Maximal regularity and a singular limit problem for the Patlak–Keller–Segel system in the scaling critical space involving BMO

Masaki Kurokiba¹ · Takayoshi Ogawa²

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Abstract

We consider a singular limit problem of the Cauchy problem for the Patlak–Keller–Segel equation in a scaling critical function space. It is shown that a solution to the Patlak–Keller–Segel system in a scaling critical function space involving the class of bounded mean oscillations converges to a solution to the drift-diffusion system of parabolic-elliptic type (simplified Keller–Segel model) strongly as the relaxation time parameter $\tau \rightarrow \infty$. For the proof, we show generalized maximal regularity for the heat equation in the homogeneous Besov spaces and the class of bounded mean oscillations and we utilize them systematically as well as the continuous embeddings between the interpolation spaces $\dot{B}_{q,\sigma}^s(\mathbb{R}^n)$ and $\dot{F}_{q,\sigma}^s(\mathbb{R}^n)$ for the proof of the singular limit. In particular, end-point maximal regularity in BMO and space time modified class introduced by Koch–Tataru is utilized in our proof.

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1 Singular limit problem for the Keller–Segel model

1.1 Critical solvability

We consider the Cauchy problem of the Keller–Segel equation in the n -dimensional Euclidean space $\mathbb{R}^n$:

Dedicated to Professor Hideo Kozono on the occasion of his sixtieth birthday.

Deceased: Masaki Kurokiba.

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✉ Takayoshi Ogawa
ogawa@math.tohoku.ac.jp

¹ Graduate School of Engineerings, Muroran Institute of Technology, Muroran 050-8585, Japan

² Mathematical Institute/Research Alliance Center for Mathematical Science, Tohoku University, Sendai 980-8578, Japan

$$\begin{cases} \partial_t u_\tau - \Delta u_\tau + \nabla \cdot (u_\tau \nabla \psi_\tau) = 0, & t > 0, x \in \mathbb{R}^n, \\ \frac{1}{\tau} \partial_t \psi_\tau - \Delta \psi_\tau + \lambda \psi_\tau = u_\tau, & t > 0, x \in \mathbb{R}^n, \\ u_\tau(0, x) = u_0(x), \quad \psi_\tau(0, x) = \psi_0(x), & t = 0, x \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

where $u_\tau = u_\tau(t, x)$ and $\psi_\tau(t) = \psi_\tau(t, x)$ denotes the unknown density of molds and the distribution of the chemical substance, (u_0, ψ_0) is given pair of initial data and $\lambda \geq 0$ is a constant. The parameter $\tau > 0$ denotes the *relaxation time* and when the dynamics of the chemical substance is relatively slow the zero relaxation time limit can be considered. By passing $\tau \rightarrow \infty$, the limiting functions

$$\begin{cases} \lim_{\tau \rightarrow \infty} u_\tau(t, x) = u(t, x), \\ \lim_{\tau \rightarrow \infty} \psi_\tau(t, x) = \psi(t, x) \end{cases} \quad (1.2)$$

formally solve the initial value problem of the drift-diffusion system:

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^n, \\ -\Delta \psi + \lambda \psi = u, & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & t = 0, x \in \mathbb{R}^n. \end{cases} \quad (1.3)$$

The system (1.3) can be also found by a singular limit problem from the compressible Navier–Stokes equations [15, 25]. The system (1.1) is called as the Patlak–Keller–Segel system (Patlak [52], Keller–Segel [24]), while (1.3) is called drift-diffusion system (Mock [37]) or the simplified Keller–Segel system (Jäger–Luckhaus [22], Biler [3], Nagai [38]). A large number of literature on the research of the well-posedness, the ill-posedness and asymptotic behaviors of the solutions have been appeared ([3–6, 8, 10, 11, 19–23, 30–33, 38–43, 54, 55, 58, 61] and references therein). The existence of solutions and the well-posedness of the problems (1.1) and (1.3) are known in the scaling invariant class, namely the critical scaling class by influence of results due to Serrin and Fujita–Kato on the incompressible Navier–Stokes equations (see [3, 5, 29, 31, 32]). In particular, the systems (1.1) and (1.3) are invariant under the following scaling transformation with the restriction $\lambda = 0$: For $\mu > 0$,

$$\begin{cases} u_\mu(t, x) = \mu^2 u(\mu^2 t, \mu x), \\ \psi_\mu(t, x) = \psi(\mu^2 t, \mu x). \end{cases} \quad (1.4)$$

Then the invariant Bochner class is given by the following:

$$\begin{cases} u \in L^\theta(\mathbb{R}_+; L^q(\mathbb{R}^n)), & \frac{2}{\theta} + \frac{n}{q} = 2, \\ \psi \in L^\sigma(\mathbb{R}_+; L^{\tilde{r}}(\mathbb{R}^n)), & \frac{2}{\sigma} + \frac{n}{\tilde{r}} = 0. \end{cases} \quad (1.5)$$

It is natural and optimal to choose $\theta = \sigma = \infty$ in order to solve the problem (1.3) in the method of the heat semi-group, then we find that the invariant exponents are $q = \frac{n}{2}$ and $\tilde{r} = \infty$. However, by the elliptic and parabolic regularity theory, it is difficult to choose the solution space $L^\infty(\mathbb{R}^n)$ for the critical space of ψ_τ and ψ and one of the corresponding regularity class is the *bounded mean oscillation* $BMO(\mathbb{R}^n)$ or the *vanishing mean oscillation* $VMO(\mathbb{R}^n)$. The equation (1.3) can be solved in such critical spaces and Kozono–Sugiyama–Yahagi [29] first pointed out that the Keller–Segel system (1.3) is globally well-posed in such a critical class involving $BMO(\mathbb{R}^n)$ for small initial data. Nagai–Ogawa [41] considered the

class of BMO for constructing a global mass threshold solution in the two-dimensional drift-diffusion system (1.3) with $\lambda = 0$ (cf. Wei [60]).

In this paper, we consider the singular limit problem for the system (1.1) in the finite mass case. Raczyński [53] and Biler-Brandolese [4] studied the singular limit problem in two space dimensions. They considered the case of the vanishing initial condition $\psi_0 \equiv 0$ and showed the strong convergence (1.2) for a small data global solution in the scaling invariant spaces such as the class of pseudo-measures or the Lorentz spaces. Their topologies for convergence on $\lim_{\tau \rightarrow \infty} \psi_\tau$ are not relevant of an initial layer since they posed a time weight function in the metric so that the initial behavior is neglected in the topology of convergence. Lemarié-Rieusset [36] also showed the singular limit in the scaling invariant space based on the Morrey space. In the previous works [34,35], the authors developed the singular limit problem with a presence of the initial layer on $\psi_\tau(t)$ and showed the strong convergence in the standard space-time Lebesgue–Bochner space uniformly including a large data time-local solution. Since we need to introduce a common function space between two problems (1.1) and (1.3), we introduce an admissible class for showing the singular limit [34,35] such as

$$\begin{cases} u \in L^\theta(\mathbb{R}_+; L^q(\mathbb{R}^n)), & \frac{2}{\theta} + \frac{n}{q} = 2, \quad \frac{n}{2} < q < \theta, \\ \nabla \psi \in L^\theta(\mathbb{R}_+; L^r(\mathbb{R}^n)), & \frac{2}{\theta} + \frac{n}{r} = 1, \quad n < r < \infty. \end{cases} \quad (1.6)$$

Let us introduce a common critical space to both the problems (1.1) and (1.3) and consider the singular limit problem (1.2).

Definition (Admissible pairs). For a pair of the exponents (θ, q) , (σ, r) they are called the scaling invariant (Serrin) admissible if

$$\begin{cases} \frac{2}{\theta} + \frac{n}{q} = 2, & \frac{n}{2} < q < \theta, \quad 2 \leq \theta, \\ \frac{2}{\sigma} + \frac{n}{r} = 1, & n < r < \sigma, \quad 1 \leq \frac{rq}{r+q}. \end{cases} \quad (1.7)$$

We frequently choose $\theta = \sigma$ as in (1.6) and we regard (θ, q) and (θ, r) as the admissible in the following. Then the solution is invariant under the scaling in the Bochner spaces (1.7) above. The spatial exponents are restricted by (1.7) and

$$1 \geq \frac{1}{r} + \frac{1}{q} = \frac{3}{n} - \frac{4}{n\theta} \iff \frac{4}{\theta} \geq 3 - n, \quad (1.8)$$

which implies $2 \leq \theta \leq 4$ when $n = 2$. The end-point cases $(\theta, q) = (\infty, n/2)$ or $(2, n)$, and $(\theta, r) = (\infty, 2)$, $(2, \infty)$, in particular the case

$$\begin{cases} u \in BC(\mathbb{R}_+; L^{\frac{n}{2}}(\mathbb{R}^n)), \\ \psi \in BC(\mathbb{R}_+; BMO(\mathbb{R}^n)) \end{cases}$$

is excluded in (1.7) in [34,35], where $BC(I; X)$ denotes the class of bounded continuous functions from an interval I to a Banach space X , and such estimates are treated in a different way.

We first show the time local well-posedness and the global well-posedness with small data to both (1.1) and (1.3) in the scaling critical function space, i.e., in the Bochner space with the admissible exponents. Then we show the singular limit problem $\tau \rightarrow \infty$ for the solution to the Keller–Segel system (1.1) and the limiting function indeed solves the problem (1.3). To this end, we introduce a generalization of the space time estimate for the parabolic

equation in the critical homogeneous Besov spaces and apply such estimates to establishing the singular limit problem in the scaling invariant class (1.4). We notice that it is possible to construct a solution to (1.1) in the class indicated in (1.7) with different exponents $\theta \neq \sigma$, however this is not suitable for the singular limit problem, since the solution to the limiting problem (1.3) has to have the same exponent for the time integration. We also note that a slightly general system including the medical models involving the chemotaxis structure are considered and a very much related singular limit is also proven. See for instance [44]. The main difference of results here from the former results [34,35] is that the solution here is considered in the end-point critical function class $VMO(\mathbb{R}^n)$ for time local solution and $BMO(\mathbb{R}^n)$ for small solution. In the proof, generalized maximal regularity is extended in Theorem 2.5 (more precisely, Proposition 5.6) and it is used for that purpose. We also employ maximal regularity in the class of $BMO(\mathbb{R}^n)$ shown in Ogawa–Shimizu [48].

1.2 The local and global well-posedness

To consider the well-posedness of the system (1.1) in both the time local and the time global cases, one of difficulty lying in the problem is regularity for the limiting solutions ψ since it must solve either the Poisson or the Helmholtz equations and under the critical setting; $u \in L^{\frac{n}{2}}(\mathbb{R}^n)$, ψ does not necessarily belong to $L^\infty(\mathbb{R}^n)$, where $L^\infty(\mathbb{R}^n)$ is the corresponding scaling critical space observed in (1.5). To avoid such a difficulty, we introduce the class of bounded mean oscillation (BMO).

For a measurable function $f = f(x)$ with $x \in \mathbb{R}^n$,

$$BMO = BMO(\mathbb{R}^n) = \{f \in L^1_{loc}(\mathbb{R}^n); \|f\|_{BMO} < \infty\},$$

where

$$\|f\|_{BMO} \equiv \sup_{x \in \mathbb{R}^n, R > 0} \left(\frac{1}{|B_R|} \int_{B_R(x)} |f(y) - \overline{f_{B_R}}|^2 dy \right)^{1/2},$$

$$\overline{f_{B_R}} = \frac{1}{|B_R|} \int_{B_R(x)} f(y) dy$$

and $B_R(x)$ denotes n -dimensional ball centered at x with a radius $R > 0$. For $s > 0$, let

$$B\dot{M}O^s = B\dot{M}O^s(\mathbb{R}^n) \equiv \left\{ f \in L^1_{loc}(\mathbb{R}^n); |\nabla|^s f \in BMO(\mathbb{R}^n) \right\}.$$

We set $VMO = VMO(\mathbb{R}^n)$ (vanishing mean oscillation) by a completion of $C_0^\infty(\mathbb{R}^n)$ by BMO semi-norm, i.e., $VMO(\mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R}^n)}^{BMO}$.

The classes BMO and VMO are quasi-Banach spaces and if we identify the elements of BMO up to constant, they are regarded as the Banach space. We also introduce the space-time space $\widetilde{L^2(I; BMO(\mathbb{R}^n))}$ that is essentially (but only inhomogeneous version) used by Koch–Tataru [26] for solving the incompressible Navier–Stokes equation in the limiting scaling invariant class.

Definition. For $I = (0, T)$ with $T \leq \infty$,

$$\widetilde{L^2(I; BMO(\mathbb{R}^n))} \equiv \left\{ f = f(t, x); I \times \mathbb{R}^n \rightarrow \mathbb{R}; \|f\|_{\widetilde{L^2(I; BMO)}} < \infty \right\},$$

with

$$\begin{aligned} & \|f\|_{\widetilde{L^2(I; BMO)}} \\ & \equiv \sup_{x_0 \in \mathbb{R}^n, R > 0} \left(\int_{I \cap (0, R^2)} \frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |f(t, x) - f(t, y)|^2 dx dy dt \right)^{1/2} \\ & \simeq \sup_{x_0 \in \mathbb{R}^n, R > 0} \left(\int_{I \cap (0, R^2)} \frac{1}{|B_R|} \int_{B_R(x_0)} |f(t, x) - \overline{f_{B_R}}(t)|^2 dx dt \right)^{1/2}, \end{aligned}$$

where

$$\overline{f_{B_R}}(t) = \frac{1}{|B_R|} \int_{B_R(x_0)} f(t, x) dx.$$

The corresponding space $\widetilde{L^2(I; VMO(\mathbb{R}^n))}$ is similarly introduced by applying the closure. The equivalent norm in the above can be identify from this definition of mean average immediately.

We also introduce the homogeneous Besov spaces (cf. [59]). Let $\{\phi_j\}_{j \in \mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity, i.e., $\hat{\phi}$ is the Fourier transform of a smooth radial function ϕ with $\hat{\phi}(\xi) \geq 0$,

$$\text{supp } \hat{\phi} \subset \{\xi \in \mathbb{R}^n \mid 2^{-1} < |\xi| < 2\}$$

and

$$\sum_{j \in \mathbb{Z}} \hat{\phi}_j(\xi) = 1, \quad \hat{\phi}_j(\xi) = \hat{\phi}(2^{-j}\xi)$$

for all $\xi \neq 0$. For $s \in \mathbb{R}$ and $1 \leq p, \sigma \leq \infty$, we define the homogeneous Besov space $\dot{B}_{p,\sigma}^s(\mathbb{R}^n)$ by

$$\dot{B}_{p,\sigma}^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' ; f = c_n^{-1} \sum_{j \in \mathbb{Z}} \phi_j * f \text{ in } \mathcal{S}', \|f\|_{\dot{B}_{p,\sigma}^s} < \infty \right\}$$

with $c_n = (2\pi)^{-n/2}$ and the norm

$$\|f\|_{\dot{B}_{p,\sigma}^s} \equiv \begin{cases} \left(\sum_{j \in \mathbb{Z}} 2^{js\sigma} \|\phi_j \hat{*} f\|_p^\sigma \right)^{1/\sigma}, & 1 \leq \sigma < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\phi_j \hat{*} f\|_p, & \sigma = \infty, \end{cases} \quad (1.9)$$

where $f \hat{*} g = c_n^{-1} f * g$ denotes the convolution with a constant correction. We also define the inhomogeneous Besov space $B_{p,\sigma}^s(\mathbb{R}^n)$ by

$$B_{p,\sigma}^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' ; \|f\|_{B_{p,\sigma}^s} < \infty \right\}$$

with sn the norm

$$\|f\|_{B_{p,\sigma}^s} \equiv \begin{cases} \left(\|\phi_0 \hat{*} f\|_p^\sigma + \sum_{j \geq 0} 2^{js\sigma} \|\phi_j \hat{*} f\|_p^\sigma \right)^{1/\sigma}, & 1 \leq \sigma < \infty, \\ \sup_{j \geq 0, j=\hat{0}} 2^{js} \|\phi_j \hat{*} f\|_p, & \sigma = \infty, \end{cases} \quad (1.10)$$

where $\widehat{\phi}_0 = 1 - \sum_{j \geq 0} \widehat{\phi}_j$.

We define the mild (strong) solution of system (1.1) and (1.3). Let $e^{t\Delta}$ denote the heat evolution operator given by

$$e^{t\Delta}u_0 \equiv \int_{\mathbb{R}^n} G_t(x-y)u_0(y)dy \quad \text{for } u_0 \in C_0(\mathbb{R}^n),$$

where $G_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right)$ is the Gauss kernel for $t > 0$.

Definition. Let $\tau > 0$ and $\lambda \geq 0$. For initial data $(u_0, \psi_0) \in L^{\frac{n}{2}}(\mathbb{R}^n) \times VMO(\mathbb{R}^n)$, (u_τ, ψ_τ) is a (mild) solution to (1.1) if the following integral equation is solved:

$$\begin{cases} u_\tau(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} \nabla \cdot (u_\tau(s) \nabla \psi_\tau(s)) ds, \\ \psi_\tau(t) = e^{\tau(\Delta-\lambda)} \psi_0 + \int_0^t e^{\tau(t-s)(\Delta-\lambda)} \tau u_\tau(s) ds \end{cases} \quad (1.11)$$

in $C(I; L^{\frac{n}{2}}(\mathbb{R}^n)) \times C(I; VMO(\mathbb{R}^n))$.

For initial data $u_0 \in L^{\frac{n}{2}}(\mathbb{R}^n)$, (u, ψ) is a (mild) solution to (1.3) if the following integral equation is solved:

$$\begin{cases} u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} \nabla \cdot (u(s) \nabla \psi(s)) ds, \\ \psi(t) = (\lambda - \Delta)^{-1} u(t) = \int_0^\infty e^{s(\Delta-\lambda)} u(t) ds \end{cases} \quad (1.12)$$

in $C(I; L^{\frac{n}{2}}(\mathbb{R}^n)) \times C(I; BMO(\mathbb{R}^n))$, where

$$(-\Delta)^{-1}u \equiv \begin{cases} \omega_n |x|^{-(n-2)} * u(x), & n \geq 3, \\ -\frac{1}{2\pi} \log |x| * u(x), & n = 2, \end{cases}$$

and

$$(\lambda - \Delta)^{-1}u \equiv B_\lambda * u,$$

with the Bessel potential given by

$$B_\lambda \equiv \int_0^\infty e^{-\lambda t} G_t(x) dt$$

and the heat kernel $G_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$. We choose pairs of exponents for the solution class as (θ, q) and (σ, r) defined in (1.7). Then a natural class for the common initial data is $\theta = \sigma$ and it is indeed given by the sharp trace estimate from the semi-group representation in the real interpolation theory such as

$$\left\{ \begin{array}{l} \|e^{t\Delta}u_0\|_{L^\theta(I;L^q)} < \infty, \\ \|\nabla e^{t\Delta}\psi_0\|_{L^\theta(I;L^r)} < \infty, \end{array} \right\} \iff (u_0, \psi_0) \in \dot{B}_{q,\theta}^{-\frac{2}{\theta}}(\mathbb{R}^n) \times \dot{B}_{r,\theta}^{1-\frac{2}{\theta}}(\mathbb{R}^n). \quad (1.13)$$

Or one can restrict the class of ψ_0 itself by choosing at $\theta = 2$ and $r = \infty$ to have

$$\|\psi_0\|_{\dot{B}_{\infty,2}^0} \simeq \|\nabla \psi_0\|_{\dot{B}_{r,\theta}^{1-\frac{2}{\theta}}}. \quad (1.14)$$

Hence we introduce a common class for the initial data and consider the equation in the class

$$(u_\tau(t), \psi_\tau(t)) \in \left(C(I; L^{\frac{n}{2}}(\mathbb{R}^n) \cap \dot{B}_{q,\theta}^{-\frac{2}{\theta}}(\mathbb{R}^n)) \right) \times \left(C(I; VMO(\mathbb{R}^n) \cap \dot{B}_{r,\theta}^{1-\frac{2}{\theta}}(\mathbb{R}^n)) \right).$$

One can find regularity of the solution ψ in Proposition 1.2 as $\psi \in BMO(\mathbb{R}^n)$ (cf. [28,29]). On the other hand the solution to (1.1), it is not clear if the similar regularity can be obtained. We first illustrate that such a common space is possible for both of system (1.1) and (1.3).

Theorem 1.1 *Let $n \geq 2$, $\tau > 0$, $\lambda \geq 0$ and (θ, q) and (θ, r) be admissible pairs defined in (1.7).*

1. *Let $(u_0, \psi_0) \in (L^{\frac{n}{2}}(\mathbb{R}^n) \cap \dot{B}_{q,\theta}^{-\frac{2}{\theta}}(\mathbb{R}^n)) \times (VMO(\mathbb{R}^n) \cap \dot{B}_{r,\theta}^{1-\frac{2}{\theta}}(\mathbb{R}^n))$. Then there exist $T = T(u_0, \psi_0) > 0$ and the unique strong solution (u_τ, ψ_τ) to (1.1) in*

$$\begin{aligned} u_\tau &\in C([0, T]; L^{\frac{n}{2}}(\mathbb{R}^n)) \cap L^\theta(0, T; L^q(\mathbb{R}^n)) \cap L^2(0, T; L^2(\mathbb{R}^n)), \\ \psi_\tau &\in C([0, T]; VMO(\mathbb{R}^n)) \cap L^\theta(0, T; \dot{W}^{1,r}(\mathbb{R}^n)) \cap \widetilde{L^2([0, T]; V\dot{M}O^1(\mathbb{R}^n))}. \end{aligned}$$

Furthermore, the solution satisfies the regularity estimates: For any admissible pairs (θ, q) and (θ, r) , there exists $M > 0$ independent of $\tau > 0$ and $\lambda \geq 0$ such that

$$\begin{aligned} &\sup_{t \in [0, T]} \|u_\tau(t)\|_{L^{\frac{n}{2}}} + \sup_{1 < q \leq 2} \|u_\tau\|_{L^\theta(0, T; L^q)} \\ &\quad + \sup_{t \in [0, T]} \|\psi_\tau\|_{VMO} + \|\nabla \psi_\tau\|_{L^\theta(0, T; L^r)} + \|\nabla \psi_\tau\|_{\widetilde{L^2(0, T; VMO)}} \quad (1.15) \\ &\leq M. \end{aligned}$$

2. *Let $(u_0, \psi_0) \in (L^{\frac{n}{2}}(\mathbb{R}^n) \cap \dot{B}_{q,\theta}^{-\frac{2}{\theta}}(\mathbb{R}^n)) \times (BMO(\mathbb{R}^n) \cap \dot{B}_{r,\theta}^{1-\frac{2}{\theta}}(\mathbb{R}^n))$ and for some $\varepsilon_0 > 0$,*
- $$\|u_0\|_{L^{\frac{n}{2}}} + \|\psi_0\|_{BMO} \leq \varepsilon_0, \quad (1.16)$$

then there exists a unique global solution (u_τ, ψ_τ) to (1.1) such that

$$\begin{aligned} u_\tau &\in BC(\mathbb{R}_+; L^{\frac{n}{2}}(\mathbb{R}^n)) \cap L^\theta(\mathbb{R}_+; L^q(\mathbb{R}^n)) \cap L^2(\mathbb{R}_+; L^2(\mathbb{R}^n)), \\ \psi_\tau &\in L^\infty(\mathbb{R}_+; BMO(\mathbb{R}^n)) \cap L^\theta(\mathbb{R}_+; \dot{W}^{1,r}(\mathbb{R}^n)) \cap \widetilde{L^2(\mathbb{R}_+; B\dot{M}O^1(\mathbb{R}^n))} \end{aligned}$$

with

$$\psi_\tau(t, x) \rightarrow \psi_0 \quad * \text{-weakly in } BMO \text{ as } t \rightarrow 0,$$

where $\mathbb{R}_+ = (0, \infty)$. Furthermore, the solution satisfies the a priori estimate: For any admissible pairs (θ, q) and (θ, r) ,

$$\begin{aligned} &\sup_{t > 0} \|u_\tau(t)\|_{L^{\frac{n}{2}}} + \sup_{1 < q \leq 2} \|u_\tau\|_{L^\theta(\mathbb{R}_+; L^q)} \\ &\quad + \sup_{t > 0} \|\psi_\tau(t)\|_{BMO} + \|\nabla \psi_\tau\|_{L^\theta(\mathbb{R}_+; L^r)} + \|\nabla \psi_\tau\|_{\widetilde{L^2(\mathbb{R}_+; BMO)}} \leq \tilde{\varepsilon}_0, \end{aligned}$$

where $\tilde{\varepsilon}_0$ is independent of $\tau > 0$.

The extra assumption $(u_0, \psi_0) \in \dot{B}_{q,\theta}^{-\frac{2}{\theta}}(\mathbb{R}^n) \times \dot{B}_{r,\theta}^{1-\frac{2}{\theta}}(\mathbb{R}^n)$ on the initial data is required for estimates involving maximal regularity. Indeed, such an assumption on the initial data can

be removed if we employ the weaker topology for constructing the solution such as

$$\begin{aligned} \sup_{t>0} \|u_\tau(t)\|_{L^{\frac{n}{2}}} + \sup_{t>0} t^{\frac{1}{\theta}} \|u_\tau(t)\|_{L^q} &< \infty, \\ \sup_{t>0} \|\psi_\tau(t)\|_{VMO} + \|\nabla \psi_\tau\|_{\widetilde{L^2(I; VMO)}} &< \infty. \end{aligned}$$

Then the assumption on the initial data can be relaxed into the simplest way $(u_0, \psi_0) \in L^{\frac{n}{2}}(\mathbb{R}^n) \times VMO(\mathbb{R}^n)$. However those function spaces are not suitable for applying the singular limit problem directly as we see below (see Sect. 1.3). Then we modify the existence class as following: For any small $\eta_0 > 0$

$$\begin{aligned} u_\tau &\in C([0, T]; L^{\frac{n}{2}}(\mathbb{R}^n)) \cap C(0, T; L^2(\mathbb{R}^n)) \cap L^\theta(\eta_0, T; L^q(\mathbb{R}^n)), \\ \psi_\tau &\in C([0, T]; VMO(\mathbb{R}^n)) \cap \widetilde{L^2([0, T]; V\dot{M}O^1(\mathbb{R}^n))} \cap L^\theta(\eta_0, T; \dot{W}^{1,r}(\mathbb{R}^n)). \end{aligned}$$

This is possible because the solution is getting smoother after $t > 0$ and $(u_\tau, \psi_\tau) \in \dot{B}_{q,\theta}^{-\frac{2}{\theta}}(\mathbb{R}^n) \times \dot{B}_{r,\theta}^{1-\frac{2}{\theta}}(\mathbb{R}^n)$.

