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ORIGINAL PAPER

On the initial-boundary value problem for a non-local elliptic-hyperbolic system related to the short pulse equation

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Abstract
In this paper, we prove the well-posedness of the initial-boundary value problem for a non-local elliptic-hyperbolic system related to the short pulse equation. Our arguments are based on energy estimates and passing to the limit in a vanishing viscosity approximation of the problem.

Keywords Existence · Uniqueness · Stability · Short pulse equation · Non-local formulation · Initial-boundary value problem

Mathematics Subject Classification 35G25 · 35L65 · 35L05

1 Introduction

This paper is dedicated to the well-posedness analysis of the following initial boundary value problem

$$\begin{cases} \partial_t u - q^2 \partial_x (uv) = bP, & t > 0,\ x > 0, \\ \partial_x P = u, & t > 0,\ x > 0, \\ \alpha \partial_x^2 v + \beta \partial_x v + \gamma v = \kappa u^2, & t > 0,\ x > 0, \\ \partial_x u(t,0) = g(t), & t > 0, \\ P(t,0) = 0, & t > 0, \\ v(t,0) = h(t), & t > 0, \\ u(0,x) = u_0(x) & x > 0. \end{cases} \tag{1.1}$$

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The equations in (1.1) belonging to the large class of systems of the form

$$\begin{cases} \partial_t u + g\partial_x u^3 - a\partial_x^3 u + q\partial_x(uv) = bP, \\ \partial_x P = u, \\ \alpha\partial_x^2 v + \beta\partial_x v + \gamma v = \kappa u^2, \end{cases} \tag{1.2}$$

are termed continuum spectrum pulse equations [7–9, 55, 60, 61, 68, 79]. They describe the dynamics of the electrical field u of linearly polarized continuum spectrum pulses in optical waveguides, including fused-silica telecommunication-type or photonic-crystal fibers, as well as hollow capillaries filled with transparent gases or liquids.

The constants $a,\ b,\ g,\ q,\ \alpha,\ \kappa,\ \beta,\ \gamma$, in (1.2), take into account the frequency dispersion of the effective linear refractive index and the nonlinear polarization response, the excitation efficiency of the vibrations, the frequency and the decay time (see [8, 9, 79]).

From a mathematical point of view, in [29], the well-posedness of the classical solutions of the Cauchy associated with (1.2) is proven.

Taking $b = \alpha = \beta = 0$, (1.2) reads

$$\partial_t u + \left(g + \frac{q\kappa}{\gamma}\right)\partial_x u^3 - a\partial_x^3 u = 0, \tag{1.3}$$

which is known as modified Korteweg–de Vries equation (see [23, 45, 59, 75, 81]).

In [6, 7, 10, 63–65], it is proven that (1.3) is a non-slowly-varying envelope approximation model that describes the physics of few-cycle-pulse optical solitons. In [34, 59], the Cauchy problem for (1.3) is studied, while, in [23, 75], the convergence of the solution of (1.3) as $a \to 0$ to the unique entropy solution of the following scalar conservation law

$$\partial_t u + \left(g + \frac{q\kappa}{\gamma}\right)\partial_x u^3 = 0 \tag{1.4}$$

is proven.

On the other hand, taking $a = \alpha = \beta = 0$ in (1.2), we have the following equation

$$\begin{cases} \partial_t u + \left(g + \frac{q\kappa}{\gamma}\right)\partial_x u^3 = bP, \\ \partial_x P = u. \end{cases} \tag{1.5}$$

It was introduced by Kozlov and Sazonov [61] as a model equation describing the nonlinear propagation of optical pulses of a few oscillations duration in dielectric media, and Schäfer and Wayne [76] as a model equation describing the propagation of ultra-short light pulses in silica optical fibers.

In [3, 4, 30, 63–65], the authors show that (1.5) is also a non-slowly-varying envelope approximation model that describes the physics of few-cycle-pulse optical solitons. Meanwhile, [5, 24, 72, 74] show that (1.5) is a particular Rabelo equation which describes pseudospherical surfaces.

System (1.5) is also deduced in [82] to describe the short pulse propagation in nonlinear metamaterials characterized by a weak Kerr-type nonlinearity in their dielectric response.

It also is interesting to remind that equation (1.5) was proposed earlier in [69] in the context of plasma physic and that similar equations describe the dynamics of radiating gases [62, 77]. Moreover, [31, 52–54] show that (1.5) is also a model for ultrafast pulse propagation in a mode-locked laser cavity in the few-femtosecond pulse regime. Finally, an interpretation of (1.5) in the context of Maxwell equations is given in [71].

From a mathematical point of view, wellposedness results for the Cauchy problem of (1.5) are proven in the context of energy spaces (see [50, 70, 80]). Similar results are proven in [20, 27, 41, 51] in the context of entropy solutions, while, in [19, 33, 43, 73], the wellposedness of the homogeneous initial boundary value problem is studied. Finally, the convergence of a finite difference scheme is studied in [42].

Observe that, taking $\alpha = \beta = 0$ and $a \neq 0$, (1.2) reads

$$\begin{cases} \partial_t u + \left(g + \dfrac{q\kappa}{\gamma}\right)\partial_x u^3 - a\partial_x^3 u = bP, \\ \partial_x P = u. \end{cases} \tag{1.6}$$

It was derived by Costanzino, Manukian and Jones [48] in the context of the nonlinear Maxwell equations with high-frequency dispersion. Kozlov and Sazonov [61] show that (1.6) is an more general equation than (1.5) to describe the nonlinear propagation of optical pulses of a few oscillations duration in dielectric media.

Mathematical properties of (1.6) are studied in many different contexts, including the local and global well-posedness in energy spaces [48, 70] and stability of solitary waves [48, 67], while, in [34], the well-posedness of the classical solutions is proven.

In analogy with the convergence result of (1.3) to (1.4), in [27, 28], the convergence of the solution of (1.6) as $a \to 0$ to the unique entropy solution of (1.5) was proved.

Taking $g = a = 0$ in (1.2), we have the following system:

$$\begin{cases} \partial_t u + q\partial_x(uv) = bP, \\ \partial_x P = u, \\ \alpha\partial_x^2 v + \beta\partial_x v + \gamma v = \kappa u^2, \end{cases} \tag{1.7}$$

which represents a non-local formulation of (1.5) (see also [37]).

Conservation laws with non-local flux can be found in the context of traffic flow modeling [1, 12, 14, 15, 39, 44, 46, 47, 49, 56–58], in the context of sedimentation dynamic modeling [11] and in the context of slow erosion modeling [2, 16, 78].

In [32, 37], the well-posedness of the classical solution of the Cauchy associated with (1.7) is proven. Here we continue the analysis started in [37], studying the initial boundary value problem associated to (1.7).

Since our argument does not depend on the sign of the coefficient q here we assume

$$q < 0$$

and for the sake of notational simplicity from now on we write $-q^2$ instead of q

$$\begin{cases} \partial_t u - q^2\partial_x(uv) = bP, & t > 0,\ x > 0, \\ \partial_x P = u, & t > 0,\ x > 0, \\ \alpha\partial_x^2 v + \beta\partial_x v + \gamma v = \kappa u^2, & t > 0,\ x > 0, \\ \partial_x u(t,0) = g(t), & t > 0, \\ P(t,0) = 0, & t > 0, \\ v(t,0) = h(t), & t > 0, \\ u(0,x) = u_0(x), & x > 0, \end{cases} \tag{1.8}$$

where $\partial_x u(t,0)$ is the trace of $\partial_x u(t,x)$ at $x = 0$.

We can rewrite the problem as a boundary value problem for a single integro-differential equation.

$$\begin{cases} \partial_t u - q^2 \partial_x (u \mathcal{V}_t[u(t,\cdot)]) = b \int_0^x u(t,y)dy, & t > 0,\ x > 0, \\ \partial_x u(t,0) = g(t), & t > 0, \\ u(0,x) = u_0(x), & x > 0, \end{cases}$$

where $v(t,x) = \mathcal{V}_t[u(t,\cdot)](x)$ is the solution of the problem

$$\begin{cases} \alpha \partial_x^2 v + \beta \partial_x v + \gamma v = \kappa u^2, & t > 0,\ x > 0, \\ v(t,0) = h(t), & t > 0, \end{cases}$$

On the function g, we assume

$$g \in W^{1,\infty}(0,\infty), \tag{1.9}$$

while on the function h, we assume

$$0 < \kappa_1^2 \le h(t), \quad h \in L^\infty(0,\infty), \tag{1.10}$$

for some constant $\kappa_1 > 0$. On the initial datum, we assume that

$$u_0 \in L^1(0,\infty) \cap H^1(0,\infty), \quad \int_0^\infty u_0(x)dx = 0, \tag{1.11}$$

$$\|P_0\|^2_{L^2(0,\infty)} = \int_0^\infty \left(\int_0^x u_0(y)dy \right)^2 dx < \infty, \tag{1.12}$$

$$\text{where } P_0(x) = \int_0^x u_0(y)dy, \qquad x > 0. \tag{1.13}$$

The zero mean requirement in (1.11) and the L^2 one in (1.13) imply that the solution u of (1.8) satisfies the same conditions at every time $t > 0$ (see [17, 22, 35]), i.e.

$$\int_0^\infty u(t,x)dx = 0, \qquad \|P(t,\cdot)\|^2_{L^2(0,\infty)} < \infty, \qquad t \ge 0.$$

Those properties will play a key role in the estimates of the next sections.

In addition, on the constants $\alpha,\ \beta,\ \kappa$, we assume

$$\frac{\alpha}{\kappa} > 0, \quad \frac{\beta}{\kappa} \le 0, \tag{1.14}$$

and either (1.15) or (1.16)

$$b,\ \beta,\ \gamma \ne 0, \quad \alpha\beta < 0, \quad \text{or}, \tag{1.15}$$

$$b \ne 0, \quad \beta = 0, \quad \alpha\gamma < 0. \tag{1.16}$$

Observe that, in all cases, $\alpha \ne 0$. Therefore, we may set it equal to 1 and work with only three constants.

We use the following definition fo solutions.

Definition 1.1 A triplet of real valued functions $(u,\ v,\ P)$ defined on $[0,\infty) \times [0,\infty)$ is a distributional solution of (1.8) if

- $u \in L^2(0,\infty; H^2(0,\infty)),\ P \in L^2(0,\infty; H^1(0,\infty)),\ v \in L^2(0,\infty; H^1(0,\infty))$;
- $\partial_x u(\cdot,0) = g,\ P(\cdot,0) = 0,\ v(\cdot,0) = h$ in the sense of traces;

- for every test function $\varphi \in C^\infty(\mathbb{R} \times (0,\infty))$ with compact support

$$\int_0^\infty \int_0^\infty \left(u\partial_t\varphi - q^2 uv\partial_x\varphi + bP\varphi\right) dtdx + \int_0^\infty u_0(x)\varphi(0,x)dx = 0$$
$$\int_0^\infty \int_0^\infty (P\partial_x\varphi + u\varphi)\, dtdx = 0, \tag{1.17}$$
$$\int_0^\infty \int_0^\infty \left(\alpha v\partial_x^2\varphi - \beta v\partial_x\varphi + \gamma v\varphi - \kappa u^2\varphi\right) dtdx = 0.$$

The assumptions (1.14), (1.15) and (1.16) on the constants guarantee the boundedness of the L^2 norm of u in time (see Lemma 2.3 below).

The main results of this paper are the following theorem.

Theorem 1.1 *Assume* (1.9), (1.10), (1.11), (1.13), (1.12), (1.14) *and either* (1.15) *or* (1.16). *Given* $T > 0$, *there exists a unique distributional solution* (u, v, P) *of* (1.8) *in the sense of Definition 1.1 such that*

$$u \in H^1((0,T)\times(0,\infty)) \cap L^\infty(0,T;H^1(0,\infty)), \tag{1.18}$$
$$v \in L^\infty(0,T;H^3(0,\infty)), \tag{1.19}$$
$$P \in L^\infty(0,T;H^2(0,\infty)), \tag{1.20}$$
$$\int_0^\infty u(t,x)dx = 0, \quad t \geq 0. \tag{1.21}$$

Moreover, if (u_1, v_1, P_1) *and* (u_2, v_2, P_2) *are two solutions of* (1.8), *we have that*

$$\begin{aligned} \|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(0,\infty)} &\leq e^{C(T)t}\left\|u_{1,0} - u_{2,0}\right\|_{L^2(0,\infty)}, \\ \|v_1(t,\cdot) - v_2(t,\cdot)\|_{H^2(0,\infty)} &\leq e^{C(T)t}\left\|u_{1,0} - u_{2,0}\right\|_{L^2(0,\infty)}, \end{aligned} \tag{1.22}$$

for some suitable $C(T) > 0$, *and every* $0 \leq t \leq T$.

Finally, we decided to study only the stability with respect to the initial datum in order to shorten the arguments.

In the next theorem we give a necessary condition on the constants that gives some additional regularity on the solutions.

Theorem 1.2 *Given* $T > 0$ *assuming either* (1.14) *or* (1.15), *if*

$$\beta\gamma < 0, \tag{1.23}$$

there exists an unique distributional solution (u, v, P) *of* (1.8) *in the sense of Definition 1.1 such that* (1.18), (1.20) *and* (1.21) *hold, while*

$$\begin{aligned} &v \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^3(\mathbb{R})) \cap W^{1,\infty}((0,T)\times\mathbb{R}) \\ &\partial_{tx}^2 v \in L^\infty(0,T;L^2(0,\infty)) \cap L^\infty((0,T)\times(0,\infty)), \quad \partial_t\partial_x^2 v \in L^\infty(0,T;L^2(0,\infty)), \end{aligned} \tag{1.24}$$

for every $0 \leq t \leq T$. *Moreover, assuming* (1.14), *one between* (1.15) *or* (1.16), *and*

$$0 < \kappa_1^2 \leq h(t), \quad h \in W^{1,\infty}(0,\infty), \tag{1.25}$$

there exists an unique distributional solution (u, v, P) *of* (1.8) *such that* (1.18), (1.20), (1.21), (1.24) *hold. Finally, if* (u_1, v_1, P_1) *and* (u_2, v_2, P_2) *are two solutions of* (1.8), (1.22) *holds.*

The paper is organized as follows. In Sect. 2, we prove several a priori estimates on a vanishing viscosity approximation of (1.8). Those play a key role in the proof of our main result, that is given in Sect. 3. In Sects. 4 and 5, we prove Theorem 1.2, under Assumptions (1.23) and (1.25), respectively.

2 Vanishing viscosity approximation

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.8)

$$\begin{cases} \partial_t u_\varepsilon - q^2\partial_x(u_\varepsilon v_\varepsilon) = bP_\varepsilon + \varepsilon\partial_x^2 u_\varepsilon, & t>0,\ x>0,\\ \partial_x P_\varepsilon = u_\varepsilon, & t>0,\ x>0,\\ \alpha\partial_x^2 v_\varepsilon + \beta\partial_x v_\varepsilon + \gamma v_\varepsilon = \kappa u_\varepsilon^2, & t>0,\ x>0,\\ \partial_x u_\varepsilon(t,0) =\geq (t), & t>0,\\ P_\varepsilon(t,0)=0, & t>0,\\ v_\varepsilon(t,0)=h_\varepsilon(t), & t>0,\\ u_\varepsilon(0,x)=u_{\varepsilon,0}(x), & x>0, \end{cases} \tag{2.1}$$

where $0<\varepsilon<1$ and $u_{\varepsilon,0},\, g_\varepsilon\ \geq,\ h_\varepsilon$ are C^∞ approximations of $u_0,\ g,\ h$ such that

$$\begin{aligned} &\left\|u_{\varepsilon,0}\right\|_{H^1(0,\infty)} \leq \|u_0\|_{H^1(0,\infty)}, \quad \int_0^\infty u_{\varepsilon,0}(x)dx = 0, \quad \varepsilon\left\|\partial_x^2 u_{\varepsilon,0}\right\|_{L^2(0,\infty)} \leq C_0,\\ &\|g_\varepsilon\|_{W^{1,\infty}(0,\infty)} \leq C_0, \quad 0<\kappa_1^2 \leq h_\varepsilon(t), \quad \|h_\varepsilon\|_{L^\infty(0,\infty)} \leq C_0,\\ &\left\|P_{\varepsilon,0}\right\|_{L^2(0,\infty)} \leq \|P_0\|_{L^2(0,\infty)}, \text{ with } P_{\varepsilon,0}(x) = \int_0^x u_{\varepsilon,0}(y)dy, \end{aligned} \tag{2.2}$$

and C_0 is a constant independent on ε. The existence of a unique smooth solution

$$u_\varepsilon \in C^\infty([0,\infty)\times[0,\infty)) \cap H^2((0,\infty)\times(0,\infty))$$

can be proved using the same arguments as in [25, 36, 38]. Finally, we want to point put that the requirement $\varepsilon\left\|\partial_x^2 u_{\varepsilon,0}\right\|_{L^2(0,\infty)} \leq C_0$ is quite common in the arguments based on vanishing viscosity and it states the fact that the H^2 norm of $u_{\varepsilon,0}$ is not uniformly bounded with respect to ε but it blows-up as $\varepsilon\to 0$.

