, but finite cube. Once the potential V is restricted to the cube, it is a bounded function. This is the situation for which Section 3 of [42] provides appropriate interpolation and spectral inequality from equidistributed sets. Note that [44] extends these results to unbounded domains including \mathbb{R}^d , which is of relevance to us since the operator H is defined on \mathbb{R}^d .

Our proof of Theorem 1.1 relies on an adaptation of these estimates, which we present next.

4.1 Ghost dimension

We make use of the so-called *ghost dimension* construction, which was first introduced in [25]. Following the proofs in [12, 42, 44], we denote by $(\mathcal{F}_t)_{t \in \mathbb{R}}$ the family of unbounded selfadjoint operators

$$\mathcal{F}_t = \int_{\mathbb{R}} s_t(\mu) \, dP_{\mu}(H), \quad s_t(\mu) = \begin{cases} \frac{\sinh(\sqrt{\mu}t)}{\sqrt{\mu}}, & \mu > 0, \\ t, & \mu = 0, \end{cases}$$

in $L^2(\mathbb{R}^d)$. For fixed $f \in \text{Ran } P_{\lambda}(H)$, $\lambda \geq 0$, we then define $F: \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{C}$ by

$$F(\cdot, t) = \mathcal{F}_t f \in \text{Ran } P_{\lambda}(H) \subset \mathcal{D}(H). \tag{4.1}$$

Expanding f as in (3.8) we clearly have

$$F(x, t) = \sum_{k=1}^{N(\lambda)} f_k(x) s_t(\lambda_k), \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}. \tag{4.2}$$

From this, we easily see that F is measurable and belongs to $H^2_{\text{loc}}(\mathbb{R}^{d+1})$. Moreover, we clearly have $\partial_t F(\cdot, t) \in \text{Ran } P_{\lambda}(H)$. Taking into account that $\partial_t s_t(\mu)|_{t=0} = 1$ and $\partial_t^2 s_t(\mu) = \mu s_t(\mu)$ for all $\mu \geq 0$, it also follows that

$$(\partial_t F)(\cdot, 0) = f, \tag{4.3}$$

as well as

$$H(F(\cdot, t)) = (\partial_t^2 F)(\cdot, t) \quad \text{for all } t \in \mathbb{R}. \tag{4.4}$$

The following lemma is an analogue to [42, Proposition 3.6], [44, Proposition 2.9], and [12, Lemma 6.1] and connects the extended function F to the original function f .

Lemma 4.1 *Let $f \in \text{Ran } P_{\lambda}(H)$, and let $F: \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{C}$ be defined as in (4.1). Then, for every $\varrho > 0$ the restriction of F to $\mathbb{R}^d \times (-\varrho, \varrho)$ belongs to the Sobolev space $H^1(\mathbb{R}^d \times (-\varrho, \varrho))$ with*

$$2\varrho \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \|F\|_{H^1(\mathbb{R}^d \times (-\varrho, \varrho))}^2 \leq 2\varrho(1 + (1 + \lambda)\varrho^2)e^{2\varrho\sqrt{\lambda}} \|f\|_{L^2(\mathbb{R}^d)}^2.$$

Proof Using $|\sinh(\sqrt{\mu}t)| \leq |t|\sqrt{\mu} \cosh(\sqrt{\mu}t)$, $1 \leq \cosh(\sqrt{\mu}t) \leq e^{|t|\sqrt{\mu}}$, and the identity $\partial_t s_t(\mu) = \cosh(\sqrt{\mu}t)$ for all $t \in \mathbb{R}$ and $\mu \geq 0$, we easily obtain from the expansion (4.2) that

$$\|F(\cdot, t)\|_{L^2(\mathbb{R}^d)}^2 \leq t^2 e^{2|t|\sqrt{\lambda}} \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \varrho^2 e^{2\varrho\sqrt{\lambda}} \|f\|_{L^2(\mathbb{R}^d)}^2, \tag{4.5}$$

as well as

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq \|\partial_t F(\cdot, t)\|_{L^2(\mathbb{R}^d)}^2 \leq e^{2|t|\sqrt{\lambda}} \|f\|_{L^2(\mathbb{R}^d)}^2 \leq e^{2\varrho\sqrt{\lambda}} \|f\|_{L^2(\mathbb{R}^d)}^2 \tag{4.6}$$

for all $t \in (-\varrho, \varrho)$.