The corresponding solvability of the initial value problem (1.3) has already known in non-critical space (Kurokiba–Ogawa [31,32]), and the critical space (Nagai–Ogawa [41]). Here we show the result as a summary:

Proposition 1.2 [31,41] *Let $n \geq 2$ and (θ, q) and (θ, r) be admissible pairs defined in (1.7) and $\lambda \geq 0$.*

1. Assume that $u_0 \in L^{\frac{n}{2}}(\mathbb{R}^n) \cap \dot{B}_{q,\theta}^{-\frac{2}{\theta}}(\mathbb{R}^n)$. If $n = 2$, then assume further that

$$\log(1 + |x|)u_0 \in L^1(\mathbb{R}^2). \quad (1.17)$$

Then there exists $T = T(u_0) > 0$ such that the unique strong solution (u, ψ) to (1.3) exists and

$$\begin{aligned} u &\in C([0, T]; L^{\frac{n}{2}}(\mathbb{R}^n)) \cap L^\theta(0, T; L^q(\mathbb{R}^n)), \\ \psi &\in C([0, T]; VMO(\mathbb{R}^n)) \cap L^\theta([0, T]; \dot{W}^{1,r}(\mathbb{R}^n)) \cap \widetilde{L^2([0, T]; V\dot{M}O^1(\mathbb{R}^n))}. \end{aligned}$$

2. Assume that $u_0 \in L^{\frac{n}{2}}(\mathbb{R}^n) \cap \dot{B}_{q,\theta}^{-\frac{2}{\theta}}(\mathbb{R}^n)$ with (1.17) for $n = 2$. Then there exists $\varepsilon_0 > 0$ such that for any $u_0 \in L^{\frac{n}{2}}(\mathbb{R}^n)$ with

$$\|u_0\|_{L^{\frac{n}{2}}} \leq M_*, \quad (1.18)$$

there exists a unique global solution (u, ψ) to (1.3) such that

$$\begin{aligned} u &\in BUC(\mathbb{R}_+; L^{\frac{n}{2}}(\mathbb{R}^n)) \cap L^\theta(\mathbb{R}_+; L^q(\mathbb{R}^n)), \\ \psi &\in L^\infty(\mathbb{R}_+; BMO(\mathbb{R}^n)) \cap L^\theta(\mathbb{R}_+; \dot{W}^{1,r}(\mathbb{R}^n)) \cap \widetilde{L^2(\mathbb{R}_+; B\dot{M}O^1(\mathbb{R}^n))}. \end{aligned}$$

Furthermore, the solution satisfies the a priori estimate: If $\|u_0\|_1 \leq \varepsilon_0$,

$$\sup_{t>0} \|u(t)\|_{L^{\frac{n}{2}}} + \|u\|_{L^\theta(\mathbb{R}_+; L^q)} + \sup_{t>0} \|\psi(t)\|_{BMO} + \|\nabla \psi\|_{\widetilde{L^2(\mathbb{R}_+; BMO)}} \leq \tilde{\varepsilon}_0.$$

Remark. When $n = 2$, the extra-assumption on the initial data $u_0 \in \dot{B}_{q,\theta}^{-\frac{2}{\theta}}(\mathbb{R}^2)$ with the admissible relation

$$\frac{1}{\theta} + \frac{1}{q} = 1$$

in Theorem 1.1 and Proposition 1.2 prevents the non-negative data $u_0 \geq 0$ unless it is trivial, since the regularity restriction necessarily implies the mean value zero of its elements (cf. [21]). In order to avoid such inconvenience, we assume that the initial data u_0 belongs to the inhomogeneous Besov space $VMO(\mathbb{R}^2) \cap B_{q,\theta}^{-\frac{2}{\theta}}(\mathbb{R}^2)$. In this case, all the statement remains valid over the time local interval $I = (0, T)$ for arbitrary $T < \infty$. However the usage of inhomogeneous Besov space does not allows to apply maximal regularity uniformly in time. The constant appears in the estimate of maximal regularity depends on T as $C \simeq O((\log T)^{1/\theta})$ as $T \rightarrow \infty$ (see Appendix below for the detail). Hence the statements should be restricted for arbitrary finite interval and the bound in (1.15) depends on T .

As is well-known, the threshold mass M_* in (1.18) is known as 8π if $n = 2$ and u_0 is non-negative [3, 5, 41, 43]. For the large initial data $\|u_0\|_1 > 8\pi$, the positive solution blows up in a finite time (Bilar [3], Nagai [38, 39], Kurokiba–Ogawa [31], Nagai–Ogawa [40], Wei [60]). If $n \geq 3$ then the $M_* \geq \frac{8}{nS_b^2}$ where S_b is the best possible constant of the Sobolev inequality

$$\|f\|_{L^{\frac{2n}{n-2}}}^2 \leq S_b^2 \|\nabla f\|_{L^2}^2.$$

However we expect that the threshold for the global existence may be

$$M_* = \left(\frac{2n}{n-2} C_{HLS}^{-1} \right),$$

where C_{HLS} is the best possible constant of the Hardy–Littlewood–Sobolev inequality

$$\int_{\mathbb{R}^n} f(-\Delta)^{-1} f dx \leq C_{HLS} \|f\|_{L^1} \|f\|_{L^{\frac{n}{2}}}. \quad (1.19)$$

For the large data case, the positive solutions blow up in a finite time, namely there exists a maximal existence time $0 < T_* < \infty$ such that

$$\limsup_{t \rightarrow T_*} \|u(t)\|_{\frac{n}{2}} = \infty,$$

if the initial data satisfies

$$H[u_0] \leq \frac{n}{2} \|u_0\|_1 \log \left(\frac{n \|u_0\|_1^{1+\frac{2}{n}}}{2\pi e^{\frac{n}{n-2}} \int_{\mathbb{R}^n} |x - x_0|^2 u_0(x) dx} \right), \quad (1.20)$$

where

$$H[u_0] \equiv \int_{\mathbb{R}^n} u_0(x) \log u_0(x) dx - \frac{1}{2} \int_{\mathbb{R}^n} u_0(x) (\lambda - \Delta)^{-1} u_0(x) dx$$

which stands for the Helmholtz free energy (see Calvez–Corrius–Ebdes [9], Ogawa–Wakui [51]). Indeed, one can observe that a solution has a concentration phenomena when the solution blows up in finite time (see [50]). We should like to notice that the initial data satisfies

$$\|u_0\|_{\frac{n}{2}} \geq \left(\frac{2n}{n-2} C_{HLS}^{-1} \right)$$

under the restriction (1.20). The existence and the uniqueness of the solution to (1.3) for $u_0 \in L^{\frac{n}{2}}(\mathbb{R}^n)$ is considered in Kozono–Sugiyama–Yahagi [29], where $\psi \in C(I; BMO(\mathbb{R}^n))$.

The main difference between the previous result [34] and ours is that ψ_τ nor ψ do not belong to $\dot{W}^{1,2}(\mathbb{R}^2)$ when $n = 2$ with $\lambda = 0$. Therefore we avoid to choose function space $L^\theta(0, T; \dot{W}^{1,2}(\mathbb{R}^2))$. To treat them in a unified way, we do not use the class $\dot{W}^{1,n}(\mathbb{R}^n)$ here. Instead of that we employ the class of $BMO(\mathbb{R}^n)$ and $VMO(\mathbb{R}^n)$, where we have natural regularity for ψ_τ component of the system. To this end, we employ maximal regularity for the heat equation in the class of $BMO(\mathbb{R}^n)$ and $VMO(\mathbb{R}^n)$ which is shown in Ogawa–Shimizu [48].

1.3 Singular limit problem

The solution to (1.1) is formulated by the Duhamel formula as is defined in (1.11). One can find that the singular limit problem accompanies with the initial layer because of the presence of the initial data ψ_0 . Since the system (1.1) and (1.3) have the common structure in the equation for u_τ and u , the main issue is how to formulate for the equation of ψ_τ and ψ . Indeed, noticing

$$\lim_{t' \rightarrow \infty} \int_0^{t'} e^{s\Delta} u(t) ds = (-\Delta)^{-1} u(t),$$

compare the equations ψ_τ and ψ . By employing analogous argument for proving the existence of solution via the construction mapping theorem, we show the difference of those equation converges to 0 as $\tau \rightarrow \infty$ except the initial layer. The following singular limit result is shown for the case $n = 2$ in Kurokiba–Ogawa [35] (cf. for a higher dimensional non- BMO result [34]). We generalized it for higher dimensional case:

Theorem 1.3 *For $n \geq 2$, let (θ, q) and (θ, r) be admissible pairs defined in (1.7) and assume that $(u_0, \psi_0) \in L^{\frac{n}{2}}(\mathbb{R}^n) \cap B_{q,\theta}^{-\frac{2}{\theta}}(\mathbb{R}^n) \times VMO(\mathbb{R}^n) \cap B_{r,\theta}^{1-\frac{2}{\theta}}(\mathbb{R}^n)$ with $\log(1+|x|)u_0 \in L^1(\mathbb{R}^n)$ if $n = 2$. For $\tau > 0$, let (u_τ, ψ_τ) be a unique strong solution to (1.1) in $(C(I; L^{\frac{n}{2}}(\mathbb{R}^n)) \cap L^\theta(I; L^q(\mathbb{R}^n))) \times (C(I; BMO(\mathbb{R}^n)) \cap \widetilde{L^2(I; B\dot{M}O^1(\mathbb{R}^n))})$, where $I = (0, T)$ with $T < \infty$. For $T = \infty$, the corresponding regularity conditions on the initial data (u_0, ψ_0) , i.e., $(u_0, \psi_0) \in (L^{\frac{n}{2}}(\mathbb{R}^n) \cap \dot{B}_{q,\theta}^{-\frac{2}{\theta}}(\mathbb{R}^n)) \times (BMO(\mathbb{R}^n) \cap \dot{B}_{r,\theta}^{1-\frac{2}{\theta}}(\mathbb{R}^n))$ with the smallness (1.16) be assumed.*

- (The existence of the limit solution) Then for the same initial data u_0 , there exists a unique strong solution (u, ψ) to (1.3) as

$$u \in C([0, T]; L^{\frac{n}{2}}(\mathbb{R}^n)) \cap L^\theta(0, T; L^q(\mathbb{R}^n)),$$

$$\psi \in C([0, T]; VMO(\mathbb{R}^n)) \cap L^\theta([0, T]; L^r(\mathbb{R}^n)) \cap L^2(\widetilde{[0, T]}; B\dot{M}O^1(\mathbb{R}^n)).$$

If $T < \infty$, then $\psi \in (BC(I; VMO(\mathbb{R}^n)) \cap L^\theta(I; \dot{W}^{1,r}(\mathbb{R}^n)))$.

- (Singular limit) For any admissible pairs (θ, q) and (θ, r) defined in (1.7),

$$\lim_{\tau \rightarrow \infty} \left(\|u_\tau - u\|_{L^\theta(I; L^q)} + \|\nabla(\psi_\tau - \psi)\|_{L^\theta(I; L^r)} + \|\nabla(\psi_\tau - \psi)\|_{\widetilde{L^2(I; BMO)}} \right) = 0.$$

- (Initial layer) For any $\eta_0 > 0$,

$$\sup_{t \in [\eta_0, \infty) \cap I} \|u_\tau(t) - u(t)\|_{L^{\frac{n}{2}}} + \sup_{t \in [\eta_0, \infty) \cap I} \|\psi_\tau(t) - \psi(t)\|_{VMO} \rightarrow 0, \quad \tau \rightarrow \infty. \quad (1.21)$$

Besides, $\psi_\tau(t)$ has the initial layer as $\tau \rightarrow \infty$. Namely it never converges to $\psi = (-\Delta)^{-1}u(0) = (-\Delta)^{-1}u_0$ for $t < \eta_0$.

We should like to emphasize that the solution to drift-diffusion equation (1.1) is in $C([0, T]; L^1(\mathbb{R}^n)) \times C([0, T]; VMO(\mathbb{R}^n))$ and the solution for (1.3) is in $C([0, T]; L^1(\mathbb{R}^n)) \times C([0, T]; BMO(\mathbb{R}^n))$ at most. Namely the class of the solutions ψ_τ and ψ of two system are different from each other. Nevertheless, the convergence is shown in the topology $C(I_{t_0}; L^1(\mathbb{R}^n)) \times C(I_{t_0}; VMO(\mathbb{R}^n))$ except the initial layer, where $I_{t_0} = (t_0, \infty) \cap I$ with $t_0 > 0$ because the solution u_τ has a higher regularity over $t \in (0, T)$ which enable us to treat the solutions ψ_τ and ψ are both in $VMO(\mathbb{R}^n)$.

Let us consider the formulation how to prove the singular limit problem (1.2). The external term of the ψ_τ -equation in (1.1) can be regarded by changing variable $\tau t - \tau s = s'$ ($s = t - \frac{1}{\tau}s'$)

$$\int_0^t e^{(t-s)\tau\Delta} \tau u_\tau(s) ds = \int_0^{\tau t} e^{s\Delta} u_\tau(t - \tau^{-1}s') ds'$$

and thus

$$\begin{aligned} & \int_0^t e^{\tau(t-s)\Delta} \tau u_\tau(s) ds - (-\Delta)^{-1}u(t) \\ &= \int_0^{\tau t} e^{s\Delta} \left(u_\tau(t - \tau^{-1}s) - u(t - \tau^{-1}s) \right) ds \\ & \quad + \int_0^{\tau t} e^{s\Delta} \left(u(t - \tau^{-1}s) - u(t) \right) ds - \int_{\tau t}^\infty e^{s\Delta} u(t) ds \\ & \equiv I_1 + I_2 + I_3. \end{aligned} \quad (1.22)$$

Then by using the dissipative estimates for the heat equation of u , we want to see that

$$\left\| u(t - \tau^{-1}t) - u(t) \right\|_{L^1} \rightarrow 0, \quad \tau \rightarrow \infty \quad \text{a.a. } t.$$

Such a formal computation can be justified by employing the similar argument to construct the solution. In particular, in order to justify the above procedure, one may introduce a typical metric induced from the norm such as

$$\|u\|_{\theta,q} \equiv \sup_{t>0} t^{1/\theta} \|u(t)\|_{L^q}, \quad \|\nabla \psi\|_{\theta,r} \equiv \sup_{t>0} t^{1/\theta} \|\nabla \psi\|_{L^r},$$

where (θ, q) and (θ, r) are the Serrin admissible given in (1.7). However such a choice of metric does not work well for the critical cases since the integrability conditions and the stability conditions are not consistent in the following estimate: For the first term of the right

hand side of (1.22),

$$\begin{aligned}
 & \|\nabla I_1\|_{\dot{W}^{1,r}} \\
 & \leq C \int_0^{\tau t} |s|^{-\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}} \|u_\tau(t-\frac{s}{\tau}) - u(t-\frac{s}{\tau})\|_{L^q} ds \\
 & \leq C \int_0^{\tau t} |s|^{-\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}} \left|t-\frac{s}{\tau}\right|^{-\frac{1}{\theta}} ds \|u_\tau(\cdot) - u(\cdot)\|_{\theta,q} \\
 & = C t^{-\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}-\frac{1}{\theta}} \int_0^{\tau t} \left|\frac{s}{t}\right|^{-\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}} \left|1-\frac{s}{\tau t}\right|^{-\frac{1}{\theta}} ds \|u_\tau(\cdot) - u(\cdot)\|_{\theta,q} \\
 & \leq C \tau^{\frac{1}{2}-\left(\frac{1}{q}-\frac{1}{r}\right)} t^{\frac{1}{2}-\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{\theta}} B\left(\frac{1}{2}-\left(\frac{1}{q}-\frac{1}{r}\right), 1-\frac{1}{\theta}\right) \|u_\tau(\cdot) - u(\cdot)\|_{\theta,q},
 \end{aligned} \tag{1.23}$$

where $B(p, q)$ denotes the Beta function given by

$$B(p, q) = \int_0^1 (1-s)^{p-1} s^{q-1} ds.$$

Now we notice that the condition on the convergence of the Beta function requires

$$\frac{1}{2} - \left(\frac{1}{q} - \frac{1}{r}\right) > 0,$$

while the stable condition on the exponent τ should be given by

$$\frac{1}{2} - \left(\frac{1}{q} - \frac{1}{r}\right) \leq 0$$

to justify the singular limit $\tau \rightarrow \infty$. Unfortunately those conditions do not hold simultaneously.

1.4 Usage of generalized maximal regularity

To overcome such a difficulty, we employ maximal or generalized maximal regularity for the heat equation and use the space time estimates. In general, for the solution of the Cauchy problem of the heat equation;

$$\begin{cases} \partial_t v - \Delta v = f, & t > 0, x \in \mathbb{R}^n, \\ v(0) = 0, \end{cases}$$

maximal regularity is given by the following inequality: For any $1 \leq \rho \leq \theta$ with (θ, q) as Serrin's admissible exponent, it holds that

$$\|\partial_t |\nabla|^{-2} v\|_{L^\theta(I; L^q)} + \|v\|_{L^\theta(I; L^q)} \leq C \|f\|_{L^\rho(I; \dot{B}_{q,\theta}^{-2+\frac{2}{\rho}-\frac{2}{\theta}})}, \tag{1.24}$$

where $\dot{B}_{q,\theta}^{-2+\frac{2}{\rho}-\frac{2}{\theta}}$ denotes the homogeneous Besov space given by (1.9). Applying such an estimate to the difference of the solutions to (1.1) and (1.3); $\psi_\tau(t) - \psi(t)$. Since the difference

solves the following equation:

$$\begin{cases} \frac{1}{\tau} \partial_t \psi_\tau - \Delta(\psi_\tau - \psi) + \lambda(\psi_\tau - \psi) = u_\tau - u, \\ (\psi_\tau - \psi)(0) = \psi_0 - (-\Delta + \lambda)^{-1} u_0. \end{cases}$$

Since $\dot{B}_{\infty,2}^0(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$, we apply the estimate (1.24) to the corresponding integral equation to the above equation with $\theta = 2$ and $q = \infty$ for the main part I_1 that

$$\begin{aligned} \|\nabla I_1\|_{\widetilde{L^2(I; VMO)}} &\equiv \left\| \nabla \int_0^\tau e^{s(\Delta-\lambda)} (u_\tau(t-\tau^{-1}s) - u(t-\tau^{-1}s)) ds \right\|_{L^2(I; \dot{B}_{\infty,2}^0)} \\ &\leq C \|u_\tau((1-\tau^{-1})t) - u((1-\tau^{-1})t)\|_{L^2(I; \dot{B}_{\infty,2}^{-1})} \\ &\leq C (1-\tau^{-1})^{-1/\theta} \|u_\tau(t) - u(t)\|_{L^2(I; \dot{B}_{2,2}^0)} \\ &\leq C \|u_\tau - u\|_{L^2(I; L^2)} \end{aligned}$$

for all $\tau \geq 2$, where we used the Sobolev embedding $L^q(\mathbb{R}^n) \subset \dot{W}^{-1,r}(\mathbb{R}^n)$ for $1/r = 1/q - 1/n$. Hence we can avoid the difficulty appeared in (1.23) and can show the singular limit.

In what follows, for $1 \leq p, r, \theta \leq \infty$ and $s \in \mathbb{R}$, let ℓ_p^s be a real sequence space determined by

$$\|\{a_k\}_{k \in \mathbb{Z}}\|_{\ell_p^s} \equiv \left(\sum_{k \in \mathbb{Z}} 2^{spk} |a_k|^p \right)^{\frac{1}{p}} < \infty,$$

and let $L^p = L^p(\mathbb{R}^n)$ and $\dot{W}^{s,p} = \dot{W}^{s,p}(\mathbb{R}^n)$ be the Lebesgue and the homogeneous Sobolev spaces in the variable x whose norm is written by $\|f\|_p \equiv \|f\|_{L^p}$ and $\|f\|_{\dot{W}^{s,p}} \equiv \| |\nabla|^s f \|_p$. Let $L^\theta(I; X)$ be a Bochner class on the Banach space X over the time interval $I = (0, T)$ ($T \leq \infty$). For $s \in \mathbb{R}$ and $1 \leq \sigma \leq \infty$, let $\dot{F}_{p,\sigma}^s = \dot{F}_{p,\sigma}^s(\mathbb{R}^n)$ the homogeneous Lizorkin-Triebel spaces and the norms of the space is given by the following: For $1 \leq p, \sigma \leq \infty$ and $s \in \mathbb{R}$,

$$\|f\|_{\dot{F}_{p,\sigma}^s} \equiv \left\| \|\{\phi_j \hat{*} f\}_{j \in \mathbb{Z}}\|_{\ell_\sigma^s} \right\|_p = \left\| \left(\sum_{j \in \mathbb{Z}} 2^{s\sigma j} |\phi_j \hat{*} f|^\sigma \right)^{1/\sigma} \right\|_p,$$

where $\{\phi_j\}$ denotes the Littlewood-Paley dyadic decomposition of unity and $\phi_j \hat{*} f$ denotes the convolution operation with ϕ_j and f with a correction of a constant. In particular, we notice that $\dot{F}_{p,2}^s \simeq \dot{W}^{s,p}$ by the well-known Littlewood-Paley theorem. We also denote the homogeneous Besov spaces $\dot{B}_{p,\sigma}^s(\mathbb{R}^n)$ as $\dot{B}_{p,\sigma}^s$ for simplicity. We should notice that the inclusion of the sequence space $\ell_\theta \subset \ell_\sigma$ directly gives that $\dot{B}_{p,\theta}^s \subset \dot{B}_{p,\sigma}^s$ and $\dot{F}_{p,\theta}^s \subset \dot{F}_{p,\sigma}^s$ if $\theta \leq \sigma$. Let the Hardy space $\mathcal{H}^1 \equiv \mathcal{H}^1(\mathbb{R}^n)$ be defined by

$$\mathcal{H}^1 = \mathcal{H}^1(\mathbb{R}^n) = \left\{ f \in L_{loc}^1(\mathbb{R}^2); \quad \|f\|_{\mathcal{H}^1} \equiv \left\| \sup_{\lambda > 0} |\varphi_\lambda * f| \right\|_1 < \infty \right\}$$

for a test function $\varphi \in \mathcal{S}$ with $\varphi_\lambda(x) = \lambda^n \varphi(\lambda x)$ for $\lambda > 0$, and the Hardy-Sobolev spaces $\dot{\mathcal{H}}^{s,1}$ is analogously introduced in a similar way as

$$\dot{\mathcal{H}}^{s,1} = \dot{\mathcal{H}}^{s,1}(\mathbb{R}^n) = \{f \in \mathcal{S}^*(\mathbb{R}^n)/\mathcal{P}; \quad \|f\|_{\dot{\mathcal{H}}^{s,1}} \equiv \| |\nabla|^s f \|_{\mathcal{H}^1} < \infty\}$$

for all $s \in \mathbb{R}$. Note that $\dot{\mathcal{H}}^s(\mathbb{R}^n)$ is the Banach spaces if $s < n$.

2 Preliminary estimates

2.1 Inequalities and embeddings in the critical spaces

In general, the function inequality in two dimensional Euclidean space is different from higher dimensions. Here we summarized the Sobolev type inequalities in two dimensions.