Let us prove some a priori estimates on u_ε, P_ε and v_ε. We denote with C all the the constants which depend only on the initial data, and with $C(T)$, the constants which depend also on T. Moreover, we always assume that $\varepsilon\in(0,1)$ is given,l $(u_\varepsilon,\ v_\varepsilon,\ P_\varepsilon)$ is a solution of (2.1), and that (2.2) holds.

We begin by proving the following lemma.

Lemma 2.1 *For each $t>0$, we have that*

$$P_\varepsilon(t,\infty)=0, \tag{2.3}$$

$$\int_0^\infty u_\varepsilon(t,x)dx = 0. \tag{2.4}$$

Proof Arguing as in [32, Lemma 2.2], or [18, Lemma 2.1], we have (2.3).

We prove (2.4). Integrating the second equation of (2.1) on $(0,x)$ and using the boundary conditions, we have that

$$P_\varepsilon(t,x) = \int_0^x u_\varepsilon(t,y)dy. \tag{2.5}$$

By (2.3) and (2.5), we get

$$\int_0^\infty u_\varepsilon(t,x)dx = P_\varepsilon(t,\infty) = 0,$$

which gives (2.4). □

Lemma 2.2 *For each $t > 0$, we have that,*

$$\int_0^\infty u_\varepsilon^2 \partial_x v_\varepsilon dx = -\frac{\alpha}{2\kappa}(\partial_x v_\varepsilon(t,0))^2 + \frac{\beta}{\kappa}\|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} - \frac{\gamma}{2\kappa}h_\varepsilon^2(t). \tag{2.6}$$

Proof Multiplying the third equation of (2.1) by $\partial_x v_\varepsilon$, thanks to (2.1), integrating on $(0,\infty)$, we get

$$\begin{aligned}\kappa\int_0^\infty u_\varepsilon^2\partial_x v_\varepsilon dx &= \alpha\int_0^\infty \partial_x v_\varepsilon \partial_x^2 v_\varepsilon dx + \beta\|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + \gamma\int_0^\infty v_\varepsilon\partial_x v_\varepsilon dx\\ &= -\frac{\alpha}{2}(\partial_x v_\varepsilon(t,0))^2 + \beta\|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} - \frac{\gamma}{2}v_\varepsilon^2(t,0)\\ &= -\frac{\alpha}{2}(\partial_x v_\varepsilon(t,0))^2 + \beta\|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} - \frac{\gamma}{2}h_\varepsilon^2(t),\end{aligned}$$

which gives (2.6). □

We continue by proving an L^2-estimate unform in ε.

Lemma 2.3 *Fix $T > 0$ and assume (1.15), or (1.16). Then, there exists a constant $C(T) > 0$, independent on ε, such that*

$$\begin{aligned}&\|u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + 2\varepsilon\int_0^\infty \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)}ds + \frac{q^2\kappa_1^2}{2}\int_0^t u_\varepsilon^2(s,0)ds\\ &\quad + \frac{q^2\alpha}{2\kappa}\int_0^t(\partial_x v_\varepsilon(s,0))^2 ds - \frac{q^2\beta}{\kappa}\int_0^t \|\partial_x v_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)}ds \le C(T),\end{aligned} \tag{2.7}$$

for every $0 \le t \le T$.

Proof Let $0 \le t \le T$. Multiplying the first equation of (2.1) by $2u_\varepsilon$, thanks to (2.1), an integration on $(0,\infty)$ gives

$$\begin{aligned}\frac{d}{dt}\|u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} &= 2\int_0^\infty u_\varepsilon\partial_t u_\varepsilon dx\\ &= 2q^2\int_0^\infty u_\varepsilon\partial_x(u_\varepsilon v_\varepsilon)dx + 2b\int_0^\infty P_\varepsilon u_\varepsilon dx + 2\varepsilon\int_0^\infty u_\varepsilon\partial_x^2 u_\varepsilon dx\\ &= 2q^2\int_0^\infty u_\varepsilon\partial_x(u_\varepsilon v_\varepsilon)dx + 2b\int_0^\infty P_\varepsilon u_\varepsilon dx\\ &\quad - 2\varepsilon u_\varepsilon(t,0)\partial_x u_\varepsilon(t,0) - 2\varepsilon\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\\ &= 2q^2\int_0^\infty u_\varepsilon\partial_x(u_\varepsilon v_\varepsilon)dx + 2b\int_0^\infty P_\varepsilon u_\varepsilon dx\\ &\quad - 2\varepsilon g_\varepsilon(t)u_\varepsilon(t,0) - 2\varepsilon\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}&\frac{d}{dt}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}+2\varepsilon\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\&\quad=2q^2\int_0^\infty u_\varepsilon\partial_x(u_\varepsilon v_\varepsilon)dx+2b\int_0^\infty P_\varepsilon u_\varepsilon dx-2\varepsilon\geq(t)u_\varepsilon(t,0).\end{aligned}\tag{2.8}$$

Observe that, by the boundary condition on P_ε and (2.3),

$$2b\int_0^\infty P_\varepsilon u_\varepsilon dx=2b\int_0^\infty P_\varepsilon\partial_x P_\varepsilon dx=0.\tag{2.9}$$

Moreover, by (2.1) and (2.6),

$$\begin{aligned}2q^2\int_0^\infty u_\varepsilon\partial_x(u_\varepsilon v_\varepsilon)dx&=-2q^2u_\varepsilon^2(t,0)v_\varepsilon(t,0)-2q^2\int_0^\infty u_\varepsilon\partial_x u_\varepsilon v_\varepsilon dx\\&=-q^2u_\varepsilon^2(t,0)v_\varepsilon(t,0)+q^2\int_0^\infty\partial_x v_\varepsilon u_\varepsilon^2dx\\&=-q^2u_\varepsilon^2(t,0)v_\varepsilon(t,0)-\frac{q^2\alpha}{2\kappa}(\partial_x v_\varepsilon(t,0))^2+\frac{q^2\beta}{\kappa}\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}-\frac{q^2\gamma}{2\kappa}h_\varepsilon^2(t)\\&=-q^2h_\varepsilon(t)u_\varepsilon^2(t,0)-\frac{q^2\alpha}{2\kappa}(\partial_x v_\varepsilon(t,0))^2+\frac{q^2\beta}{\kappa}\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}-\frac{q^2\gamma}{2\kappa}h_\varepsilon^2(t)\\&\leq-q^2\kappa_1^2u_\varepsilon^2(t,0)-\frac{q^2\alpha}{2\kappa}(\partial_x v_\varepsilon(t,0))^2+\frac{q^2\beta}{\kappa}\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}-\frac{q^2\gamma}{2\kappa}h_\varepsilon^2(t).\end{aligned}\tag{2.10}$$

Using (2.9) and (2.10) in (2.8)

$$\begin{aligned}&\frac{d}{dt}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}+2\varepsilon\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}+q^2\kappa_1^2u_\varepsilon^2(t,0)+\frac{q^2\alpha}{2\kappa}(\partial_x v_\varepsilon(t,0))^2\\&\quad-\frac{q^2\beta}{\kappa}\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\leq-2\varepsilon g_\varepsilon(t)u_\varepsilon(t,0)-\frac{q^2\gamma}{2\kappa}h_\varepsilon^2(t).\end{aligned}\tag{2.11}$$

Since $0<\varepsilon<1$, thanks to (1.10) and the Young's inequality,

$$2\varepsilon|g_\varepsilon(t)||u_\varepsilon(t,0)|\leq C|u_\varepsilon(t,0)|\leq C+\frac{q^2\kappa_1^2}{2}u_\varepsilon^2(t,0),\qquad\left|\frac{q^2\gamma}{2\kappa}\right|h_\varepsilon^2(t)\leq C.$$

Consequently, (2.11) becomes,

$$\begin{aligned}&\frac{d}{dt}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}+2\varepsilon\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}+\frac{q^2\kappa_1^2}{2}u_\varepsilon^2(t,0)\\&\qquad+\frac{q^2\alpha}{2\kappa}(\partial_x v_\varepsilon(t,0))^2-\frac{q^2\beta}{\kappa}\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\leq C.\end{aligned}$$

Integrating on $(0,t)$, by (2.2) we get

$$\begin{aligned}&\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}+2\varepsilon\int_0^t\left\|\partial_x u_\varepsilon(s,\cdot)\right\|^2_{L^2(0,\infty)}ds+\frac{q^2\kappa_1^2}{2}\int_0^t u_\varepsilon^2(s,0)ds\\&\quad+\frac{q^2\alpha}{2\kappa}\int_0^t(\partial_x v_\varepsilon(s,0))^2ds-\frac{q^2\beta}{\kappa}\int_0^t\left\|\partial_x v_\varepsilon(s,\cdot)\right\|^2_{L^2(0,\infty)}ds\\&\leq\left\|u_{\varepsilon,0}\right\|^2_{L^2(0,\infty)}C+Ct\leq C+Ct\leq C(T),\end{aligned}$$

which gives (2.7). □

We continue by proving the Lipschitz continuity of v_ε

Lemma 2.4 *Given $T > 0$. Assume either (1.15) or (1.16). There exists a constant $C(T) > 0$, independent on ε, such that*

$$\|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))} \le C(T), \tag{2.12}$$

$$\|\partial_x v_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \le C(T), \tag{2.13}$$

$$\|v_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \le C(T), \tag{2.14}$$

$$\|v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))} \le C(T), \tag{2.15}$$

for every $0 \le t \le T$.

Proof (Proof assuming (1.15)) Let $0 \le t \le T$. We begin by proving that

$$2\beta^2 \|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} - \alpha\beta(\partial_x v_\varepsilon(t,0))^2 \le C(T)\left(1 + \|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))}\right). \tag{2.16}$$

Multiplying the third equation of (2.1) by $2\beta\partial_x v_\varepsilon$, we have that

$$2\beta\alpha\partial_x v_\varepsilon\partial_x^2 v_\varepsilon + 2\beta^2(\partial_x v_\varepsilon)^2 + 2\beta\gamma v_\varepsilon\partial_x v_\varepsilon = 2\beta\kappa u_\varepsilon^2\partial_x v_\varepsilon. \tag{2.17}$$

Observe that

$$\begin{aligned} 2\beta\alpha\int_0^\infty \partial_x v_\varepsilon\partial_x^2 v_\varepsilon dx &= -\beta\alpha(\partial_x v_\varepsilon(t,0))^2,\\ 2\beta\gamma\int_0^\infty v_\varepsilon\partial_x v_\varepsilon dx &= -\beta\gamma v_\varepsilon^2(t,0) = -\beta\gamma h_\varepsilon^2(t). \end{aligned} \tag{2.18}$$

Thanks to (1.15) and (2.18), an integration of (2.17) on $(0,\infty)$ gives

$$2\beta^2 \|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} - \alpha\beta(\partial_x v_\varepsilon(t,0))^2 = 2\beta\kappa\int_0^\infty u_\varepsilon^2\partial_x v_\varepsilon dx + \beta\gamma h_\varepsilon^2(t). \tag{2.19}$$

Since, using Lemma 2.3,

$$\begin{aligned} &2|\beta\kappa|\int_0^\infty u_\varepsilon^2|\partial_x v_\varepsilon|dx + |\beta\gamma|h_\varepsilon^2(t)\\ &\le 2|\beta\kappa|\,\|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))}\,\|u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + |\beta\gamma|h_\varepsilon^2(t)\\ &\le C(T)\left(1 + \|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))}\right), \end{aligned} \tag{2.20}$$

(2.16) follows from (2.19) and (2.20).

We prove that for every $t \in [0,T]$

$$\|v_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \le C(T)\sqrt{1 + \|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))}}, \tag{2.21}$$

$$\|v_\varepsilon(t,\cdot)\|_{L^\infty(0,\infty)} \le C(T)\sqrt{1 + \|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))}}. \tag{2.22}$$

Multiplying the third equation of (2.1) by $2\gamma v_\varepsilon$, we get

$$2\gamma\alpha v_\varepsilon\partial_x^2 v_\varepsilon + 2\gamma\beta v_\varepsilon\partial_x v_\varepsilon + 2\gamma^2 v_\varepsilon^2 = 2\gamma\kappa u_\varepsilon^2 v_\varepsilon. \tag{2.23}$$

Observe that, thanks to (2.1),

$$\begin{aligned} 2\gamma\alpha\int_0^\infty v_\varepsilon\partial_x^2 v_\varepsilon dx =& -2\gamma\alpha v_\varepsilon(t,0)\partial_x v_\varepsilon(t,0)-2\gamma\alpha\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ =& -2\gamma\alpha h_\varepsilon(t)\partial_x v_\varepsilon(t,0)-2\gamma\alpha\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}, \\ 2\gamma\beta\int_0^\infty v_\varepsilon\partial_x v_\varepsilon dx =& -\gamma\beta v_\varepsilon^2(t,0)=-\gamma\beta h_\varepsilon^2(t). \end{aligned} \tag{2.24}$$

Therefore, integrating (2.23) on $(0,\infty)$, thanks to (2.24),

$$\begin{aligned} 2\gamma^2\left\|v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} =& 2\gamma\kappa\int_0^\infty u_\varepsilon^2 v_\varepsilon dx+\gamma\beta h_\varepsilon^2(t)\\ &+2\gamma\alpha h_\varepsilon(t)\partial_x v_\varepsilon(t,0)+2\gamma\alpha\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}. \end{aligned} \tag{2.25}$$

Due to (2.2), (2.7) and the Young's inequality,

$$\begin{aligned} 2\left|\gamma\kappa\right|\int_0^\infty u_\varepsilon^2\left|v_\varepsilon\right|dx \le& 2\left|\gamma\kappa\right|\left\|v_\varepsilon(t,\cdot)\right\|_{L^\infty(0,\infty)}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ \le& C(T)\left\|v_\varepsilon(t,\cdot)\right\|_{L^\infty(0,\infty)}\le\frac{C(T)}{D_1}+D_1\left\|v_\varepsilon(t,\cdot)\right\|^2_{L^\infty(0,\infty)}, \end{aligned}$$

$$\gamma\beta h_\varepsilon^2(t)+2\left|\gamma\alpha\right|\left|h_\varepsilon(t)\right|\left|\partial_x v_\varepsilon(t,0)\right|\le C+2C\left|\partial_x v_\varepsilon(t,0)\right|\le C+(\partial_x v_\varepsilon(t,0))^2,$$

where D_1 is an arbitrary positive constant, which will be specified later. It follows from (2.25) that

$$\begin{aligned} 2\gamma^2\left\|v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} \le& \frac{C(T)}{D_1}+D_1\left\|v_\varepsilon(t,\cdot)\right\|^2_{L^\infty(0,\infty)}\\ &+C+(\partial_x v_\varepsilon(t,0))^2+2\left|\gamma\alpha\right|\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}. \end{aligned} \tag{2.26}$$

Due to (2.2) and the Young's inequality,

$$\begin{aligned} v_\varepsilon^2(t,x) =& 2\int_0^x v_\varepsilon\partial_x v_\varepsilon dy+h_\varepsilon^2(t)\le 2\int_0^\infty\left|v_\varepsilon\right|\left|\partial_x v_\varepsilon\right|dx+C\\ \le& \left\|v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}+\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}+C. \end{aligned}$$

Hence,

$$\left\|v_\varepsilon(t,\cdot)\right\|^2_{L^\infty(0,\infty)}\le\left\|v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}+\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}+C. \tag{2.27}$$

Since $D_1>0$, (2.26) implies

$$\begin{aligned} 2\gamma^2\left\|v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} \le& \frac{C(T)}{D_1}+D_1\left\|v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &+\left(1+\frac{1}{D_1}\right)C+(\partial_x v_\varepsilon(t,0))^2+\left(2\left|\gamma\alpha\right|+\frac{1}{D_1}\right)\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}, \end{aligned}$$

that is

$$\begin{aligned} \left(2\gamma^2-D_1\right)\left\|v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} \le& \frac{C(T)}{D_1}+\left(1+\frac{1}{D_1}\right)C\\ &+\left(1+\frac{1}{D_1}\right)C+(\partial_x v_\varepsilon(t,0))^2\\ &+\left(2\left|\gamma\alpha\right|+\frac{1}{D_1}\right)\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}. \end{aligned}$$

Choosing $D_1 = \gamma^2$, we have that

$$\gamma^2 \|v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} \le C(T)\left(1 + (\partial_x v_\varepsilon(t,0))^2 + \|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\right). \tag{2.28}$$

(2.21) follows from (2.16) and (2.28), while (2.16), (2.21) and (2.27) give (2.22).