Taking into account that $F(\cdot, t) \in \text{Ran } P_\lambda(H) \subset \mathcal{D}(H)$ for all $t \in \mathbb{R}$, we now clearly have

$$\begin{aligned} \sum_{k=1}^d \|\partial_k F(\cdot, t)\|_{L^2(\mathbb{R}^d)}^2 &\leq \langle HF(\cdot, t), F(\cdot, t) \rangle_{L^2(\mathbb{R}^d)} \leq \lambda \|F(\cdot, t)\|_{L^2(\mathbb{R}^d)}^2 \\ &\leq \lambda \varrho^2 e^{2\varrho\sqrt{\lambda}} \|f\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

Combining the latter with (4.5) and the upper bound in (4.6) and integrating over $t \in (-\varrho, \varrho)$ proves that $F|_{\mathbb{R}^d \times (-\varrho, \varrho)}$ belongs to $H^1(\mathbb{R}^d \times (-\varrho, \varrho))$ satisfying the upper bound in the claim.

For the lower bound, we simply observe that

$$\|F\|_{H^1(\mathbb{R}^d \times (-\varrho, \varrho))}^2 \geq \int_{-\varrho}^{\varrho} \|\partial_t F(\cdot, t)\|_{L^2(\mathbb{R}^d)}^2 dt \geq 2\varrho \|f\|_{L^2(\mathbb{R}^d)}^2$$

by the lower bound in (4.6), which completes the proof. □

4.2 Proof of Theorem 1.1

Let $\lambda \geq 1$ and $f \in \text{Ran } P_\lambda(H) \setminus \{0\}$, and define F as in (4.2). We infer from Theorem 1.4 that there is a constant $C' > 0$, depending on τ_1, c_1, ν, M_ν , and d , such that

$$\|g\|_{H^1(\mathbb{R}^d)}^2 \leq 2\|g\|_{H^1(B(0, C'\lambda^{1/\tau_1}))}^2 \quad \text{and} \quad \|g\|_{L^2(\mathbb{R}^d)}^2 \leq 2\|g\|_{L^2(B(0, C'\lambda^{1/\tau_1}))}^2$$

for all $g \in \text{Ran } P_\lambda(H)$. Applying the latter for each $t \in (-1, 1)$ to $g = F(\cdot, t)$ and $g = \partial_t F(\cdot, t)$, respectively, yields

$$\|F\|_{H^1(\mathbb{R}^d \times (-1, 1))}^2 \leq 2\|F\|_{H^1(B(0, C'\lambda^{1/\tau_1}) \times (-1, 1))}^2. \tag{4.7}$$

Let Λ be the smallest cube of integer sidelength centered at the origin that contains $B(0, C'\lambda^{1/\tau_1})$. For technical reasons, we from now on suppose that $C' \geq 5$, so that Λ has sidelength at least 5. Hence $\Lambda := (-L, L)^d$ with $2L \in \{5, 6, 7, \dots\}$ and $C'\lambda^{1/\tau_1} \leq L \leq C'\lambda^{1/\tau_1} + 1$. Set $\mathcal{K} := \mathcal{K}(\lambda) := \{k \in \mathbb{Z}^d : k \in \Lambda\}$. Then, $|k| \leq \sqrt{d}C'\lambda^{1/\tau_1} + \sqrt{d}$ for all $k \in \mathcal{K}$, so that

$$\delta^{1+|k|^\alpha} \geq \delta^{2(2\sqrt{d}C')^\alpha \lambda^{\alpha/\tau_1}} =: \theta \quad \text{for all } k \in \mathcal{K}. \tag{4.8}$$

Moreover, the closure of Λ agrees with the union $\bigcup_{k \in \mathcal{K}} (k + [-1/2, +1/2]^d)$, and the hypothesis on ω implies that for each $k \in \mathcal{K}$ the intersection $\omega \cap (k + (-1/2, +1/2)^d)$ contains a ball of radius θ . In particular, $\omega \cap \Lambda$ is θ -equidistributed (in Λ) in the sense of [12, 42, 44].