Lemma 2.1 *Let $n \geq 2$ and let $f = f(x)$ be a measurable function on $\mathbb{R}^n$. There exists a constant $C = C(n) > 0$ such that the following inequality hold:*

$$\|f\|_{\dot{B}^{\frac{n}{n-1}}} \leq C \|\nabla f\|_1. \quad (2.1)$$

$$\|f\|_{\dot{B}^{\frac{n}{n-1}, \sigma}} \leq C \|f\|_{\dot{B}^0_{1, \sigma}}, \quad 1 \leq \sigma \leq \infty. \quad (2.2)$$

$$\|\nabla g\|_{\dot{B}^{\frac{n}{n-1}, 2}} \leq C \|\nabla g\|_1. \quad (2.3)$$

$$\| |\nabla|^{-1} \nabla g \|_{\dot{B}^{\frac{n}{n-1}}} \leq C \|\nabla g\|_1. \quad (2.4)$$

$$\|f\|_{BMO} \leq C \|\nabla f\|_n. \quad (2.5)$$

$$\| |\nabla|^{-1} f \|_{BMO} \leq C \|f\|_n. \quad (2.6)$$

$$\| |\nabla|^{-1} f \|_{\dot{B}^0_{\infty, 2}} \leq C \|f\|_n. \quad (2.7)$$

Proof of Lemma 2.1 The inequality (2.1) is due to Gagliardo and Nash, and is obtained by a straightforward computation: Integrating the both side of the following identity

$$|f(x)|^{\frac{n}{n-1}} = \left(\left| \int_{-\infty}^{x_1} \partial_1 f(y_1, x_2, \dots, x_n) dy_1 \right| \cdots \left| \int_{-\infty}^{x_n} \partial_n f(x_1, x_2, \dots, y_n) dy_n \right| \right)^{\frac{1}{n-1}},$$

and applying the Hölder inequality in each variable by n times, we obtain (2.1) (see e.g., Brezis [7]). The embedding (2.2) and (2.7) are direct consequences of the Bernstein type lemma and Hausdorff-Young's inequality. (2.3) and (2.4) follow from (2.1) and the boundedness of the singular integral operator in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$. The inequality (2.5) follows from the Poincaré inequality in n -dimensions:

$$\int_{B_R(x)} |f - \bar{f}_{B_R}|^n dx \leq C |B_R| \|\nabla f\|_n^n$$

with

$$\bar{f}_{B_R} \equiv \frac{1}{|B_R|} \int_{B_R(x_0)} f(y) dy.$$

The inequality (2.6) is a consequence from (2.5) and the boundedness of the singular integral operators in BMO . $\square$

We notice that the following inequalities generally fail:

$$\begin{aligned} \|f\|_{\infty} &\leq C \|\nabla f\|_n, \\ \|f\|_{\dot{W}^{-1, \infty}} &\leq C \|f\|_n. \end{aligned}$$

Lemma 2.2 *Let $n \geq 1$ and $f \in \dot{B}^0_{\infty, 2}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that*

$$\|f\|_{BMO} \leq C \|f\|_{\dot{B}^0_{\infty, 2}}. \quad (2.8)$$

Proof of Lemma 2.2 For $f \in BMO(\mathbb{R}^n)$, it is known that

$$\|f\|_{BMO} \simeq \sup_{x \in \mathbb{R}^n, k \in \mathbb{Z}} \left(\sum_{j \geq k} \frac{1}{|B_{2^{-k}}|} \int_{B_{2^{-k}}(x)} |\phi_j * f(x)|^2 dx \right)^{1/2} < \infty, \quad (2.9)$$

(cf. Frazier-Jawerth [16]) and the inequality (2.8) follows immediately from (2.9) that

$$\begin{aligned} & \sup_{x \in \mathbb{R}^n, k \in \mathbb{Z}} \left(\sum_{j \geq k} \frac{1}{|B_{2^{-k}}|} \int_{B_{2^{-k}}(x)} |\phi_j * f(x)|^2 dx \right)^{1/2} \\ & \leq \sup_{k \in \mathbb{Z}} \left(\sum_{j \geq k} \|\phi_j * f\|_{\infty}^2 \right)^{1/2} \leq \|f\|_{\dot{B}_{\infty,2}^0}. \end{aligned}$$

□

Lemma 2.3 Let (θ, q) and (θ, r) be admissible pairs defined in (1.6) with $1 \leq q \leq \theta$ and $2 \leq \theta < \infty$. Then the following continuous embeddings hold:

$$L^{\frac{qr}{q+r}}(\mathbb{R}^n) \subset \dot{W}^{-1+\frac{2}{\theta},q}(\mathbb{R}^n) \simeq \dot{F}_{q,2}^{-1+\frac{2}{\theta}}(\mathbb{R}^n) \subset \dot{F}_{q,\theta}^{-1+\frac{2}{\theta}}(\mathbb{R}^n) \subset \dot{B}_{q,\theta}^{-1+\frac{2}{\theta}}(\mathbb{R}^n). \quad (2.10)$$

Proof of Lemma 2.3 Noticing the relations

$$\frac{2}{\theta} + \frac{n}{q} = 2, \quad \frac{2}{\theta} + \frac{n}{r} = 1,$$

the first embedding is due to the Sobolev inequality

$$\|f\|_{\dot{W}^{-1+\frac{2}{\theta},q}} \leq S_b \|f\|_{\frac{rq}{r+q}}$$

with

$$\frac{1}{q} - \frac{1}{n} \left(-1 + \frac{2}{\theta} \right) = \frac{1}{r} + \frac{1}{q}.$$

The second relation is due to the well-known theorem by Littlewood–Paley: $\dot{F}_{q,2}^0(\mathbb{R}^n) \simeq L^q(\mathbb{R}^n)$ for any $1 < q < \infty$ (see Stein [56]). The third embedding is due to the property of the sequence spaces $\ell_2 \subset \ell_\theta$ under $2 \leq \theta$. The last embedding is given by the Minkowski inequality such as

$$\begin{aligned} \|f\|_{\dot{B}_{q,\theta}^{-1+\frac{2}{\theta}}} &= \left(\sum_{j \in \mathbb{Z}} 2^{(-1+\frac{2}{\theta})j\theta} \|\phi_j * f\|_q^\theta \right)^{1/\theta} \\ &\leq \left\| \left(\sum_{j \in \mathbb{Z}} 2^{(-1+\frac{2}{\theta})j\theta} |\phi_j * f|^\theta \right)^{1/\theta} \right\|_q = \|f\|_{\dot{F}_{q,\theta}^{-1+\frac{2}{\theta}}} \end{aligned}$$

under the restriction on q and θ as $q \leq \theta < \infty$. □

2.2 The heat evolution on VMO

It is well-known that the heat kernel has a dissipative estimate of L^p - L^q type: For $1 \leq q \leq p \leq \infty$,

$$\|e^{t\Delta}u_0\|_p \leq Ct^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u_0\|_q \quad (2.11)$$

for any $u_0 \in L^q(\mathbb{R}^n)$. Here is a dissipative estimate for the heat kernel on BMO and VMO.

Lemma 2.4 *Let $u_0 \in BMO(\mathbb{R}^n)$. Then $e^{t\Delta}$ is a bounded operator from $BMO(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$. If $u_0 \in VMO(\mathbb{R}^n)$, then*

$$\|e^{t\Delta}u_0\|_{VMO} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (2.12)$$

Proof of Lemma 2.4 Let $A_k = B_{k+1} \setminus B_k$ be annulus with $B_k = B_{2^k R}(x_0)$ as the ball centered at x_0 . Since $e^{t\Delta}$ is a bounded operator on $L^2(\mathbb{R}^n)$, for $u_0 \in BMO(\mathbb{R}^n)$, we see that

$$\begin{aligned} & \|e^{t\Delta}u_0\|_{BMO} \\ &= C \sup_{x_0, R>0} \left(\frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |e^{t\Delta_x}u_0(x) - e^{t\Delta_y}u_0(y)|^2 dx dy \right)^{1/2} \\ &\leq C \sup_{x_0, R>0} \left(\frac{1}{|B_R|^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |e^{t\Delta_x}e^{t\Delta_y} \right. \\ &\quad \left. \times \chi_{B_R \times B_R}(x, y)(u_0(x) - u_0(y))|^2 dx dy \right)^{1/2} \\ &\quad + 2C \sup_{x_0, R>0} \sum_{k \geq 1} \left(\frac{1}{|B_R|^2} \iint_{\mathbb{R}^n \times B_R(x_0)} |e^{t\Delta_x}e^{t\Delta_y} \right. \\ &\quad \left. \times \chi_{B_R \times A_k}(x, y)(u_0(x) - u_0(y))|^2 dx dy \right)^{1/2} \\ &\quad + C \sup_{x_0, R>0} \sum_{k \geq 1} \left(\frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |e^{t\Delta_x}e^{t\Delta_y} \right. \\ &\quad \left. \times \chi_{A_k \times A_k}(x, y)(u_0(x) - u_0(y))|^2 dx dy \right)^{1/2} \\ &\leq C \sup_{x_0, R>0} \left(\frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |u_0(x) - u_0(y)|^2 dx dy \right)^{1/2} \\ &\quad + C \sup_{x_0, R>0} \sum_{k \geq 1} \left(\frac{1}{|B_R|^2} \iint_{B_k \times B_k} \left(\iint_{A_k \times A_k} \frac{t^{1/2}}{(t^{1/2} + |x - x_1|)^{n+1}} \right. \right. \\ &\quad \left. \left. \times \frac{t^{1/2}}{(t^{1/2} + |y - y_1|)^{n+1}} |u_0(x_1) - u_0(y_1)| dx_1 dy_1 \right)^2 dx dy \right)^{1/2} \\ &\leq C \|u_0\|_{BMO} \\ &\quad + C \sum_{k \geq 1} 2^{-k} \sup_{x_0, R>0} \left(\frac{1}{|B_k|^2} \iint_{B_k \times B_k} |u_0(x_1) - u_0(y_1)| dx_1 dy_1 \right)^{1/2} \\ &\leq C \|u_0\|_{BMO}. \end{aligned} \quad (2.13)$$

See for the details Stein [57, p. 159]. Besides for any $u_0 \in VMO(\mathbb{R}^n)$ there exists a sequence $\{u_{0,n}\}_{n=1}^\infty \subset C_0(\mathbb{R}^n)$ such that for any $\varepsilon > 0$ there exists $n \gg 1$ such that

$$\|u_{0,n} - u_0\|_{VMO} < \varepsilon.$$

Then from the dissipative estimate (2.11) and (2.13),

$$\begin{aligned} \|e^{t\Delta} u_0\|_{VMO} &\leq \|e^{t\Delta} u_{0,n}\|_{VMO} + \|e^{t\Delta} (u_{0,n} - u_0)\|_{VMO} \\ &\leq \|e^{t\Delta} u_{0,n}\|_\infty + \|u_{0,n} - u_0\|_{VMO} \\ &\leq Ct^{-\frac{n}{2}} \|u_{0,n}\|_1 + \varepsilon. \end{aligned} \quad (2.14)$$

By passing $t \rightarrow \infty$, we obtain (2.12). $\square$

2.3 Generalized maximal regularity

We consider the Cauchy problem of the heat equation: For $\mu > 0$,

$$\begin{cases} \partial_t v - \mu \Delta v = f, & t > 0, x \in \mathbb{R}^n, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^n. \end{cases} \quad (2.15)$$

Then maximal regularity is given by the following way [12,46]: For any $1 \leq \rho, p \leq \infty$,

$$\|\partial_t v\|_{L^\rho(I; \dot{B}_{p,\rho}^0)} + \mu \|\nabla^2 v\|_{L^\rho(I; \dot{B}_{q,\rho}^0)} \leq C \left(\|v_0\|_{\dot{B}_{p,\rho}^{\frac{2}{\rho}}} + \|f\|_{L^\rho(I; \dot{B}_{p,\rho}^0)} \right). \quad (2.16)$$

The first estimate is well-known result from the general framework [2,13,14,18] and the cases $\rho = 1, \infty$ are generally excluded since such spaces are not UMD (unconditional Martingale difference) and it is not covered by the general theory of UMD. The remarkable feature of the latter estimates is that the estimate (2.16) allows the case $\rho = \infty$ and it is useful to estimate for applying the integral equation. On the other hand the latter estimate involves the homogeneous Besov spaces and it is not easy to make clear the relation between the Legesgue spaces since

$$\dot{B}_{p,2}^0(\mathbb{R}^n) \subset \dot{F}_{p,2}^0(\mathbb{R}^n) \simeq L^p(\mathbb{R}^n) \subset \dot{B}_{p,\infty}^0(\mathbb{R}^n)$$

if $2 \leq p < \infty$, and

$$\dot{B}_{p,1}^0(\mathbb{R}^n) \subset \dot{F}_{p,2}^0(\mathbb{R}^n) \simeq L^p(\mathbb{R}^n) \subset \dot{B}_{p,2}^0(\mathbb{R}^n)$$

if $1 \leq p \leq 2$.

We state the following slightly general form of maximal regularity.

Theorem 2.5 (Generalized maximal regularity) *Let $\mu > 0$, $1 < v \leq \rho \leq \infty$, or $1 = v = \rho$, $1 \leq p \leq \infty$, $s \in \mathbb{R}$ and let $I = (0, T) \subset \mathbb{R}_+$ be an interval (possibly $I = \mathbb{R}_+$). Given initial data $v_0 \in \dot{B}_{p,\rho}^{s+2-2/\rho}(\mathbb{R}^n)$ and external force $f \in L^v(I; \dot{B}_{p,\sigma}^{s+2/v-2/\rho}(\mathbb{R}^n))$, the solution v of the Cauchy problem (2.15) fulfills the following estimates:*

- (1) *Suppose that $f \equiv 0$. Then for any $1 \leq \sigma \leq \infty$, there exists a constant $C > 0$ independent of T and μ such that*

$$\|\partial_t v\|_{L^\rho(I; \dot{B}_{p,\sigma}^s)} + \mu \|\nabla^2 v\|_{L^\rho(I; \dot{B}_{p,\sigma}^s)} \leq C \|v_0\|_{\dot{B}_{p,\rho}^{s+2-\frac{2}{\rho}}}, \quad (2.17)$$

where $\nabla^2 = \partial_{x_i} \partial_{x_j}$.

- (2) Suppose that $u_0 \equiv 0$, then there exists a constant $C > 0$ independent of $T > 0$ and μ such that for any $v \leq \sigma \leq \rho$,

$$\|\partial_t v\|_{L^\rho(I; \dot{B}_{p,\sigma}^s)} + \mu \|\nabla^2 v\|_{L^\rho(I; \dot{B}_{p,\sigma}^s)} \leq C \|f\|_{L^v(I; \dot{B}_{p,\sigma}^{s+\frac{2}{v}-\frac{2}{\rho}})}. \quad (2.18)$$

- (3) Suppose further that $1 < v < \rho \leq \infty$ and $0 < r < 2$ with $u_0 \equiv 0$. Then there exists a constant $C > 0$ independent of $T > 0$ and μ such that

$$\|\partial_t |\nabla|^{r-2} v\|_{L^\rho(I; \dot{B}_{p,1}^s)} + \mu \| |\nabla|^r v \|_{L^\rho(I; \dot{B}_{p,1}^s)} \leq C \|f\|_{L^v(I; \dot{B}_{p,\infty}^{s+\frac{2}{v}-\frac{2}{\rho}})}. \quad (2.19)$$

The analogous estimate of (2.19) between the inhomogeneous Besov spaces also holds with the bound $C = C_T$ is depending on $T < \infty$.

The proof of Theorem 2.5 (1) (2) are given in [34] (cf. [47]). For the completeness, we show the out-lined proof in Appendix. The proof of (2.19) is in Proposition 5.6 in Appendix, too.

We also show the maximal regularity estimate in BMO in the composite space-time space $\widetilde{L^2(I; BMO(\mathbb{R}^n))}$ which is essentially due to Stein [57, p. 161] and explicitly appeared in Koch–Tataru [26] and Ogawa–Shimizu [48].

We consider maximal regularity for (2.15) in non-UMD Besov spaces (cf. Ogawa–Shimizu [48]). Koch–Tataru [26] obtained the homogeneous estimate in BMO by using the Carleson measure:

$$\left(\sup_{x_0, R>0} \int_0^{R^2} \left(\frac{1}{|B_R|} \int_{B_R(x_0)} |\nabla_x e^{t\Delta_x} u_0|^2 dx dy \right) dt \right)^{1/2} \leq C_0 \|u_0\|_{BMO}. \quad (2.20)$$

Proposition 2.6 [26] Let $e^{t\Delta}$ be the heat kernel and $u_0 \in BMO(\mathbb{R}^n)$. Then

$$\|\nabla e^{t\Delta} u_0\|_{\widetilde{L^2(I; BMO)}} \leq C_0 \|u_0\|_{BMO}, \quad (2.21)$$

where C_0 is independent of $T > 0$. The corresponding estimate from VMO to $\widetilde{L^2(I; VMO(\mathbb{R}^n))}$ also holds.

We show a sketch of alternative proof in Sect. 1 of Appendix below (see [26]).

Lemma 2.7 Let $I = (0, T)$ for some $T > 0$ and $e^{t\Delta}$ be the heat semigroup for $t > 0$. Then for any $u_0 \in VMO(\mathbb{R}^n)$,

$$\|\nabla e^{t\Delta} u_0\|_{\widetilde{L^2(I; VMO)}} \rightarrow 0 \quad \text{as } T \rightarrow 0. \quad (2.22)$$

Analogously for $u_0 \in V\dot{M}O^{-1}$,

$$\|e^{t\Delta} u_0\|_{\widetilde{L^2(I; VMO)}} \rightarrow 0 \quad \text{as } T \rightarrow 0. \quad (2.23)$$

Proof of Lemma 2.7 We show result for the case $u_0 \in V\dot{M}O^{-1}$. The other case (2.22) follows in a similar way. Then we see the convergence for $u_0 \in C_0(\mathbb{R}^n)$ first, where $C_0(\mathbb{R}^n)$ denotes the set of all compact supported continuous functions.

$$\begin{aligned} & \|e^{t\Delta} u_0\|_{\widetilde{L^2(I; VMO)}} \\ &= C \sup_{x_0, R>0} \left(\int_{I \cap (0, R^2)} \frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |e^{t\Delta_x} u_0(x) - e^{t\Delta_y} u_0(y)|^2 dx dy dt \right)^{1/2}. \end{aligned}$$

Then it is clear if $u_0 \in C_0(\mathbb{R}^n)$ then for some $q > n$,

$$\begin{aligned} & \int_{I \cap (0, R^2)} \frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |e^{t\Delta_x} u_0(x) - e^{t\Delta_y} u_0(y)|^2 dx dy dt \\ & \leq 2 \int_I \|e^{t\Delta} u_0(\cdot)\|_\infty^2 dt \\ & \leq C_n \int_I t^{-2 \cdot \frac{n}{2q}} dt \|u_0\|_q^2 = C_n T^{1-\frac{n}{q}} \|u_0\|_q^2 \rightarrow 0 \end{aligned} \quad (2.24)$$

as $T \rightarrow 0$ uniformly in $x_0 \in \mathbb{R}^n$ and $R > 0$.

For the general case, let $u_0 \in V\dot{M}O^{-1}(\mathbb{R}^n)$. Then there exists $\{u_{n,0}\}_{n \in \mathbb{N}} \in C_0(\mathbb{R}^n)$ such that for any $\varepsilon > 0$, there exists a large $N \in \mathbb{N}$, $n \geq N$,

$$\|u_{n,0} - u_0\|_{V\dot{M}O^{-1}} < \varepsilon.$$

It follows that

$$\begin{aligned} \|e^{t\Delta} u_0\|_{\widetilde{L^2(I; VMO)}} & \leq \|e^{t\Delta} u_{n,0}\|_{\widetilde{L^2(I; VMO)}} + \|e^{t\Delta} u_0 - e^{t\Delta} u_{n,0}\|_{\widetilde{L^2(I; VMO)}} \\ & \leq \|e^{t\Delta} u_{n,0}\|_{\widetilde{L^2(I; VMO)}} + \|u_0 - u_{n,0}\|_{V\dot{M}O^{-1}} \\ & \leq \|e^{t\Delta} u_{n,0}\|_{\widetilde{L^2(I; VMO)}} + \varepsilon. \end{aligned}$$

Fixing n and passing $|I| \rightarrow 0$ and applying the estimate (2.24), we obtain the result. $\square$

The following estimates are related to maximal regularity for the Cauchy problem of the heat equation: We consider the Cauchy problem of the heat equation: For $\mu > 0$ and $\lambda \geq 0$,

$$\begin{cases} \partial_t v - \mu \Delta v + \lambda v = f, & t > 0, x \in \mathbb{R}^n, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^n. \end{cases} \quad (2.25)$$

The estimate for the homogeneous equation (i.e. the case for $f \equiv 0$ and $\lambda = 0$) is shown by Koch–Tataru [26] and applied to the well-posedness issue for the incompressible Navier–Stokes equations. As a generalization of the homogeneous estimate, maximal regularity for the Cauchy problem of the heat equation can be shown in Ogawa–Shimizu [48] for $\lambda = 0$ including its optimality of trace estimate (cf. Iwabuchi–Nakamura [20]).

Theorem 2.8 (Maximal regularity in BMO [48]) *Let $\mu > 0$ and $\lambda \geq 0$ be constants. For all $f \in \widetilde{L^2(\mathbb{R}_+; BMO(\mathbb{R}^n))}$ and $\nabla u_0 \in BMO(\mathbb{R}^n)$, the Cauchy problem (2.25) admits a unique solution*

$$v \in \widetilde{W^{1,2}(\mathbb{R}_+; BMO(\mathbb{R}^n))} \cap \widetilde{L^2(\mathbb{R}_+; B\dot{M}O^2(\mathbb{R}^n))}$$

that satisfies the following estimate: There exists a constant $C > 0$ independent of T , μ and λ such that

$$\begin{aligned} & \|\partial_t v\|_{\widetilde{L^2(\mathbb{R}_+; BMO)}} + \mu \|\Delta v\|_{\widetilde{L^2(\mathbb{R}_+; BMO)}} + \lambda \|v\|_{\widetilde{L^2(\mathbb{R}_+; BMO)}} \\ & \leq C (\|\nabla u_0\|_{BMO} + \|f\|_{\widetilde{L^2(\mathbb{R}_+; BMO)}}). \end{aligned} \quad (2.26)$$

The corresponding estimate in $\widetilde{L^2(I; VMO(\mathbb{R}^n))}$ to (2.26) also holds under suitable conditions on u_0 and f .

The proof of Theorem 2.8 can be seen in [48]. For completeness, we show the out-lined proof of Theorem 2.8 in Appendix below.

3 Proof of local and global well-posedness

In this section, we show the local and global well-posedness of the solution to (1.1). The proof of Proposition 1.2 for the limiting equation (1.3) is indeed simpler than the proof of Theorem 1.1 and we do not show the proof for Proposition 1.2 (cf. [29,32]).

Proof of Theorem 1.1 We first show the local existence of solution for $I = (0, T)$. Consider the mild solution to the corresponding integral equation:

$$\begin{cases} u_\tau(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\nabla \cdot (u_\tau(s)\nabla\psi_\tau(s))ds, \\ \psi_\tau(t) = e^{\tau t(\Delta-\lambda)}\psi_0 + \int_0^t e^{\tau(t-s)(\Delta-\lambda)}\tau u_\tau(s)ds. \end{cases} \quad (3.1)$$

(Step 1) (The local well-posedness): Let $\tau > 0$. We show the local in time existence and well-posedness of the solutions for the large initial data $(u_0, \psi_0) \in L^{\frac{n}{2}} \cap B_{q,\theta}^{-\frac{2}{\theta}} \times VMO \cap B_{r,\theta}^{1-\frac{2}{\theta}}$. Let (θ, q) and (θ, r) satisfy the Serrin admissible conditions with additional assumptions as follows:

$$\begin{cases} \frac{2}{\theta} + \frac{n}{q} = 2, & \frac{n}{2} < q \leq \theta, & 2 < \theta \leq 4, \\ \frac{2}{\theta} + \frac{n}{r} = 1, & n < r, & 1 \leq \frac{rq}{r+q} < \theta, & 1 - \frac{2}{\theta} + 2 - \frac{2}{\theta} = \frac{n}{r} + \frac{n}{q} > \frac{1}{\theta} \end{cases} \quad (3.2)$$

and set $I = (0, T)$ for some $0 < T < \infty$ chosen later and let

$$X_T = \left\{ \begin{array}{l} u \in L^\theta(I; L^q) \cap \widetilde{L^2(I; V\dot{M}O^{-1})}, \\ \psi \in BC(I; VMO) \cap L^\theta(I; \dot{W}^{1,r}) \cap \widetilde{L^2(I; V\dot{M}O^1)}; \\ \|\psi\|_{L^2(I; VMO)} \leq M, \\ \|u\|_{L^\theta(I; L^q)}, \|u\|_{\widetilde{L^2(I; V\dot{M}O^{-1})}}, \|\nabla\psi\|_{L^\theta(I; L^r)}, \|\nabla\psi\|_{\widetilde{L^2(I; VMO)}} \leq N \end{array} \right\},$$

where

$$M = 4C_0(\|u_0\|_{\frac{n}{2}}^q + \|\psi_0\|_{VMO}^2) \quad (3.3)$$

and $N > 0$ will be determined later. Introducing the metric on X_T by

$$\|(u, \psi) - (\tilde{u}, \tilde{\psi})\|_T \equiv \|u - \tilde{u}\|_{L^\theta(I; L^q)} + \|\nabla(\psi - \tilde{\psi})\|_{L^\theta(I; L^r)},$$

we can show that X_M is a complete metric space. Indeed, since $\widetilde{L^2(I; BMO)}$ can be identified by $BMO(\mathbb{R}^n; L^2(I))$, the pre-dual of $\widetilde{L^2(I; BMO)}$ coincides $\mathcal{H}^1(\mathbb{R}^n; L^2(I)) = \widetilde{L^2(I; \mathcal{H}^1)}$ which is separable. Hence the Banach-Alaoglu theorem implies that the weak-* sequence compactness holds for any bounded set in $\widetilde{L^2(I; BMO)}$ (Alaoglu [1], Brezis [7, Cor. III.26.]). By this fact, any Cauchy sequences converges a limit (u, ψ) in $L^\theta(I; L^q) \times L^\theta(I; \dot{W}^{1,r})$ and weak-* compactness ensures that the limit indeed belongs to X_T .

We then introduce a pair of the solution maps $(\Phi[u_\tau, \psi_\tau], \Psi[u_\tau, \psi_\tau])$ as follows: For $(u_0, \psi_0) \in (L^{\frac{n}{2}} \cap \dot{B}_{q,\theta}^{-\frac{2}{n}}) \times (BMO \cap \dot{B}_{r,\theta}^{1-\frac{2}{\theta}})$ and $(u_\tau, \psi_\tau) \in X_M$, let

$$\begin{cases} \Phi[u_\tau, \psi_\tau](t) \equiv e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} \nabla \cdot (u_\tau(s) \nabla \psi_\tau(s)) ds, \\ \Psi[u_\tau, \psi_\tau](t) \equiv e^{\tau t(\Delta-\lambda)} \psi_0 + \int_0^{\tau t} e^{(\tau t-s)(\Delta-\lambda)} \Phi[u_\tau, \psi_\tau](\tau^{-1}s) ds \end{cases} \quad (3.4)$$

and claim that the map (Φ, Ψ) is contraction in the critical space X_T .

First we claim that (Φ, Ψ) is onto X_T . By maximal regularity (2.17) in Theorem 2.5 with $s = -2, \sigma = 1$ with the embeddings $B_{q,1}^0 \subset L^q$ to see that

$$\begin{aligned} \|e^{t\Delta}u_0\|_{L^\theta(I;L^q)} &\leq \|e^{t\Delta}u_0\|_{L^\theta(I;\dot{F}_{q,2}^0)} \\ &\leq C\|u_0\|_{B_{q,\theta}^{-2+\frac{2}{\theta}}} \leq C\|u_0\|_{B_{q,\theta}^{-\frac{2}{\theta}}}, \end{aligned} \quad (3.5)$$

and by Proposition 2.6,

$$\begin{aligned} \|e^{t\Delta}u_0\|_{\widetilde{L^2(I;V\dot{M}O^{-1})}} &\leq C\|u_0\|_{V\dot{M}O^{-2}} \\ &\leq C\|u_0\|_{\dot{W}^{-1,n}} \leq C\|u_0\|_{\frac{n}{2}}. \end{aligned} \quad (3.6)$$

It also follows for ψ_0 that

$$\begin{aligned} \|\nabla e^{\tau t(\Delta-\lambda)}\psi_0\|_{L^\theta(I;L^r)} &\leq \|\nabla e^{\tau t\Delta}\psi_0\|_{L^\theta(I;L^r)} \\ &\leq C\|\psi_0\|_{B_{r,\theta}^{1-\frac{2}{\theta}}}. \end{aligned} \quad (3.7)$$

From (2.21) in Proposition 2.6 that for $\psi_0 \in VMO$,

$$\|\nabla e^{\tau t\Delta}\psi_0\|_{\widetilde{L^2(I;VMO)}} \leq C\|\psi_0\|_{VMO}. \quad (3.8)$$

Hence from (3.5)–(3.8) and noting Lemma 2.7, we may choose the time interval $|I| \leq T$ sufficiently small such that for some small $\varepsilon_0 > 0$,¹

$$\begin{cases} \|e^{t\Delta}u_0\|_{L^\theta(I;L^q)} \leq \varepsilon_0, \\ \|e^{t\Delta}u_0\|_{\widetilde{L^2(I;V\dot{M}O^{-1})}} \leq \varepsilon_0, \\ \|\nabla e^{\tau t\Delta}\psi_0\|_{L^\theta(I;L^r)} \leq \varepsilon_0, \\ \|\nabla e^{\tau t\Delta}\psi_0\|_{\widetilde{L^2(I;VMO)}} \leq \varepsilon_0 \end{cases} \quad (3.9)$$

for any $\tau > 1$ and the choice of T is independent of $\tau > 1$. Let (θ, q) and (θ, r) be admissible pairs with $1 < q < 2 < \theta$. We apply the bound for the initial data (3.5) and (2.10), (3.9), maximal regularity (2.18) and Lemma 2.3 with $\sigma = \theta$ to have

$$\begin{aligned} &\|\Phi[u_\tau, \psi_\tau]\|_{L^\theta(I;L^q)} \\ &\leq \|e^{t\Delta}u_0\|_{L^\theta(I;L^q)} + \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot (u_\tau(s) \nabla \psi_\tau(s)) ds \right\|_{L^\theta(I;L^q)} \\ &\leq \varepsilon_0 + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I;\dot{B}_{q,\theta}^{-1+\frac{2}{\theta}})} \end{aligned}$$

¹ The choice of T is independent of $\tau > 1$.

$$\begin{aligned}
&\leq \varepsilon_0 + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; L^{\frac{rq}{r+q}})} \\
&\leq \varepsilon_0 + C \left\| u_\tau(\cdot) \right\|_{L^\theta(I; L^q)} \left\| \nabla \psi_\tau(\cdot) \right\|_{L^\theta(I; L^r)} \\
&\leq \varepsilon_0 + CN^2 \leq N
\end{aligned} \tag{3.10}$$

for any $2 < \theta$ and $1 < q < \theta$, and for some small ε_0 , $CN^2 \leq \frac{1}{2}N$.