We prove (2.12). Multiplying the third equation of (2.1) by $2\alpha\partial_x v_\varepsilon$, we have

$$2\alpha^2 \partial_x v_\varepsilon \partial_x^2 v_\varepsilon + 2\alpha\beta(\partial_x v_\varepsilon)^2 + 2\alpha\gamma v_\varepsilon \partial_x v_\varepsilon = 2\alpha\kappa u_\varepsilon^2 \partial_x v_\varepsilon. \tag{2.29}$$

Observe that,

$$\begin{aligned} 2\alpha^2 \int_0^x \partial_x v_\varepsilon \partial_x^2 v_\varepsilon dy &= \alpha^2(\partial_x v_\varepsilon(t,x))^2 - \alpha^2(\partial_x v_\varepsilon(t,0))^2, \\ 2\alpha\gamma \int_0^x v_\varepsilon \partial_x v_\varepsilon dy &= \alpha\gamma v_\varepsilon^2(t,x) - \alpha\gamma v_\varepsilon^2(t,0) = \alpha\gamma v_\varepsilon^2(t,x) - \alpha\gamma h_\varepsilon^2(t). \end{aligned} \tag{2.30}$$

Integrating (2.29) on $(0,x)$, thanks to (2.2), (2.7) and (2.30), we have

$$\begin{aligned} \alpha^2(\partial_x v_\varepsilon(t,x))^2 &= 2\alpha\kappa \int_0^x u_\varepsilon^2 \partial_x v_\varepsilon dx + \alpha^2(\partial_x v_\varepsilon(t,0))^2 - 2\alpha\beta \int_0^x (\partial_x v_\varepsilon)^2 dy \\ &\quad - \alpha\gamma v_\varepsilon^2(t,x) - \alpha\gamma h_\varepsilon^2(t) \\ &\le 2|\alpha\kappa| \int_0^\infty u_\varepsilon^2 |\partial_x v_\varepsilon| dx + \alpha^2(\partial_x v_\varepsilon(t,0))^2 + 2|\alpha\beta| \|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} \\ &\quad + |\alpha\gamma| \|v_\varepsilon(t,\cdot)\|^2_{L^\infty(0,\infty)} + |\alpha\gamma| h_\varepsilon^2(t) \\ &\le 2|\alpha\kappa| \|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))} \|u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + 2\alpha^2(\partial_x v_\varepsilon(t,0))^2 \\ &\quad + 2|\alpha\beta| \|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + |\alpha\gamma| \|v_\varepsilon(t,\cdot)\|^2_{L^\infty(0,\infty)} + C \\ &\le C(T)\left(1 + \|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))}\right) + 2\alpha^2(\partial_x v_\varepsilon(t,0))^2 \\ &\quad + 2|\alpha\beta| \|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + |\alpha\gamma| \|v_\varepsilon(t,\cdot)\|^2_{L^\infty(0,\infty)}. \end{aligned}$$

Consequently, by (2.16) and (2.22), we have that

$$\alpha^2(\partial_x v_\varepsilon(t,x))^2 \le C(T)\left(1 + \|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))}\right). \tag{2.31}$$

Hence,

$$\alpha^2 \|\partial_x v_\varepsilon\|^2_{L^\infty((0,T)\times(0,\infty))} - C(T)\|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))} - C(T) \le 0, \tag{2.32}$$

which gives (2.12).

Finally, (2.13), (2.14) and (2.15) follows from (2.12), (2.13), (2.14) and (2.15), respectively. □

Proof (Proof assuming (1.16)) Let $0 \le t \le T$. We begin by observing that, thanks to (1.16), the third equation of (2.1) reads

$$\alpha\partial_x^2 v_\varepsilon + \gamma v_\varepsilon = \kappa^2 u_\varepsilon^2. \tag{2.33}$$

Following [18], or [33], in order to work with homogeneous boundary conditions we define

$$W_1(t,x) = v_\varepsilon(t,x) - h_\varepsilon(t)e^{-x}, \quad x \ge 0. \tag{2.34}$$

We have that

$$\begin{aligned} \partial_x W_1(t,x) &= \partial_x v_\varepsilon(t,x) + h_\varepsilon(t)e^{-x}, \\ \partial_x^2 W_1(t,x) &= \partial_x^2 v_\varepsilon(t,x) - h_\varepsilon(t)e^{-x}, \end{aligned} \tag{2.35}$$

and

$$W_1(t,0) = v_\varepsilon(t,0) - h_\varepsilon(t) = 0. \tag{2.36}$$

Due to (2.34) and (2.35), (2.33) is equivalent to the following one:

$$\alpha\partial_x^2 W_1 + \gamma W_1 = \kappa^2 u_\varepsilon^2 + \alpha h_\varepsilon(t)e^{-x} + \gamma h_\varepsilon(t)e^{-x}. \tag{2.37}$$

Multiplying (2.37) by $2\gamma W_1$, we have that

$$2\alpha\gamma W_1\partial_x^2 W_1 + 2\gamma^2 W_1^2 = 2\gamma\kappa^2 u_\varepsilon^2 W_1 + 2\alpha\gamma h_\varepsilon(t)e^{-x}W_1 + 2\gamma^2 h_\varepsilon(t)e^{-x}W_1. \tag{2.38}$$

Observe that, thanks to (2.36),

$$2\alpha\gamma\int_0^\infty W_1\partial_x^2 W_1 dx = -2\alpha\gamma\left\|\partial_x W_1(t,\cdot)\right\|^2_{L^2(0,\infty)}. \tag{2.39}$$

Consequently, integrating (2.38) on $(0,\infty)$, by (1.16) and (2.39), we have that

$$\begin{aligned} &-2\alpha\gamma\left\|\partial_x W_1(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2\gamma^2\left\|W_1(t,\cdot)\right\|^2_{L^2(0,\infty)} \\ &\quad= 2\gamma\kappa\int_0^\infty u_\varepsilon^2 W_1 dx + 2\alpha\gamma h_\varepsilon(t)\int_0^\infty e^{-x}W_1 dx + 2\gamma^2 h_\varepsilon(t)\int_0^\infty e^{-x}W_1 dx. \end{aligned} \tag{2.40}$$

Observe that

$$\int_0^\infty e^{-2x}dx = \frac{1}{2}. \tag{2.41}$$

Thanks to (2.2), (2.41), the fact that $\gamma \neq 0$, and the Young's inequality,

$$\begin{aligned} 2|\alpha\gamma||h_\varepsilon(t)|\int_0^\infty e^{-x}|W_1|dx &\le C\int_0^\infty e^{-x}|W_1|dx \\ &= \int_0^\infty \left|\frac{Ce^{-x}}{\gamma}\right| |\gamma W_1|\,dx \le C + \frac{\gamma^2}{2}\left\|W_1(t,\cdot)\right\|^2_{L^2(0,\infty)}, \\ 2\gamma^2|h_\varepsilon(t)|\int_0^\infty e^{-x}|W_1|dx &\le C\gamma^2\int_0^\infty e^{-x}|W_1|dx \\ &\le C + \frac{\gamma^2}{2}\left\|W_1(t,\cdot)\right\|^2_{L^2(0,\infty)}. \end{aligned}$$

The L^2 estimate stated in (2.7) and (2.40) gives

$$\begin{aligned} -\alpha\gamma&\left\|\partial_x W_1(t,\cdot)\right\|^2_{L^2(0,\infty)} + \gamma^2\left\|W_1(t,\cdot)\right\|^2_{L^2(0,\infty)} \\ &\le |\gamma\kappa|\int_0^\infty u_\varepsilon^2|W_1|dx + C \\ &\le |\gamma\kappa|\left\|W_1\right\|_{L^\infty((0,T)\times(0,\infty))}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + C \\ &\le C(T)\left(1 + \left\|W_1\right\|_{L^\infty((0,T)\times(0,\infty))}\right). \end{aligned} \tag{2.42}$$

We prove that

$$\left\|W_1\right\|_{L^\infty((0,T)\times(0,\infty))} \le C(T). \tag{2.43}$$

Due to (2.36), (2.42) and the Hölder inequality,

$$\begin{aligned} W_1^2(t,x) &= 2\int_0^x W_1\partial_x W_1 dy \le 2\int_0^\infty |W_1||\partial_x W_1|dx \\ &\le 2\left\|W_1(t,\cdot)\right\|_{L^2(0,\infty)}\left\|\partial_x W_1(t,\cdot)\right\|_{L^2(0,\infty)} \le C(T)\left(1 + \left\|W_1\right\|_{L^\infty((0,T)\times(0,\infty))}\right), \end{aligned}$$

for every $x \geq 0$. Hence,

$$\|W_1\|^2_{L^\infty((0,T)\times(0,\infty))} - C(T)\,\|W_1\|_{L^\infty((0,T)\times(0,\infty))} - C(T) \leq 0,$$

which gives (2.43).

Observe that, by (2.42) and (2.43), we have that

$$-\alpha\gamma\,\|\partial_x W_1(t,\cdot)\|^2_{L^2(0,\infty)} + \gamma^2\,\|W_1(t,\cdot)\|^2_{L^2(0,\infty)} \leq C(T). \tag{2.44}$$

We prove (2.15). Thanks to (2.2) and (2.34),

$$\begin{aligned}|v_\varepsilon(t,x)| &\leq \left|W_1(t,x) + h_\varepsilon(t)e^{-x}\right| \leq |W_1(t,x)| + |h_\varepsilon(t)e^{-x}|\\ &\leq \|W_1\|_{L^\infty((0,T)\times(0,\infty))} + C \leq C(T).\end{aligned}$$

Therefore,

$$\|v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))} \leq C(T),$$

that is (2.15).

We prove (2.14). Thanks to (2.2), (2.34) and the Young's inequality,

$$v_\varepsilon^2(t,x) = \left(W_1(t,x) + h_\varepsilon(t)e^{-x}\right)^2 \leq 2W_1^2(t,x) + 2h_\varepsilon^2(t)e^{-2x} \leq 2W_1^2(t,x) + Ce^{-2x}. \tag{2.45}$$

Integrating (2.45) on $(0,\infty)$, by (2.41) and (2.44), we have (2.14).

In a similar way, thanks to (2.2), (2.35), (2.44) and the Young's inequality, we have (2.13).

Finally, we prove (2.12). We begin by proving that

$$\alpha^2(\partial_x v_\varepsilon(t,0)^2) \leq C(T)\left(1 + \|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))}\right). \tag{2.46}$$

Multiplying (2.33) by $-2\alpha\partial_x v_\varepsilon$, we have that

$$-2\alpha^2\partial_x v_\varepsilon\partial_x^2 v_\varepsilon - 2\alpha\gamma v_\varepsilon\partial_x v_\varepsilon = -2\alpha\kappa^2 u_\varepsilon^2\partial_x v_\varepsilon. \tag{2.47}$$

Observe that, thanks to (1.8), we have that

$$\begin{aligned}-2\alpha^2\int_0^\infty \partial_x v_\varepsilon\partial_x^2 v_\varepsilon dx &= \alpha^2(\partial_x v_\varepsilon(t,0))^2,\\ -2\alpha\gamma\int_0^\infty v_\varepsilon\partial_x v_\varepsilon dx &= \alpha\gamma v_\varepsilon^2(t,0) = \alpha\gamma h_\varepsilon^2(t,0).\end{aligned} \tag{2.48}$$

Therefore, integrating (2.47) on $(0,\infty)$, by (1.16), (2.2), (2.7) and (2.48), we get

$$\begin{aligned}\alpha^2(\partial_x v_\varepsilon(t,0))^2 &= -\alpha\kappa\int_0^\infty u_\varepsilon^2\partial_x v_\varepsilon dx - \alpha\gamma h_\varepsilon^2(t,0)\\ &\leq |\alpha\kappa|\int_0^\infty u_\varepsilon^2|\partial_x v_\varepsilon|dx + C\\ &\leq |\alpha\kappa|\,\|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))}\,\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + C\\ &\leq C(T)\left(1 + \|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))}\right),\end{aligned}$$

which gives (2.46).

Now, we prove (2.12). Multiplying (2.33) by $2\alpha\partial_x v_\varepsilon$, thanks to (2.1), an integration on $(0,x)$ gives

$$\alpha^2(\partial_x v_\varepsilon(t,x))^2 = 2\alpha\kappa^2\int_0^x u_\varepsilon^2\partial_x v_\varepsilon dx + \alpha^2(\partial_x v_\varepsilon(t,0))^2 - \alpha\gamma v_\varepsilon^2(t,x) + \alpha\gamma h_\varepsilon^2(t,0).$$

Therefore, by (2.2), (2.7), (2.14) and (2.46),

$$\begin{aligned}\alpha^2(\partial_x v_\varepsilon(t,x))^2 \le& 2|\alpha\kappa| \int_0^x u_\varepsilon^2 |\partial_x v_\varepsilon| dx + \alpha^2(\partial_x v_\varepsilon(t,0))^2 \\ &+ |\alpha\gamma| v_\varepsilon^2(t,x) + |\alpha\gamma| h_\varepsilon^2(t,0) \\ \le& 2|\alpha\kappa| \int_0^\infty u_\varepsilon^2 |\partial_x v_\varepsilon| dx + \alpha^2(\partial_x v_\varepsilon(t,0))^2 \\ &+ |\alpha\gamma| v_\varepsilon^2(t,x) + |\alpha\gamma| h_\varepsilon^2(t,0) \\ \le& 2|\alpha\kappa| \, \|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))} \|u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} \\ &+ C(T)\left(1 + \|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))}\right) + |\alpha\gamma| \, \|v_\varepsilon\|^2_{L^\infty((0,T)\times(0,\infty))} \\ \le& C(T)\left(1 + \|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))}\right).\end{aligned}$$

Therefore, we have (2.32), which gives (2.12). □

Lemma 2.5 *Assume* (1.15)*, or* (1.16)*. Define*

$$F_\varepsilon(t,x) = \int_0^x P_\varepsilon(t,y)dy, \qquad t,\ x \ge 0. \tag{2.49}$$

Then for every $t \ge 0$ *we have*

$$bF_\varepsilon(t,\infty) = b\int_0^\infty P_\varepsilon(t,x)dx = q^2 u_\varepsilon(t,0)h_\varepsilon(t) + \varepsilon g_\varepsilon(t), \tag{2.50}$$

$$F_\varepsilon^2(t,\infty) \le C u_\varepsilon^2(t,0) + C, \tag{2.51}$$

where

$$F_\varepsilon(t,\infty) = \lim_{x\to\infty} F_\varepsilon(t,x).$$

Proof Integrating the first equation of (2.1) on $(0,x)$, thanks to (2.1) and (2.49), we have that

$$\begin{aligned}bF_\varepsilon(t,x) =& \int_0^x \partial_t u_\varepsilon(t,y)dy - q^2 u_\varepsilon(t,x)v_\varepsilon(t,x) + q^2 u_\varepsilon(t,0)h_\varepsilon(t) \\ &- \varepsilon\partial_x u_\varepsilon(t,x) + \varepsilon g_\varepsilon(t),\end{aligned} \tag{2.52}$$

for every $x \ge 0$. Observe that, differentiating (2.4) with respect to t, we have that

$$\frac{d}{dt}\int_0^\infty u_\varepsilon(t,x)dx = \int_0^\infty \partial_t u_\varepsilon(t,0) = 0, \tag{2.53}$$

therefore (2.50) follows from (2.52) and the fact that $u_\varepsilon,\ v_\varepsilon,\ \partial_x v_\varepsilon$ vanish at infinity.