The next interpolation result is a consequence of the Carleman estimates from [25, 33, 41] and relies on the fact that by the upper bound in part (i) of Assumption A the potential V can be bounded on Λ (or a suitably scaled version thereof) by a multiple of $(\lambda^{1/\tau_1})^{\tau_2}$. The situation therefore morally boils down to the one of bounded potentials, so that the proof can be extracted from [42, 44] and is omitted here. We refer, however, to [10, Proposition 6.14] for a detailed presentation.

Proposition 4.2 *Let θ be as in (4.8), and set $R := 9e\sqrt{d}$. Then, there is $\kappa \in (0, 1)$ satisfying $1/\kappa \lesssim_d \log(1/\theta)$ and a constant $D \geq 1$, depending on c_2, τ_2, C' , and the dimension d , such that*

$$\|F\|_{H^1(\Lambda \times (-1, 1))} \leq \theta^{-\kappa D \lambda^{2\tau_2/3\tau_1}} \|F\|_{H^1(\mathbb{R}^d \times (-R, R))}^{1-\kappa/2} \|f\|_{L^2(\omega \cap \Lambda)}^{\kappa/2}.$$

Remark 4.3 The appearance of the power $\lambda^{2\tau_2/3\tau_1}$ in Proposition 4.2 is due to the following: Once we insert in the upper bound $\|V\|_{\infty,\Lambda} \leq c_2 d^{\tau_2/2} L^{\tau_2}$ on the potential on the box $\Lambda := (-L, L)^d$ the bound on the sidelength $L \leq C'\lambda^{1/\tau_1} + 1$, we obtain $\|V\|_{\infty,\Lambda} \lesssim \lambda^{\tau_2/\tau_1}$. The sup of the potential enters the interpolation inequality via the Carleman estimate and appears in the scaling $\|V\|_{\infty,\Lambda}^{2/3}$, giving rise to the term $\lambda^{2\tau_2/3\tau_1}$.

With Proposition 4.2 at hand, we are finally in position to prove the main result of this note.

Proof of Theorem 1.1 We adopt the notation established in the preceding part of the current section.

The lower bound in Lemma 4.1 for $\varrho = R = 9e\sqrt{d}$ gives

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{1}{2R} \|F\|_{H^1(\mathbb{R}^d \times (-R, R))}^2.$$

In order to estimate the right-hand side further, we combine the lower bound in Lemma 4.1 for $\varrho = 1$ with the corresponding upper bound for $\varrho = R$ and obtain

$$\frac{\|F\|_{H^1(\mathbb{R}^d \times (-R, R))}^2}{\|F\|_{H^1(\mathbb{R}^d \times (-1, 1))}^2} \leq R(1 + (1 + \lambda)R^2)e^{2R\sqrt{\lambda}} \leq e^{K\lambda^{1/2}}$$

with some constant K depending only on the dimension. Together with (4.7) and the bound from Proposition 4.2 this yields

$$\begin{aligned} \|F\|_{H^1(\mathbb{R}^d \times (-R, R))}^2 &\leq e^{K\lambda^{1/2}} \|F\|_{H^1(\mathbb{R}^d \times (-1, 1))}^2 \leq 2e^{K\lambda^{1/2}} \|F\|_{H^1(\Lambda \times (-1, 1))}^2 \\ &\leq e^{2K\lambda^{1/2}} \theta^{-2\kappa D\lambda^{2\tau_2/3\tau_1}} \|F\|_{H^1(\mathbb{R}^d \times (-R, R))}^{2-\kappa} \|f\|_{L^2(\omega \cap \Lambda)}^\kappa, \end{aligned}$$

that is,

$$\|F\|_{H^1(\mathbb{R}^d \times (-R, R))} \leq e^{2K\lambda^{1/2}/\kappa} \theta^{-2D\lambda^{2\tau_2/3\tau_1}} \|f\|_{L^2(\omega \cap \Lambda)}.$$

In light of $1/\kappa \lesssim_d \log(1/\theta)$, $2\tau_2/3\tau_1 \geq 2/3 > 1/2$, and the definition of θ in (4.8), the claim follows upon a suitable choice of the constant C depending on C' , D , and the dimension d . □

5 Partial confinement potentials

The above proof can easily be adapted to more general anisotropic potentials V , such as certain partial confinement potentials. By the latter we mean potentials that behave like in Assumption A only with respect to certain coordinate directions. For simplicity, we demonstrate this in the case

$$H = -\Delta + V \quad \text{with} \quad V(x_1, x_2) = W(x_1), \quad (x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d-d_1}, \quad (5.1)$$

where $d_1 \in \mathbb{N}$, $d_1 < d$, and where $W \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^{d_1})$ satisfies Assumption A with d replaced by d_1 .