By adding the both sides of

$$\frac{2}{\theta} + \frac{n}{q} = 2, \quad \frac{2}{\theta} + \frac{n}{r} = 1,$$

the following embedding is realized:

$$L^{\frac{rq}{r+q}} \subset \dot{B}_{\infty,2}^{-3+\frac{4}{\theta}}, \quad \frac{1}{q} + \frac{1}{r} = -\frac{1}{n} \left(-3 + \frac{4}{\theta} \right)$$

and we have from Lemma 2.2, (2.23) in Lemma 2.7 and (2.18) in Theorem 2.5 with (3.9) that for small $T < 1$,

$$\begin{aligned}
&\left\| \Phi[u_\tau, \psi_\tau] \right\|_{\widetilde{L^2(I; \dot{VMO}^{-1})}} \\
&\leq \|e^{t\Delta} u_0\|_{\widetilde{L^2(I; \dot{VMO}^{-1})}} + \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot (u_\tau(s) \nabla \psi_\tau(s)) ds \right\|_{\widetilde{L^2(I; \dot{VMO}^{-1})}} \\
&\leq \varepsilon_0 + \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot (u_\tau(s) \nabla \psi_\tau(s)) ds \right\|_{L^2(I; \dot{B}_{\infty,2}^{-1})} \\
&\leq \varepsilon_0 + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; \dot{B}_{\infty,2}^{-2-\frac{2}{\theta}+\frac{4}{\theta}})} \\
&\leq \varepsilon_0 + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; \dot{B}_{\frac{rq}{r+q},2}^0)} \\
&\leq \varepsilon_0 + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; L^{\frac{rq}{r+q}})} \\
&\leq \varepsilon_0 + C \left\| u_\tau \right\|_{L^\theta(I; L^q)} \left\| \nabla \psi_\tau \right\|_{L^\theta(I; L^r)} \\
&\leq \varepsilon_0 + CN^2 \leq N.
\end{aligned} \tag{3.11}$$

We proceed the estimate for the map Ψ . From Theorem 2.8, (3.9) and (3.11),

$$\begin{aligned}
&\left\| \nabla \Psi[u_\tau, \psi_\tau] \right\|_{\widetilde{L^2(I; VMO)}} \\
&\leq \|e^{\tau t(\Delta-\lambda)} \nabla \psi_0\|_{\widetilde{L^2(I; VMO)}} + \left\| \nabla \int_0^{\tau t} e^{(\tau t-s)(\Delta-\lambda)} \Phi[u_\tau, \psi_\tau](\tau^{-1}s) ds \right\|_{\widetilde{L^2(I; VMO)}} \\
&\leq \varepsilon_0 + C \left\| \Phi[u_\tau, \psi_\tau](\tau^{-1}s)|_{s=\tau t} \right\|_{\widetilde{L^2(I; \dot{VMO}^{-1})}} \\
&\leq \varepsilon_0 + C \left\| \Phi[u_\tau, \psi_\tau](t) \right\|_{\widetilde{L^2(I; \dot{VMO}^{-1})}} \\
&\leq \varepsilon_0 + C(\varepsilon_0 + 2N^2) < N.
\end{aligned} \tag{3.12}$$

From (2.18) in Theorem 2.5 and (3.10),

$$\begin{aligned}
&\left\| \nabla \Psi[u_\tau, \psi_\tau] \right\|_{L^\theta(I; L^r)} \\
&\leq \|e^{\tau t(\Delta-\lambda)} \nabla \psi_0\|_{L^\theta(I; L^r)}
\end{aligned}$$

$$\begin{aligned}
 & + \left\| \nabla \int_0^{\tau t} e^{(\tau t-s)(\Delta-\lambda)} \Phi[u_\tau, \psi_\tau](\tau^{-1}s) ds \right\|_{L^\theta(I; \dot{F}_{r,2}^0)} \\
 & \leq \varepsilon_0 + C \left\| \Phi[u_\tau, \psi_\tau](\tau^{-1}s)|_{s=\tau t} \right\|_{L^\theta(I; \dot{B}_{r,\theta}^{-1})} \\
 & \leq \varepsilon_0 + C \left\| \Phi[u_\tau, \psi_\tau](\tau^{-1}s)|_{s=\tau t} \right\|_{L^\theta(I; \dot{F}_{r,2}^{-1})} \\
 & \leq \varepsilon_0 + C \left\| \Phi[u_\tau, \psi_\tau] \right\|_{L^\theta(I; \dot{W}^{-n(\frac{1}{q}-\frac{1}{r}),r})} \\
 & \leq \varepsilon_0 + C \left\| \Phi[u_\tau, \psi_\tau] \right\|_{L^\theta(I; L^q)} \\
 & \leq \varepsilon_0 + C(\varepsilon_0 + 2N^2) < N.
 \end{aligned} \tag{3.13}$$

From (3.11) and (3.13), and choosing $N > 0$ small enough so that for some small $\varepsilon_0 > 0$,

$$\left\{ \begin{aligned} & \left\| \Phi[u_\tau, \psi_\tau] \right\|_{L^\theta(I; L^q)}, \left\| \Phi[u_\tau, \psi_\tau] \right\|_{\widetilde{L^2(I; VMO^{-1})}}, \\ & \left\| \nabla \Psi[u_\tau, \psi_\tau] \right\|_{L^\theta(I; L^r)}, \left\| \nabla \Psi[u_\tau, \psi_\tau] \right\|_{\widetilde{L^2(I; VMO)}} \end{aligned} \right\} \leq N. \tag{3.14}$$

On the other hand, since the heat kernel is a bounded operator in $VMO(\mathbb{R}^n)$, we again use generalized maximal regularity (2.18), Lemma 2.2 and (3.10) to see that

$$\begin{aligned}
 & \left\| \Psi[u_\tau, \psi_\tau] \right\|_{L^\infty(I; VMO)} \\
 & \leq \|e^{\tau t(\Delta-\lambda)} \psi_0\|_{VMO} + \left\| \int_0^{\tau t} e^{(\tau t-s)(\Delta-\lambda)} \Phi[u_\tau, \psi_\tau](\tau^{-1}s) ds \right\|_{L^\infty(I; VMO)} \\
 & \leq C_0 \|\psi_0\|_{VMO} + \left\| \int_0^{\tau t} e^{(\tau t-s)(\Delta-\lambda)} \Phi[u_\tau, \psi_\tau](\tau^{-1}s) ds \right\|_{L^\infty(I; \dot{B}_{\infty,2}^0)} \\
 & \leq \frac{1}{4}M + C \left\| \Phi[u_\tau, \psi_\tau](\tau^{-1}s)|_{s=\tau t} \right\|_{L^\theta(I; \dot{B}_{\infty,2}^{-2+\frac{2}{\theta}})} \\
 & \leq \frac{1}{4}M + C \left\| \Phi[u_\tau, \psi_\tau](\tau^{-1}s)|_{s=\tau t} \right\|_{L^\theta(I; \dot{B}_{q,2}^0)} \\
 & \text{(since } q < 2, L^q(\mathbb{R}^n) \simeq \dot{F}_{q,2}^0(\mathbb{R}^n) \subset \dot{B}_{q,2}^0(\mathbb{R}^n)) \\
 & \leq \frac{1}{4}M + C \left\| \Phi[u_\tau, \psi_\tau](t) \right\|_{L^\theta(I; L^q)} \\
 & \leq \frac{1}{4}M + C(\varepsilon_0 + CN^2) \leq M.
 \end{aligned} \tag{3.15}$$

Hence we obtain from (3.15) that

$$\left\| \Psi[u_\tau, \psi_\tau] \right\|_{L^\infty(I; VMO)} \leq M. \tag{3.16}$$

Combining (3.14) and (3.16), we obtain $(\Phi[u_\tau, \psi_\tau], \Psi[u_\tau, \psi_\tau]) \in X_T$. Analogously from (3.10), we have for the difference of solutions that

$$\begin{aligned}
 & \left\| \Phi[u_\tau, \psi_\tau] - \Phi[\tilde{u}_\tau, \tilde{\psi}_\tau] \right\|_{L^\theta(I; L^q)} \\
 & \leq C \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot (u_\tau(s) \nabla \psi_\tau(s) - \tilde{u}_\tau(s) \nabla \tilde{\psi}_\tau(s)) ds \right\|_{L^\theta(I; \dot{F}_{q,2}^0)} \\
 & \leq C \left\| u_\tau(s) \nabla \psi_\tau(s) - \tilde{u}_\tau(s) \nabla \tilde{\psi}_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; \dot{B}_{q,\theta}^{-1+\frac{2}{\theta}})}
 \end{aligned}$$

$$\begin{aligned}
&\leq C \left\| u_\tau(s) \nabla(\psi_\tau(s) - \tilde{\psi}_\tau(s)) \right\|_{L^{\frac{\theta}{2}}(I; L^{\frac{rq}{r+q}})} \\
&\quad + C \left\| (u_\tau(s) - \tilde{u}_\tau(s)) \nabla \tilde{\psi}_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; L^{\frac{rq}{r+q}})} \\
&\leq C \left\| u_\tau(s) \right\|_{L^\theta(I; L^q)} \left\| \nabla(\psi_\tau(s) - \tilde{\psi}_\tau(s)) \right\|_{L^\theta(I; L^r)} \\
&\quad + C \left\| u_\tau(s) - \tilde{u}_\tau(s) \right\|_{L^\theta(I; L^q)} \left\| \nabla \tilde{\psi}_\tau(s) \right\|_{L^\theta(I; L^r)} \\
&\leq C_1 N \left\| (u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau) \right\|_T.
\end{aligned} \tag{3.17}$$

Analogously from (3.10) and (3.17), we have

$$\begin{aligned}
&\left\| \nabla(\Psi[u_\tau, \psi_\tau] - \Psi[\tilde{u}_\tau, \tilde{\psi}_\tau]) \right\|_{L^\theta(I; L^r)} \\
&\leq C \left\| \int_0^{\tau t} \nabla e^{(\tau t-s)(\Delta-\lambda)} (\Phi[u_\tau, \psi_\tau](\tau^{-1}s) - \Phi[\tilde{u}_\tau, \tilde{\psi}_\tau](\tau^{-1}s)) ds \right\|_{L^\theta(I; \dot{B}_{r,\theta}^0)} \\
&\leq C \left\| \Phi[u_\tau, \psi_\tau](\tau^{-1}s)|_{s=\tau t} - \Phi[\tilde{u}_\tau, \tilde{\psi}_\tau](\tau^{-1}s)|_{s=\tau t} \right\|_{L^\theta(I; \dot{B}_{r,\theta}^{-1})} \\
&\leq C \left\| \Phi[u_\tau, \psi_\tau](t) - \Phi[\tilde{u}_\tau, \tilde{\psi}_\tau](t) \right\|_{L^\theta(I; \dot{B}_{r,\theta}^{-n(\frac{1}{q}-\frac{1}{r})})} \\
&\leq C \left\| \Phi[u_\tau, \psi_\tau](t) - \Phi[\tilde{u}_\tau, \tilde{\psi}_\tau](t) \right\|_{L^\theta(I; L^q)} \\
&\leq C_1^2 N \left\| (u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau) \right\|_T.
\end{aligned} \tag{3.18}$$

Choosing ε_0 smaller as

$$C_1 N \leq \frac{1}{4}, \tag{3.19}$$

if necessary, (3.17) and (3.18) with (3.19) yield that

$$\left\| (\Phi[u_\tau, \psi_\tau], \Psi[u_\tau, \psi_\tau]) - (\Phi[\tilde{u}_\tau, \tilde{\psi}_\tau], \Psi[\tilde{u}_\tau, \tilde{\psi}_\tau]) \right\|_T \leq \frac{1}{2} \left\| (u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau) \right\|_T$$

under the smallness assumption (3.9) on the interval. Thus the map (Φ, Ψ) is contraction onto X_T and the Banach fixed point theorem implies that there exists a unique fixed point $(u_\tau, \psi_\tau) \in X_T$ that solves the equation (1.1) in the critical space. In particular by (3.15), it follows that for $u_0 \in L^{\frac{n}{2}}(\mathbb{R}^n) \cap \dot{B}_{q,\theta}^{-\frac{2}{\theta}}(\mathbb{R}^n)$ and $\psi_0 \in BMO(\mathbb{R}^n) \cap \dot{B}_{r,\theta}^{1-\frac{2}{\theta}}(\mathbb{R}^n)$,

$$\|u_\tau\|_{\widetilde{L^2(I; VMO^{-1})}} + \|\psi_\tau\|_{L^\infty(I; BMO)} + \|\nabla \psi_\tau\|_{\widetilde{L^2(I; BMO)}} \leq 3M,$$

where the constant M given by (3.3) does not depend on the parameter $\tau > 0$. Next, we show the continuous dependence of the initial data in $L^\theta(I; L^q) \times (L^\infty(I; BMO) \cap L^\theta(I; \dot{W}^{1,r}) \cap \widetilde{L^2(I; BMO^1)})$. Let (u_τ, ψ_τ) and $(\tilde{u}_\tau, \tilde{\psi}_\tau)$ be a solution of (1.1) corresponding to the initial data (u_0, ψ_0) and $(\tilde{u}_0, \tilde{\psi}_0)$, respectively. Then very much similar estimate of (3.17) and (3.18), we obtain

$$\begin{aligned}
 & \|u_\tau - \tilde{u}_\tau\|_{L^\theta(I; L^q)} \\
 & \leq \|e^{t\Delta}u_0 - e^{t\Delta}\tilde{u}_0\|_{L^\theta(I; L^q)} \\
 & \quad + \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot (u_\tau(s) \nabla \psi_\tau(s) - \tilde{u}_\tau(s) \nabla \tilde{\psi}_\tau(s)) ds \right\|_{L^\theta(I; \dot{B}_{q,2}^0)} \\
 & \leq C_0 \|u_0 - \tilde{u}_0\|_{\dot{B}_{q,\theta}^{-\frac{2}{\theta}}} + C \|u_\tau(s)\|_{L^\theta(I; L^q)} \|\nabla(\psi_\tau(s) - \tilde{\psi}_\tau(s))\|_{L^\theta(I; L^r)} \\
 & \quad + C \|u_\tau(s) - \tilde{u}_\tau(s)\|_{L^\theta(I; L^q)} \|\nabla \tilde{\psi}_\tau(s)\|_{L^\theta(I; L^r)} \\
 & \leq C_0 \|u_0 - \tilde{u}_0\|_{\dot{B}_{q,\theta}^{-\frac{2}{\theta}}} + C_1 N \|(u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau)\|_T
 \end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
 & \|\nabla(\psi_\tau - \tilde{\psi}_\tau)\|_{L^\theta(I; L^r)} \\
 & \leq \|\nabla(e^{\tau t(\Delta-\lambda)}\psi_0 - e^{\tau t(\Delta-\lambda)}\tilde{\psi}_0)\|_{L^\theta(I; L^r)} \\
 & \quad + \left\| \int_0^{\tau t} \nabla e^{(\tau t-s)(\Delta-\lambda)} (u_\tau(\tau^{-1}s) - \tilde{u}_\tau(\tau^{-1}s)) ds \right\|_{L^\theta(I; L^r)} \\
 & \leq C_0 \|\psi_0 - \tilde{\psi}_0\|_{\dot{B}_{r,\theta}^{1-\frac{2}{\theta}}} + C \|u_\tau(t) - \tilde{u}_\tau(t)\|_{L^\theta(I; L^q)} \\
 & \leq C_0 \|\psi_0 - \tilde{\psi}_0\|_{\dot{B}_{r,\theta}^{1-\frac{2}{\theta}}} + C_1^2 N \|(u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau)\|_T.
 \end{aligned} \tag{3.21}$$

From Theorem 2.8,

$$\begin{aligned}
 & \|\nabla(\psi_\tau - \tilde{\psi}_\tau)\|_{\widetilde{L^2(I; BMO)}} \\
 & \leq \|\nabla(e^{\tau t(\Delta-\lambda)}\psi_0 - e^{\tau t(\Delta-\lambda)}\tilde{\psi}_0)\|_{\widetilde{L^2(I; BMO)}} \\
 & \quad + \left\| \int_0^{\tau t} \nabla e^{(\tau t-s)(\Delta-\lambda)} (u_\tau(\tau^{-1}s) - \tilde{u}_\tau(\tau^{-1}s)) ds \right\|_{\widetilde{L^2(I; BMO)}} \\
 & \leq C_0 \|\psi_0 - \tilde{\psi}_0\|_{BMO} + C \|u_\tau(t) - \tilde{u}_\tau(t)\|_{L^2(I; BMO^{-1})} \\
 & \leq C_0 \|\psi_0 - \tilde{\psi}_0\|_{BMO} + C \|u_\tau(t) - \tilde{u}_\tau(t)\|_{L^2(I; L^n)} \\
 & \leq C_0 \|\psi_0 - \tilde{\psi}_0\|_{BMO} + C_1^2 N \|(u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau)\|_T.
 \end{aligned} \tag{3.22}$$

By (3.20)–(3.22), the solution (u_τ, ψ_τ) converges to $(\tilde{u}_\tau, \tilde{\psi}_\tau)$ in

$L^\theta(I; L^q) \times (L^\theta(I; \dot{W}^{1,r}) \cap \widetilde{L^2(I; BMO^1)})$ as $(u_0, \psi_0) \rightarrow (\tilde{u}_0, \tilde{\psi}_0)$ in $\dot{B}_{q,\theta}^{-\frac{2}{\theta}} \times \dot{B}_{r,\theta}^{1-\frac{2}{\theta}} \cap BMO$.

Finally we show that the solution (u_τ, ψ_τ) obtained above is in $BC(I; L^{\frac{n}{2}}) \times BC(I; VMO)$, where $I = (0, T)$ with $T < \infty$. We employ generalized maximal regularity (2.19) in Theorem 2.5 with $r = 1$ and $\rho = \infty$ to have

$$\begin{aligned}
 \|u_\tau\|_{L^\infty(I; L^{\frac{n}{2}})} &\leq C_0 \|u_0\|_{B_{\frac{n}{2}, \infty}^0} + \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot (u_\tau(s) \nabla \psi_\tau(s)) ds \right\|_{L^\infty(I; \dot{B}_{\frac{n}{2}, 1}^0)} \\
 &\leq C_0 \|u_0\|_{\frac{n}{2}} + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; \dot{B}_{\frac{n}{2}, \infty}^{-1+\frac{4}{\theta}})} \\
 &\leq C_0 \|u_0\|_{\frac{n}{2}} + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; \dot{B}_{\frac{n}{2}, \infty}^0)} \\
 &\leq C_0 \|u_0\|_{\frac{n}{2}} + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; L^{\frac{rq}{q+r}})} \\
 &\leq \frac{1}{4} M + C \|u_\tau(s)\|_{L^\theta(I; L^q)} \|\nabla \psi_\tau(s)\|_{L^\theta(I; L^r)} \leq M.
 \end{aligned} \tag{3.23}$$

Since $\Phi[u_\tau, \psi_\tau](t)$ defined in (3.4) is a continuous function on $L^{\frac{n}{2}}$, $u_\tau \in BC(I; L^{\frac{n}{2}}(\mathbb{R}^n))$. Besides let (u_τ, ψ_τ) and $(\tilde{u}_\tau, \tilde{\psi}_\tau)$ are two solutions of (1.1) corresponding the initial data (u_0, ψ_0) and $(\tilde{u}_0, \tilde{\psi}_0)$, respectively. Then,

$$\begin{aligned}
 \|u_\tau - \tilde{u}_\tau\|_{L^\infty(I; L^{\frac{n}{2}})} &\leq C_0 \|u_0 - \tilde{u}_0\|_{\dot{B}_{\frac{n}{2}, \infty}^0} \\
 &\quad + \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot (u_\tau(s) \nabla \psi_\tau(s) - \tilde{u}_\tau(s) \nabla \tilde{\psi}_\tau(s)) ds \right\|_{L^\infty(I; \dot{B}_{\frac{n}{2}, 1}^0)} \\
 &\leq C_0 \|u_0 - \tilde{u}_0\|_{\frac{n}{2}} + C \left\| (u_\tau(s) - \tilde{u}_\tau(s)) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; \dot{B}_{\frac{n}{2}, \infty}^{-1+\frac{4}{\theta}})} \\
 &\quad + C \left\| \tilde{u}_\tau(s) (\nabla \psi_\tau(s) - \nabla \tilde{\psi}_\tau(s)) \right\|_{L^{\frac{\theta}{2}}(I; \dot{B}_{\frac{n}{2}, \infty}^{-1+\frac{4}{\theta}})} \\
 &\leq C_0 \|u_0 - \tilde{u}_0\|_1 + C \|u_\tau(s) - \tilde{u}_\tau(s)\|_{L^\theta(I; L^q)} \|\nabla \psi_\tau(s)\|_{L^\theta(I; L^r)} \\
 &\quad + C \|u_\tau(s)\|_{L^\theta(I; L^q)} \|\nabla \psi_\tau(s) - \nabla \tilde{\psi}_\tau(s)\|_{L^\theta(I; L^r)}.
 \end{aligned} \tag{3.24}$$

From (3.23) and (3.24), the continuous dependence of the solution u_τ in $BC(I; L^{\frac{n}{2}})$ is also shown. For the convergence of ψ_τ in $L^\infty(I; BMO)$ is also shown in a similar way. This show the local well-posedness of (1.1).

(Step 2) (Global existence for small data). Since our function space in the previous step is scaling invariant, the global existence for (1.1) also follows almost similar (but somewhat simpler) way to the case for local well-posedness. Let (θ, q) , (θ, r) be admissible given by (3.2). We also call

$$(\theta, q) \in \left\{ \left(\infty, \frac{n}{2} \right), (2, n) \right\}, \quad (\theta, r) \in \left\{ (2, \infty), (\infty, n) \right\}$$

the end-point admissible pairs. Fixing the admissible pair for $I = \mathbb{R}_+$ as (θ, q) , (θ, r) , we introduce the complete metric space:

$$\begin{aligned} X_{\tilde{\varepsilon}_0} = & \left\{ u \in BC(I; L^{\frac{n}{2}}) \cap L^\theta(I; L^q) \cap \widetilde{L^2(I; BMO^{-1})}, \right. \\ & \psi \in BC(I; BMO) \cap L^\theta(I; \dot{W}^{1,r}) \cap \widetilde{L^2(I; BMO^1)}; \\ & \|u\|_{L^\infty(I; L^{\frac{n}{2}})}, \|u\|_{L^\theta(I; L^q)}, \\ & \left. \|\psi\|_{L^\infty(I; BMO)}, \|\nabla \psi\|_{L^\theta(I; L^r)}, \|\nabla \psi\|_{\widetilde{L^2(I; BMO)}} \leq \tilde{\varepsilon}_0 \right\}, \end{aligned}$$

where

$$\tilde{\varepsilon}_0 \equiv 8C_0 \left(\|u_0\|_{\frac{n}{2}} + \|u_0\|_{\dot{B}_{q,\theta}^{-1+\frac{4}{\theta}}} + \|\psi_0\|_{BMO}^2 \right)$$

is chosen small later. For any admissible exponents (θ, q) and (σ, r) (not the end-point exponents), we define the metric on $X_{\tilde{\varepsilon}_0}$ by

$$\|(u, \psi) - (\tilde{u}, \tilde{\psi})\|_X \equiv \|u - \tilde{u}\|_{L^\theta(I; L^q)} + \|\nabla(\psi - \tilde{\psi})\|_{\widetilde{L^2(I; BMO)}}.$$

By this metric, $X_{\tilde{\varepsilon}_0}$ is a complete metric space. For $(u_0, \psi_0) \in L^{\frac{n}{2}} \times BMO$ and $(u_\tau, \psi_\tau) \in X_{\tilde{\varepsilon}_0}$, we define a pair of the solution maps $(\Phi[u_\tau, \psi_\tau], \Psi[u_\tau, \psi_\tau])$ by (3.4) and claim that the map (Φ, Ψ) is contraction in the critical space $X_{\tilde{\varepsilon}_0}$. Noticing $1 < q < \theta < \infty$ and the embedding (2.10),

$$\dot{F}_{\frac{rq}{r+q}, \infty}^0 \subset \dot{B}_{\frac{n}{2}, \infty}^{-1+\frac{4}{\theta}}$$

with

$$\frac{1}{q} + \frac{1}{r} = \frac{2}{n} - \frac{1}{n} \left(-1 + \frac{4}{\theta} \right)$$

we apply generalized maximal regularity

$$\begin{aligned} & \|\Phi[u_\tau, \psi_\tau]\|_{L^\infty(I; L^{\frac{n}{2}})} \\ & \leq C_0 \|u_0\|_{\dot{B}_{\frac{n}{2}, \infty}^0} + \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot (u_\tau(s) \nabla \psi_\tau(s)) ds \right\|_{L^\infty(I; \dot{B}_{\frac{n}{2}, 1}^0)} \\ & \leq C_0 \|u_0\|_{\frac{n}{2}} + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; \dot{B}_{\frac{n}{2}, \infty}^{-1+\frac{4}{\theta}})} \\ & \leq C_0 \|u_0\|_{\frac{n}{2}} + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; \dot{F}_{\frac{rq}{r+q}, 2}^0)} \\ & \leq \frac{1}{4} \tilde{\varepsilon}_0 + C \|u_\tau\|_{L^\theta(I; L^q)} \|\nabla \psi_\tau\|_{L^\theta(I; L^r)} \leq \frac{1}{4} \tilde{\varepsilon}_0 + C \tilde{\varepsilon}_0^2 \leq \tilde{\varepsilon}_0 \end{aligned} \quad (3.25)$$

under $C\tilde{\varepsilon}_0 \leq \frac{1}{2}$. All the above estimate in (3.25) is true for a finite time interval and we may prolong such an estimate for all time.