Finally, we prove (2.51). Since $0 < \varepsilon < 1$, by (2.2) and the Young's inequality,

$$\begin{aligned}b^2 F_\varepsilon^2(t,\infty) =& \left(q^2 u_\varepsilon(t,0)h_\varepsilon(t) + \varepsilon \ge (t)\right)^2 \le 2q^4 u_\varepsilon^2(t,0)h_\varepsilon^2(t) + 2\varepsilon^2 g_\varepsilon^2(t) \\ \le& C u_\varepsilon^2(t,0) + 2g_\varepsilon^2(t) \le C u_\varepsilon^2(t,0) + C,\end{aligned}$$

which gives (2.51). □

Lemma 2.6 *Fix* $T > 0$ *and assume* (1.15)*, or* (1.16)*. There exists a constant* $C(T) > 0$*, independent on* ε*, such that*

$$\|P_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + 2e^{C(T)t}\varepsilon\int_0^t e^{-C(T)s}\|u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)}ds \le C(T), \tag{2.54}$$

for every $0 \leq t \leq T$. *Moreover, we have that*

$$\|P_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))} \leq C(T). \tag{2.55}$$

Proof Let $0 \leq t \leq T$ and $x \geq 0$. We begin by observing that, differentiating (2.5) with respect to t, we have that

$$\partial_t P_\varepsilon(t,x) = \frac{d}{dt}\int_0^x u_\varepsilon(t,y)dy = \int_0^x \partial_t u_\varepsilon(t,y)dy. \tag{2.56}$$

It follows from (2.52) and (2.56) that

$$\partial_t P_\varepsilon - q^2 u_\varepsilon v_\varepsilon + q^2 u_\varepsilon(t,0)h_\varepsilon(t) - \varepsilon\partial_x u_\varepsilon(t,x) + \varepsilon \geq (t) = bF_\varepsilon. \tag{2.57}$$

Observe that, thanks to (2.1) and (2.49),

$$\begin{aligned}
-2q^2\int_0^\infty P_\varepsilon u_\varepsilon v_\varepsilon dx &= -2q^2\int_0^\infty P_\varepsilon \partial_x P_\varepsilon v_\varepsilon dx = q^2\int_0^\infty \partial_x v_\varepsilon P_\varepsilon^2 dx,\\
-2\varepsilon\int_0^\infty P_\varepsilon\partial_x u_\varepsilon(t,x) &= 2\varepsilon\int_0^\infty \partial_x P_\varepsilon u_\varepsilon dx = 2\varepsilon\|u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)},\\
2b\int_0^\infty F_\varepsilon P_\varepsilon dx &= 2b\int_0^\infty F_\varepsilon\partial_x F_\varepsilon dx = bF_\varepsilon^2(t,\infty).
\end{aligned} \tag{2.58}$$

Therefore, thanks to (2.49), (2.50) and (2.58), multiplying (2.57) by $2P_\varepsilon$, an integration on $(0,\infty)$ gives

$$\begin{aligned}
&\frac{d}{dt}\|P_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + 2\varepsilon\|u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\\
&\quad = -q^2\int_0^\infty \partial_x v_\varepsilon P_\varepsilon^2 dx + bF_\varepsilon^2(t,\infty)\\
&\qquad - q^2 u_\varepsilon(t,0)h_\varepsilon(t)F_\varepsilon(t,\infty) - \varepsilon g_\varepsilon(t)F_\varepsilon(t,\infty).
\end{aligned} \tag{2.59}$$

Since $0 < \varepsilon < 1$, due to Lemma 2.4, (2.2), (2.51) and the Young's inequality,

$$\begin{aligned}
q^2\int_0^\infty |\partial_x v_\varepsilon|P_\varepsilon^2 dx &\leq q^2\|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))}\|P_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\\
&\leq C(T)\|P_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)},\\
|b|F_\varepsilon^2(t,\infty) &\leq Cu_\varepsilon^2(t,0) + C,\\
q^2|u_\varepsilon(t,0)||h_\varepsilon(t)||F_\varepsilon(t,\infty)| &\leq C|u_\varepsilon(t,0)||F_\varepsilon(t,\infty)| \leq Cu_\varepsilon^2(t,0) + CF_\varepsilon^2(t,\infty)\\
&\leq Cu_\varepsilon^2(t,0) + C,\\
\varepsilon| \geq (t)||F_\varepsilon(t,\infty)| &\leq C|F_\varepsilon(t,\infty)| \leq Cu_\varepsilon^2(t,0) + C.
\end{aligned}$$

Consequently, by (2.59),

$$\frac{d}{dt}\|P_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + 2\varepsilon\|u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} \leq C(T)\|P_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + Cu_\varepsilon^2(t,0) + C.$$

The Gronwall Lemma and (2.7) give

$$\begin{aligned}
\|P_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + 2e^{C(T)t}\varepsilon\int_0^t e^{-C(T)s}\|u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)}ds\\
\leq C + Ce^{C(T)t}\int_0^t e^{-C(T)s}u_\varepsilon^2(s,0)ds\\
\leq C(T) + C(T)\int_0^t u_\varepsilon^2(s,0)ds \leq C(T),
\end{aligned}$$

that is (2.54).

Finally, we prove (2.55). Due to (2.1), (2.7), (2.54) and the Hölder inequality, for every $x \geq 0$

$$\begin{aligned} P_\varepsilon^2(t,x) &= 2\int_0^x P_\varepsilon \partial_x P_\varepsilon dy = 2\int_0^x P_\varepsilon u_\varepsilon dy \leq 2\int_0^\infty |P_\varepsilon||u_\varepsilon| dx \\ &\leq 2\left\| P_\varepsilon(t,\cdot)\right\|_{L^2(0,\infty)} \left\| u_\varepsilon(t,\cdot)\right\|_{L^2(0,\infty)} \leq C(T). \end{aligned}$$

Hence,

$$\| P_\varepsilon\|^2_{L^\infty((0,T)\times(0,\infty))} \leq C(T),$$

which gives (2.55). □

Lemma 2.7 *Fix $T > 0$ and assume (1.15), or (1.16). There exists a constant $C(T) > 0$, independent on ε, such that*

$$\|u_\varepsilon(t,\cdot)\|_{L^\infty(0,\infty)} \leq C(T)\sqrt{\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}}, \tag{2.60}$$

$$\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|_{L^\infty(0,\infty)} \leq C(T)\left(1 + \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}\right), \tag{2.61}$$

for every $0 \leq t \leq T$.

Proof Let $0 \leq t \leq T$. We prove (2.60). We begin by observing that, by (2.7) and the Hölder inequality,

$$\begin{aligned} u_\varepsilon^2(t,x) &= 2\int_0^x u_\varepsilon \partial_x u_\varepsilon dy \leq 2\int_0^\infty |u_\varepsilon||\partial_x u_\varepsilon| dx \\ &\leq 2\|u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \leq C(T)\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}. \end{aligned}$$

Hence,

$$\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(0,\infty)} \leq C(T)\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)},$$

which gives (2.60).

Finally, we prove (2.61). By the third equation of (2.1) and Lemma 2.4, we have that

$$\begin{aligned} |\alpha||\partial_x^2 v_\varepsilon| &= \left|\kappa u_\varepsilon^2 - \beta\partial_x v_\varepsilon - \gamma v_\varepsilon\right| \\ &\leq |\kappa| u_\varepsilon^2 + |\beta||\partial_x v_\varepsilon| + |\gamma||v_\varepsilon| \\ &\leq |\kappa|\,\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(0,\infty)} + |\beta|\,\|\partial_x v_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} + \gamma\,\|v_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \\ &\leq C(T)\left(1 + \|u_\varepsilon(t,\cdot)\|^2_{L^\infty(0,\infty)}\right). \end{aligned}$$

Therefore, since $\alpha \neq 0$ (see (1.14)),

$$\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|_{L^\infty(0,\infty)} \leq C(T)\left(1 + \|u_\varepsilon(t,\cdot)\|^2_{L^\infty(0,\infty)}\right). \tag{2.62}$$

(2.61) follows from (2.60) and (2.62). □

Following [18], or [33], in order to work with homogeneous boundary conditions we define

$$W_2(t,x) = u_\varepsilon(t,x) + g_\varepsilon(t)e^{-x}, \quad t,\ x \geq 0. \tag{2.63}$$

Thanks to (2.63), we have that

$$\begin{aligned} \partial_t W_2(t,x) &= \partial_t u_\varepsilon(t,x) + g'_\varepsilon(t)e^{-x},\\ \partial_x W_2(t,x) &= \partial_x u_\varepsilon(t,x) - g_\varepsilon(t)e^{-x},\\ \partial_x^2 W_2(t,x) &= \partial_x^2 u_\varepsilon(t,x) + g_\varepsilon(t)e^{-x}. \end{aligned} \tag{2.64}$$

Moreover, by (2.1), (2.2) and (2.64),

$$\partial_x W_2(t,0) = \partial_x u_\varepsilon(t,0) - g_\varepsilon(t) = 0, \quad \|W_2(0,\cdot)\|_{L^2(0,\infty)} \le C. \tag{2.65}$$

Thanks to (2.63) and (2.64), the first equation of (2.1) reads

$$\begin{aligned} &\partial_t W_2 - q^2 v_\varepsilon \partial_x W_2 - q^2 W_2 \partial_x v_\varepsilon \\ &\quad = bP_\varepsilon + \varepsilon\partial_x^2 W_2 - \varepsilon g_\varepsilon(t)e^{-x} + g'_\varepsilon(t)e^{-x} + q^2 g_\varepsilon(t)e^{-x} v_\varepsilon - q^2 g_\varepsilon(t)e^{-x}\partial_x v_\varepsilon. \end{aligned} \tag{2.66}$$

We prove the following lemma.

Lemma 2.8 *Fix $T > 0$ and assume (1.15), or (1.16). There exists a constant $C(T) > 0$, independent on ε, such that*

$$\|W_2(t,\cdot)\|_{L^2(0,\infty)} \le C(T), \tag{2.67}$$

$$\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \le C\left(\|\partial_x W_2(t,\cdot)\|_{L^2(0,\infty)} + 1\right), \tag{2.68}$$

$$\|u_\varepsilon(t,\cdot)\|_{L^\infty(0,\infty)} \le C(T)\sqrt{1 + \|\partial_x W_2(t,\cdot)\|_{L^2(0,\infty)}}, \tag{2.69}$$

$$\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|_{L^\infty(0,\infty)} \le C(T)\left(1 + \|\partial_x W_2(t,\cdot)\|_{L^2(0,\infty)}\right), \tag{2.70}$$

for every $0 \le t \le T$.

Proof Let $0 \le t \le T$ and $x \ge 0$. We prove (2.67). We begin by observing that, by (2.2), (2.63) and the Young's inequality,

$$W_2^2(t,x) = \left(u_\varepsilon(t,x) + g_\varepsilon(t)e^{-x}\right)^2 \le u_\varepsilon^2(t,x) + 4g_\varepsilon^2(t)e^{-2x} \le u_\varepsilon^2(t,x) + Ce^{-2x}.$$

Integrating on $(0,\infty)$, by (2.7), we get

$$\|W_2(t,\cdot)\|^2_{L^2(0,\infty)} \le \|u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + C\int_0^\infty e^{-2x}dx \le C(T) + C \le C(T),$$

which gives (2.67).

We prove (2.68). By (2.2), (2.64) and the Young's inequality,

$$\begin{aligned} (\partial_x u_\varepsilon(t,x))^2 &= \left(\partial_x W_2(t,x) + g_\varepsilon(t)e^{-x}\right)^2 \le (\partial_x W_2(t,x))^2 + g_\varepsilon^2(t)e^{-2x}\\ &\le (\partial_x W_2(t,x))^2 + Ce^{-2x}. \end{aligned}$$

(2.41) and an integration on $(0,\infty)$ gives

$$\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} \le C\left(\|\partial_x W_2(t,\cdot)\|^2_{L^2(0,\infty)} + 1\right)$$

Therefore, we get

$$\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \le C\left(\|\partial_x W_2(t,\cdot)\|_{L^2(0,\infty)} + 1\right),$$

which gives (2.68).

Finally, (2.69) follows from (2.60) and (2.68), while (2.61) and (2.68) give (2.70). □

Lemma 2.9 *Fix $T > 0$ and assume (1.15), or (1.16). There exists a constant $C(T) > 0$, independent on ε, such that*

$$\|\partial_x W_2(t,\cdot)\|^2_{L^2(0,\infty)} + 2\varepsilon e^{C(T)t}\int_0^t e^{-C(T)s}\left\|\partial_x^2 W_2(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \le C(T), \tag{2.71}$$

$$\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)},\ \|u_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))},\ \left\|\partial_x^2 v_\varepsilon\right\|_{L^\infty((0,T)\times(0,\infty))} \le C(T), \tag{2.72}$$

$$\varepsilon\int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \le C(T), \tag{2.73}$$

for every $0 \le t \le T$.

Proof Let $0 \le t \le T$. We begin by proving (2.71). Multiplying (2.66) by $-2\partial_x^2 W_2$, thanks to (2.1) and (2.65), an integration on $(0,\infty)$ gives

$$\begin{aligned}
-2\int_0^\infty \partial_x^2 W_2 \partial_t W_2 dx &= \frac{d}{dt}\|\partial_x W_2(t,\cdot)\|^2_{L^2(0,\infty)}\\
&= -2q^2\int_0^\infty v_\varepsilon \partial_x W_2 \partial_x^2 W_2 dx - 2q^2\int_0^\infty W_2\partial_x^2 W_2 \partial_x v_\varepsilon dx\\
&\quad - 2b\int_0^\infty P_\varepsilon \partial_x^2 W_2 dx - 2\varepsilon\left\|\partial_x^2 W_2(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad + 2\varepsilon g_\varepsilon(t)\int_0^\infty e^{-x}\partial_x^2 W_2 dx - 2g'_\varepsilon(t)\int_0^\infty e^{-x}\partial_x^2 W_2 dx\\
&\quad - 2q^2 g_\varepsilon(t)\int_0^\infty e^{-x} v_\varepsilon \partial_x^2 W_2 dx + 2q^2 g_\varepsilon(t)\int_0^\infty e^{-x}\partial_x v_\varepsilon \partial_x^2 W_2 dx\\
&= 3q^2\int_0^\infty \partial_x v_\varepsilon (\partial_x W_2)^2 dx + 2q^2\int_0^\infty W_2\partial_x W_2 \partial_x^2 v_\varepsilon dx\\
&\quad + 2b\int_0^\infty u_\varepsilon \partial_x W_2 dx - 2\varepsilon\left\|\partial_x^2 W_2(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad + 2\varepsilon g_\varepsilon(t)\int_0^\infty e^{-x}\partial_x W_2 dx - 2g'_\varepsilon(t)\int_0^\infty e^{-x}\partial_x W_2 dx\\
&\quad + 2q^2 g_\varepsilon(t)\int_0^\infty e^{-x} v_\varepsilon \partial_x W_2 dx + 4q^2 g_\varepsilon(t)\int_0^\infty e^{-x}\partial_x v_\varepsilon \partial_x W_2 dx\\
&\quad - 2q^2 g_\varepsilon(t)\int_0^\infty e^{-x}\partial_x^2 v_\varepsilon \partial_x W_2 dx.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x W_2(t,\cdot)\|^2_{L^2(0,\infty)} + 2\varepsilon\left\|\partial_x^2 W_2(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad = 3q^2\int_0^\infty \partial_x v_\varepsilon (\partial_x W_2)^2 dx + 2q^2\int_0^\infty W_2\partial_x W_2 \partial_x^2 v_\varepsilon dx\\
&\qquad + 2b\int_0^\infty u_\varepsilon \partial_x W_2 dx - 2\varepsilon g_\varepsilon(t)\int_0^\infty e^{-x}\partial_x W_2 dx\\
&\qquad - 2g'_\varepsilon(t)\int_0^\infty e^{-x}\partial_x W_2 dx - 2q^2 g_\varepsilon(t)\int_0^\infty e^{-x} v_\varepsilon \partial_x W_2 dx\\
&\qquad + 4q^2 g_\varepsilon(t)\int_0^\infty e^{-x}\partial_x v_\varepsilon \partial_x W_2 dx - 2q^2 g_\varepsilon(t)\int_0^\infty e^{-x}\partial_x^2 v_\varepsilon \partial_x W_2 dx.
\end{aligned} \tag{2.74}$$

Since $0 < \varepsilon < 1$, using Lemma 2.4, (2.2), (2.7), (2.41), (2.67), (2.70) and the Young's inequality, we estimate separaltely each term in (2.74)

$$
\begin{aligned}
&3q^2\int_0^\infty |\partial_x v_\varepsilon|(\partial_x W_2)^2dx \le 3q^2\left\|\partial_x v_\varepsilon\right\|_{L^\infty((0,T)\times(0,\infty))}\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le C(T)\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2q^2\int_0^\infty |W_2\partial_x^2 v_\varepsilon||\partial_x W_2|dx \le \int_0^\infty W_2^2(\partial_x^2 v_\varepsilon)^2dx + q^4\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le \left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^\infty(0,\infty)}\left\|W_2(t,\cdot)\right\|^2_{L^2(0,\infty)} + q^4\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le C(T)\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^\infty(0,\infty)} + q^4\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le C(T)\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)} + C(T),\\
&2|b|\int_0^\infty |u_\varepsilon||\partial_x W_2|dx \le b^2\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + \left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le C(T) + \left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2(\varepsilon|g_\varepsilon(t)| + |g'_\varepsilon(t)|)\int_0^\infty e^{-x}|\partial_x W_2|dx \le 2C\int_0^\infty e^{-x}|\partial_x W_2|dx\\
&\quad\le C + C\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2q^2|g_\varepsilon(t)|\int_0^\infty e^{-x}|v_\varepsilon||\partial_x W_2|dx \le 2C\int_0^\infty e^{-x}|v_\varepsilon||\partial_x W_2|dx\\
&\quad\le C\int_0^\infty e^{-2x}v_\varepsilon^2dx + C\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le C\left\|v_\varepsilon\right\|^2_{L^\infty((0,T)\times(0,\infty))}\int_0^\infty e^{-2x}dx + C\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le C(T) + C\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&4q^2|g_\varepsilon(t)|\int_0^\infty e^{-x}|\partial_x v_\varepsilon||\partial_x W_2|dx \le 2C\int_0^\infty e^{-x}|\partial_x v_\varepsilon||\partial_x W_2|\\
&\quad\le C\int_0^\infty e^{-2x}dx + \int_0^\infty (\partial_x v_\varepsilon)^2(\partial_x W_2)^2dx\\
&\quad\le C + \left\|\partial_x v_\varepsilon\right\|^2_{L^\infty((0,T)\times(0,\infty))}\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C + C(T)\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&2q^2|g_\varepsilon(t)|\int_0^\infty e^{-x}|\partial_x^2 v_\varepsilon||\partial_x W_2|dx \le 2C\int_0^\infty e^{-x}|\partial_x^2 v_\varepsilon||\partial_x W_2|dx\\
&\quad\le C\int_0^\infty e^{-2x}(\partial_x^2 v_\varepsilon)^2dx + C\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le C\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^\infty(0,\infty)} + C\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le C(T)\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)} + C(T).
\end{aligned}
$$

It follows from (2.74) that

$$
\frac{d}{dt}\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2\varepsilon\left\|\partial_x^2 W_2(t,\cdot)\right\|^2_{L^2(0,\infty)} \le C(T)\left\|\partial_x W_2(t,\cdot)\right\|^2_{L^2(0,\infty)} + C(T).
$$

The Gronwall Lemma and (2.65) give

$$\begin{aligned}\|\partial_x W_2(t,\cdot)\|^2_{L^2(0,\infty)} &+ 2\varepsilon e^{C(T)t}\int_0^t e^{-C(T)s}\left\|\partial_x^2 W_2(s,\cdot)\right\|^2_{L^2(0,\infty)} ds\\ &\le C + C(T)e^{C(T)t}\int_0^t e^{-C(T)s} ds \le C(T),\end{aligned}$$

that is (2.71).

