A note on 2D Navier–Stokes equations

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Abstract

In this note, we prove a new L^4 -estimate of the velocity by the technique of Hardy space $\mathcal{H}^1$ and BMO .

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1 Introduction

In this note, we consider the Cauchy problem of the 2D Navier-Stokes equations:

$$\partial_t u + (u \cdot \nabla)u + \nabla \pi - \Delta u = 0, \quad (1.1)$$

$$\operatorname{div} u = 0, \quad (1.2)$$

$$u(\cdot, 0) = u_0, \operatorname{div} u_0 = 0 \text{ in } \mathbb{R}^2. \quad (1.3)$$

Here u is the velocity field and π is the pressure.

In [1], Beirão da Veiga showed the well-known estimate:

$$\|u(\cdot, t)\|_{L^p} \leq \|u_0\|_{L^p} \exp\{C\|u\|_{L^q(0,t;L^p)}^q\} \quad (1.4)$$

with the space dimension $n < p < \infty$ and $\frac{2}{q} + \frac{n}{p} = 1$ and $C > 0$ is an absolute constant by the standard L^p -energy method.

Dedicated to Professor Hideo Kozono on the occasion of his sixtieth birthday.

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The Navier–Stokes equation (1.1) can be written in the form of the following nonlinear heat equation

$$\partial_t u - \Delta u = -\operatorname{div}(u \otimes u) - \nabla \pi. \quad (1.5)$$

Using the L^∞ -estimate of the heat equation, we have

$$\begin{aligned} \|u\|_{L^\infty(\mathbb{R}^n \times (0, T))} &\lesssim \|u \otimes u\|_{L^s(\mathbb{R}^n \times (0, T))} + \|\pi\|_{L^s(\mathbb{R}^n \times (0, T))} + \|u_0\|_{L^\infty} \\ &\lesssim \|u\|_{L^{2s}(\mathbb{R}^n \times (0, T))}^2 + \|u_0\|_{L^\infty} \end{aligned} \quad (1.6)$$

with $s > n + 2$ and $u_0 \in L^2 \cap L^\infty$.

For other L^∞ -estimates of the velocity, we refer to [2, 3].

The aim of this note is to prove a different estimate. We will prove

Theorem 1.1 *Let $n = 2$ and $u_0 \in L^2 \cap L^4$. Then solutions of the Navier–Stokes equations satisfy the L^4 -estimate of the form*

$$\|u(\cdot, t)\|_{L^4} \leq \|u_0\|_{L^4} \exp\left(C \int_0^t \|\nabla u\|_{L^2}^2 d\tau\right). \quad (1.7)$$

In the following proof, we will use the Hardy-BMO duality and the following div-curl lemma [4]:

$$u \cdot \nabla \pi \in \mathcal{H}^1 \quad (1.8)$$

and

$$\|u \cdot \nabla \pi\|_{\mathcal{H}^1} \lesssim \|u\|_{L^4} \|\nabla \pi\|_{L^{\frac{4}{3}}} \quad (1.9)$$

with

$$\|\nabla \pi\|_{L^{\frac{4}{3}}} \lesssim \|u \cdot \nabla u\|_{L^{\frac{4}{3}}} \lesssim \|u\|_{L^4} \|\nabla u\|_{L^2}, \quad (1.10)$$

where we have used the well-known relation

$$\pi = \sum_{j,k=1}^2 R_j R_k u_j u_k$$

with the Riesz operator R_j and its boundedness in $L^{\frac{4}{3}}$.

2 Proof of Theorem 1.1

This section is devoted to the proof Theorem 1.1. We assume that the solution is smooth and only need to show the a priori estimates.

Testing (1.1) by u and using (1.2), we see that

$$\frac{1}{2} \int_{\mathbb{R}^2} |u|^2 dx + \int_0^t \int_{\mathbb{R}^2} |\nabla u|^2 dx d\tau = \frac{1}{2} \int_{\mathbb{R}^2} |u_0|^2 dx. \quad (2.1)$$

Testing (1.1) by $|u|^2 u$ and using (1.2), (1.8), (1.9) and (1.10), we find that

$$\begin{aligned} & \frac{1}{4} \frac{d}{dt} \int_{\mathbb{R}^2} |u|^4 dx + \int_{\mathbb{R}^2} |u|^2 |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} |\nabla |u|^2|^2 dx \\ &= - \int_{\mathbb{R}^2} (u \cdot \nabla \pi) |u|^2 dx \\ &\lesssim \|u \cdot \nabla \pi\|_{\mathcal{H}^1} \| |u|^2 \|_{BMO} \\ &\lesssim \|u\|_{L^4} \|\nabla \pi\|_{L^{\frac{4}{3}}} \|\nabla |u|^2\|_{L^2} \\ &\lesssim \|u\|_{L^4}^2 \|\nabla u\|_{L^2} \|\nabla |u|^2\|_{L^2} \\ &\leq \frac{1}{4} \|\nabla |u|^2\|_{L^2}^2 + C \|u\|_{L^4}^4 \|\nabla u\|_{L^2}^2, \end{aligned}$$

which gives (1.7) by the Gronwall lemma.

This completes the proof. $\square$

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