For any admissible exponents (θ, q) and (θ, r) ,

$$\begin{aligned}
 & \|\Phi[u_\tau, \psi_\tau]\|_{L^\theta(I; L^q)} \\
 & \leq \|e^{t\Delta} u_0\|_{L^\theta(I; L^q)} + \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot (u_\tau(s) \nabla \psi_\tau(s)) ds \right\|_{L^\theta(I; \dot{F}_{q,2}^0)} \\
 & \leq C_0 \|u_0\|_{\dot{B}_{q,\theta}^{2-\frac{2}{\theta}}} + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; \dot{B}_{q,\theta}^{-1+\frac{4}{\theta}-\frac{2}{\theta}})} \\
 & \leq C_0 \|u_0\|_{\dot{B}_{q,\theta}^{2-\frac{2}{\theta}}} + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; \dot{F}_{q,\theta}^{-1+\frac{2}{\theta}})} \\
 & \leq C_0 \|u_0\|_{\dot{B}_{q,\theta}^{2-\frac{2}{\theta}}} + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; L^{\frac{rq}{r+q}})} \\
 & \leq C_0 \|u_0\|_{\dot{B}_{q,\theta}^{2-\frac{2}{\theta}}} + C \|u_\tau(\cdot)\|_{L^\theta(I; L^q)} \|\nabla \psi_\tau(\cdot)\|_{L^\theta(I; L^r)} \\
 & \leq \frac{1}{4} \tilde{\varepsilon}_0 + C \tilde{\varepsilon}_0^2 \leq \tilde{\varepsilon}_0
 \end{aligned} \tag{3.26}$$

for $C \tilde{\varepsilon}_0 \leq \frac{1}{2}$.

In particular for the end-point admissible case $(\theta, q) = (2, n)$, by the same choice of the embedding classes, we see that

$$\begin{aligned}
 & \|\Phi[u_\tau, \psi_\tau]\|_{\widetilde{L^2(I; B\dot{M}O^{-1})}} \\
 & \leq \|e^{t\Delta} u_0\|_{\widetilde{L^2(I; B\dot{M}O^{-1})}} + \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot (u_\tau(s) \nabla \psi_\tau(s)) ds \right\|_{\widetilde{L^2(I; B\dot{M}O^{-1})}} \\
 & \leq \|e^{t\Delta} u_0\|_{\widetilde{L^2(I; B\dot{M}O^{-1})}} + \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot (u_\tau(s) \nabla \psi_\tau(s)) ds \right\|_{L^2(I; B\dot{M}O^{-1})} \\
 & \leq C \|u_0\|_{B\dot{M}O^{-2}} + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; \dot{B}_{\infty,2}^{-2-\frac{2}{\theta}+\frac{4}{\theta}})} \\
 & \leq C_0 \|u_0\|_{\dot{W}^{-1,n}} + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; \dot{B}_{\infty,2}^{-3+\frac{4}{\theta}})} \\
 & \leq C_0 \|u_0\|_{\frac{n}{2}} + C \left\| u_\tau(s) \nabla \psi_\tau(s) \right\|_{L^{\frac{\theta}{2}}(I; \dot{B}_{\frac{rq}{r+q},2}^0)} \\
 & \leq C_0 \|u_0\|_{\frac{n}{2}} + C \left(\|u_\tau\|_{L^\theta(I; L^q)} \|\nabla \psi_\tau\|_{L^\theta(I; L^r)} \right. \\
 & \quad \left. + \|u_\tau\|_{L^\theta(I; L^q)} \|\nabla \psi_\tau\|_{L^\theta(I; L^r)} \right) \\
 & \leq \frac{1}{4} \tilde{\varepsilon}_0 + 2C \tilde{\varepsilon}_0^2 \leq \tilde{\varepsilon}_0
 \end{aligned} \tag{3.27}$$

under $2C \tilde{\varepsilon}_0 \leq \frac{1}{2}$. Here we need one more restriction $\frac{rq}{q+r} \leq 2$ and by (1.8), it restrict ourselves for $2 \leq n \leq 4$.

It follows from (3.25)–(3.27) that

$$\max \left\{ \|\Phi[u_\tau, \psi_\tau]\|_{L^\infty(I; L^{\frac{n}{2}})}, \|\Phi[u_\tau, \psi_\tau]\|_{L^\theta(I; L^q)}, \|\Phi[u_\tau, \psi_\tau]\|_{\widetilde{L^2(I; B\dot{M}O^{-1})}} \right\} \leq \tilde{\varepsilon}_0. \tag{3.28}$$

Meanwhile by embedding $\dot{B}_{\infty,2}^0(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$ and maximal regularity

$$\begin{aligned}
 & \|\Psi[u_\tau, \psi_\tau]\|_{L^\infty(I; BMO)} \\
 & \leq \|e^{\tau t(\Delta-\lambda)}\psi_0\|_{BMO} + \left\| \int_0^{\tau t} e^{(\tau t-s)(\Delta-\lambda)} \Phi[u_\tau, \psi_\tau](\tau^{-1}s) ds \right\|_{L^\infty(I; BMO)} \\
 & \leq C_0 \|\psi_0\|_{BMO} + \left\| \int_0^{\tau t} e^{(\tau t-s)(\Delta-\lambda)} \Phi[u_\tau, \psi_\tau](\tau^{-1}s) ds \right\|_{L^\infty(I; \dot{B}_{\infty,2}^0)} \\
 & \leq \frac{1}{4} \tilde{\varepsilon}_0 + C \left\| \Phi[u_\tau, \psi_\tau](\tau^{-1}s) \Big|_{s=\tau t} \right\|_{L^\theta(I; \dot{B}_{\infty,\infty}^{-2+\frac{2}{\theta}})} \\
 & \leq \frac{1}{4} \tilde{\varepsilon}_0 + C \left\| \Phi[u_\tau, \psi_\tau](t) \right\|_{L^\theta(I; \dot{B}_{\infty,\infty}^{-2+\frac{2}{\theta}})} \\
 & \quad (\text{since } q < 2, L^q(\mathbb{R}^n) \simeq \dot{F}_{q,2}^0(\mathbb{R}^n) \subset \dot{B}_{q,\infty}^0(\mathbb{R}^n) \subset \dot{B}_{\infty,\infty}^{-2+\frac{2}{\theta}}(\mathbb{R}^n)) \\
 & \leq \frac{1}{4} \tilde{\varepsilon}_0 + C \left\| \Phi[u_\tau, \psi_\tau](t) \right\|_{L^\theta(I; L^q)} \\
 & \leq \frac{1}{4} \tilde{\varepsilon}_0 + \left(\frac{1}{4} + C\tilde{\varepsilon}_0\right) \tilde{\varepsilon}_0 \leq \tilde{\varepsilon}_0
 \end{aligned} \tag{3.29}$$

for small $C\tilde{\varepsilon}_0 \leq \frac{1}{4}$.

$$\begin{aligned}
 & \|\nabla \Psi[u_\tau, \psi_\tau]\|_{L^\theta(I; L^r)} \\
 & \leq \|\nabla e^{\tau t(\Delta-\lambda)}\psi_0\|_{L^\theta(I; L^r)} \\
 & \quad + \left\| \nabla \int_0^{\tau t} e^{(\tau t-s)(\Delta-\lambda)} \Phi[u_\tau, \psi_\tau](\tau^{-1}s) ds \right\|_{L^\theta(I; L^r)} \\
 & \leq C \|u_0\|_{\dot{B}_{r,\theta}^{1-\frac{2}{\theta}}} + C \left\| \Phi[u_\tau, \psi_\tau](\tau^{-1}s) \Big|_{s=\tau t} \right\|_{L^\theta(I; \dot{B}_{r,\theta}^{-2+\frac{2}{\theta}})} \\
 & \leq C \|u_0\|_{\dot{B}_{r,\theta}^{1-\frac{2}{\theta}}} + C \left\| \Phi[u_\tau, \psi_\tau](\tau^{-1}s) \Big|_{s=\tau t} \right\|_{L^\theta(I; \dot{B}_{r,\theta}^{-2+\frac{2}{\theta}})} \\
 & \leq C \|u_0\|_{\dot{B}_{r,\theta}^{1-\frac{2}{\theta}}} + C \left\| \Phi[u_\tau, \psi_\tau](t) \right\|_{L^\theta(I; \dot{F}_{r,\theta}^{-2+\frac{2}{\theta}})} \\
 & \leq \frac{1}{4} \tilde{\varepsilon}_0 + C \left\| \Phi[u_\tau, \psi_\tau](t) \right\|_{L^\theta(I; L^q)} \\
 & \leq \frac{1}{4} \tilde{\varepsilon}_0 + \left(\frac{1}{4} + C\tilde{\varepsilon}_0\right) \tilde{\varepsilon}_0 \\
 & \leq \tilde{\varepsilon}_0
 \end{aligned} \tag{3.30}$$

and by Theorem 2.8,

$$\begin{aligned}
 & \|\nabla \Psi[u_\tau, \psi_\tau]\|_{\widetilde{L^2(I; BMO)}} \\
 & \leq \|e^{\tau t(\Delta-\lambda)}\nabla \psi_0\|_{\widetilde{L^2(I; BMO)}} \\
 & \quad + \left\| \nabla \int_0^{\tau t} e^{(\tau t-s)(\Delta-\lambda)} \Phi[u_\tau, \psi_\tau](\tau^{-1}s) ds \right\|_{\widetilde{L^2(I; BMO)}} \\
 & \leq C_0 \|\psi_0\|_{BMO} + \left\| \nabla \int_0^{\tau t} e^{(\tau t-s)\Delta} \Phi[u_\tau, \psi_\tau](\tau^{-1}s) ds \right\|_{\widetilde{L^2(I; BMO)}}
 \end{aligned}$$

$$\begin{aligned}
&\leq C_0 \|\psi_0\|_{BMO} + C \left\| \Phi[u_\tau, \psi_\tau](\tau^{-1}s) \Big|_{s=\tau t} \right\|_{\widetilde{L^2(I; BMO^{-1})}} \\
&\leq C_0 \|\psi_0\|_{BMO} + C \left\| \Phi[u_\tau, \psi_\tau](t) \right\|_{\widetilde{L^2(I; BMO^{-1})}} \\
&\leq \frac{1}{4} \widetilde{\varepsilon}_0 + \frac{1}{4} \widetilde{\varepsilon}_0 + 2C \widetilde{\varepsilon}_0^2 \\
&\leq \widetilde{\varepsilon}_0
\end{aligned} \tag{3.31}$$

for $2C\widetilde{\varepsilon}_0 \leq \frac{1}{2}$. From (3.29) and (3.31),

$$\max \left\{ \|\Psi[u_\tau, \psi_\tau]\|_{L^\infty(I; BMO)}, \|\nabla \Psi[u_\tau, \psi_\tau]\|_{\widetilde{L^2(I; BMO)}} \right\} \leq \widetilde{\varepsilon}_0. \tag{3.32}$$

Combining (3.28) and (3.32), we obtain that $(\Phi[u_\tau, \psi_\tau], \Psi[u_\tau, \psi_\tau]) \in X_{\widetilde{\varepsilon}_0}$. Analogously from (3.17) and (3.18) for the difference of solutions

$$\begin{aligned}
&\|\Phi[u_\tau, \psi_\tau] - \Phi[\tilde{u}_\tau, \tilde{\psi}_\tau]\|_{L^\theta(I; L^q)} \leq C\widetilde{\varepsilon}_0 \| (u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau) \|_X, \\
&\|\nabla(\Psi[u_\tau, \psi_\tau] - \Psi[\tilde{u}_\tau, \tilde{\psi}_\tau])\|_{\widetilde{L^2(I; BMO)}} \leq C\widetilde{\varepsilon}_0 \| (u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau) \|_X.
\end{aligned} \tag{3.33}$$

Choosing M smaller as

$$CM \leq \frac{1}{4},$$

if necessary, we have from (3.33) that

$$\|(\Phi[u_\tau, \psi_\tau], \Psi[u_\tau, \psi_\tau]) - (\Phi[\tilde{u}_\tau, \tilde{\psi}_\tau], \Psi[\tilde{u}_\tau, \tilde{\psi}_\tau])\|_X \leq \frac{1}{2} \| (u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau) \|_X$$

under the smallness assumption on the initial data. Thus the map $(\Phi, \Psi[\Phi])$ is contraction onto $X_{\widetilde{\varepsilon}_0}$ and the Banach fixed point theorem implies that there exists a unique fixed point $(u_\tau, \psi_\tau) \in X_{\widetilde{\varepsilon}_0}$ that solves the Eq. (1.1) in the critical space. In particular, from (3.28) and (3.32), the a priori estimate holds:

$$\begin{aligned}
&\max \left(\|u_\tau\|_{L^\infty(I; L^{\frac{n}{2}})}, \|u_\tau\|_{L^\theta(I; L^q)}, \|u_\tau\|_{\widetilde{L^2(I; BMO^{-1})}} \right), \\
&\|\psi_\tau\|_{L^\infty(I; BMO)}, \|\nabla \psi_\tau\|_{L^\theta(I; L^r)}, \|\nabla \psi_\tau\|_{\widetilde{L^2(I; BMO)}} \leq \widetilde{\varepsilon}_0,
\end{aligned}$$

where the constant M does not depend on the parameter $\tau > 0$. For $n = 2$ we need a similar modification to the local well-posedness theory. $\square$

4 Proof for the singular limit

Proof of Theorem 1.3 We first show Theorem 1.3 for the small data case:

In what follows we assume that $\lambda = 0$ for simplicity. For the other cases $\lambda > 0$, all the proof can be applicable in the same way, since the evolution $e^{\tau t(\Delta - \lambda)}$ satisfies the same estimate with the case $\lambda = 0$.

(Step 1): For any $\eta_0 > 0$, let $I = (0, \infty)$ and we consider the difference of solutions between (3.1) and (1.12) as the following: For any $t \in I$,

$$\left\{ \begin{array}{l} u_\tau(t) - u(t) = \int_0^t e^{(t-s)\Delta} \nabla \cdot ((u_\tau(s) - u(s)) \nabla \psi_\tau(s)) ds \\ \quad + \int_0^t e^{(t-s)\Delta} \nabla \cdot (u(s) (\nabla \psi_\tau(s) - \nabla \psi(s))) ds, \quad t \in I, \\ \psi_\tau(t) - \psi(t) = e^{\tau t \Delta} \psi_0 + \int_0^{\tau t} e^{s\Delta} (u_\tau(t - \tau^{-1}s) - u(t - \tau^{-1}s)) ds \\ \quad + \int_0^{\tau t} e^{s\Delta} (u(t - \tau^{-1}s) - u(t)) ds + \int_{\tau t}^\infty e^{s\Delta} u(t) ds. \end{array} \right. \quad (4.1)$$

Choose the admissible exponents (θ, q) (θ, r) such as

$$\begin{aligned} \frac{2}{\theta} + \frac{n}{q} &= 2, \quad \frac{n}{2} < q \leq \theta, \\ \frac{2}{\theta} + \frac{n}{r} &= 1, \quad n < r \leq \theta. \end{aligned}$$

Let the time interval $I = (\eta_0 \tau^{-1}, \infty)$ for any $\eta_0 > 0$. Introducing the difference of solution as

$$\begin{cases} w_u(t) \equiv u_\tau(t) - u(t), \\ w_\psi(t) \equiv \psi_\tau(t) - \psi(t), \end{cases}$$

we apply the similar estimate in (3.17), it follows from Lemma 2.3 that

$$\begin{aligned} \|u_\tau(t) - u(t)\|_{L^\theta(I; L^q)} &\leq C \|w_u(t) \nabla \psi_\tau\|_{L^{\frac{\theta}{2}}(I; \dot{B}_{q, \theta}^{-2})} + C \|u(t) \nabla w_\psi\|_{L^{\frac{\theta}{2}}(I; \dot{B}_{q, \theta}^{-2})} \\ &\leq C \|w_u \nabla \psi_\tau\|_{L^{\frac{\theta}{2}}(I; L^{\frac{rq}{r+q}})} + C \|u \nabla w_\psi\|_{L^{\frac{\theta}{2}}(I; L^{\frac{rq}{r+q}})} \\ &\leq C \|w_u\|_{L^\theta(I; L^q)} \|\nabla \psi_\tau\|_{L^\theta(I; L^r)} + C \|u\|_{L^\theta(\mathbb{R}_+; L^q)} \|\nabla w_\psi\|_{L^\theta(I; L^r)} \\ &\leq CM \left(\|u_\tau(t) - u(t)\|_{L^\theta(I; L^q)} + \|\nabla \psi_\tau - \nabla \psi(t)\|_{L^\theta(I; L^r)} \right). \end{aligned} \quad (4.2)$$

Meanwhile,

$$\begin{aligned} \|\nabla \psi_\tau(t) - \nabla \psi(t)\|_{L^\theta(I; L^r)} &= \|\nabla e^{\tau t \Delta} \psi_0\|_{L^\theta(I; L^r)} \\ &\quad + \left\| \int_0^{\tau t} \nabla e^{s\Delta} (u_\tau(t - \tau^{-1}s) - u(t - \tau^{-1}s)) ds \right\|_{L^\theta(I; L^r)} \\ &\quad + \left\| \int_0^{\tau t} \nabla e^{s\Delta} (u_\infty(t - \tau^{-1}s) - u(t)) ds \right\|_{L^\theta(I; L^r)} \\ &\quad + \left\| \int_{\tau t}^\infty \nabla e^{s\Delta} u(t) ds \right\|_{L^\theta(I; L^r)} \\ &\equiv I_0 + I_1 + I_2 + I_3. \end{aligned} \quad (4.3)$$

Let $t \in (0, \infty)$, we see under the assumption $\psi_0 \in \dot{B}_{r,\theta}^{1-\frac{2}{\theta}}$ that

$$\begin{aligned} I_0 &= \left(\int_0^\infty \|e^{\tau t \Delta} \nabla \psi_0\|_{L^r}^\theta dt \right)^{1/\theta} \\ &= \tau^{-1/\theta} \left(\int_0^\infty \|e^{s \Delta} \nabla \psi_0\|_{L^r}^\theta ds \right)^{1/\theta} \\ &\leq \tau^{-1/\theta} \left(\int_0^\infty \|e^{s \Delta} \nabla \psi_0\|_{L^r}^\theta ds \right)^{1/\theta} \\ &\rightarrow 0 \quad \text{as } \tau \rightarrow \infty. \end{aligned} \quad (4.4)$$

We have from the second equation of (4.1), (2.18) in Theorem 2.5 and (θ, r) is the admissible, i.e., $2 < r \leq \theta$ that

$$\begin{aligned} I_1 &= \left\| \int_0^{\tau t} \nabla e^{s \Delta} (u_\tau(t - \tau^{-1}s) - u(t - \tau^{-1}s)) ds \right\|_{L^\theta(I; L^r)} \\ &\leq C \|u_\tau(t - \tau^{-1}t) - u(t - \tau^{-1}t)\|_{L^\theta(I; \dot{B}_{r,\theta}^{-1})} \\ &\leq C \|u_\tau((1 - \tau^{-1})t) - u((1 - \tau^{-1})t)\|_{L^\theta(I; \dot{F}_{r,\theta}^{-1})} \\ &\leq C(1 - \tau^{-1})^{-1/\theta} \|u_\tau(t) - u(t)\|_{L^\theta(I; \dot{F}_{r,2}^{-1})} \leq C \|u_\tau(t) - u(t)\|_{L^\theta(I; \dot{W}^{-1,r})} \\ &\leq C \|u_\tau - u\|_{L^\theta(I; L^q)} \end{aligned} \quad (4.5)$$

for all $\tau \geq 2$, where $C > 0$ is independent of $\tau > 0$. The third term of (4.3), we apply the Sobolev inequality and the mean value theorem to see that

$$\begin{aligned} I_2 &= \left\| \int_0^{\tau t} \nabla e^{s \Delta} \left(u\left(t - \frac{s}{\tau}\right) - u(t) \right) ds \right\|_{L^\theta(I; L^r)} \\ &\leq S_b \left\| \int_0^{\tau t} \Delta e^{s \Delta} \left(u\left(t - \frac{s}{\tau}\right) - u(t) \right) ds \right\|_{L^\theta(I; L^{\frac{nr}{n+r}})} \\ &= S_b \left\| \int_0^{\tau t} \Delta e^{s \Delta} \left(\int_0^s \frac{\partial}{\partial r} u\left(t - \frac{r}{\tau}\right) dr \right) ds \right\|_{L^\theta(I; L^q)} \\ &= S_b \left\| \int_0^{\tau t} \left(\int_r^{\tau t} \Delta e^{s \Delta} ds \right) \frac{\partial}{\partial r} u\left(t - \frac{r}{\tau}\right) dr \right\|_{L^\theta(I; L^q)} \\ &= S_b \left\| \int_0^{\tau t} (e^{r \Delta} - e^{\tau t \Delta}) \frac{\partial}{\partial r} u\left(t - \frac{r}{\tau}\right) dr \right\|_{L^\theta(I; L^q)} \\ &= S_b \left\| \int_0^{\tau t} e^{r \Delta} \frac{\partial}{\partial r} u\left(t - \frac{r}{\tau}\right) dr - e^{\tau t \Delta} \left[u\left(t - \frac{r}{\tau}\right) \right]_{r=0}^{\tau t} \right\|_{L^\theta(I; L^q)} \\ &\leq S_b \left\| \int_0^{\tau t} e^{r \Delta} \frac{\partial}{\partial r} u\left(t - \frac{r}{\tau}\right) dr \right\|_{L^\theta(I; L^q)} \\ &\quad + S_b \|e^{\tau t \Delta} u_0\|_{L^\theta(I; L^q)} + S_b \|e^{\tau t \Delta} u(t)\|_{L^\theta(I; L^q)} \\ &= I_{2,1} + I_{2,2} + I_{2,3}. \end{aligned} \quad (4.6)$$

For treating the first term of the right hand side of (4.6), we proceed by changing the variable $r = \tau s$

$$\begin{aligned} I_{2,1} &= \left\| \int_0^t e^{\tau s \Delta} \frac{\partial}{\partial s} u(t-s) ds \right\|_{L^\theta(I; L^q)} \\ &\leq \left\| \int_0^t e^{\tau s \Delta} \Delta u(t-s) ds \right\|_{L^\theta(I; L^q)} + \left\| \int_0^t e^{\tau s \Delta} \nabla \cdot (u(t-s) \nabla \psi(t-s)) ds \right\|_{L^\theta(I; L^q)} \\ &\equiv J_1 + J_2. \end{aligned} \quad (4.7)$$

Then applying (2.18) in Theorem 2.5, we see by regarding $\mu \rightarrow \tau$, that

$$J_1 = \left\| \int_0^t e^{\tau(t-r)\Delta} \Delta u(r) dr \right\|_{L^\theta(I; L^q)} \leq C \tau^{-1} \|u\|_{L^\theta(I; L^q)}. \quad (4.8)$$

Similarly analogous estimate (3.26), it follows that

$$\begin{aligned} J_2 &= \left\| \int_0^t e^{\tau(t-r)\Delta} \nabla \cdot (u \nabla \psi)(r) dr \right\|_{L^\theta(I; L^q)} \\ &\leq C \tau^{-1} \|u \nabla \psi\|_{L^{\frac{\theta}{2}}(I; L^{\frac{rq}{r+q}})} \\ &\leq C \tau^{-1} \|u\|_{L^\theta(I; L^q)} \|\nabla \psi\|_{L^{\frac{\theta}{2}}(I; L^r)}. \end{aligned} \quad (4.9)$$

On the other hand, since u_0 and $u(t) \in L^q$ for almost everywhere,

$$\lim_{\tau \rightarrow \infty} \|e^{\tau t \Delta} \{u_0 - u(t)\}\|_q = 0$$

and $e^{\tau t \Delta}$ is a bounded operator from L^q to itself. Since

$$\begin{aligned} \|e^{t \Delta} u_0\|_{L^\theta(I; L^q)} &\leq C \|u_0\|_{\dot{B}_{q, \theta}^{-\frac{2}{\theta}}} \leq C \|u_0\|_{\frac{n}{2}}, \\ \|e^{t \Delta} u(t)\|_{L^\theta(I; L^q)} &\leq C \|u\|_{L^\theta(I; L^q)}, \end{aligned}$$

we see for all $\tau > 1$ that

$$\begin{aligned} \|e^{\tau t \Delta} u_0\|_q^\theta &\leq C \|e^{t \Delta} u_0\|_q^\theta \in L^1(I), \\ \|e^{\tau t \Delta} u(t)\|_q^\theta &\leq C \|e^{t \Delta} u(t)\|_q^\theta \in L^1(I). \end{aligned}$$

Hence by the Lebesgue dominated convergence theorem, for any $\varepsilon > 0$, we may choose sufficiently large $\tau > 0$ such that

$$I_{2,2} + I_{2,3} < \varepsilon. \quad (4.10)$$

Combining the estimates (4.6), (4.7), (4.8), (4.9) and (4.10), we obtain that for any $\varepsilon > 0$, there exists a large $\tau > 0$ such that

$$I_2 \leq \varepsilon. \quad (4.11)$$

Lastly for the forth term, setting $I = (\eta_0 \tau^{-1}, \infty)$, employing the Sobolev embedding:

$$\|f\|_r \leq S_b \|\nabla f\|_{\frac{2r}{2+r}},$$

we see

$$\begin{aligned}
 I_3 &= \left\| \int_{\tau t}^{\infty} \nabla e^{s\Delta} ds u(t) \right\|_{L^\theta(I; L^r)} \\
 &= \left\| \nabla (-\Delta)^{-1} e^{\tau t \Delta} u(t) \right\|_{L^\theta(I; L^r)} \\
 &\leq S_b \left\| \nabla^2 (-\Delta)^{-1} e^{\tau t \Delta} u(t) \right\|_{L^\theta(I; L^{\frac{nr}{r+n}})} \\
 &= S_b \left(\int_0^\infty \|e^{\tau t \Delta} u(t)\|_{\frac{2r}{r+2}}^\theta dt \right)^{1/\theta} \\
 &= S_b \left(\int_0^\infty \|u(s)\|_{\frac{2r}{r+2}}^\theta ds \right)^{1/\theta}.
 \end{aligned} \tag{4.12}$$