(2.72), follows from (2.68), (2.69), (2.70) and (2.71).

Finally, we prove (2.73). By (2.2), (2.64) and the Young's inequality, we have that

$$\begin{aligned}(\partial_x^2 u_\varepsilon(t,x))^2 = \left(\partial_x^2 W_2(t,x) - g_\varepsilon(t)e^{-x}\right)^2 &\le 2(\partial_x^2 W_2(t,x))^2 + 2g_\varepsilon^2(t)e^{-2x}\\ &\le 2(\partial_x^2 W_2(t,x))^2 + Ce^{-2x}.\end{aligned} \tag{2.75}$$

Multiplying (2.75) by ε and integrating on $(0,\infty)$, thanks to (2.41), we have that

$$\begin{aligned}\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} &\le 2\varepsilon\left\|\partial_x^2 W_2(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2C\varepsilon\int_0^\infty e^{-2x}dx\\ &\le 2\varepsilon\left\|\partial_x^2 W_2(t,\cdot)\right\|^2_{L^2(0,\infty)} + C\varepsilon.\end{aligned}$$

Being $0<\varepsilon<1$, integrating on $(0,t)$, by (2.71), we get

$$\begin{aligned}\varepsilon\int_0^t\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(0,\infty)} ds &\le 2\varepsilon\int_0^t\left\|\partial_x^2 W_2(s,\cdot)\right\|^2_{L^2(0,\infty)} ds + \varepsilon Ct\\ &\le 2\varepsilon e^{C(T)t}\int_0^t e^{-C(T)s}\left\|\partial_x^2 W_2(s,\cdot)\right\|^2_{L^2(0,\infty)} ds + Ct\\ &\le C(T),\end{aligned}$$

which gives (2.73). □

Lemma 2.10 *Fix $T>0$ and assume (1.15), or (1.16). There exists a constant $C(T)>0$, independent on ε, such that*

$$\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|_{L^2(0,\infty)} \le C(T), \tag{2.76}$$

$$\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|_{L^2(0,\infty)} \le C(T), \tag{2.77}$$

for every $0\le t\le T$.

Proof Let $0\le t\le T$. Multiplying the third equation of (2.1) by $2\alpha\partial_x^2 v_\varepsilon$, an integration on $(0,\infty)$ gives

$$\begin{aligned}2\alpha^2\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} =& 2\kappa\alpha\int_0^\infty u_\varepsilon^2\partial_x^2 v_\varepsilon dx - 2\beta\alpha\int_0^\infty \partial_x v_\varepsilon\partial_x^2 v_\varepsilon dx\\ &- 2\gamma\alpha\int_0^\infty v_\varepsilon\partial_x^2 v_\varepsilon dx.\end{aligned} \tag{2.78}$$

Due to Lemma 2.4, (2.7), (2.72) and the Young's inequality, we obtain

$$\begin{aligned}2|\kappa||\alpha|\int_0^\infty u_\varepsilon^2|\partial_x^2 v_\varepsilon|dx &\le 2\kappa^2\int_0^\infty u_\varepsilon^4 dx + \frac{\alpha^2}{2}\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &\le 2\kappa^2\|u_\varepsilon\|^2_{L^\infty((0,T)\times(0,\infty))}\|u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\alpha^2}{2}\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\end{aligned}$$

$$\leq C(T) + \frac{\alpha^2}{2} \left\| \partial_x^2 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)},$$

$$2|\beta||\alpha| \int_0^\infty |\partial_x v_\varepsilon||\partial_x^2 v_\varepsilon| dx \leq 2\beta^2 \left\| \partial_x v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} + \frac{\alpha^2}{2} \left\| \partial_x^2 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)}$$

$$\leq C(T) + \frac{\alpha^2}{2} \left\| \partial_x^2 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)},$$

$$2|\gamma||\alpha| \int_0^\infty |v_\varepsilon||\partial_x^2 v_\varepsilon| dx \leq 2\gamma^2 \left\| v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} + \frac{\alpha^2}{2} \left\| \partial_x^2 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)}$$

$$\leq C(T) + \frac{\alpha^2}{2} \left\| \partial_x^2 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)}.$$

Consequently, by (2.78),

$$\frac{\alpha^2}{2} \left\| \partial_x^2 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} \leq C(T), \tag{2.79}$$

which gives (2.76).

Finally, we prove (2.77). Differentiating the third equation of (2.1) with respect to x, we have that

$$\alpha \partial_x^3 v_\varepsilon + \beta \partial_x^2 v_\varepsilon + \gamma \partial_x v_\varepsilon = 2\kappa u_\varepsilon \partial_x u_\varepsilon. \tag{2.80}$$

Multiplying (2.80) by $2\alpha \partial_x^3 v_\varepsilon$, an integration on $(0,\infty)$ gives

$$\begin{aligned} 2\alpha^2 \left\| \partial_x^3 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} =& 4\alpha\kappa \int_0^\infty u_\varepsilon \partial_x u_\varepsilon \partial_x^3 v_\varepsilon dx - 2\alpha\beta \int_0^\infty \partial_x^2 v_\varepsilon \partial_x^3 v_\varepsilon dx \\ &- 2\alpha\gamma \int_0^\infty \partial_x v_\varepsilon \partial_x^3 v_\varepsilon dx. \end{aligned} \tag{2.81}$$

Due to Lemma 2.4, (2.72) and the Young's inequality,

$$4|\alpha||\kappa| \int_0^\infty |u_\varepsilon \partial_x u_\varepsilon||\partial_x^3 v_\varepsilon| dx \leq 8\kappa^2 \int_0^\infty u_\varepsilon^2 (\partial_x u_\varepsilon)^2 dx + \frac{\alpha^2}{2} \left\| \partial_x^3 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)}$$

$$\leq 8\kappa^2 \left\| u_\varepsilon \right\|^2_{L^\infty((0,\infty)\times(0,\infty))} \left\| \partial_x u_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} + \frac{\alpha^2}{2} \left\| \partial_x^3 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)}$$

$$\leq C(T) + \frac{\alpha^2}{2} \left\| \partial_x^3 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)},$$

$$2|\alpha||\beta| \int_0^\infty |\partial_x^2 v_\varepsilon||\partial_x^3 v_\varepsilon| dx \leq 4\beta^2 \left\| \partial_x^2 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} + \frac{\alpha^2}{2} \left\| \partial_x^3 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)}$$

$$\leq C(T) + \frac{\alpha^2}{2} \left\| \partial_x^3 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)},$$

$$2|\alpha||\gamma| \int_0^\infty |\partial_x v_\varepsilon||\partial_x^3 v_\varepsilon| dx \leq 2\gamma^2 \left\| \partial_x v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} + \frac{\alpha^2}{2} \left\| \partial_x^3 u_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)}$$

$$\leq C(T) + \frac{\alpha^2}{2} \left\| \partial_x^3 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)}.$$

Therefore, by (2.81),

$$\frac{\alpha^2}{2} \left\| \partial_x^3 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} \leq C(T),$$

which gives (2.77). □

Following [26, Lemma 3.2], or [40, Lemma 2.2], we prove the following H^2 estimate on u_ε.

Lemma 2.11 *Fix $T > 0$ and assume (1.15), or (1.16). There exists a constant $C(T) > 0$, independent on ε, such that*

$$\begin{aligned}&\varepsilon^2 \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + 3\varepsilon^2 \left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &\quad+\frac{\varepsilon}{2}\int_0^t \left\|\partial_t\partial_x u_\varepsilon(s,\cdot)\right\|^2_{L^2(0,\infty)} ds + \varepsilon\int_0^t \left\|\partial_t u_\varepsilon(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \le C(T),\end{aligned} \tag{2.82}$$

for every $0 \le t \le T$.

Proof Let $0 \le t \le T$ and $x \ge 0$. Let A be a positive constant, which will be specified later. Multiplying the first equation of (2.1) by

$$-2\varepsilon\partial_t\partial_x^2 u_\varepsilon + 2A\varepsilon\partial_t u_\varepsilon$$

we have that

$$\begin{aligned}&\left(-2\varepsilon\partial_t\partial_x^2 u_\varepsilon + 2A\varepsilon\partial_t u_\varepsilon\right)\partial_t u_\varepsilon - q^2\left(-2\varepsilon\partial_t\partial_x^2 u_\varepsilon + 2A\varepsilon\partial_t u_\varepsilon\right) v_\varepsilon\partial_x u_\varepsilon\\ &\qquad - q^2\left(-2\varepsilon\partial_t\partial_x^2 u_\varepsilon + 2A\varepsilon\partial_t u_\varepsilon\right) u_\varepsilon\partial_x v_\varepsilon - b\left(-2\varepsilon\partial_t\partial_x^2 u_\varepsilon + 2A\varepsilon\partial_t u_\varepsilon\right) P_\varepsilon\\ &\quad= \varepsilon\left(-2\varepsilon\partial_t\partial_x^2 u_\varepsilon + 2A\varepsilon\partial_t u_\varepsilon\right)\partial_x^2 u_\varepsilon.\end{aligned} \tag{2.83}$$

Observe that, since the traces of $\partial_t\partial_x u_\varepsilon$ and P at $x = 0$ are g'_ε and 0 respectively, by (2.1), we have that

$$\begin{aligned}&\int_0^\infty \left(-2\varepsilon\partial_t\partial_x^2 u_\varepsilon + A\varepsilon\partial_t u_\varepsilon\right)\partial_t u_\varepsilon dx\\ &\quad= 2\varepsilon g'_\varepsilon(t)\partial_t u_\varepsilon(t,0) + 2\varepsilon\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2A\varepsilon\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)},\end{aligned} \tag{2.84}$$

$$\begin{aligned}&-q^2\int_0^\infty \left(-2\varepsilon\partial_t\partial_x^2 u_\varepsilon + 2A\varepsilon\partial_t u_\varepsilon\right) v_\varepsilon\partial_x u_\varepsilon dx\\ &\quad= -q^2\varepsilon h_\varepsilon(t) g'_\varepsilon(t) g_\varepsilon(t) - 2q^2\varepsilon\int_0^\infty \partial_x v_\varepsilon\partial_x u_\varepsilon\partial_t\partial_x u_\varepsilon dx\\ &\qquad - 2q^2\varepsilon\int_0^\infty v_\varepsilon\partial_x^2 u_\varepsilon\partial_t\partial_x u_\varepsilon dx - 2q^2 A\varepsilon\int_0^\infty v_\varepsilon\partial_x u_\varepsilon\partial_t u_\varepsilon dx,\end{aligned}$$

$$\begin{aligned}&-q^2\int_0^\infty \left(-2\varepsilon\partial_t\partial_x^2 u_\varepsilon + 2A\varepsilon\partial_t u_\varepsilon\right) u_\varepsilon\partial_x v_\varepsilon dx\\ &\quad= -2q^2\varepsilon u_\varepsilon(t,0)\partial_x v_\varepsilon(t,0) g'_\varepsilon(t) - 2q^2\varepsilon\int_0^\infty \partial_x u_\varepsilon\partial_x v_\varepsilon\partial_t\partial_x u_\varepsilon dx\\ &\qquad - 2q^2\varepsilon\int_0^\infty u_\varepsilon\partial_x^2 v_\varepsilon\partial_t\partial_x u_\varepsilon dx - 2q^2 A\varepsilon\int_0^\infty u_\varepsilon\partial_x v_\varepsilon\partial_t u_\varepsilon dx,\end{aligned}$$

$$\begin{aligned}&-b\int_0^\infty \left(-2\varepsilon\partial_t\partial_x^2 u_\varepsilon + 2A\varepsilon\partial_t u_\varepsilon\right) P_\varepsilon dx\\ &\quad= -2b\varepsilon\int_0^\infty \partial_x P_\varepsilon\partial_t\partial_x u_\varepsilon dx - 2bA\varepsilon\int_0^\infty P_\varepsilon\partial_t u_\varepsilon dx\\ &\quad= -2b\varepsilon\int_0^\infty u_\varepsilon\partial_t\partial_x u_\varepsilon dx - 2bA\varepsilon\int_0^\infty P_\varepsilon\partial_t u_\varepsilon dx,\end{aligned}$$

$$\begin{aligned}&\varepsilon\int_0^\infty \left(-2\varepsilon\partial_t\partial_x^2 u_\varepsilon + 2A\varepsilon\partial_t u_\varepsilon\right)\partial_x^2 u_\varepsilon dx\\ &\quad= -\frac{d}{dt}\left(\varepsilon^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + A\varepsilon^2\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\right)\end{aligned}$$

$$- 2A\varepsilon^2 g_\varepsilon(t)\partial_t u_\varepsilon(t,0).$$

Therefore, integrating (2.83) on $(0,\infty)$, thanks to (2.84), we have

$$\begin{aligned}
&\frac{d}{dt}\left(\varepsilon^2 \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + A\varepsilon^2 \left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\right)\\
&\qquad + 2\varepsilon \left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2A\varepsilon \left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} \qquad (2.85)\\
&\quad = -2\varepsilon g'_\varepsilon(t)\partial_t u_\varepsilon(t,0) + 2q^2\varepsilon h_\varepsilon(t) g'_\varepsilon(t) g_\varepsilon(t) + 4q^2\varepsilon\int_0^\infty \partial_x v_\varepsilon \partial_x u_\varepsilon \partial_t\partial_x u_\varepsilon dx\\
&\qquad + 2q^2\varepsilon\int_0^\infty v_\varepsilon \partial_x^2 u_\varepsilon \partial_t\partial_x u_\varepsilon dx + 2q^2A\varepsilon\int_0^\infty v_\varepsilon\partial_x u_\varepsilon\partial_t u_\varepsilon dx\\
&\qquad + 2q^2\varepsilon u_\varepsilon(t,0)\partial_x v_\varepsilon(t,0) g'_\varepsilon(t) + 2q^2\varepsilon\int_0^\infty u_\varepsilon\partial_x^2 v_\varepsilon\partial_t\partial_x u_\varepsilon dx\\
&\qquad + 2q^2A\varepsilon\int_0^\infty u_\varepsilon\partial_x v_\varepsilon\partial_t u_\varepsilon dx + 2b\varepsilon\int_0^\infty u_\varepsilon\partial_t\partial_x u_\varepsilon dx\\
&\qquad + 2bA\varepsilon\int_0^\infty P_\varepsilon\partial_t u_\varepsilon dx - 2A\varepsilon^2 g_\varepsilon(t)\partial_t u_\varepsilon(t,0).
\end{aligned}$$

Since $0<\varepsilon<1$, due to Lemma 2.4, (2.2), (2.7), (2.54), (2.72), and the Young's inequality, we can estimate all the previous terms in (2.85) as follows.