Since the operator H no longer has purely discrete spectrum, an expansion as in (4.2) for the extension F of $f \in \text{Ran } P_\lambda(H)$ via the ghost dimension construction is not available. However, straightforward adaptations of the arguments in [12, 44] show that we still have $F \in H_{\text{loc}}^2(\mathbb{R}^{d+1})$, $F(\cdot, t)$, $\partial_t F(\cdot, t) \in \text{Ran } P_\lambda(H)$ for all $t \in \mathbb{R}$, as well as (4.3) and (4.4). Also an analogue to Lemma 4.1 remains valid verbatim.

Following the reasoning in the proof of [16, Lemma A.2], we see that H admits the tensor representation

$$H = H_1 \otimes I_2 + I_1 \otimes H_2,$$

where $H_1 = -\Delta + W$ in $L^2(\mathbb{R}^{d_1})$ and $H_2 = -\Delta$ in $L^2(\mathbb{R}^{d-d_1})$, and where I_1 and I_2 denote the identity operators in $L^2(\mathbb{R}^{d_1})$ and $L^2(\mathbb{R}^{d-d_1})$, respectively. Consequently, as in [16, Corollary A.5] (cf. also the proof of [18, Lemma 2.3]), every function $h \in \text{Ran } P_\lambda(H)$ can be written as a finite sum

$$h = \sum_k \phi_k \otimes \psi_k$$

with suitable $\phi_k \in \text{Ran } P_\lambda(H_1)$ and $\psi_k \in \text{Ran } P_\lambda(H_2)$. Applying this to $h = F(\cdot, t)$ and $\partial_t F(\cdot, t)$ implies that $\partial_t F(\cdot, x_2, t)$ and $\partial_{x_2}^\alpha F(\cdot, x_2, t)$, $|\alpha| \leq 1$, belong to $\text{Ran } P_\lambda(H_1)$ for all $t \in \mathbb{R}$ and (almost) all $x_2 \in \mathbb{R}^{d-d_1}$. Theorem 1.4 for H_1 instead of H therefore provides an analogue to (4.7) with respect to the first d_1 coordinates, that is,

$$\|F\|_{H^1(\mathbb{R}^d \times (-1,1))}^2 \leq 2\|F\|_{H^1(B^{(d_1)}(0, C'\lambda^{1/\tau_1}) \times \mathbb{R}^{d-d_1} \times (-1,1))}^2, \tag{5.2}$$

where $B^{(d_1)}(0, C'\lambda^{1/\tau_1}) \subset \mathbb{R}^{d_1}$. After that, we may follow our proof of Theorem 1.1 verbatim to get a statement analogous to Theorem 1.1, which also covers Theorem 1.5 above. We here just give the corresponding result and refer to [10] for more details.

Theorem 5.1 *Let H be as in (5.1), and let $\omega \in \mathbb{R}^d$ be measurable such that for some $\delta \in (0, 1/2)$ and $\alpha \geq 0$ each intersection $\omega \cap (k + (-1/2, 1/2)^d)$, $k = (k_1, k_2) \in \mathbb{Z}^d$, contains a ball of radius $\delta^{1+|k_1|^\alpha}$. Then there is a constant $C > 0$ depending only on the dimension d and the parameters $\tau_1, \tau_2, c_1, c_2, \nu, M_\nu$ connected to W such that for all $\lambda \geq 1$ and all $f \in \text{Ran } P_\lambda(H)$ we have*

$$\|f\|_{L^2(\mathbb{R}^d)} \leq \left(\frac{1}{\delta}\right)^{C^{1+\alpha} \cdot \lambda^{(\alpha+2\tau_2/3)/\tau_1}} \|f\|_{L^2(\omega)}.$$

Note that this theorem relates to the spectral inequality for the partial harmonic oscillator from [16] in the same way as Theorem 1.1 relates to the one for the harmonic oscillator from [15].

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