For the Serrin-admissible (θ, q) , the limiting solution u is integrable in $L^\theta([\eta_0, \infty); L^q)$ and especially, by

$$\frac{2}{\theta} + \frac{r+2}{r} = \frac{2}{\theta} + \frac{2}{r} + 1 = 2$$

we find that $(\theta, \frac{2r}{r+2})$ is also the Serrin-admissible for u and

$$\left(\int_0^\infty \|u(s)\|_{\frac{2r}{r+2}}^\theta ds \right)^{1/\theta} < \infty.$$

Hence from the fourth line of (4.12), the integrant

$$\|e^{\tau t \Delta} u(t)\|_{\frac{2r}{r+2}}^\theta$$

is $L^1(\mathbb{R}_+)$ and it is dominated by the integrable function $\|u(s)\|_{\frac{2r}{r+2}}^\theta$ as;

$$\|e^{\tau t \Delta} u(t)\|_{\frac{2r}{r+2}}^\theta \leq \|u(t)\|_{\frac{2r}{r+2}}^\theta.$$

Besides for almost all $t > 0$,

$$\lim_{\tau \rightarrow \infty} \|e^{\tau t \Delta} u(t)\|_{\frac{2r}{r+2}}^\theta = 0. \tag{4.13}$$

Applying the Lebesgue dominated convergence theorem, it follows from (4.13) that for any $\varepsilon > 0$

$$\begin{aligned}
 I_3 &= \left\| \int_{\tau t}^{\infty} \nabla e^{s\Delta} ds u(t) \right\|_{L^\theta(I; L^r)} \\
 &\leq S_b \left(\int_0^\infty \|e^{\tau t \Delta} u(t)\|_{\frac{2r}{r+2}}^\theta dt \right)^{1/\theta} < \varepsilon,
 \end{aligned} \tag{4.14}$$

as $\tau \rightarrow \infty$. Combining all the estimates (4.3), (4.4), (4.5), (4.11) and (4.14), we obtain

$$\|\nabla w_\psi(t)\|_{L^\theta(I; L^r)} \leq C \|w_u\|_{L^\theta(I; L^q)} + \varepsilon. \tag{4.15}$$

Gathering (4.2) and (4.15), we see that for any $\varepsilon > 0$, choosing τ sufficiently large such that

$$\|w_u\|_{L^\theta(I; L^q)} \leq CM \left(\|w_u\|_{L^\theta(I; L^q)} + \varepsilon \right). \tag{4.16}$$

In particular, from (4.15) and (4.16), for small any $\varepsilon > 0$

$$\|u_\tau - u\|_{L^\theta(I; L^q)} + \|\nabla \psi_\tau - \nabla \psi\|_{L^\theta(I; L^r)} \leq \varepsilon. \quad (4.17)$$

(Step 2): For any $\eta_0 > 0$, let $I = (0, T) \cap (\eta_0, \infty)$. From maximal regularity (2.18) in Theorem 2.5, we estimate the first component of the integral equation (4.1) as follows:

$$\begin{aligned} & \|u_\tau - u\|_{L^\infty(I; L^{\frac{n}{2}})} \\ & \leq \left\| \int_0^t \nabla e^{(t-s)\Delta} \left(u_\tau(s) \nabla \psi_\tau(s) - u(s) \nabla \psi(s) \right) ds \right\|_{L^\infty(I; L^{\frac{n}{2}})} \\ & \leq C \|u_\tau(s) \nabla \psi_\tau(s) - u(s) \nabla \psi(s)\|_{L^{\frac{\theta}{2}}(I; \dot{B}_{1,2}^{-1+\frac{4}{\theta}})} \\ & \leq C \left(\| (u_\tau(s) - u(s)) \nabla \psi_\tau(s) \|_{L^{\frac{n}{2}}(I; \dot{W}^{-1+\frac{4}{\theta}, 1})} \right. \\ & \quad \left. + \| u(s) (\nabla \psi_\tau(s) - \nabla \psi(s)) \|_{L^{\frac{\theta}{2}}(I; \dot{W}^{-1+\frac{4}{\theta}, 1})} \right) \\ & \leq C \left(\| (u_\tau(s) - u(s)) \nabla \psi_\tau(s) \|_{L^{\frac{\theta}{2}}(I; L^{\frac{rq}{r+q}})} \right. \\ & \quad \left. + \| u(s) (\nabla \psi_\tau(s) - \nabla \psi(s)) \|_{L^{\frac{\theta}{2}}(I; L^{\frac{rq}{r+q}})} \right) \\ & \leq C \left(\|u_\tau - u\|_{L^\theta(I; L^q)} \|\nabla \psi_\tau\|_{L^\theta(I; L^r)} + \|\nabla \psi_\tau - \nabla \psi\|_{L^\theta(I; L^r)} \|u\|_{L^\infty(I; L^q)} \right), \quad (4.18) \end{aligned}$$

where

$$\|f\|_{\dot{W}^{-1+\frac{4}{\theta}, 1}} \leq C \|f\|_{L^{\frac{rq}{r+q}}}, \quad \frac{2}{2} - \frac{1}{2} \left(-1 + \frac{4}{\theta} \right) = \frac{1}{r} + \frac{1}{q}.$$

On the other hand, we apply the Sobolev embedding

$$L^q \simeq \dot{F}_{q,2}^0 \subset \dot{B}_{q,\infty}^0 \subset \dot{B}_{\infty,\infty}^{-2+\frac{2}{\theta}}$$

generalized maximal regularity (2.18) to the second integral equation in (4.1). Then from estimates (4.5), (4.6), (4.11) and (4.14), we see for any $\varepsilon > 0$, choosing large $\tau > 0$ that

$$\begin{aligned} & \|\psi_\tau - \psi\|_{L^\infty(I; BMO)} \\ & \leq \|e^{\tau t \Delta} \psi_0\|_{L^\infty(I; BMO)} \\ & \quad + \left\| \int_0^{\tau t} e^{s\Delta} (u_\tau(t - \tau^{-1}s) - u(t - \tau^{-1}s)) ds \right\|_{L^\infty(I; BMO)} \\ & \quad + \left\| \int_0^{\tau t} e^{s\Delta} (u(t - \tau^{-1}s) - u(t)) ds \right\|_{L^\infty(I; BMO)} \\ & \quad + \left\| \int_{\tau t}^\infty e^{s\Delta} u(t) ds \right\|_{L^\infty(I; BMO)} \\ & \leq \|e^{\tau \eta_0 \Delta} \psi_0\|_{BMO} + C \|u_\tau(t - \tau^{-1}t) - u(t - \tau^{-1}t)\|_{L^\theta(I; \dot{B}_{\infty,\infty}^{-2+\frac{2}{\theta}})} \\ & \quad + C \|u(t - \tau^{-1}t) - u(t)\|_{L^\theta(I; \dot{B}_{\infty,\infty}^{-2+\frac{2}{\theta}})} + C \|e^{\tau \eta_0 \Delta} u(t)\|_{L^\theta(I; \dot{B}_{\infty,\infty}^{-2+\frac{2}{\theta}})} \\ & \leq \|e^{\tau \eta_0 \Delta} \psi_0\|_{BMO} + CS_b \|u_\tau(t - \tau^{-1}t) - u(t - \tau^{-1}t)\|_{L^\theta(I; L^q)} \end{aligned}$$

$$\begin{aligned}
& + CS_b \|u(t - \tau^{-1}t) - u(t)\|_{L^\theta(I; L^q)} + CS_b \|e^{\tau\eta_0\Delta} u(t)\|_{L^\theta(I; L^q)} \\
& \leq \|e^{\tau\eta_0\Delta} \psi_0\|_{BMO} + CS_b \|u_\tau - u\|_{L^\theta(I; L^q)} + 2\varepsilon.
\end{aligned} \quad (4.19)$$

Since

$$\|e^{\tau\eta_0\Delta} \psi_0\|_{BMO} \rightarrow 0 \quad (4.20)$$

as $\tau \rightarrow \infty$ for $\psi_0 \in VMO(\mathbb{R}^n)$, by passing $\tau \rightarrow \infty$ in both (4.18) and (4.19), we conclude from (4.17) that the convergence (1.21) holds. This completes the proof. $\square$

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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Appendix

We show the out-lined proof for generalized maximal regularity for the heat equation Theorem 2.5.

Proposition 5.1 [46] *For $0 < T \leq \infty$, we set $I = [0, T)$. Let $1 \leq p, \sigma \leq \infty$, $1 \leq \rho \leq \infty$ and $s \in \mathbb{R}$. For $u_0 \in \dot{B}_{p,\rho}^{s-2/\rho}$, the solution of the heat equation $u(t) = e^{t\Delta} u_0$ satisfies the following estimate: For any $0 < T \leq \infty$ and $s \in \mathbb{R}$,*

$$\left(\int_0^T \|e^{t\Delta} u_0\|_{\dot{B}_{p,\sigma}^s}^\rho dt \right)^{1/\rho} \leq C \|u_0\|_{\dot{B}_{p,\rho}^{s-2/\rho}}. \quad (5.1)$$

where C is independent of T .

See for the proof, Ogawa–Shimizu [46] for $1 < \rho \leq \infty$. The end-point case $\rho = 1$ is directly obtained in [12, 47].

We now state the inhomogeneous version of Proposition 5.1, which is shown in [47] for $\rho = 1$ and it is applied for the time local well-posedness for a positive solutions.

Proposition 5.2 *For $0 < T \leq \infty$, we set $I = [0, T)$. Let $1 \leq p, \sigma \leq \infty$, $1 \leq \rho \leq \infty$ and $s \in \mathbb{R}$. Then there exists a constant $C_T > 0$ such that for any $u_0 \in \dot{B}_{p,\rho}^{s-2/\rho}$ with $s \in \mathbb{R}$ the*

solution of the heat equation $u(t) = e^{t\Delta}u_0$ satisfies

$$\left(\int_0^T \|e^{t\Delta}u_0\|_p^\rho dt \right)^{1/\rho} \leq C_T \|u_0\|_{B_{p,\rho}^{-\frac{2}{p}}}, \quad (5.2)$$

where $C_T = O((\log T)^{1/\rho})$ as $T \rightarrow \infty$.

For $u_0 \in \mathcal{B}_{\infty,\rho}^{s-\frac{2}{\rho}} = \overline{C_0^\infty}^{B_{\infty,\rho}^{s-\frac{2}{\rho}}}$, then there exists a constant $C_T > 0$ such that

$$\left(\int_0^T \|e^{t\Delta}u_0\|_{\infty}^\rho dt \right)^{1/\rho} \leq C_T \|u_0\|_{B_{\infty,\rho}^{-\frac{2}{\rho}}}, \quad (5.3)$$

where $C_T = O((\log T)^{1/\rho})$ as $T \rightarrow \infty$.

We recall the following lemma which is a variant of the estimate shown in [47].

Lemma 5.3 *Let $1 \leq \rho < \infty$ and $1 \leq p \leq \infty$. For $\ell \in \mathbb{Z}$, set the interval $I_\ell = [2^{-2\ell}, 2^{-2\ell+2})$. Let $\{\phi_j\}_{j \in \mathbb{Z}}$ be the Littlewood-Paley decomposition of unity and $u_0 \in \dot{B}_{p,\rho}^0$. Then there exists a constant $C > 0$ such that for any $k, k' \in \mathbb{Z}$ with $k' < k$,*

$$\sum_{k' \leq \ell \leq k} \int_{I_\ell} \|\nabla|^{2/\rho} e^{s\Delta} u_0\|_p^\rho ds \leq C \sum_{j \in \mathbb{Z}} \min(2^{2(j-k')}, e^{-\rho 2^{2(j-k)-2}}) \|\phi_j * u_0\|_p^\rho \quad (5.4)$$

holds, where the constant $C > 1$ is depending on n, p and ϕ_0 .

To prove Lemma 5.3, we invoke the following estimate (see [47, Lemma 3.4]):

Lemma 5.4 [47] *Let $1 \leq \rho < \infty$ and $1 \leq p \leq \infty$. Then for any $s \in I_\ell \equiv [2^{-2\ell}, 2^{-2\ell+2})$, it holds*

$$\|(s\Delta)^{1/\rho} e^{s\Delta} \phi_j * f\|_p \leq C 2^{2(j-\ell)/\rho} (1 + 2^{2(n+3)(j-\ell)}) e^{-2^{2(j-\ell)}} \|\phi_j * f\|_p \quad (5.5)$$

Proof of Lemma 5.3 Then we see that

$$\begin{aligned} & \sum_{k' \leq \ell \leq k} \int_{I_\ell} \|\nabla|^{2/\rho} e^{s\Delta} u_0\|_p^\rho ds \\ & \leq \sum_{k' \leq \ell \leq k} \int_{I_\ell} \sum_{j \in \mathbb{Z}} \|(s\Delta)^{1/\rho} e^{s\Delta} \phi_j * f\|_p^\rho \frac{ds}{s} \\ & \leq C \sum_{k' \leq \ell \leq k} \sum_{j \leq \ell} (2^{2(j-\ell)/\rho} e^{-2^{2(j-\ell)}} \|\phi_j * f\|_p)^\rho \\ & \quad + C \sum_{k' \leq \ell \leq k} \sum_{j > \ell} (2^{2(n+4)(j-\ell)} e^{-2^{2(j-\ell)}} \|\phi_j * f\|_p)^\rho \\ & \leq C \sum_{j \leq k'} \sum_{k' \leq \ell \leq k} 2^{2(j-\ell)} e^{-\rho 2^{2(j-\ell)}} \|\phi_j * f\|_p^\rho \\ & \quad + C \sum_{k' \leq j \leq k} \sum_{j \leq \ell \leq k} 2^{2(j-\ell)} e^{-\rho 2^{2(j-\ell)}} \|\phi_j * f\|_p^\rho \\ & \quad + C \sum_{k' \leq j \leq k} \sum_{k' \leq \ell \leq j} 2^{2(n+4)(j-\ell)} e^{-\rho 2^{2(j-\ell)}} \|\phi_j * f\|_p^\rho \end{aligned}$$

$$\begin{aligned}
& + C \sum_{j>k} \sum_{k' \leq \ell \leq k} 2^{2(n+4)(j-\ell)} e^{-\rho 2^{2(j-\ell)}} \|\phi_j * f\|_p^\rho \\
& \leq C \sum_{j \leq k'} 2^{2(j-k')} \|\phi_j * f\|_p^\rho \\
& \quad + C \sum_{k' \leq j \leq k} \|\phi_j * f\|_p^\rho \\
& \quad + C \sum_{j>k} e^{-\rho 2^{2(j-k)-2}} \|\phi_j * f\|_p^\rho \\
& \leq C \sum_{j \in \mathbb{Z}} \min(2^{2(j-k')}, e^{-\rho 2^{2(j-k)-2}}) \|\phi_j * f\|_p^\rho.
\end{aligned}$$

□

Proof of Proposition 5.2 For $T > 0$, choose $\tilde{\ell} \in \mathbb{N}$ such that $T \in I_{\tilde{\ell}} = [2^{-2\tilde{\ell}}, 2^{-2\tilde{\ell}+2})$ and assume that $\tilde{\ell} < 0$. From (5.4),

$$\begin{aligned}
\int_0^T \|\nabla|^{2/\rho} e^{s\Delta} u_0\|_p^\rho ds & \leq C \sum_{j \in \mathbb{Z}} \min(2^{2(j-\tilde{\ell})}, 1) \|\phi_j * u_0\|_p^\rho \\
& \leq C \sum_{j \geq 1} \min(2^{2(j-\tilde{\ell})}, 1) \|\phi_j * u_0\|_p^\rho \\
& \quad + C \sum_{\tilde{\ell} \leq j \leq 0} \min(2^{2(j-\tilde{\ell})}, 1) \|\phi_j * u_0\|_p^\rho \\
& \quad + C \sum_{j \leq \tilde{\ell}} \min(2^{2(j-\tilde{\ell})}, 1) \|\phi_j * u_0\|_p^\rho \\
& \leq C \sum_{j \geq 1} \|\phi_j * u_0\|_p^\rho + C \sum_{\tilde{\ell} \leq j \leq 0} \|\phi_j * u_0\|_p^\rho \\
& \quad + C \sum_{j \leq \tilde{\ell}} \min(2^{2(j-\tilde{\ell})}, 1) \|\phi_j * u_0\|_p^\rho \\
& \leq C \sum_{j \geq 1} \|\phi_j * u_0\|_p^\rho + C \sum_{\tilde{\ell} \leq j \leq 0} \|\phi_j\|_1 \|\psi * u_0\|_p^\rho \\
& \quad + C \sum_{j \leq \tilde{\ell}} 2^{2(j-\tilde{\ell})} \|\phi_j\|_1 \|\psi * u_0\|_p^\rho \\
& \leq C \sum_{j \geq 1} \|\phi_j * u_0\|_p^\rho + C(|\tilde{\ell}| + 1) \|\psi * u_0\|_p^\rho \\
& \quad + C \|\psi * u_0\|_p^\rho \\
& \leq (2 + |\tilde{\ell}|) \|u_0\|_{B_{p,\rho}^0}^\rho,
\end{aligned}$$

Since $|\tilde{\ell}| \simeq \frac{\log T}{2 \log 2}$, we conclude the resulting estimate (5.2). □

We next consider the solution of the inhomogeneous heat equation with 0-initial data:

$$\begin{cases} \partial_t v - \Delta v = f, & t > 0, x \in \mathbb{R}^n, \\ v(0, x) = 0, & x \in \mathbb{R}^n. \end{cases}$$

The following proposition is the key to proving Theorem 2.5.

Proposition 5.5 [34] *Let $\{e^{t\Delta}\}_{t \geq 0}$ be a heat semi group in $\mathbb{R}^n$ and $I = (0, T)$ for any $0 < T \leq \infty$. Then for $1 \leq p \leq \infty$ and $1 \leq v \leq \sigma \leq \rho \leq \infty$, it holds that*

$$\left\| \int_0^t e^{(t-s)\Delta} f(s) ds \right\|_{L^\rho(I; \dot{B}_{p,\sigma}^{s+\frac{2}{p}})} \leq C \|f\|_{L^v(I; \dot{B}_{p,\sigma}^{s-2+\frac{2}{p}})}. \quad (5.6)$$

Proof of Proposition 5.5 Since we show the result by using the duality argument, we only treat the case when $1 < \mu \leq \sigma \leq \rho \leq \infty$ and $1 < p \leq \infty$. For the case $p = 1$, the proof requires a similar argument involving the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. This is because the base space $L^1(\mathbb{R}^n)$ is not the dual of $L^\infty(\mathbb{R}^n)$. We also show the inequality for the case $T = \infty$. The other case is obtained by letting $f(t)$ by $\chi_{(0,T)}(t)f(t)$. Let

$$\chi_j(r) = \begin{cases} 1, & 2^j < r \leq 2^{j+1}, \\ 0, & \text{otherwise} \end{cases}$$

and $g(t) \in C^\infty(I; S(\mathbb{R}^n))$. We consider the dual coupling: For $j \in \mathbb{Z}$,

$$\begin{aligned} & \left| \int_0^\infty \left(\int_0^t e^{(t-s)\Delta} f(s) ds, g(t) \right)_{L^2} dt \right| \\ & \leq \iint_{t>s>0} \left| \left(e^{(t-s)\Delta} f(s), g(t) \right)_{L^2} \right| ds dt \\ & \leq \sum_{j \in \mathbb{Z}} \iint_{t>s>0} \chi_j(t-s) \left| \left(e^{(t-s)\Delta} f(s), g(t) \right)_{L^2} \right| ds dt. \end{aligned}$$

Introducing

$$T_j(f, g) \equiv \iint_{t>s>0} \chi_j(t-s) \left| \left(e^{(t-s)\Delta} f(s), g(t) \right)_{L^2} \right| ds dt,$$

we claim that $T : (f \times g) \rightarrow \{T_j(f, g)\}_j$ is a bilinear bounded form:

$$L^v(I; (\dot{B}_{p,\gamma}^{s-2}, \dot{B}_{p,\gamma}^s)_{1/v,\rho}) \times L^{\rho'}(I; (\dot{B}_{p',\gamma'}^{-s-2}, \dot{B}_{p',\gamma'}^{-s})_{1-1/\rho,\rho'}) \rightarrow \ell_1,$$

where $1 \leq p, \gamma \leq \infty$, $1 < \rho \leq \infty$, $1 \leq v < \infty$ with $v \leq \rho$. From the dissipative estimate for the heat evolution operator,

$$\|e^{t\Delta} u_0\|_{\dot{B}_{p,\gamma}^s} \leq C t^{-\frac{s-s'}{2}} \|u_0\|_{\dot{B}_{p,\gamma}^{s'}}, \quad (5.7)$$

we have for $l = 0, 2$ and $m = 0, 2$ that

$$\begin{aligned} T_j(f, g) & \leq \iint_{t>s>0} \chi_j(t-s) \|e^{(t-s)\Delta} f(s)\|_{\dot{B}_{p,\gamma}^{s+l}} \|g(t)\|_{\dot{B}_{p',\gamma'}^{-s-l}} ds dt \\ & \leq C \iint_{t>s>0} \chi_j(t-s) |t-s|^{-(l+m)/2} \|f(s)\|_{\dot{B}_{p,\gamma}^{s-m}} \|g(t)\|_{\dot{B}_{p',\gamma'}^{-s-l}} ds dt \\ & \leq C 2^{-(l+m)j/2} \iint_{t>s>0} \chi_j(t-s) \|f(s)\|_{\dot{B}_{p,\gamma}^{s-m}} \|g(t)\|_{\dot{B}_{p',\gamma'}^{-s-l}} ds dt. \end{aligned} \quad (5.8)$$

We decompose that $f = f_0 + f_1$ with $f_0 \in \dot{B}_{p,\gamma}^{s-2}$ and $f_1 \in \dot{B}_{p,\gamma}^s$, and $g = g_0 + g_1$ with $g_0 \in \dot{B}_{p',\gamma'}^{-s-2}$ and $g_1 \in \dot{B}_{p',\gamma'}^{-s}$, and taking the infimum over all representations of $g \in \dot{B}_{p',\gamma'}^{-s-2} + \dot{B}_{p',\gamma'}^{-s}$

and $f \in \dot{B}_{p,\gamma}^{s-2} + \dot{B}_{p,\gamma}^s$,

$$\begin{aligned} |T_j(f, g)| &\leq \inf_{f=f_0+f_1} \left\{ \inf_{g=g_0+g_1} (|T_j(f_0, g_0)| + |T_j(f_0, g_1)|) \right. \\ &\quad \left. + \inf_{g=g_0+g_1} (|T_j(f_1, g_0)| + |T_j(f_1, g_1)|) \right\}. \end{aligned}$$

We then let

$$\begin{cases} F_j(s) \equiv \|f(s)\|_{\{\dot{B}_{p,\gamma}^{s-2}, \dot{B}_{p,\gamma}^s\}_{1/v, j}} \\ \quad \equiv \inf_{f=f_0+f_1} (2^{-\frac{j}{v}} \|f_0(s)\|_{\dot{B}_{p,\gamma}^{s-2}} + 2^{j(1-\frac{1}{v})} \|f_1(s)\|_{\dot{B}_{p,\gamma}^s}), \\ G_j(t) \equiv \|g(t)\|_{\{\dot{B}_{p',\gamma'}^{s-2}, \dot{B}_{p',\gamma'}^s\}_{1-1/\rho, j}} \\ \quad \equiv \inf_{g=g_0+g_1} (2^{-j(1-\frac{1}{\rho})} \|g_0(t)\|_{\dot{B}_{p',\gamma'}^{s-2}} + 2^{\frac{j}{\rho}} \|g_1(t)\|_{\dot{B}_{p',\gamma'}^s}). \end{cases}$$

It follows by letting $\frac{1}{\mu} + \frac{1}{\rho'} + \frac{1}{v} = 2$ and it is $\frac{1}{\mu} + 2 + \frac{1}{v} - \frac{1}{\rho} - 1 = 2$ and also $\frac{1}{\mu} + 1 - \frac{1}{\rho} + 1 - \frac{1}{v'} = 2$ i.e., $\frac{1}{\rho} + \frac{1}{v'} = \frac{1}{\mu}$ and using the Hölder inequality (including the case $v = 1$) from (5.8) that we have

$$\begin{aligned} \sum_{j \in \mathbb{N}} |T_j(f, g)| &\leq C \sum_{j \in \mathbb{Z}} 2^{\frac{j}{v} - \frac{j}{\rho} - j} \\ &\quad \times \inf_{f=f_0+f_1} \left\{ 2^{-\frac{j}{v}} \inf_{g=g_0+g_1} \left(2^{-j(1-\frac{1}{\rho})} \iint_{t>s>0} \chi_j(t-s) \|f_0(s)\|_{\dot{B}_{p,\gamma}^{s-2}} \right. \right. \\ &\quad \times \|g_0(t)\|_{\dot{B}_{p',\gamma'}^{s-2}} ds dt + 2^{\frac{j}{\rho}} \iint_{t>s>0} \chi_j(t-s) \|f_0(s)\|_{\dot{B}_{p,\gamma}^{s-2}} \|g_1(t)\|_{\dot{B}_{p',\gamma'}^s} ds dt \\ &\quad + 2^{j(1-\frac{1}{v})} \inf_{g=g_0+g_1} \left(2^{-j(1-\frac{1}{\rho})} \iint_{t>s>0} \chi_j(t-s) \|f_1(s)\|_{\dot{B}_{p,\gamma}^s} \|g_0(t)\|_{\dot{B}_{p',\gamma'}^{s-2}} ds dt \right. \\ &\quad \left. \left. + 2^{\frac{j}{\rho}} \iint_{t>s>0} \chi_j(t-s) \|f_1(s)\|_{\dot{B}_{p,\gamma}^s} \|g_1(t)\|_{\dot{B}_{p',\gamma'}^s} ds dt \right) \right\} \\ &\leq C \sum_{j \in \mathbb{Z}} 2^{\frac{j}{v} - \frac{j}{\rho} - j} \iint_{t>s>0} \chi_j(t-s) F_j(s) G_j(t) ds dt \\ &\leq C \sum_{j \in \mathbb{Z}} 2^{j(\frac{1}{v} - \frac{1}{\rho} - 1)} \left(\int_0^\infty \chi_j^\mu(r) dr \right)^{1/\mu} \left(\int_0^\infty F_j(s)^v ds \right)^{1/v} \\ &\quad \times \left(\int_0^\infty G_j(t)^{\rho'} dt \right)^{1/\rho'} \\ &\leq C \sum_{j \in \mathbb{Z}} \left(\int_0^\infty F_j(s)^v ds \right)^{\frac{1}{v}} \left(\int_0^\infty G_j(t)^{\rho'} dt \right)^{\frac{1}{\rho'}}. \end{aligned}$$