$$\begin{aligned}
&2\varepsilon|g'_\varepsilon(t)||\partial_t u_\varepsilon(t,0)| \le 2\varepsilon C|\partial_t u_\varepsilon(t,0)| \le C+\varepsilon^2(\partial_t u_\varepsilon(t,0))^2 \le C+\varepsilon(\partial_t u_\varepsilon(t,0))^2,\\
&q^2\varepsilon|h_\varepsilon(t)||g'_\varepsilon(t)||g_\varepsilon(t)| \le q^2|h_\varepsilon(t)||g'_\varepsilon(t)||g_\varepsilon(t)| \le C,\\
&4q^2\varepsilon\int_0^\infty |\partial_x v_\varepsilon\partial_x u_\varepsilon||\partial_t\partial_x u_\varepsilon|dx = 2\varepsilon\int_0^\infty \left|\frac{2q^2\partial_x v_\varepsilon\partial_x u_\varepsilon}{\sqrt{D_2}}\right|\left|\sqrt{D_2}\partial_t\partial_x u_\varepsilon dx\right|dx\\
&\quad\le \frac{4q^4\varepsilon}{D_2}\int_0^\infty(\partial_x v_\varepsilon)^2(\partial_x u_\varepsilon)^2dx + D_2\varepsilon\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le \frac{4q^4}{D_2}\left\|\partial_x v_\varepsilon\right\|^2_{L^\infty((0,T)\times(0,\infty))}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_2\varepsilon\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le \frac{C(T)}{D_2} + D_2\varepsilon\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2q^2\varepsilon\int_0^\infty|v_\varepsilon\partial_x^2 u_\varepsilon||\partial_t\partial_x u_\varepsilon|dx = 2\varepsilon\int_0^\infty\left|\frac{q^2 v_\varepsilon\partial_x^2 u_\varepsilon}{\sqrt{D_2}}\right|\left|\sqrt{D_2}\partial_t\partial_x u_\varepsilon\right|dx\\
&\quad\le\frac{q^4\varepsilon}{D_2}\int_0^\infty v_\varepsilon^2(\partial_x^2 u_\varepsilon)^2dx + D_2\varepsilon\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le\frac{q^4\varepsilon}{D_2}\left\|v_\varepsilon\right\|^2_{L^\infty((0,T)\times(0,\infty))}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_2\varepsilon\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le\frac{C(T)\varepsilon}{D_2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_2\varepsilon\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2q^2A\varepsilon\int_0^\infty|v_\varepsilon\partial_x u_\varepsilon||\partial_t u_\varepsilon|dx = 2A\varepsilon\int_0^\infty\left|\frac{q^2 v_\varepsilon\partial_x u_\varepsilon}{\sqrt{D_3}}\right|\left|\sqrt{D_3}\partial_t u_\varepsilon\right|dx\\
&\quad\le\frac{q^4A\varepsilon}{D_3}\int_0^\infty v_\varepsilon^2(\partial_x u_\varepsilon)^2dx + AD_3\varepsilon\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le\frac{q^4A}{D_3}\left\|v_\varepsilon\right\|^2_{L^\infty((0,\infty)\times(0,\infty))}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + AD_3\varepsilon\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C(T)A}{D_3} + AD_3\varepsilon \left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2q^2\varepsilon|u_\varepsilon(t,0)||\partial_x v_\varepsilon(t,0)||g'_\varepsilon(t)| \leq C|u_\varepsilon(t,0)||\partial_x v_\varepsilon(t,0)| \leq Cu^2_\varepsilon(t,0) + C(\partial_x v_\varepsilon(t,0))^2\\
&\leq Cu^2_\varepsilon(t,0) + C\left\|\partial_x v_\varepsilon(\cdot,0)\right\|^2_{L^\infty(0,T)} \leq Cu^2_\varepsilon(t,0) + C(T),\\
&2q^2\varepsilon\int_0^\infty |u_\varepsilon\partial_x^2 v_\varepsilon||\partial_t\partial_x u_\varepsilon|dx = 2\varepsilon\int_0^\infty \left|\frac{q^2u_\varepsilon\partial_x^2 v_\varepsilon}{\sqrt{D_2}}\right|\left|\sqrt{D_2}\partial_t\partial_x u_\varepsilon\right|dx\\
&\leq \frac{q^4\varepsilon}{D_2}\int_0^\infty u^2_\varepsilon(\partial_x^2 v_\varepsilon)^2dx + D_2\varepsilon\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\leq \frac{q^4}{D_2}\left\|\partial_x^2 v_\varepsilon\right\|^2_{L^\infty((0,T)\times(0,\infty))}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_2\varepsilon\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\leq \frac{C(T)}{D_2} + D_2\varepsilon\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2q^2A\varepsilon\int_0^\infty |u_\varepsilon\partial_x v_\varepsilon||\partial_t u_\varepsilon|dx = 2A\varepsilon\int_0^\infty \left|\frac{q^2u_\varepsilon\partial_x v_\varepsilon}{\sqrt{D_3}}\right|\left|\sqrt{D_3}\partial_t u_\varepsilon\right|dx\\
&\leq \frac{q^4A\varepsilon}{D_3}\int_0^\infty u^2_\varepsilon(\partial_x v_\varepsilon)^2dx + AD_3\varepsilon\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\leq \frac{q^4A}{D_3}\left\|\partial_x v_\varepsilon\right\|^2_{L^\infty((0,T)\times(0,\infty))}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + AD_3\varepsilon\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\leq \frac{AC(T)}{D_3} + AD_3\varepsilon\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2|b|\varepsilon\int_0^\infty |u_\varepsilon||\partial_t\partial_x u_\varepsilon|dx = 2\varepsilon\int_0^\infty\left|\frac{bu_\varepsilon}{\sqrt{D_2}}\right|\left|\sqrt{D_2}\partial_t\partial_x u_\varepsilon\right|dx\\
&\leq \frac{b^2\varepsilon}{D_2}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_2\varepsilon\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\leq \frac{C(T)}{D_2} + D_2\varepsilon\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2|b|A\varepsilon\int_0^\infty |P_\varepsilon||\partial_t u_\varepsilon|dx = 2A\varepsilon\int_0^\infty\left|\frac{bP_\varepsilon}{\sqrt{D_3}}\right|\left|\sqrt{D_3}\partial_t u_\varepsilon\right|dx\\
&\leq \frac{Ab^2\varepsilon}{D_3}\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + AD_3\varepsilon\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\leq \frac{AC(T)}{D_3} + AD_3\varepsilon\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2A\varepsilon^2| \geq (t)||\partial_t u_\varepsilon(t,0)| \leq 2AC\varepsilon|\partial_t u_\varepsilon(t,0)| \leq A^2C\varepsilon + \varepsilon(\partial_t u_\varepsilon(t,0))^2\\
&\leq AC + \varepsilon(\partial_t u_\varepsilon(t,0))^2,
\end{aligned}
$$

where $D_2,\ D_3$ are two positive constants, which will be specified later. It follows from (2.85) that

$$
\begin{aligned}
&\frac{d}{dt}\left(\varepsilon^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + A\varepsilon^2\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\right)\\
&\quad + 2\left(1-2D_3\right)\varepsilon\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + A\left(2-3D_3\right)\varepsilon\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad \leq C(T)\left(1 + A + \frac{1}{D_2} + \frac{A}{D_3}\right) + Cu^2_\varepsilon(t,0) + 2\varepsilon(\partial_t u_\varepsilon(t,0))^2
\end{aligned}
$$

$$+\frac{C(T)\varepsilon}{D_2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}.$$

Choosing $D_2 = \frac{1}{4}$ and $D_3 = \frac{1}{3}$, we have that

$$\begin{aligned}
&\frac{d}{dt}\left(\varepsilon^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + A\varepsilon^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\right)\\
&\quad+\varepsilon\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + A\varepsilon\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\\
&\le C(T)(1+A) + Cu_\varepsilon^2(t,0) + 2\varepsilon(\partial_t u_\varepsilon(t,0))^2 + C(T)\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}\tag{2.86}$$

Observe that, thanks to the Hölder inequality and the Young's inequality,

$$\begin{aligned}
2\varepsilon(\partial_t u_\varepsilon(t,0))^2 &= -2\varepsilon\int_0^\infty \partial_t u_\varepsilon \partial_t\partial_x u_\varepsilon dx \le 2\varepsilon\int_0^\infty |\partial_t u_\varepsilon||\partial_t\partial_x u_\varepsilon| dx\\
&\le 2\varepsilon\|\partial_t u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}\\
&\le 2\varepsilon\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\varepsilon}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}.
\end{aligned}$$

Consequently, by (2.86),

$$\begin{aligned}
&\frac{d}{dt}\left(\varepsilon^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + A\varepsilon^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\right)\\
&\quad+\frac{\varepsilon}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + (A-2)\,\varepsilon\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\\
&\le C(T)(1+A) + Cu_\varepsilon^2(t,0) + C(T)\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}$$

Taking $A = 3$, we have that

$$\begin{aligned}
&\frac{d}{dt}\left(\varepsilon^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + 3\varepsilon^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\right)\\
&\quad+\frac{\varepsilon}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + \varepsilon\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\\
&\le C(T) + Cu_\varepsilon^2(t,0) + C(T)\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}$$

Integrating on $(0,\infty)$, by (2.2), (2.7) and (2.73), we get

$$\begin{aligned}
&\varepsilon^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + 3\varepsilon^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\\
&\quad+\frac{\varepsilon}{2}\int_0^t\|\partial_t\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)} ds + \varepsilon\int_0^t\|\partial_t u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)} ds\\
&\le C + C(T)t + C\int_0^t u_\varepsilon^2(s,0)ds + 2C(T)\varepsilon\int_0^t\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(0,\infty)} ds\\
&\le C(T),
\end{aligned}$$

which gives (2.82). □

In order to prove the compactness of the family u_ε we prove the following estimate on the time derivative of u_ε.

Lemma 2.12 *Fix $T > 0$ and assume (1.15), or (1.16). There exists a constant $C(T) > 0$, independent on ε, such that*

$$\|\partial_t u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \le C(T),\tag{2.87}$$

for every $0 \le t \le T$.

Proof Let $0 \le t \le T$. Multiplying the first equation in (2.1) by $2\partial_t u_\varepsilon$, an integration on $(0,\infty)$ gives

$$
\begin{aligned}
2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} =& 2q^2\int_0^\infty \partial_x u_\varepsilon v_\varepsilon \partial_t u_\varepsilon dx + 2q^2\int_0^\infty u_\varepsilon \partial_x v_\varepsilon \partial_t u_\varepsilon dx \\
&+ 2b\int_0^\infty P_\varepsilon \partial_t u_\varepsilon dx + 2\varepsilon\int_0^\infty \partial_x^2 u_\varepsilon \partial_t u_\varepsilon dx.
\end{aligned} \tag{2.88}
$$

Due to Lemma 2.4, (2.7), (2.54), (2.72), (2.82) and the Young's inequality, we can estimate the previous terms as follows

$$
\begin{aligned}
&2q^2\int_0^\infty |\partial_x u_\varepsilon v_\varepsilon||\partial_t u_\varepsilon| dx \le 2q^4\int_0^\infty (\partial_x u_\varepsilon)^2 v_\varepsilon^2 dx + \frac{1}{2}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} \\
&\quad\le 2q^4\left\|v_\varepsilon\right\|^2_{L^\infty((0,T)\times(0,\infty))}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{1}{2}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} \\
&\quad\le C(T) + \frac{1}{2}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}, \\
&2q^2\int_0^\infty |u_\varepsilon \partial_x v_\varepsilon||\partial_t u_\varepsilon| dx \le 2q^4\int_0^\infty u_\varepsilon^2(\partial_x v_\varepsilon)^2 dx + \frac{1}{2}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} \\
&\quad\le 2q^4\left\|\partial_x v_\varepsilon\right\|^2_{L^\infty((0,T)\times(0,\infty))}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{1}{2}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} \\
&\quad\le C(T) + \frac{1}{2}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}, \\
&2|b|\int_0^\infty |P_\varepsilon||\partial_t u_\varepsilon| dx \le 2b^2\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{1}{2}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} \\
&\quad\le C(T) + \frac{1}{2}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}, \\
&2\varepsilon\int_0^\infty |\partial_x^2 u_\varepsilon||\partial_t u_\varepsilon| dx = 2\int_0^\infty \left|\sqrt{3}\varepsilon\partial_x^2 u_\varepsilon\right|\left|\frac{\partial_t u_\varepsilon}{\sqrt{3}}\right| dx \\
&\quad\le 3\varepsilon^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{1}{3}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} \\
&\quad\le C(T) + \frac{1}{3}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}
$$

It follows from (2.88) that

$$
\frac{1}{6}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|_{L^2(0,\infty)} \le C(T),
$$

which gives (2.87). □

Lemma 2.13 *Fix $T > 0$ and assume (1.15), or (1.16). There exists a constant $C(T) > 0$, independent on ε, such that*

$$
\left\|\partial_t v_\varepsilon(t,\cdot)\right\|_{L^2(0,\infty)} \le C(T), \tag{2.89}
$$

for every $0 \le t \le T$.

Proof Thanks to the estimates (2.60) and (2.87), the claim follows differentiating with respect to t the third equation in (2.1) and using the same argument developed for (2.14). □

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Using the Sobolev Immersion Theorem [13], we begin by proving the following result.

Lemma 3.1 *Fix $T > 0$. There exist a subsequence $\{(u_{\varepsilon_k}, v_{\varepsilon_k}, P_{\varepsilon_k})\}_{k\in\mathbb{N}}$ of $\{(u_\varepsilon, v_\varepsilon, P_\varepsilon)\}_{\varepsilon>0}$ and a limit triplet (u, v, P) which satisfies (1.18), (1.19) and (1.20) such that*

$$\begin{aligned} &u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{loc}((0,\infty)\times(0,\infty)),\ 1\le p<\infty,\\ &u_{\varepsilon_k} \rightharpoonup u \text{ in } H^1((0,T)\times(0,\infty)),\\ &v_{\varepsilon_k} \rightharpoonup v \text{ in } H^1((0,T)\times(0,\infty)),\\ &P_{\varepsilon_k} \to \int_0^x u(t,y)dy \text{ a.e. and in } L^p_{loc}((0,\infty)\times(0,\infty)),\ 1\le p<\infty. \end{aligned} \tag{3.1}$$

Moreover, (u, v, P) is solution of (1.8)satisfying (1.21).

Proof Let $0 \le t \le T$. We begin by observing that, thanks to Lemmas 2.3, 2.8, and 2.12

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } H^1((0,T)\times(0,\infty)). \tag{3.2}$$

Therefore, there exists a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and an function u such that

$$u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{loc}((0,\infty)\times(0,\infty)),\ 1\le p<\infty, \tag{3.3}$$

$$u_{\varepsilon_k} \rightharpoonup u \text{ in } H^1((0,T)\times(0,\infty)). \tag{3.4}$$

Observe that by (2.1), we have that

$$P_{\varepsilon_k}(t,x) = \int_0^x u_{\varepsilon_k}(t,y)dy. \tag{3.5}$$

Therefore, by (3.3), (3.5) and the Hölder inequality, we have that

$$P_{\varepsilon_k} \to \int_0^x u(t,y)dy \text{ a.e. and in } L^p_{loc}((0,\infty)\times(0,\infty)),\ 1\le p<\infty. \tag{3.6}$$

Moreover, by Lemmas 2.4 and 2.13, we have that

$$v_{\varepsilon_k} \rightharpoonup v \text{ in } H^1((0,T)\times(0,\infty)). \tag{3.7}$$

(3.3), (3.6), (3.7) and (3.4) give (3.1).

Observe that, by Lemmas 2.7, 2.8 and 2.9 , we have that

$$u \in L^\infty(0,T; H^1(0,\infty)), \tag{3.8}$$

while by Lemmas 2.7, 2.6, 2.8 and 2.9,

$$P \in L^\infty(0,T; H^2(0,\infty)).$$

Additionally, by Lemmas 2.4 and 2.10,

$$v \in L^\infty(0,T; H^3(\mathbb{R})). \tag{3.9}$$

Therefore, (1.17), (1.18), (1.19) and (1.20) holds and (u, v, P) is solution of (1.8).

Finally, (1.21) follows from (3.1) and Lemma 2.4. □

Following [25, Theorem 1.1], we prove Theorem 1.1.

Proof of Theorem 1.1. Lemma 3.1 gives the existence of a solution of (1.8) such that (1.18), (1.19) and (1.20) hold.

We prove (1.22). Given $t \geq 0,\ x \geq 0$. Let $(u_1,\ v_1,\ P_1)$ and $(u_2,\ v_2,\ P_2)$ be two solutions of (1.8), which satisfy (1.18), that is

$$\begin{cases} \partial_t u_i - q^2\partial_x(u_i v_i) = bP_i, & t > 0,\ x > 0,\\ \partial_x P_i = u_i, & t > 0,\ x > 0,\\ \alpha\partial_x^2 v_i + \beta\partial_x v_i + \gamma v_i = \kappa u_i^2, & t > 0,\ x > 0,\\ \partial_x u_i(t,0) = g(t), & t > 0, \\ P_i(t,0) = 0, & t > 0,\\ v_i(t,0) = h(t), & t > 0,\\ u_i(0,x) = u_{i,0}(x), & x > 0, \end{cases} \qquad i = 1, 2.$$

Then, the triplet $(U,\ V,\ U)$ defined by

$$\begin{aligned} &U(t,x) = u_1(t,x) - u_2(t,x), \quad V(t,x) = v_1(t,x) - v_2(t,x),\\ &\Omega(t,x) = \int_0^x U(t,y)dy = \int_0^x u_1(t,y)dy - \int_0^x u_2(t,y)dy, \end{aligned} \tag{3.10}$$

is solution of the following Cauchy problem:

$$\begin{cases} \partial_t U - q^2\partial_x(u_1v_1 - u_2v_2) = b\Omega, & t > 0,\ x > 0,\\ \partial_x\Omega = U, & t > 0,\ x > 0,\\ \alpha\partial_x^2 V + \beta\partial_x V + \gamma V = \kappa(u_1^2 - u_2^2), & t > 0,\ x > 0,\\ \partial_x U(t,0) = 0, & t > 0,\\ \Omega(t,0) = 0, & t > 0,\\ V(t,0) = 0, & t > 0,\\ U(0,x) = u_{1,0}(x) - u_{2,0}(x), & x > 0. \end{cases} \tag{3.11}$$

Observe that, thanks to (1.21) and (3.10),

$$\Omega(t,\infty) = \int_0^\infty U(t,x)dx = \int_0^\infty u_1(t,x)dx - \int_0^\infty u_2(t,x)dy = 0. \tag{3.12}$$

Moreover, since $u_1,\ u_2 \in L^\infty(0,T;H^1(0,\infty))$, for very $0 \leq t \leq T$, we can define

$$C(T) = \operatorname*{ess\,sup}_{(0,T)\times\mathbb{R}} \left\{|u_1| + |u_2|\right\}. \tag{3.13}$$

We prove that

$$\|V(t,\cdot)\|^2_{H^2(0,\infty)} \leq C(T)\,\|U(t,\cdot)\|^2_{L^2(0,\infty)}\,. \tag{3.14}$$

We need to distinguish two cases. We begin by assuming (1.15). Multiplying the third equation of (3.11) by $2\beta\partial_x V$, an integration on $(0,\infty)$ and (3.10) give

$$\begin{aligned} &2\beta\alpha\int_0^\infty \partial_x^2 V\partial_x V dx + 2\beta^2\,\|\partial_x V(t,\cdot)\|^2_{L^2(0,\infty)}\\ &\qquad + 2\gamma\beta\int_0^\infty V\partial_x V dx = 2\beta\kappa\int_0^\infty \partial_x V(u_1 + u_2)U dx. \end{aligned} \tag{3.15}$$

Since,

$$2\beta\alpha\int_0^\infty \partial_x^2 V\partial_x V = -\alpha\beta(\partial_x V(t,0))^2, \quad 2\gamma\beta\int_0^\infty V\partial_x V dx = 0,$$

it follows from (1.15) and (3.15) that

$$2\beta^2 \|\partial_x V(t,\cdot)\|^2_{L^2(0,\infty)} - \alpha\beta(\partial_x V(t,0))^2 = 2\beta\kappa \int_0^\infty (u_1+u_2)U\partial_x V dx. \tag{3.16}$$

Due to (3.13) and the Young's inequality,

$$\begin{aligned} 2|\beta||\kappa| \int_0^\infty |u_1+u_2||U||\partial_x V|dx &\le 2|\beta|C(T)\int_0^\infty |U||\partial_x V|dx \\ &\le C(T)\|U(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2 \|\partial_x V(t,\cdot)\|^2_{L^2(0,\infty)}. \end{aligned}$$

Consequently, by (3.16),

$$\beta^2 \|\partial_x V(t,\cdot)\|^2_{L^2(0,\infty)} - \alpha\beta(\partial_x V(t,0))^2 \le C(T)\|U(t,\cdot)\|^2_{L^2(0,\infty)}. \tag{3.17}$$

Squaring the equation for V in (3.11), by (3.10), we get

$$\begin{aligned} &\alpha^2(\partial_x^2 V)^2 + \beta^2(\partial_x V)^2 + \gamma^2 V^2 + 2\alpha\beta\partial_x V\partial_x^2 V \\ &\quad + 2\alpha\gamma V\partial_x^2 V + 2\beta\gamma V\partial_x V = \kappa^2(u_1+u_2)^2U^2. \end{aligned} \tag{3.18}$$

Observe that, by (3.11),

$$2\alpha\gamma \int_0^\infty V\partial_x^2 V dx = -2\alpha\gamma \|\partial_x V(t,\cdot)\|^2_{L^2(0,\infty)},$$

an integration on $(0,\infty)$ gives

$$\begin{aligned} &\alpha^2 \left\|\partial_x^2 V(t,\cdot)\right\|^2_{L^2(0,\infty)} + \beta^2 \|\partial_x V(t,\cdot)\|^2_{L^2(0,\infty)} + \gamma^2 \|V(t,\cdot)\|^2_{L^2(0,\infty)} \\ &\quad = \kappa^2 \int_0^\infty (u_1+u_2)^2U^2dx + 2\alpha\gamma \|\partial_x V(t,\cdot)\|^2_{L^2(0,\infty)} - \alpha\beta(\partial_x V(t,0))^2. \end{aligned} \tag{3.19}$$

Due to (1.15), (3.13) and (3.17),

$$\begin{aligned} &\kappa^2 \int_0^\infty (u_1+u_2)^2U^2dx + 2|\alpha\gamma| \|\partial_x V(t,\cdot)\|^2_{L^2(0,\infty)} \\ &\qquad + |\alpha\beta|(\partial_x V(t,0))^2 \le C(T)\|U(t,\cdot)\|^2_{L^2(0,\infty)}. \end{aligned}$$

Therefore, by (3.19),

$$\alpha^2 \left\|\partial_x^2 V(t,\cdot)\right\|^2_{L^2(0,\infty)} + \beta^2 \|\partial_x V(t,\cdot)\|^2_{L^2(0,\infty)} + \gamma^2 \|V(t,\cdot)\|^2_{L^2(0,\infty)} \le C(T)\|U(t,\cdot)\|^2_{L^2(0,\infty)}.$$

Defining

$$\tau^2 = \min\{\alpha^2,\ \beta^2,\ \gamma^2\},$$

we have that

$$\tau^2 \|V(t,\cdot)\|^2_{H^2(0,\infty)} \le C(T)\|U(t,\cdot)\|^2_{L^2(0,\infty)},$$

which gives (3.14).

We continue by assuming (1.16). Since $\beta = 0$, by (3.10), the third equation of (3.11) reads

$$\alpha\partial_x^2 V + \gamma V = \kappa(u_1+u_2)U. \tag{3.20}$$

Squaring (3.20), we have that

$$\alpha^2(\partial_x^2 V)^2 + \gamma^2 V^2 + 2\alpha\gamma V\partial_x^2 V = \kappa^2(u_1+u_2)^2U^2. \tag{3.21}$$

Since, by (3.11)

$$2\alpha \int_0^\infty \gamma V \partial_x^2 V dx = -2\alpha\gamma \left\| \partial_x V(t,\cdot) \right\|^2_{L^2(0,\infty)}, \tag{3.22}$$

it follows from an integration of (3.21) on $(0, \infty)$, (1.16) and (3.13) that

$$\alpha^2 \left\| \partial_x^2 V(t,\cdot) \right\|^2_{L^2(0,\infty)} - 2\alpha\gamma \left\| \partial_x V(t,\cdot) \right\|^2_{L^2(0,\infty)} + \gamma^2 \left\| V(t,\cdot) \right\|^2_{L^2(0,\infty)} \le C(T) \left\| U(t,\cdot) \right\|^2_{L^2(0,\infty)}.$$

Reminding that $\alpha\gamma < 0$ we define

$$\tau_1^2 = \min\{\alpha^2, \; -2\alpha\gamma, \; \gamma^2\},$$

we have that

$$\tau_1^2 \left\| V(t,\cdot) \right\|^2_{H^2(0,\infty)} \le C(T) \left\| U(t,\cdot) \right\|^2_{L^2(0,\infty)},$$

which gives (3.14).

We prove that

$$\left\| V(t,\cdot) \right\|^2_{L^\infty(0,\infty)} \le C(T) \left\| U(t,\cdot) \right\|^2_{L^2(0,\infty)}, \tag{3.23}$$

for every $0 \le t \le T$.

Due to (3.11) and the Hölder inequality,

$$V^2(t,x) = 2\int_0^x V \partial_x V dy \le 2\int_0^\infty |V||\partial_x v| dx \le 2 \left\| V(t,\cdot) \right\|_{L^2(0,\infty)} \left\| \partial_x V(t,\cdot) \right\|_{L^2(0,\infty)}.$$

Consequently, thanks to (3.14),

$$V^2(t,x) \le C(T) \left\| U(t,\cdot) \right\|^2_{L^2(0,\infty)},$$

which gives (3.23).

In a similar way, we have that

$$\left\| \partial_x V(t,\cdot) \right\|^2_{L^\infty(\mathbb{R})} \le C(T) \left\| U(t,\cdot) \right\|^2_{L^2(0,\infty)}. \tag{3.24}$$

Observe that, by (3.10),

$$\partial_x(u_1 v_1 - u_2 v_2) = \partial_x(u_1 v_1 - u_2 v_1 + u_2 v_1 - u_2 v_2) = \partial_x(v_1 U) + \partial_x(u_2 V).$$

Therefore, the first equation of (3.11) is equivalent to the following one:

$$\partial_t U = b\Omega + q^2 \partial_x(v_1 U) + q^2 \partial_x(u_2 V). \tag{3.25}$$

Multiplying (3.25) by $2U$, an integration on $(0, \infty)$ gives

$$\frac{d}{dt} \left\| U(t,\cdot) \right\|^2_{L^2(0,\infty)} = 2b\int_0^\infty \Omega U dx + 2q^2 \int_{\mathbb{R}} U \partial_x(v_1 U) dx + 2q^2 \int_0^\infty U \partial_x(u_2 V) dx. \tag{3.26}$$

Observe that by (3.12) and the second equation of (3.11),

$$2b\int_0^\infty \Omega U dx = 2b\int_0^\infty \Omega \partial_x \Omega dx = b\Omega^2(t,\infty) = 0. \tag{3.27}$$

Moreover, since $v_1(t,0) = h(t)$,

$$\begin{aligned} 2q^2 \int_0^\infty U \partial_x(v_1 U) dx =& -2q^2 h(t) U^2(t,0) - 2q^2 \int_0^\infty U \partial_x U v_1 dx \\ =& -q^2 h(t) U^2(t,0) + q^2 \int_0^\infty \partial_x v_1 U^2 dx. \end{aligned} \tag{3.28}$$

It follows from (1.10), (3.26), (3.27) and (3.28) that

$$\begin{aligned}&\frac{d}{dt}\|U(t,\cdot)\|^2_{L^2(0,\infty)}+q^2\kappa_1^2U^2(t,0)\\&\quad\le q^2\int_0^\infty\partial_x v_1U^2dx+2q^2\int_0^\infty\partial_x u_2UVdx+2q^2\int_0^\infty u_2U\partial_xVdx.\end{aligned}\tag{3.29}$$

Fix $T>0$. By (3.8) and (3.9), there exists a constant $C(T)>0$ such that

$$\|u_2(t,\cdot)\|_{L^2(0,\infty)},\ \|\partial_x u_2(t,\cdot)\|_{L^2(0,\infty)},\ \|\partial_x v_1(t,\cdot)\|_{L^\infty(\mathbb{R})}\le C(T),\tag{3.30}$$

Due to (3.23), (3.24), (3.30) and the Young's inequality,

$$\begin{aligned}&q^2\int_0^\infty|\partial_x v_1|U^2dx\le C(T)\|U(t,\cdot)\|^2_{L^2(0,\infty)},\\&2q^2\int_0^\infty|\partial_x u_2||U||V|dx\le q^4\|U(t,\cdot)\|^2_{L^2(0,\infty)}+\int_0^\infty(\partial_x u_2)^2V^2dx\\&\quad\le q^4\|U(t,\cdot)\|^2_{L^2(0,\infty)}+\|V(t,\cdot)\|^2_{L^\infty(0,\infty)}\|\partial_x u_1(t,\cdot)\|^2_{L^2(0,\infty)}\\&\quad\le q^4\|U(t,\cdot)\|^2_{L^2(0,\infty)}+C(T)\|V(t,\cdot)\|^2_{L^\infty(0,\infty)}\le C(T)\|U(t,\cdot)\|^2_{L^2(0,\infty)},\\&2q^2\int_0^\infty|u_2||U||\partial_xV|dx\le q^4\|U(t,\cdot)\|^2_{L^2(0,\infty)}+\int_0^\infty u_2^2(\partial_xV)^2dx\\&\quad\le q^4\|U(t,\cdot)\|^2_{L^2(0,\infty)}+\|u_2(t,\cdot)\|^2_{L^2(0,\infty)}\|\partial_xV(t,\cdot)\|^2_{L^\infty(0,\infty)}\\&\quad\le q^4\|U(t,\cdot)\|^2_{L^2(0,\infty)}+C(T)\|\partial_xV(t,\cdot)\|^2_{L^\infty(0,\infty)}\le C(T)\|U(t,\cdot)\|^2_{L^2(0,\infty)}.\end{aligned}$$

It follows from (3.29) that

$$\frac{d}{dt}\|U(t,\cdot)\|^2_{L^2(0,\infty)}+q^2\kappa_1^2U^2(t,0)\le C(T)\|U(t,\cdot)\|^2_{L^2(0,\infty)}.$$

The Gronwall Lemma and (3.11) give

$$\|U(t,\cdot)\|^2_{L^2(0,\infty)}+q^2\kappa_1^2e^{C(T)t}\int_0^te^{-C(T)s}U^2(s,0)ds\le e^{C(T)t}\|U_0\|^2_{L^2(0,\infty)}.\tag{3.31}$$

(1.22) follows from (3.10), (3.14) and (3.31). □

4 Proof of Theorem 1.2 assuming (1.23)

In this section, we prove Theorem 1.2 assuming (1.23). We consider (2.1), which is an approximation of (1.8), such that (2.2) holds.

Arguing as in Section 2, we have Lemmas 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11 and 2.12.

We prove the following result.

Lemma 4.1 *Fix $T>0$. Assume (1.15) and (1.23). There exists a constant $C(T)>0$, independent on ε, such that*

$$\begin{aligned}&\frac{\alpha^2}{2}\left\|\partial_t\partial_x^2v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}+\beta^2\|\partial_t\partial_xv_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\\&\qquad+\frac{\gamma^2}{2}\|\partial_tv_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\le C(T),\end{aligned}\tag{4.1}$$

for every $0 \le t \le T$. *In particular, we have that*

$$\|\partial_t v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))} \le C(T), \tag{4.2}$$
$$\|\partial_t \partial_x v_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))} \le C(T). \tag{4.3}$$

Proof Let $0 \le t \le T$. We begin by proving that

$$\beta^2 \|\partial_x \partial_t v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} - \beta\alpha(\partial_t\partial_x v_\varepsilon(t,0))^2 - \gamma\beta(\partial_t v_\varepsilon(t,0))^2 \le C(T). \tag{4.4}$$

Differentiating the third equation of (2.1) with respect to t, we have that

$$\alpha\partial_t\partial_x^2 v_\varepsilon + \beta\partial_t\partial_x v_\varepsilon + \gamma\partial_t v_\varepsilon = 2\kappa u_\varepsilon \partial_t u_\varepsilon. \tag{4.5}$$

Multiplying (4.5) by $2\beta\partial_t\partial_x v_\varepsilon$, an integration on $(0,\infty)$ gives

$$\begin{aligned} 2\beta^2 \|\partial_t\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} &+ 2\beta\alpha \int_0^\infty \partial_t\partial_x v_\varepsilon \partial_t\partial_x^2 v_\varepsilon dx \\ &+ 2\beta\gamma \int_0^\infty \partial_t v_\varepsilon \partial_t\partial_x v_\varepsilon dx = 4\kappa\beta \int_0^\infty u_\varepsilon \partial_t u_\varepsilon \partial_t\partial_x v_\varepsilon dx. \end{aligned} \tag{4.6}$$

Observe that

$$\begin{aligned} 2\beta\alpha \int_0^\infty \partial_t\partial_x v_\varepsilon \partial_t\partial_x^2 v_\varepsilon dx &= -\beta\alpha(\partial_t\partial_x v_\varepsilon(t,0))^2, \\ 2\beta\gamma \int_0^\infty \partial_t v_\varepsilon \partial_t\partial_x v_\varepsilon dx &= -\beta\gamma(\partial_t v_\varepsilon(t,0))^2. \end{aligned}$$

Consequently, by (1.15), (1.23) and (4.6), we have that

$$\begin{aligned} 2\beta^2 \|\partial_t\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} &- \beta\alpha(\partial_t\partial_x v_\varepsilon(t,0))^2 \\ &- \beta\gamma(\partial_t v_\varepsilon(t,0))^2 = 4\kappa\beta \int_0^\infty u_\varepsilon \partial_t u_\varepsilon \partial_t\partial_x v_\varepsilon dx. \end{aligned} \tag{4.7}$$

Due to (2.72), (2.87) and the Young's inequality,

$$\begin{aligned} 2|\kappa||\beta| \int_0^\infty |u_\varepsilon\partial_t u_\varepsilon||\partial_t\partial_x v_\varepsilon| dx &\le \kappa^2 \int_0^\infty u_\varepsilon^2(\partial_t u_\varepsilon)^2 dx + \beta^2 \|\partial_t\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} \\ &\le \kappa^2 \|u_\varepsilon\|^2_{L^\infty((0,t)\times(0,\infty))} \|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2 \|\partial_t\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} \\ &\le C(T) + \beta^2 \|\partial_t\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}. \end{aligned} \tag{4.8}$$

Therefore, by (4.7),

$$\beta^2 \|\partial_t\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} - \beta\alpha(\partial_t\partial_x v_\varepsilon(t,0))^2 - \beta\gamma(\partial_t v_\varepsilon(t,0))^2 \le C(T),$$

which gives (4.4).