Here we used the fact that $\frac{1}{\mu} = 2 - \frac{1}{v} - \frac{1}{\rho'} = 1 - \frac{1}{v} + \frac{1}{\rho}$. By using the fact that $\int_{2^j < \lambda \leq 2^{j+1}} \frac{d\lambda}{\lambda} = \int_{\mathbb{R}_+} \chi_j(\lambda) \frac{d\lambda}{\lambda} = \log 2$, we apply the Hölder inequality for j in $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$ and noting $\sigma' \geq \rho'$ and $v \leq \sigma$, we obtain

$$\begin{aligned} \sum_{j \in \mathbb{N}} |T_j(f, g)| &\leq C \sum_{j \in \mathbb{Z}} 2^{\frac{j}{v} - \frac{j}{\rho} - j} \\ &\quad \times \inf_{f=f_0+f_1} \left\{ 2^{-\frac{j}{v}} \inf_{g=g_0+g_1} \left(2^{-j(1-\frac{1}{\rho})} \iint_{t>s>0} \chi_j(t-s) \|f_0(s)\|_{\dot{B}_{p,\gamma}^{s-2}} \right. \right. \end{aligned}$$

$$\begin{aligned}
 & \times \|g_0(t)\|_{\dot{B}_{p',\gamma'}^{s-2}} ds dt + 2^{\frac{j}{\rho}} \iint_{t>s>0} \chi_j(t-s) \|f_0(s)\|_{\dot{B}_{p,\gamma}^{s-2}} \|g_1(t)\|_{\dot{B}_{p',\gamma'}^{s-2}} ds dt \\
 & + 2^{j(1-\frac{1}{\nu})} \inf_{g=g_0+g_1} \left(2^{-j(1-\frac{1}{\rho})} \iint_{t>s>0} \chi_j(t-s) \|f_1(s)\|_{\dot{B}_{p,\gamma}^s} \|g_0(t)\|_{\dot{B}_{p',\gamma'}^{s-2}} ds dt \right. \\
 & \left. + 2^{\frac{j}{\rho}} \iint_{t>s>0} \chi_j(t-s) \|f_1(s)\|_{\dot{B}_{p,\gamma}^s} \|g_1(t)\|_{\dot{B}_{p',\gamma'}^{s-2}} ds dt \right) \Big\} \\
 & \leq C \sum_{j \in \mathbb{Z}} 2^{\frac{j}{\nu} - \frac{j}{\rho} - j} \iint_{t>s>0} \chi_j(t-s) F_j(s) G_j(t) ds dt \\
 & \leq C \sum_{j \in \mathbb{Z}} 2^{j(\frac{1}{\nu} - \frac{1}{\rho} - 1)} \left(\int_0^\infty \chi_j^\mu(r) dr \right)^{1/\mu} \left(\int_0^\infty F_j(s)^\nu ds \right)^{1/\nu} \\
 & \quad \times \left(\int_0^\infty G_j(t)^{\rho'} dt \right)^{1/\rho'} \\
 & \leq C \sum_{j \in \mathbb{Z}} \left(\int_0^\infty F_j(s)^\nu ds \right)^{\frac{1}{\nu}} \left(\int_0^\infty G_j(t)^{\rho'} dt \right)^{\frac{1}{\rho'}}.
 \end{aligned}$$

We conclude that

$$\| \{T_j(f, g)\}_j \|_{\ell_1} \leq C \|f\|_{L^\nu(I; (\dot{B}_{p,\gamma}^{s-2}, \dot{B}_{p,\gamma}^s)_{1/\nu, \sigma})} \|g\|_{L^{\rho'}(I; (\dot{B}_{p',\gamma'}^{s-2}, \dot{B}_{p',\gamma'}^{s-2})_{1-1/\rho, \sigma'})}.$$

Noting for $1 < \nu < \infty$, $1 \leq \gamma' < \infty$ and $1 \leq \rho' < \infty$,

$$\begin{aligned}
 ((\dot{B}_{p',\gamma'}^{s-2}, \dot{B}_{p',\gamma'}^{s-2})_{1-1/\rho, \sigma'})^* &= (\dot{B}_{p,\gamma}^{s+2}, \dot{B}_{p,\gamma}^s)_{1-1/\rho, \sigma} = \dot{B}_{p,\sigma}^{s+\frac{2}{\rho}}, \\
 (\dot{B}_{p,\gamma}^{s-2}, \dot{B}_{p,\gamma}^s)_{1/\nu, \sigma} &= \dot{B}_{p,\sigma}^{s-2+\frac{2}{\nu}},
 \end{aligned}$$

we obtain the desired estimate (5.10) by a duality argument.

For the case $p = 1$, we employ the $\mathcal{H}^1$ - VMO duality (cf. [17]). By the dissipative estimate for the heat evolution operator,

$$\|e^{t\Delta} u_0\|_{\dot{\mathcal{H}}^{s,1}} \leq C t^{-\frac{s-s'}{2}} \|u_0\|_{\dot{\mathcal{H}}^{s',1}},$$

and we decompose that $f = f_0 + f_1$ with $f_0 \in \dot{\mathcal{H}}^{s-2,1}$ and $f_1 \in \dot{\mathcal{H}}^{s,1}$, and $g = g_0 + g_1$ with $g_0 \in V\dot{MO}^{-s-2}$ and $g_1 \in V\dot{MO}^{-s}$ and taking the infimum over all representations to have

$$\begin{aligned}
 |T_j(f, g)| &\leq \inf_{f=f_0+f_1} \left\{ \inf_{g=g_0+g_1} (|T_j(f_0, g_0)| + |T_j(f_0, g_1)|) \right. \\
 &\quad \left. + \inf_{g=g_0+g_1} (|T_j(f_1, g_0)| + |T_j(f_1, g_1)|) \right\}.
 \end{aligned}$$

Then letting

$$\left\{ \begin{aligned} \widetilde{F}_j(s) &\equiv \|f(s)\|_{\{\dot{\mathcal{H}}^{s-2,1}, \dot{\mathcal{H}}^{s,1}\}_{1/\nu, j}} \\ &\equiv \inf_{f=f_0+f_1} (2^{-\frac{j}{\nu}} \|f_0(s)\|_{\dot{\mathcal{H}}^{s-2,1}} + 2^{j(1-\frac{1}{\nu})} \|f_1(s)\|_{\dot{\mathcal{H}}^{s,1}}), \\ \widetilde{G}_j(t) &\equiv \|g(t)\|_{\{V\dot{MO}^{-s-2}, V\dot{MO}^{-s}\}_{1-1/\rho, j}} \\ &\equiv \inf_{g=g_0+g_1} (2^{-j(1-\frac{1}{\rho})} \|g_0(t)\|_{\dot{\mathcal{H}}^{s-2,1}} + 2^{\frac{j}{\rho}} \|g_1(t)\|_{\dot{\mathcal{H}}^{s,1}}). \end{aligned} \right.$$

Adding in j and applying the estimates (5.8), it follows by letting $\frac{1}{\mu} + 2 + \frac{1}{\nu} - \frac{1}{\rho} - 1 = 2$ and also $\frac{1}{\mu} + 1 - \frac{1}{\rho} + 1 - \frac{1}{\nu'} = 2$ i.e., $\frac{1}{\rho} + \frac{1}{\nu'} = \frac{1}{\mu}$ and using the Hölder inequality (including the case $\nu = 1$) that we have similarly before that

$$\begin{aligned}
 & \sum_{j \in \mathbb{N}} |T_j(f, g)| \\
 & \leq C \sum_{j \in \mathbb{Z}} 2^{\frac{j}{\nu} - \frac{j}{\rho} - j} \iint_{t>s>0} \chi_j(t-s) F_j(s) G_j(t) ds dt \\
 & \leq C \sum_{j \in \mathbb{Z}} 2^{j(\frac{1}{\nu} - \frac{1}{\rho} - 1)} \left(\int_0^\infty \chi_j^\mu(r) dr \right)^{1/\mu} \left(\int_0^\infty \widetilde{F}_j(s)^\nu ds \right)^{1/\nu} \left(\int_0^\infty \widetilde{G}_j(t)^{\rho'} dt \right)^{1/\rho'} \\
 & \leq C \left(\sum_{j \in \mathbb{Z}} \|\widetilde{F}_j(s)\|_{L^\nu(I)}^\sigma \right)^{1/\sigma} \left(\sum_{j \in \mathbb{Z}} \|\widetilde{G}_j(t)\|_{L^{\rho'}(I)}^{\sigma'} \right)^{1/\sigma'} \\
 & \quad (\text{Minkovski's inequality by } \nu \leq \sigma \text{ and } \rho' \leq \sigma') \\
 & \leq C \left(\sum_{j \in \mathbb{Z}} \left(\int_0^\infty \widetilde{F}_j(s)^\nu ds \right)^{\frac{\sigma}{\nu}} \right)^{\frac{1}{\sigma}} \left(\sum_{j \in \mathbb{Z}} \left(\int_0^\infty \widetilde{G}_j(t)^{\rho'} dt \right)^{\frac{\sigma'}{\rho'}} \right)^{\frac{1}{\sigma'}} \\
 & \leq C \left(\int_0^\infty \left(\sum_{j \in \mathbb{Z}} \int_{2^j < \lambda \leq 2^{j+1}} \|f(s)\|_{\{\dot{\mathcal{H}}^{s-2,1}, \dot{\mathcal{H}}^{s,1}\}_{1/\nu, j}}^\sigma \frac{d\lambda}{\lambda} \right)^{\nu/\sigma} ds \right)^{1/\nu} \\
 & \quad \times \left(\int_0^\infty \left(\sum_{j \in \mathbb{Z}} \int_{2^j < \lambda \leq 2^{j+1}} \|g(t)\|_{\{V\dot{M}O^{-s-2}, V\dot{M}O^{-s}\}_{1-1/\nu, j}}^{\sigma'} \frac{d\lambda}{\lambda} \right)^{\rho'/\sigma'} dt \right)^{1/\rho'} \\
 & \leq C \left(\int_0^\infty \left(\int_0^\infty \left(\inf_{f=f_0+f_1} \lambda^{-\frac{1}{\nu}} (\|f_0(s)\|_{\dot{\mathcal{H}}^{s-2,1}} + \lambda \|f_1(s)\|_{\dot{\mathcal{H}}^{s,1}}) \right)^\sigma \times \frac{d\lambda}{\lambda} \right)^{\nu/\sigma} ds \right)^{1/\nu} \\
 & \quad \times \left(\int_0^\infty \left(\int_0^\infty \left(\inf_{g=g_0+g_1} \lambda^{-(1-\frac{1}{\rho})} (\|g_0(t)\|_{V\dot{M}O^{-s-2}} + \lambda \|g_1(t)\|_{V\dot{M}O^{-s}}) \right)^{\sigma'} \right. \right. \\
 & \quad \times \left. \left. \frac{d\lambda}{\lambda} \right)^{\rho'/\sigma'} dt \right)^{1/\rho'} \\
 & = C \|f\|_{L^\nu(I; (\dot{\mathcal{H}}^{s-2,1}, \dot{\mathcal{H}}^{s,1})_{1/\nu, \sigma})} \|g\|_{L^{\rho'}(I; (V\dot{M}O^{-s-2}, V\dot{M}O^{-s})_{1-1/\rho, \sigma'})}.
 \end{aligned}$$

Noting for $1 < \nu < \infty$, $1 \leq \nu' < \infty$ and $1 \leq \rho' < \infty$ and by the Fefferman-Stein duality [17] $(V\dot{M}O^{-s})^* = \dot{\mathcal{H}}^{s,1}$, it follows that

$$\begin{aligned}
 (V\dot{M}O^{-s-2}, V\dot{M}O^{-s})_{1-1/\rho, \sigma'}^* &= (\dot{\mathcal{H}}^{s+2,1}, \dot{\mathcal{H}}^{s,1})_{1-1/\rho, \sigma} = \dot{B}_{1, \sigma}^{s+\frac{2}{\rho}}, \\
 (\dot{\mathcal{H}}^{s-2,1}, \dot{\mathcal{H}}^{s,1})_{1/\nu, \sigma} &= \dot{B}_{1, \sigma}^{s-2+\frac{2}{\nu}},
 \end{aligned}$$

(see for the details, [45]) we obtain the desired estimate (5.10) by a duality argument. $\square$

Proof of Theorem 2.5 Since the homogeneous estimate (2.17) can be obtained from Proposition 5.1, we only show the estimate (2.18). For $1 < \nu \leq \sigma \leq \rho < \infty$, we have shown

that

$$\left\| \int_0^t e^{(t-s)\Delta} f(s) ds \right\|_{L^\rho(I; \dot{B}_{p,\sigma}^{s+\frac{2}{\rho}})} \leq C \|f\|_{L^\nu(I; \dot{B}_{p,\sigma}^{s-2+\frac{2}{\nu}})} \quad (5.9)$$

from Proposition 5.5. Then the estimate (2.18) follows from (5.9), since $\partial_t u = \Delta u + f$.

Finally we treat the case $\rho = \infty$. In this case, we modify the above argument in the different interpolation parameter. Namely, we have from (5.8) that

$$\begin{aligned} \sum_{j \in \mathbb{Z}} |T_j(f, g)| &\leq C \sum_{j \in \mathbb{Z}} 2^{\frac{j}{\nu}-j} \\ &\times \inf_{f=f_0+f_1} \left\{ 2^{-j(\frac{1}{2}+\frac{1}{\nu})} \inf_{g=g_0+g_1} \left(2^{-\frac{1}{2}j} \iint_{t>s>0} \chi_j(t-s) \|f_0(s)\|_{\dot{B}_{p,\gamma}^{s-2}} \right. \right. \\ &\times \|g_0(t)\|_{\dot{B}_{p',\gamma'}^{s-2}} ds dt + 2^{\frac{1}{2}j} \iint_{t>s>0} \chi_j(t-s) \|f_0(s)\|_{\dot{B}_{p,\gamma}^{s-2}} \|g_1(t)\|_{\dot{B}_{p',\gamma'}^{s-2}} ds dt \\ &+ 2^{j(\frac{1}{2}-\frac{1}{\nu})} \inf_{g=g_0+g_1} \left(2^{-\frac{1}{2}j} \iint_{t>s>0} \chi_j(t-s) \|f_1(s)\|_{\dot{B}_{p,\gamma}^s} \|g_0(t)\|_{\dot{B}_{p',\gamma'}^{s-2}} ds dt \right. \\ &\left. \left. + 2^{\frac{1}{2}j} \iint_{t>s>0} \chi_j(t-s) \|f_1(s)\|_{\dot{B}_{p,\gamma}^s} \|g_1(t)\|_{\dot{B}_{p',\gamma'}^{s-2}} ds dt \right) \right\} \\ &\leq C \sum_{j \in \mathbb{Z}} 2^{\frac{j}{\nu}-j} \iint_{t>s>0} \chi_j(t-s) F_j(s) G_j(t) ds dt, \end{aligned}$$

where

$$\begin{cases} F_j(s) \equiv \|f(s)\|_{\{\dot{B}_{p,\gamma}^{s-2}, \dot{B}_{p,\gamma}^s\}_{\frac{1}{2}+\frac{1}{\nu}, j}} \\ \quad \equiv \inf_{f=f_0+f_1} (2^{-j(\frac{1}{2}+\frac{1}{\nu})} \|f_0(s)\|_{\dot{B}_{p,\gamma}^{s-2}} + 2^{j(\frac{1}{2}-\frac{1}{\nu})} \|f_1(s)\|_{\dot{B}_{p,\gamma}^s}), \\ G_j(t) \equiv \|g(t)\|_{\{\dot{B}_{p',\gamma'}^{s-2}, \dot{B}_{p',\gamma'}^s\}_{\frac{1}{2}, j}} \\ \quad \equiv \inf_{g=g_0+g_1} (2^{-\frac{1}{2}j} \|g_0(t)\|_{\dot{B}_{p',\gamma'}^{s-2}} + 2^{\frac{1}{2}j} \|g_1(t)\|_{\dot{B}_{p',\gamma'}^s}). \end{cases}$$

It follows by letting $\frac{1}{\mu} + \frac{1}{\nu} + 1 = 2$ and using the Hausdorff-Young inequality with $\frac{1}{\mu} = 2 - 1 - \frac{1}{\nu} = 1 - \frac{1}{\nu}$,

$$\begin{aligned} \sum_{j \in \mathbb{Z}} |T_j(f, g)| &\leq C \sum_{j \in \mathbb{Z}} 2^{j(\frac{1}{\nu}-1)} \|\chi_j(\cdot)\|_{\mu} \|F_j(\cdot)\|_{L^\nu(I)} \|G_j(\cdot)\|_{L^1(I)} \\ &= C \sum_{j \in \mathbb{Z}} \|F_j(\cdot)\|_{L^\nu(I)} \|G_j(\cdot)\|_{L^1(I)} \\ &\leq C \left(\sum_{j \in \mathbb{Z}} \|F_j(\cdot)\|_{L^\nu(I)}^\sigma \right)^{1/\sigma} \left(\sum_{j \in \mathbb{Z}} \|G_j(\cdot)\|_{L^1(I)}^{\sigma'} \right)^{1/\sigma'}. \end{aligned}$$

By using the fact that $\int_{2^j < \lambda \leq 2^{j+1}} \frac{d\lambda}{\lambda} = \int_{\mathbb{R}_+} \chi_j(\lambda) \frac{d\lambda}{\lambda} = \log 2$, we apply the Hölder and the Minkowski inequalities for j to obtain

$$\sum_{j \in \mathbb{Z}} |T_j(f, g)| \leq C \|f\|_{L^\nu(I; (\dot{B}_{p,\gamma}^{s-2}, \dot{B}_{p,\gamma}^s)_{\frac{1}{2}+\frac{1}{\nu}, \sigma})} \|g\|_{L^1(I; (\dot{B}_{p',\gamma'}^{s-2}, \dot{B}_{p',\gamma'}^s)_{\frac{1}{2}, \sigma'})}.$$

We conclude that

$$\| \{T_j(f, g)\}_j \|_{\ell_1} \leq C \|f\|_{L^v(I; (\dot{B}_{p, \gamma'}^{s-2}, \dot{B}_{p, \gamma'}^s)_{\frac{1}{2} + \frac{1}{v}, \sigma})} \|g\|_{L^1(I; (\dot{B}_{p', \gamma'}^{-s-2}, \dot{B}_{p', \gamma'}^{-s})_{\frac{1}{2}, \sigma'})}.$$

Noting for $2 < v \leq \rho = \infty$, $1 \leq \gamma' < \infty$ and $1 \leq p' < \infty$,

$$\begin{aligned} ((\dot{B}_{p', \gamma'}^{-s-2}, \dot{B}_{p', \gamma'}^{-s})_{\frac{1}{2} - \frac{1}{\rho}, \sigma'})^* &= (\dot{B}_{p, \gamma}^{s+2}, \dot{B}_{p, \gamma}^s)_{\frac{1}{2} - \frac{1}{\rho}, \sigma} = \dot{B}_{p, \sigma}^{s+1 + \frac{2}{\rho}}, \\ (\dot{B}_{p, \gamma}^{s-2}, \dot{B}_{p, \gamma}^s)_{\frac{1}{2} + \frac{1}{v}, \sigma} &= \dot{B}_{p, \sigma}^{s-1 + \frac{2}{v}} \end{aligned}$$

and we obtain the desired estimate (5.10) by letting $\rho = \infty$ and the duality argument. $\square$

If we trade off the range of exponents to slightly restricted conditions, then we may generalize Proposition 5.5 which stems from an estimate appeared in Kozono–Ogawa–Taniuchi [27] and Ogawa–Suguro [49] for the Morrey spaces.

Proposition 5.6 *Let $1 \leq p \leq \infty$, $1 < v < \rho \leq \infty$, $s \in \mathbb{R}$, $0 < r < 2$ and $\{e^{t\Delta}\}_{t \geq 0}$ be the heat semi group in $\mathbb{R}^n$. For any $0 < T \leq \infty$ let $I = (0, T)$. Then there exists a constants $C > 0$ independent of T such that*

$$\left\| |\nabla|^r \int_0^t e^{(t-s)\Delta} f(s) ds \right\|_{L^\rho(I; \dot{B}_{p,1}^s)} \leq C \|f\|_{L^v(I; \dot{B}_{p,\infty}^{s+\frac{2}{v}-\frac{2}{\rho}})}. \quad (5.10)$$

In the case of the inhomogeneous Besovs, the corresponding estimate holds with a constant C depending on T .

Proof of Proposition 5.6 Limiting the integral range $1 < \rho < \infty$, we employ the straight forward argument to show (5.10). For any $f \in L^v(I; \dot{B}_{p,\infty}^0)$, let $\chi_I(t)$ be the characteristic function of $I = (0, T)$. Then we apply the dissipative estimate for the heat kernel

$$\| |\nabla|^r e^{t\Delta} g \|_{\dot{B}_{p,1}^s} \leq C t^{-\frac{r}{2}} \|g\|_{\dot{B}_{p,\infty}^s} \quad (5.11)$$

for all $t > 0$, where $0 < r < 2$ (see [27, Lemma 2.2]) and the Hausdorff-Young inequality to obtain

$$\begin{aligned} & \left\| |\nabla|^{r+s} \int_0^t e^{(t-s)\Delta} \chi_I(s) f(s) ds \right\|_{L^\rho(I; \dot{B}_{p,1}^0)} \\ & \leq C \left\| \int_0^t |t-s|^{-\frac{r}{2}} \chi_I(s) \|f(s)\|_{\dot{B}_{p,\infty}^s} ds \right\|_{L^\rho(I)} \\ & \leq C \| |t|^{-\frac{r}{2}} * (\chi_I \|f\|_{\dot{B}_{p,\infty}^s}) \|_{L^\rho(\mathbb{R})} \\ & \leq C \| |t|^{-\frac{r}{2}} \|_{L_w^{\frac{2}{\rho}}(I)} \| \chi_I f \|_{L^v(\mathbb{R}; \dot{B}_{p,\infty}^s)} \\ & \leq C \|f\|_{L^v(I; \dot{B}_{p,\infty}^s)}, \end{aligned}$$

where $1 < v < \rho < \infty$, $0 < r < 2$ and

$$\frac{1}{\rho} = \frac{r}{2} + \frac{1}{v} - 1.$$

When $\rho = \infty$, we apply the duality argument to see from (5.8) and (5.11) that

$$\begin{aligned} T_j(\nabla f, g) &\leq \iint_{t>s>0} \chi_j(t-s) \|\nabla e^{(t-s)\Delta} f(s)\|_{\dot{B}_{p,1}^0} \|g(t)\|_{\dot{B}_{p',\infty}^0} ds dt \\ &\leq C \iint_{t>s>0} \chi_j(t-s) (|t-s|^{-1+1/\nu+1/\rho}) \|f(s)\|_{\dot{B}_{p,\infty}^{-1+\frac{2}{\nu}-\frac{2}{\rho}}} \|g(t)\|_{\dot{B}_{p',\infty}^0} ds dt \\ &\leq C \iint_{t>s>0} \chi_j(t-s) (2^{-j(1-1/\nu+1/\rho)}) \|f(s)\|_{\dot{B}_{p,\infty}^{-1+\frac{2}{\nu}-\frac{2}{\rho}}} \|g(t)\|_{\dot{B}_{p',\infty}^0} ds dt \end{aligned} \quad (5.12)$$

and by summing up by $j+1 \leq j_0 = \log T / \log 2$ for all $I_j = (2^j, 2^{j+1})$, it follows by applying Hausdorff-Young's inequality that

$$\begin{aligned} \sum_{j \leq j_0-1} T_j(\nabla f, g) &\leq C \sum_{j \leq j_0} (2^{-j(1-1/\nu+1/\rho)}) \iint_{I_j} \|f(s)\|_{\dot{B}_{p,\infty}^{-1+\frac{2}{\nu}-\frac{2}{\rho}}} \|g(t)\|_{\dot{B}_{p',\infty}^0} ds dt \\ &\leq C \sum_{j \leq j_0} (2^{-j(1-1/\nu+1/\rho)}) |I_j|^{\frac{1}{\mu}} \\ &\quad \times \left(\int_{I_j} \|f(s)\|_{\dot{B}_{p,\infty}^{-1+\frac{2}{\nu}-\frac{2}{\rho}}}^{\nu} ds \right)^{1/\nu} \left(\int_{I_j} \|g(t)\|_{\dot{B}_{p',\infty}^0}^{\rho'} ds \right)^{1/\rho'} \\ &\leq C \sup_{j \leq j_0} \left(\int_{I_j} \|f(s)\|_{\dot{B}_{p,\infty}^{-1+\frac{2}{\nu}-\frac{2}{\rho}}}^{\nu} ds \right)^{1/\nu} \sum_{j \leq j_0} \left(\int_{I_j} \|g(t)\|_{\dot{B}_{p',\infty}^0}^{\rho'} ds \right)^{1/\rho'}, \end{aligned} \quad (5.13)$$

where

$$\frac{1}{\nu} + \frac{1}{\rho'} + \frac{1}{\mu} = 2. \quad (5.14)$$

Setting $\rho = \infty$,

$$\sum_{j \leq j_0} T_j(\nabla f, g) \leq C \left(\int_I \|f(s)\|_{\dot{B}_{p,\infty}^{-1+\frac{2}{\nu}-\frac{2}{\rho}}}^{\nu} ds \right)^{1/\nu} \sum_{j \leq j_0} \int_{I_j} \|g(t)\|_{\dot{B}_{p',\infty}^0} ds, \quad (5.15)$$

and we obtain² that

$$\left| \int_0^T \left(\int_0^t e^{(t-s)\Delta} \nabla f(s) ds, g(t) \right) dt \right| \leq C \|f\|_{L^\nu(I; \dot{B}_{p,\infty}^{-1+\frac{2}{\nu}})} \|g\|_{L^1(I; \dot{B}_{p',\infty}^0)}$$

and it concludes to have

$$\left\| \nabla \int_0^t e^{(t-s)\Delta} f(s) ds \right\|_{L^\infty(I; \dot{B}_{p,1}^0)} \leq C \|f\|_{L^\nu(I; \dot{B}_{p,\infty}^{-1+\frac{2}{\nu}})},$$

where C is independent of T .