We prove (4.1). Squaring (4.5), an integration on $(0,\infty)$ gives

$$\begin{aligned} \alpha^2 \left\|\partial_t\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} &+ \beta^2 \|\partial_t\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + \gamma^2 \|\partial_t v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} \\ &\le 4\kappa^2 \int_0^\infty u_\varepsilon^2(\partial_t u_\varepsilon)^2 dx - 2\alpha\beta \int_0^\infty \partial_t\partial_x v_\varepsilon \partial_t\partial_x^2 v_\varepsilon dx \\ &- 2\alpha\gamma \int_0^\infty \partial_t v_\varepsilon \partial_t\partial_x^2 v_\varepsilon dx - 2\beta\gamma \int_0^\infty \partial_t v_\varepsilon \partial_t\partial_x v_\varepsilon dx. \end{aligned} \tag{4.9}$$

Observe that

$$-2\alpha\gamma\int_0^\infty \partial_t v_\varepsilon \partial_t \partial_x^2 v_\varepsilon dx = 2\alpha\gamma\partial_t v_\varepsilon(t,0)\partial_t\partial_x v_\varepsilon(t,0) + 2\alpha\gamma\left\|\partial_t\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}.$$

Consequently, by (1.16), (1.23) and (4.9),

$$\begin{aligned}\alpha^2\left\|\partial_t\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} &+ \beta^2\left\|\partial_t\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \gamma^2\left\|\partial_t v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &\le 4\kappa^2\int_0^\infty u_\varepsilon^2(\partial_t u_\varepsilon)^2 dx - 2\alpha\beta\int_0^\infty \partial_t\partial_x v_\varepsilon \partial_t\partial_x^2 v_\varepsilon dx\\ &\quad - 2\beta\gamma\int_0^\infty \partial_t v_\varepsilon\partial_t\partial_x v_\varepsilon dx + 2\alpha\gamma\partial_t v_\varepsilon(t,0)\partial_t\partial_x v_\varepsilon(t,0)\\ &\quad + 2\alpha\gamma\left\|\partial_t\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned} \tag{4.10}$$

Due to (2.72), (2.87), (4.4) and the Young's inequality,

$$\begin{aligned}&4\kappa^2\int_0^\infty u_\varepsilon^2(\partial_t u_\varepsilon)^2 dx \le 4\kappa^2\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times(0,\infty))}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} \le C(T),\\ &2|\alpha||\beta|\int_0^\infty |\partial_t\partial_x v_\varepsilon||\partial_t\partial_x^2 v_\varepsilon| dx \le 2\beta^2\left\|\partial_x\partial_t v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\alpha^2}{2}\left\|\partial_t\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &\quad\le C(T) + \frac{\alpha^2}{2}\left\|\partial_t\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)},\\ &2|\beta||\gamma|\int_0^\infty |\partial_t v_\varepsilon||\partial_t\partial_x v_\varepsilon| dx \le \frac{\gamma^2}{2}\left\|\partial_t v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_t\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &\quad\le \frac{\gamma^2}{2}\left\|\partial_t v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + C(T),\\ &2|\alpha||\gamma||\partial_t v_\varepsilon(t,0)||\partial_t\partial_x v_\varepsilon(t,0)| \le \alpha^2(\partial_t v_\varepsilon(t,0))^2 + \gamma^2(\partial_x\partial_t v_\varepsilon(t,0))^2 \le C(T),\\ &2|\alpha\gamma|\left\|\partial_t\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} \le C(T).\end{aligned}$$

Therefore, by (1.15), (1.23) and (4.10), we get

$$\begin{aligned}\frac{\alpha^2}{2}\left\|\partial_t\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} &+ \beta^2\left\|\partial_t\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &+ \frac{\gamma^2}{2}\left\|\partial_t v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} \le C(T),\end{aligned}$$

which gives (4.1).

We prove (4.2). Thanks to (4.1), (4.4) and the Hölder inequality, we obtain for $t \ge 0,\ x \ge 0$

$$\begin{aligned}(\partial_t v_\varepsilon(t,x))^2 &= 2\int_0^x \partial_t v_\varepsilon\partial_t\partial_x v_\varepsilon dy + (\partial_t v_\varepsilon(t,0))^2 \le 2\int_0^\infty |\partial_t v_\varepsilon||\partial_t\partial_x v_\varepsilon| dx + C(T)\\ &\le 2\left\|\partial_t v_\varepsilon(t,\cdot)\right\|_{L^2(0,\infty)}\left\|\partial_t\partial_x v_\varepsilon(t,\cdot)\right\|_{L^2(0,\infty)} + C(T) \le C(T).\end{aligned}$$

Hence,

$$\left\|\partial_t v_\varepsilon\right\|^2_{L^\infty((0,T)\times(0,\infty))} \le C(T),$$

which gives (4.2).

In a similar way, we can prove (4.3). □

Using the Sobolev immersion Theorem, we begin by proving the following result.

Lemma 4.2 *Fix $T > 0$. There exist a subsequence $\{u_{\varepsilon_k}, v_{\varepsilon_k}, P_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon, v_\varepsilon, P_\varepsilon\}_{\varepsilon>0}$ and a triplet (u, v, P) which satisfies (1.18), (1.19) and (1.24) such that*

$$\begin{aligned}
&u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{loc}((0,\infty)\times(0,\infty)),\ 1\le p<\infty,\\
&u_{\varepsilon_k} \rightharpoonup u \text{ in } H^1((0,T)\times(0,\infty)),\\
&v_{\varepsilon_k} \to v \text{ a.e. and in } L^p_{loc}((0,\infty)\times(0,\infty)),\ 1\le p<\infty,\\
&v_{\varepsilon_k} \rightharpoonup v \text{ in } H^1((0,T)\times(0,\infty)),\\
&P_{\varepsilon_k} \to \int_0^x u(t,y)dy \text{ a.e. and in } L^p_{loc}((0,\infty)\times(0,\infty)),\ 1\le p<\infty.
\end{aligned} \tag{4.11}$$

Moreover, (u, v, P) is solution of (1.8) satisfying (1.21).

Proof Let $0 \le t \le T$. Arguing as in Lemma 3.1, we have (3.2), (3.6), (3.7) and (3.4). Moreover, thanks to (3.4) and Lemma 2.1, (1.21) holds.

Observe that, thanks to Lemmas 2.4, 2.10 and 4.1, we have that

$$v_{\varepsilon_k} \to v \text{ a.e. and in } L^p_{loc}((0,\infty)\times(0,\infty)),\ 1\le p<\infty. \tag{4.12}$$

Therefore, (4.11) is proven.

Observe that, again by Lemmas 2.4, 2.10 and 4.1, we have that

$$v \in L^\infty(0,T; H^3(\mathbb{R})) \cap W^{1,\infty}((0,T)\times\mathbb{R}),$$

while, by Lemma 4.1,

$$\partial^2_{tx} v \in L^\infty(0,T; L^2(0,\infty)) \cap L^\infty((0,T)\times(0,\infty)), \quad \partial_t\partial_x^2 v \in L^\infty(0,T; L^2(0,\infty)).$$

Arguing as in Lemma 3.1, the proof is conclued. □

Proof (Proof of Theorem 1.2 assuming (1.23)) Lemma 4.2 gives the existence of a solution of (1.8), such that (1.18), (1.20) and (1.24) hold. Arguing as in Theorem 1.1, we have (1.22). □

5 Proof of Theorem 1.2 assuming (1.25)

In this section, we prove Theorem 1.2 assuming (1.25). We consider (2.1), which is an approximation of (1.8), such that (2.2) holds, and

$$\left\| h'_\varepsilon \right\|_{L^\infty(0,\infty)} \le C, \tag{5.1}$$

where C is a constant independent on ε.

Arguing as in Section 2, we have Lemmas 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11 and 2.12.

We prove the following result.

Lemma 5.1 *Fix $T > 0$ and assume (1.15) or (1.16). There exists a constant $C(T) > 0$, independent on ε, such that*

$$\left\|\partial_t\partial_x^2 v_\varepsilon(t,\cdot)\right\|_{L^2(0,\infty)}, \ \|\partial_t\partial_x v_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}, \ \|\partial_t v_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \le C(T), \tag{5.2}$$

for every $0 \le t \le T$. Moreover, we have (4.2) and (4.3).

Proof Let $0 \le t \le T$. Assume (1.15). We begin by proving that

$$\beta^2 \|\partial_t \partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} - \alpha\beta(\partial_t \partial_x v_\varepsilon(t,0))^2 \le C(T). \tag{5.3}$$

Thanks to (1.8), we have that

$$\partial_t v_\varepsilon(t,0) = h'_\varepsilon(t). \tag{5.4}$$

Consequently, by (4.7), we have that

$$\begin{aligned}2\beta^2 \|\partial_t \partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} &- \beta\alpha(\partial_t \partial_x v_\varepsilon(t,0))^2\\ &= 2\kappa\beta \int_0^\infty u_\varepsilon \partial_t u_\varepsilon \partial_t \partial_x v_\varepsilon dx - \beta\gamma h^2_\varepsilon(t).\end{aligned}$$

Therefore, by (4.8) and (5.1), we get

$$2\beta^2 \|\partial_t \partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} - \beta\alpha(\partial_t \partial_x v_\varepsilon(t,0))^2 \le C(T),$$

which gives (5.3).

Now, we prove (5.2). Arguing as in Lemma 4.1, we have (4.9). Moreover, thanks to (5.4),

$$\begin{aligned}-2\alpha\beta \int_0^\infty \partial_t \partial_x v_\varepsilon \partial_t \partial_x^2 v_\varepsilon dx &= \alpha\beta(\partial_t \partial_x v_\varepsilon(t,0)^2 dx,\\ -2\alpha\gamma \int_0^\infty \partial_t v_\varepsilon \partial_x^2 \partial_t \partial_x^2 v_\varepsilon dx &= 2\alpha\gamma h_\varepsilon(t)\partial_x v_\varepsilon(t,0) + 2\alpha\gamma \|\partial_t \partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)},\\ -2\beta\gamma \int_0^\infty \partial_t v_\varepsilon \partial_t \partial_x v_\varepsilon dx &= \beta\gamma h^2_\varepsilon(t).\end{aligned}$$

Therefore, by (1.15) and (4.9), we have that

$$\begin{aligned}\alpha^2 \left\|\partial_t \partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} &+ \beta^2 \|\partial_t \partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + \gamma^2 \|\partial_t v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\\ &\le 4\kappa^2 \int_0^\infty u^2_\varepsilon(\partial_t u_\varepsilon)^2 dx + \alpha\beta(\partial_t \partial_x v_\varepsilon(t,0))^2 + 2\alpha\gamma h_\varepsilon(t)\partial_x v_\varepsilon(t,0)\\ &\quad + 2\alpha\gamma \|\partial_t \partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + \beta\gamma h^2_\varepsilon(t).\end{aligned} \tag{5.5}$$

Due to (2.72), (2.87), (5.3) and the Young's inequality,

$$\begin{aligned}&4\kappa^2 \int_0^\infty u^2_\varepsilon(\partial_t u_\varepsilon)^2 dx \le 4\kappa^2 \|u_\varepsilon\|^2_{L^\infty((0,T)\times(0,\infty))} \|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} \le C(T),\\ &|\alpha\beta|(\partial_t \partial_x v_\varepsilon(t,0))^2 \le C(T),\\ &2|\alpha\gamma||h_\varepsilon(t)||\partial_x v_\varepsilon(t,0)| \le 2C|\partial_t \partial_x v_\varepsilon| \le C + (\partial_t \partial_x v_\varepsilon(t,0))^2 \le C(T),\\ &2|\alpha\gamma| \|\partial_t \partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + |\beta\gamma| h^2_\varepsilon(t) \le C(T) + C \le C(T).\end{aligned}$$

Consequently, by (5.5), we have that

$$\alpha^2 \left\|\partial_t \partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + \beta^2 \|\partial_t \partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + \gamma^2 \|\partial_t v_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} \le C(T),$$

which gives (5.2).

Arguing as in Lemma 4.1, we have (4.2) and (4.3).

Assume (1.16). Differentiating (2.33) with respect to t, we have that

$$\alpha \partial_t \partial_x^2 v_\varepsilon + \gamma \partial_t v_\varepsilon = 2\kappa u_\varepsilon \partial_t u_\varepsilon. \tag{5.6}$$

Squaring (5.6), an integration on $(0, \infty)$ gives

$$\begin{aligned} \alpha^2 \left\| \partial_t \partial_x^2 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} + \gamma^2 \left\| \partial_t \partial_x v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} \\ + 2\alpha\gamma \int_0^\infty \partial_t v_\varepsilon \partial_t \partial_x^2 v_\varepsilon dx = 4\kappa^2 \int_0^\infty u_\varepsilon^2 (\partial_t u_\varepsilon)^2 dx. \end{aligned} \tag{5.7}$$

Observe that, by (5.4), we have that

$$2\alpha\gamma \int_0^\infty \partial_t v_\varepsilon \partial_t \partial_x^2 v_\varepsilon dx = -2\alpha\gamma h_\varepsilon(t) \partial_t \partial_x v_\varepsilon(t,0) - 2\alpha\gamma \left\| \partial_t \partial_x v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)}.$$

Therefore, by (1.16), (5.7), we have that

$$\begin{aligned} \alpha^2 \left\| \partial_t \partial_x^2 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} - 2\alpha\gamma \left\| \partial_t \partial_x v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} + \gamma^2 \left\| \partial_t \partial_x v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} \\ = 4\kappa^2 \int_0^\infty u_\varepsilon^2 (\partial_t u_\varepsilon)^2 dx + 2\alpha\gamma h_\varepsilon(t) \partial_t \partial_x v_\varepsilon(t,0). \end{aligned} \tag{5.8}$$

Due to (2.72), (2.87) and (5.1),

$$\begin{aligned} & 4\kappa^2 \int_0^\infty u_\varepsilon^2 (\partial_t u_\varepsilon)^2 dx \le 4\kappa^2 \left\| u_\varepsilon \right\|^2_{L^\infty((0,T)\times(0,\infty))} \left\| \partial_t u_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} \le C(T), \\ & 2|\alpha\gamma||h_\varepsilon(t)||\partial_t \partial_x v_\varepsilon(t,0)| \le C|\partial_t \partial_x v_\varepsilon(t,0)|. \end{aligned}$$

Therefore, by (5.8),

$$\begin{aligned} \alpha^2 \left\| \partial_t \partial_x^2 v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} - 2\alpha\gamma \left\| \partial_t \partial_x v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} + \gamma^2 \left\| \partial_t \partial_x v_\varepsilon(t,\cdot) \right\|^2_{L^2(0,\infty)} \\ \le C(T) \left(1 + |\partial_t \partial_x v_\varepsilon(t,0)|\right). \end{aligned} \tag{5.9}$$

We prove that

$$|\partial_t \partial_x v_\varepsilon(t,0)| \le C(T), \tag{5.10}$$

for every $0 \le t \le T$. Due to (5.9) and the Hölder inequality,

$$\begin{aligned} (\partial_t \partial_x v_\varepsilon(t,0))^2 = & -2 \int_0^\infty \partial_t \partial_x u_\varepsilon \partial_t \partial_x^2 u_\varepsilon dx \le 2 \int_0^\infty |\partial_t \partial_x v_\varepsilon| |\partial_t \partial_x^2 v_\varepsilon| dx \\ & \le 2 \left\| \partial_t \partial_x v_\varepsilon(t,\cdot) \right\|_{L^2(0,\infty)} \left\| \partial_t \partial_x^2 u_\varepsilon(t,\cdot) \right\|_{L^2(0,\infty)} \le C(T) \left(1 + |\partial_t \partial_x v_\varepsilon(t,0)|\right). \end{aligned}$$

Therefore,

$$(\partial_t \partial_x v_\varepsilon(t,0))^2 - C(T)|\partial_t \partial_x v_\varepsilon(t,0)| - C(T) \le 0,$$

which gives (5.10).

(5.2) follows from (5.9) and (5.10).

Finally, arguing as in Lemma 4.1, we have (4.2) and (4.3). □

Arguing as in Sect. 4, we have Theorem 1.2 assuming (1.25).

6 Conclusions

In this paper we proved the well-posedness of a non local approximation of the short pulse equation. Since in this problem the evolutive equation is a transport equation with smooth coefficients several approaches for the well-posedness could be used (e.g. the fixed point one). We used the vanishing viscosity one because we plan to design finite difference numerical schemes for the problem. Indeed such schemes have an intrinsic diffusion similar to the one produced by vanishing viscosity. Moreover, we plan to study the boundary controllability of this nonlocal approximation of the short pulse equation.

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Data availability No datasets were generated or analyzed during the current study.

Declarations

Code availability Not applicable.

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