² This argument is due to T. Suguro. See for more general cases [49].

All the above estimates are valid even if we exchange the homogeneous Besov spaces into the inhomogeneous Besov spaces except the bound constant. In such cases, the dissipative estimate (5.11) can be modified by

$$\| |\nabla|^r e^{t\Delta} g \|_{B_{p,1}^s} \leq C(1 + t^{-\frac{r}{2}}) \|g\|_{B_{p,\infty}^s}$$

and the estimate (5.15) still remains valid if one modifies as follows:

$$\begin{aligned} & \sum_{j \leq j_0} T_j(\nabla f, g) \\ & \leq C \left(\int_I \|f(s)\|_{B_{p,\infty}^{-1+\frac{2}{p}-\frac{2}{p}}}^v ds \right)^{1/v} \sum_{j \leq j_0} (2^{j(1-1/v+1/\rho)} + 1) \int_{I_j} \|g(t)\|_{B_{p',\infty}^0} ds, \end{aligned} \quad (5.16)$$

and we obtain by setting $C_T \equiv C \max_{j \leq j_0} (2^{j(1-1/v+1/\rho)} + 1)^3$ that

$$\left| \int_0^T \left(\int_0^t e^{(t-s)\Delta} \nabla f(s) ds, g(t) \right) dt \right| \leq C_T \|f\|_{L^v(I; B_{p,\infty}^{-1+\frac{2}{p}})} \|g\|_{L^1(I; B_{p',\infty}^0)}$$

and we conclude to have

$$\left\| \nabla \int_0^t e^{(t-s)\Delta} f(s) ds \right\|_{L^\infty(I; B_{p,1}^0)} \leq C_T \|f\|_{L^v(I; B_{p,\infty}^{-1+\frac{2}{p}})},$$

where $C_T \simeq C 2^{j_0(1-1/v)} \simeq T^{1-1/v}$. □

A.1 Proof for the Koch–Tataru estimate

In this subsection, we illustrate for completeness that another proof of Proposition 2.6 under a slightly general setting.

Proof of Proposition 2.6 Let $R > 0$ and

$$\eta_R(x) = \begin{cases} 1, & |x - x_0| < R, \\ 0, & |x - x_0| > 2R \end{cases}$$

³ This argument is due to T. Suguro. See for more general cases [49].

be a smooth cut-off function. Integrate by parts, we have

$$\begin{aligned}
 & \frac{1}{|B_R|^2} \frac{d}{dt} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |e^{t(\Delta-\lambda)} u_0(x) - e^{t(\Delta-\lambda)} u_0(y)|^2 \eta_R(x) \eta_R(y) dx dy \\
 &= \frac{2}{|B_R|^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} (e^{t(\Delta-\lambda)} u_0(x) - e^{t(\Delta-\lambda)} u_0(y)) \\
 &\quad \times (\partial_t e^{t(\Delta-\lambda)} u_0(x) - \partial_t e^{t(\Delta-\lambda)} u_0(y)) \eta_R(x) \eta_R(y) dx dy \\
 &= \frac{2}{|B_R|^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} (e^{t(\Delta-\lambda)} u_0(x) - e^{t(\Delta-\lambda)} u_0(y)) \\
 &\quad \times \nabla_x \cdot (\nabla_x e^{t(\Delta-\lambda)} u_0(x) - \nabla_y e^{t(\Delta-\lambda)} u_0(y)) \eta_R(x) \eta_R(y) dx dy \\
 &\quad - \frac{2}{|B_R|^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} (e^{t(\Delta-\lambda)} u_0(x) - e^{t(\Delta-\lambda)} u_0(y)) \\
 &\quad \times \nabla_y \cdot (\nabla_y e^{t(\Delta-\lambda)} u_0(y) - \nabla_x e^{t(\Delta-\lambda)} u_0(x)) \eta_R(x) \eta_R(y) dx dy \\
 &\quad - \frac{2\lambda}{|B_R|^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |e^{t(\Delta-\lambda)} u_0(x) - e^{t(\Delta-\lambda)} u_0(y)|^2 \eta_R(x) \eta_R(y) dx dy \\
 &\leq -\frac{2}{|B_R|^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\nabla_x e^{t\Delta} u_0(x) - \nabla_y e^{t\Delta} u_0(y)|^2 \eta_R(x) \eta_R(y) dx dy \\
 &\quad - \frac{4}{|B_R|^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} (e^{t\Delta} u_0(x) - e^{t\Delta} u_0(y)) \\
 &\quad \times (\nabla_x e^{t\Delta} u_0(x) - \nabla_y e^{t\Delta} u_0(y)) \cdot \nabla_x \eta_R(x) \eta_R(y) dx dy \\
 &\equiv -J_1 + J_2.
 \end{aligned} \tag{5.17}$$

Integrate the both sides of (5.17) in $t \in (0, R^2)$. Then for $x \in \text{supp } \nabla \eta_R(x - x_0)$ and $y \in \text{supp } \eta_R(x - x_0)$,

$$\frac{|x - y|}{R} \leq 2$$

and for any $x \in \mathbb{R}^n$,

$$|\nabla \eta_R(x - x_0)| \leq \frac{C}{R}.$$

Hence

$$\begin{aligned}
 & \left| \int_0^t J_2(t) dt \right| \\
 &= \left| \int_0^t \frac{4}{|B_R|^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} (e^{t\Delta} u_0(x) - e^{t\Delta} u_0(y)) \right. \\
 &\quad \times (\nabla_x e^{t\Delta} u_0(x) - \nabla_y e^{t\Delta} u_0(y)) \cdot \nabla_x \eta_R(x) \eta_R(y) dx dy dt \left. \right| \\
 &\leq 4 \sup_{t \in (0, R^2)} \left(\frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |e^{t\Delta} u_0(x) - e^{t\Delta} u_0(y)|^2 dx dy \right)^{1/2} \\
 &\quad \times \left(\int_0^{R^2} \frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |\nabla e^{t\Delta} u_0(x) - \nabla_y e^{t\Delta} u_0(y)|^2 dx dy dt \right)^{1/2}
 \end{aligned}$$

$$\begin{aligned}
&\leq C_\varepsilon \sup_{t \in (0, R^2)} \left(\frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |e^{t\Delta} u_0(x) - e^{t\Delta} u_0(y)|^2 dx dy \right) \\
&\quad + \varepsilon \left(\int_0^{R^2} \frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |\nabla e^{t\Delta} u_0(x) - \nabla_y e^{t\Delta} u_0(y)|^2 dx dy dt \right) \\
&\equiv J_2^1 + \varepsilon \int_0^{R^2} \tilde{J}_1 dt.
\end{aligned} \tag{5.18}$$

Hence

$$\begin{aligned}
J_2^1 &= C_\varepsilon \sup_{t \in (0, R^2)} \left(\frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |e^{t\Delta} u_0(x) - e^{t\Delta} u_0(y)|^2 dx dy \right) \\
&= C_\varepsilon \sup_{t \in (0, R^2)} \|e^{t\Delta} u_0\|_{BMO}^2 \leq C_\varepsilon \|u_0\|_{BMO}^2.
\end{aligned} \tag{5.19}$$

The last inequality follows from Fefferman–Stein’s $\mathcal{H}^1$ - BMO -duality (see [17]) and regarding the heat kernel as the test function in $\mathcal{H}^1$,

$$\begin{aligned}
|(e^{t\Delta} u_0, \phi)| &= (u_0, e^{t\Delta} \phi) \leq C \|u_0\|_{BMO} \|e^{t\Delta} \phi\|_{\mathcal{H}^1} \leq C \|u_0\|_{BMO} \|\phi\|_{\mathcal{H}^1} \\
\sup_{\phi \in \mathcal{H}^1} \frac{|(e^{t\Delta} u_0, \phi)|}{\|\phi\|_{\mathcal{H}^1}} &\leq C \|u_0\|_{BMO}.
\end{aligned}$$

Namely, it holds that

$$\sup_{x_0} \sup_{R>0} J_2^1 = C v \|e^{vt\Delta} u_0\|_{BMO}^2 \leq C v \|u_0\|_{BMO}^2. \tag{5.20}$$

Therefore from (5.17), (5.18) and (5.19),

$$\begin{aligned}
&2 \int_0^{R^2} \left(\frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |\nabla_x e^{t\Delta} u_0(x) - \nabla_y e^{t\Delta} u_0(y)|^2 dx dy \right) dt \\
&= - \left[\frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |e^{t\Delta} u_0(x) - e^{t\Delta} u_0(y)|^2 \eta_R(x) \eta_R(y) dx dy \right]_{t=0}^{R^2} \\
&\quad + C_\varepsilon \sup_{t \in (0, R^2)} \left(\frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |e^{t\Delta} u_0(x) - e^{t\Delta} u_0(y)|^2 dx dy \right) \\
&\quad + \varepsilon \left(\int_0^{R^2} \frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |\nabla e^{t\Delta} u_0(x) - \nabla_y e^{t\Delta} u_0(y)|^2 dx dy dt \right) \\
&\leq \sup_{x_0, R>0} \frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |u_0(x) - u_0(y)|^2 dx dy \\
&\quad + C_\varepsilon \sup_{t \in (0, R^2)} \left(\frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |e^{t\Delta} u_0(x) - e^{t\Delta} u_0(y)|^2 dx dy \right) \\
&\quad + \varepsilon \left(\int_0^{R^2} \frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} |\nabla e^{t\Delta} u_0(x) - \nabla_y e^{t\Delta} u_0(y)|^2 dx dy dt \right).
\end{aligned} \tag{5.21}$$

From (5.17), (5.20) and (5.21), we conclude that

$$\|\nabla e^{t(\Delta-\lambda)}u_0\|_{\widetilde{L^2(I;BMO)}}^2 + \lambda\|e^{t(\Delta-\lambda)}u_0\|_{\widetilde{L^2(I;BMO)}}^2 \leq C\|u_0\|_{BMO}^2. \quad (5.22)$$

□

A direct cosequence of Proposition 2.6 is the following estimate (see for the proof, Ogawa–Shimizu [48]):

Proposition 5.7 [48] *Let $e^{t\Delta}$ be the heat kernel and $u_0 \in BMO(\mathbb{R}^n)$. Then*

$$\begin{aligned} & \sup_{x_0, R>0} \left(\int_0^{R^2} \frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} \right. \\ & \quad \times \int_s^{R^2} \left| \nabla_x \cdot e^{(t-s)(\Delta-\lambda)} f(s, x) - \nabla_y \cdot e^{(t-s)(\Delta-\lambda)} f(s, y) \right|^2 dt dx dy ds \Big)^{1/2} \\ & \quad + \lambda \sup_{x_0, R>0} \left(\int_0^{R^2} \frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} \right. \\ & \quad \times \int_s^{R^2} \left| e^{(t-s)(\Delta-\lambda)} f(s, x) - e^{(t-s)(\Delta-\lambda)} f(s, y) \right|^2 dt dx dy ds \Big)^{1/2} \\ & \leq C_0 \|f\|_{\widetilde{L^2(I;BMO)}}, \end{aligned} \quad (5.23)$$

where C_0 is independent of $T > 0$. The corresponding estimate in $\widetilde{L^2(I;VMO)}$ to (2.21) also holds.

A.2 Maximal regularity in BMO

Proposition 2.6 implies regularity estimate for the Laplacian. We extend this estimate to full maximal regularity to the initial value problem for the heat equation in BMO as in Theorem 2.8.

Proof of Theorem 2.8 The homogeneous estimate for the initial value problem with $f \equiv 0$ have seen in Proposition 2.6 and it suffices to show that the inhomogeneous estimate with $u_0 \equiv 0$. Let $T \leq \infty$ and $I = (0, T)$ and $f \in L^2(I; S)$ and $g \in L^2(I; S)$. We show the inhomogeneous estimate in $VMO(\mathbb{R}^n)$ -based space. The general case can be shown by applying the Hahn–Banach extension theorem (cf. [7]). For $R > 0$, let

$$\eta_R(x) = \begin{cases} 1, & |x - x_0| \leq \frac{1}{2}R, \\ \text{smooth, radially decreasing,} & \frac{1}{2}R < |x - x_0| \leq R, \\ 0, & R < |x - x_0| \end{cases} \quad (5.24)$$

be a smooth cut-off function around $x_0 \in \mathbb{R}^n$ and for $\lambda \geq 0$ set

$$v(t, x) = \int_0^t e^{(t-s)(\Delta-\lambda)} f(s, x) ds. \quad (5.25)$$

Then

$$\begin{aligned} & \frac{d}{dt} \frac{1}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |v(t, x) - v(t, y)|^2 \eta_R(x) \eta_R(y) dx dy \\ & = \frac{2}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} (v(t, x) - v(t, y)) \end{aligned}$$

$$\begin{aligned}
& \times \partial_t (v(t, x) - v(t, y)) \eta_R(x) \eta_R(y) dx dy \\
&= \frac{2}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} (v(t, x) - v(t, y)) \\
& \quad \times (\Delta_x v(t, x) - \Delta_y v(t, y)) \eta_R(x) \eta_R(y) dx dy \\
& - \frac{2}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \lambda |v(t, x) - v(t, y)|^2 \eta_R(x) \eta_R(y) dx dy \\
& + \frac{2}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} (v(t, x) - v(t, y)) \\
& \quad \times (f(t, x) - f(t, y)) \eta_R(x) \eta_R(y) dx dy \\
& \equiv I + II + III.
\end{aligned} \tag{5.26}$$

The first term of the right hand side of (5.26) can be treated by noticing

$$\nabla_x v(t, y) ds = \nabla_y v(t, x) ds = 0$$

and integrate by parts of each derivative for x and y variables, respectively to see

$$\begin{aligned}
I &= -\frac{2}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\nabla_x v(t, x) - \nabla_y v(t, y)|^2 \eta_R(x) \eta_R(y) dx dy \\
& - \frac{4}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} (v(t, x) - v(t, y)) (\nabla_x v(t, x) - \nabla_y v(t, y)) \\
& \quad \times \nabla_x \eta_R(x) \eta_R(y) dx dy,
\end{aligned} \tag{5.27}$$

where we note that the last term in the right hand side of (5.27) is obtained from the one line above by symmetry of x - y exchanging.

Combining (5.26) and (5.27) and using $\widetilde{\eta_R(x)} \nabla \eta_R(x) = \nabla \eta_R(x)$ we integrate it over $t \in (0, R^2)$ to see

$$\begin{aligned}
& 2 \int_0^{R^2} \left(\frac{1}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\nabla_x v(t, x) - \nabla_y v(t, y)|^2 \eta_R(x) \eta_R(y) dx dy \right) dt \\
& + 2 \int_0^{R^2} \left(\frac{\lambda}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |v(t, x) - v(t, y)|^2 \eta_R(x) \eta_R(y) dx dy \right) dt \\
& \leq - \left[\frac{1}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |v(t, x) - v(t, y)|^2 \eta_R(x) \eta_R(y) dx dy \right]_{t=0}^{R^2} \\
& + \varepsilon \int_0^{R^2} \frac{1}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\nabla_x v(t, x) - \nabla_y v(t, y)|^2 \widetilde{\eta_R(x)} \widetilde{\eta_R(y)} dx dy dt \\
& + C(\varepsilon) \int_0^{R^2} \frac{1}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |v(t, x) - v(t, y)|^2 \\
& \quad \times |\nabla_x \eta_R(x) \eta_R(y)|^2 dx dy dt \\
& + \left| \int_0^{R^2} III(t) dt \right|
\end{aligned}$$

$$\begin{aligned}
 &\leq \varepsilon \int_0^{R^2} \frac{1}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \left| \nabla_x v(t, x) - \nabla_y v(t, y) \right|^2 \widetilde{\eta}_R(x) \widetilde{\eta}_R(y) dx dy dt \\
 &\quad + C(\varepsilon) \int_0^{R^2} \frac{1}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |v(t, x) - v(t, y)|^2 \\
 &\quad \quad \quad \times |\nabla_x \eta_R(x) \eta_R(y)|^2 dx dy dt \\
 &\quad + \left| \int_0^{R^2} III(t) dt \right| \\
 &\equiv J_1 + J_2 + J_3.
 \end{aligned} \tag{5.28}$$

J_1 can be cancelled with the left hand side by choosing $\varepsilon > 0$ small.

Let $\widetilde{\eta}_R(x)$ be a smooth cut-off function whose support is a little wider than the support of the ball $B_R(x_0)$,

$$\begin{aligned}
 \eta_R(x) &= \widetilde{\eta}_R(x) \eta_R(x), \quad \nabla_x \eta_R(x) = \widetilde{\eta}_R(x) \nabla_x \eta_R(x), \\
 \frac{1}{2^n} \|\widetilde{\eta}_R\|_1 &\leq \|\eta_R\|_1 \leq \|\widetilde{\eta}_R\|_1 = |B_R|.
 \end{aligned} \tag{5.29}$$

Noticing that $|\nabla \eta_R| \leq \frac{C}{R}$ J_2 is estimated as

$$\begin{aligned}
 J_2 &\leq C \frac{1}{R^2 \|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \\
 &\quad \times \left(\int_0^{R^2} \int_0^{R^2} \left(\int_{\max(s, r)}^{R^2} (e^{(t-s)(\Delta-\lambda)} f(s, x) - e^{(t-s)(\Delta-\lambda)} f(s, y)) \right. \right. \\
 &\quad \times (e^{(t-r)(\Delta-\lambda)} f(r, x) - e^{(t-r)(\Delta-\lambda)} f(r, y)) dt \Big) dr ds \Big) \widetilde{\eta}_R(x) \widetilde{\eta}_R(y) dx dy \\
 &\leq C \int_0^{R^2} \int_0^{R^2} \frac{1}{R^2 \|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \\
 &\quad \times \left(\int_s^{R^2} |e^{(t-s)(\Delta-\lambda)} f(s, x) - e^{(t-s)(\Delta-\lambda)} f(s, y)|^2 dt \right. \\
 &\quad \times \left. \int_r^{R^2} |e^{(t-r)(\Delta-\lambda)} f(r, x) - e^{(t-r)(\Delta-\lambda)} f(r, y)|^2 dt \right)^{1/2} \widetilde{\eta}_R(x) \widetilde{\eta}_R(y) dx dy dr ds
 \end{aligned} \tag{5.30}$$

Here the integral can be split into r integration and s integration, the right hand side is expressed by a multiple integration and using the Cauchy–Schwartz inequality and $\widetilde{\eta}_R(x) \leq \chi_{B_R(x_0)}(x)$, $2^{-n}|B_R| \leq \|\eta_R\|_1$ to have the following;

$$\begin{aligned}
 J_2 &\leq C \left[\frac{1}{R} \int_0^{R^2} \left(\frac{1}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \right. \right. \\
 &\quad \times \left(\int_s^{R^2} |e^{(t-s)(\Delta-\lambda)} f(s, x) - e^{(t-s)(\Delta-\lambda)} f(s, y)|^2 dt \right) \\
 &\quad \quad \quad \times \widetilde{\eta}_R(x) \widetilde{\eta}_R(y) dx dy \Big)^{1/2} ds \Big]^2
 \end{aligned}$$

$$\begin{aligned}
&\leq C \sup_{x_0, R>0} \int_0^{R^2} \frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} \left(\int_s^{R^2} |\nabla_x \cdot e^{(t-s)(\Delta-\lambda)} R_x |\nabla|^{-1} f(s, x) \right. \\
&\quad \left. - \nabla_y \cdot e^{(t-s)(\Delta-\lambda)} R_y |\nabla|^{-1} f(s, y) \right|^2 dt \Big) dx dy ds \\
&\leq C \|\nabla|^{-1} f\|_{\widetilde{L^2(I; BMO)}}^2.
\end{aligned}$$

Here we have used the estimate in Proposition 5.7.

On the other hand, let the Riesz operator defined by $R_j f = \mathcal{F}^{-1} \left[\frac{\xi_j}{i|\xi|} \hat{f} \right]$. Then we can regard

$$f(t, x) = \nabla_x \cdot R_x |\nabla_x|^{-1} f(t, x)$$

and

$$\begin{aligned}
III &= - \frac{2}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \left(\nabla_x v(t, x) - \nabla_y v(t, y) \right) \\
&\quad \times \left(R_x |\nabla_x|^{-1} f(t, x) - R_y |\nabla_y|^{-1} f(t, y) \right) \eta_R(x) \eta_R(y) dx dy \\
&\quad - \frac{4}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \left(v(t, x) - v(t, y) \right) \\
&\quad \times \left(R_x |\nabla_x|^{-1} f(t, x) - R_y |\nabla_y|^{-1} f(t, y) \right) \nabla_x \eta_R(x) \eta_R(y) dx dy \\
&\equiv III_1 + III_2.
\end{aligned} \tag{5.31}$$

Integrating the both sides of (5.31) over $t \in (0, R^2)$,

$$\begin{aligned}
&\left| \int_0^{R^2} III(t) dt \right| \leq \int_0^{R^2} (|III_1| + |III_2|) dt \\
&\leq \varepsilon \int_0^{R^2} \left(\frac{1}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \right. \\
&\quad \times \left| \nabla_x v(t, x) - \nabla_y v(t, y) \right|^2 \eta_R(x) \eta_R(y) dx dy \Big) dt \\
&\quad + 4\varepsilon^{-1} \int_0^{R^2} \left(\frac{1}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \right. \\
&\quad \times \left| R_x |\nabla_x|^{-1} f(t, x) - R_y |\nabla_y|^{-1} f(t, y) \right|^2 \eta_R(x) \eta_R(y) dx dy \Big) dt \\
&\quad + \int_0^{R^2} \frac{2}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \left| v(t, x) - v(t, y) \right|^2 \nabla_x \eta_R(x) \eta_R(y) dx dy dt \\
&\quad + \int_0^{R^2} \frac{2}{\|\eta_R\|_1^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \left| R_x |\nabla_x|^{-1} f(t, x) - R_y |\nabla_y|^{-1} f(t, y) \right|^2 \\
&\quad \times \widetilde{\eta_R}(x) \widetilde{\eta_R}(y) dx dy dt \\
&\equiv \varepsilon L_1 + 4\varepsilon^{-1} L_2 + L_3 + L_4.
\end{aligned} \tag{5.32}$$

The first term L_1 can be cancelled by choosing $\varepsilon > 0$ small enough.

$$\begin{aligned}
 & 4\varepsilon^{-1}L_2 + L_4 \\
 & \leq C \sup_{x_0, R>0} \int_0^{R^2} \left(\frac{1}{\|\eta_R\|_1^2} \iint_{B_{2R}(x_0) \times B_{2R}(x_0)} \right. \\
 & \quad \left. \times \left| R_x |\nabla_x|^{-1} f(t, x) - R_y |\nabla_y|^{-1} f(t, y) \right|^2 dx dy \right) dt \\
 & \leq C_\varepsilon \|R_x |\nabla_x|^{-1} f\|_{L^2(I; BMO)}^2. \tag{5.33}
 \end{aligned}$$

Using $|\nabla \eta_R| \leq \frac{C}{R}$, we estimate the third term L_3 by changing the order of the time integration to see

$$\begin{aligned}
 L_3 & \leq C \int_0^{R^2} \left(\frac{1}{R^2 |B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} \right. \\
 & \quad \times \left(\int_0^t (e^{(t-s)(\Delta-\lambda)} f(s, x) - e^{(t-s)(\Delta-\lambda)} f(s, y)) ds \right) \\
 & \quad \times \left(\int_0^t (e^{(t-r)(\Delta-\lambda)} f(r, x) - e^{(t-r)(\Delta-\lambda)} f(r, y)) dr \right) dx dy \Big) dt \\
 & \leq C \left[\frac{1}{R} \int_0^{R^2} \left(\frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} \right. \right. \\
 & \quad \times \left. \left. \left(\int_s^{R^2} |e^{(t-s)(\Delta-\lambda)} f(s, x) - e^{(t-s)(\Delta-\lambda)} f(s, y)|^2 dt \right) dx dy \right)^{1/2} ds \right]^2 \\
 & \leq C \int_0^{R^2} \frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} \\
 & \quad \times \left(\int_s^{R^2} |e^{(t-s)(\Delta-\lambda)} f(s, x) - e^{(t-s)(\Delta-\lambda)} f(s, y)|^2 dt \right) dx dy ds \\
 & \leq C \sup_{x_0, R>0} \int_0^{R^2} \frac{1}{|B_R|^2} \iint_{B_R(x_0) \times B_R(x_0)} \\
 & \quad \times \left(\int_s^{R^2} \left| \nabla_x e^{(t-s)(\Delta-\lambda)} R_x |\nabla|^{-1} f(s, x) \right. \right. \\
 & \quad \left. \left. - \nabla_y e^{(t-s)(\Delta-\lambda)} R_y |\nabla|^{-1} f(s, y) \right|^2 dt \right) dx dy ds \\
 & \leq C \|\nabla|^{-1} f\|_{L^2(I; BMO)}^2. \tag{5.34}
 \end{aligned}$$

Here we have again used Proposition 5.7. Gathering the estimates (5.28), (5.30), (5.32)–(5.34), we obtain

$$\begin{aligned}
 & \sup_{x_0, R>0} \int_0^{R^2} \left(\frac{1}{|B_R|^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \left| \nabla_x v(t, x) - \nabla_y v(t, y) \right|^2 \eta_R(x) \eta_R(y) dx dy \right) dt \\
 & \leq C \|\nabla|^{-1} f\|_{L^2(I; BMO)}^2.
 \end{aligned}$$

In particular, noticing (5.23),

$$\begin{aligned} & \left\| \nabla_x^2 \int_0^t e^{(t-s)(\Delta-\lambda)} f(s) ds \right\|_{\widetilde{L^2(I; BMO)}} \\ & + \lambda \left\| \int_0^t e^{(t-s)(\Delta-\lambda)} f(s) ds \right\|_{\widetilde{L^2(I; BMO)}} \leq C \|f\|_{\widetilde{L^2(I; BMO)}}. \end{aligned}$$

Combining Proposition 2.6, we conclude the desired estimate. $\square$

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