Inhomogeneous Dirichlet boundary value problem for nonlinear Schrödinger equations in the upper half-space

Nakao Hayashi¹ · Elena I. Kaikina² · Takayoshi Ogawa³

Received: 29 June 2021 / Accepted: 30 August 2021 / Published online: 7 October 2021
© The Author(s) 2021

Abstract

We consider the inhomogeneous Dirichlet-boundary value problem for nonlinear Schrödinger equations with a power nonlinearity in general space dimension $n \geq 3$. We present some results on global existence in time and uniqueness of small solutions to integral equations associated to the original problem.

Keywords Nonlinear Schrödinger equation · Large time asymptotics · Inhomogeneous Dirichlet-boundary value problem · Upper half-line

Mathematics Subject Classification Primary 35Q35

1 Introduction

We consider the initial-boundary value problem for the nonlinear Schrödinger equations on the upper half-space with a power nonlinearity

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{p-1}u, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}_+^n, \\ u(t, x', 0) = h(t, x'), & \text{a.e. } (t, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1}, \end{cases} \quad (1.1)$$

This article is part of the section “Theory of PDEs” edited by Eduardo Teixeira.

✉ Nakao Hayashi
nakao.hayashi.e6@tohoku.ac.jp

Elena I. Kaikina
ekaikina@matmor.unam.mx

Takayoshi Ogawa
ogawa@math.tohoku.ac.jp

¹ Research Alliance Center for Mathematical Sciences, Tohoku University, Sendai 980-8578, Japan

² Centro de Ciencias Matemáticas, UNAM Campus Morelia, AP 61-3 (Xangari), CP 58089 Morelia, Michoacán, Mexico

³ Mathematical Institute, Tohoku University, Sendai 980-8578, Japan

where, $1 + \frac{4}{n+2} < p < 1 + \frac{4}{n-2}$, $n \geq 3$, $\mathbb{R}_+^n = \mathbb{R}^{n-1} \times \mathbb{R}_+$, $\mathbb{R}_+ = (0, \infty)$, $x = (x_1, \dots, x_n)$, $x' = (x_1, \dots, x_{n-1})$, $\lambda \in \mathbb{C}$. The main purpose of this paper is to show asymptotic behavior in time of small solutions to the integral equation associated with (1.1) which is defined by

$$u(t) = U_D(t)u_0 + z(t) - i\lambda \int_0^t U_D(t-\tau)|u|^{p-1}u(\tau)d\tau, \quad (1.2)$$

where

$$z(t, x) = c_1 \int_0^t x_n \tau^{-\frac{3}{2}} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) h(t-\tau) d\tau, \quad (1.3)$$

$U_D(t) = U_{nD}(t)\tilde{U}(t)$ with

$$\begin{aligned} U_{nD}(t)\phi &= c_1 t^{-1/2} \int_0^\infty \left(e^{-\frac{(x_n-y_n)^2}{2it}} - e^{-\frac{(x_n+y_n)^2}{2it}} \right) \phi(y_n) dy_n, \\ \tilde{U}(t)\phi &= c_{n-1} t^{-(n-1)/2} \int_{\mathbb{R}^{n-1}} e^{-\frac{|x'-y'|^2}{2it}} \phi(y') dy', \end{aligned}$$

and $c_n = (2i\pi)^{-n/2}$. The integral representation (1.2) was obtained in [17] and we find that z is the solution of

$$\begin{cases} i\partial_t z + \frac{1}{2}\Delta z = 0, (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+^n, \\ z(0, x) = 0, x \in \mathbb{R}_+^n, \\ z(t, x', 0) = h(t, x'), (t, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1}. \end{cases} \quad (1.4)$$

There are a lot of works for the homogeneous Dirichlet boundary value problem

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{p-1}u, t > 0, x \in \Omega, \\ u(0, x) = u_0(x), x \in \Omega, \\ u(t, x') = 0, t > 0, x' \in \partial\Omega, \end{cases}$$

where Ω is a bounded or an unbounded domain on $\mathbb{R}^n$ and $\partial\Omega$ is a boundary of Ω . The cases of smooth non trapping compact boundaries $\partial\Omega$ were considered in [11–14, 33]. In [5, 29, 30], the cases of two space dimensions and $\partial\Omega$ is a smooth boundary of Ω were studied.

In [32], the authors considered the inhomogeneous Dirichlet problem $u(t, x') = h(t, x')$, $t > 0$, $x' \in \partial\Omega$, and proved an existence of global solutions (without uniqueness) in the energy space $H^1(\Omega)$ when Ω is a smooth boundary with compact support and the data satisfy the compatibility conditions $u_0(x') = h(0, x')$, $x' \in \partial\Omega$. This solution was obtained as the limit of a sequence of solutions of the approximate problems for which an energy method provides suitable a-priori estimates of solutions.

Also there are many works devoted to the study of one-dimensional initial-boundary value problems with inhomogeneous boundary conditions. One of them can be seen in [16], in which a general approach was proposed to study the well-posedness and qualitative properties of solutions of pseudodifferential nonlinear equations on a half-line by using Laplace transform and contraction mapping principle. One-dimensional nonlinear Schrödinger equations on a half-line with inhomogeneous Dirichlet condition

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda |u|^{p-1}u, t > 0, x \in \mathbb{R}_+, \\ u(0, x) = u_0(x), x \in \mathbb{R}_+, \\ u(t, 0) = h(t), t > 0 \end{cases} \quad (1.5)$$

were studied in [4, 19]. In [19], it was shown that if the data

$$(u_0, h) \in H^s(\mathbb{R}_+) \times H_{loc}^{\frac{2s+1}{4}}(\mathbb{R}_+),$$

then (1.5) has a solution $u \in C([0, T]; H^s(\mathbb{R}_+))$ when $(s, p) \in (\frac{1}{2}, \frac{3}{2}) \times [2, \infty)$ with a compatibility condition $u_0(0) = h(0)$ or $(s, p) \in (0, \frac{1}{2}) \times [2, \frac{5-2s}{1-2s})$. Moreover when $(s, p) \in (\frac{1}{2}, \frac{3}{2}) \times [2, \infty)$, the solution is unique, where fractional derivatives in time and space are defined by Fourier transform and the extended operators. This result was improved in [4] for the uniqueness of solutions when $0 < s < \frac{1}{2}$ by using the Strichartz estimates. Also global well-posedness was established in $H^1(\mathbb{R}_+)$ for the defocusing case $\lambda < 0$, if the initial data $(u_0, h) \in H^1(\mathbb{R}_+) \times H^{\frac{3}{4}}(\mathbb{R}_+)$ and $p \in [2, \infty)$ by using a-priori estimates of solutions in $H^1(\mathbb{R}_+)$ through the energy method. In paper [9], well-posedness for the cubic nonlinear Schrödinger equation (CNLS) on the half-line with data $(u_0, h) \in H^s(\mathbb{R}_+) \times H^{\frac{2s+1}{4}}(0, T)$, $s > \frac{1}{2}$ was established via the formula obtained through the unified transform method and a contraction mapping approach. This unified transform method also provides an approach for obtaining the large time asymptotics of solutions to CNLS equation for certain particular boundary conditions called linearizable (see [10]). Another method for analyzing one dimensional initial-boundary value problems, based on the Riemann–Hilbert approach, was introduced in [20]. By this method, in [21], it was shown that long range scattering occurs in CNLS equation. The advantage of Riemann–Hilbert approach is that it can be applied to non-integrable equations with general inhomogeneous boundary data, but some technical problems have to be overcome (see [22, 24–26]). There are some papers discussing the inhomogeneous initial boundary value problem for multidimensional nonlinear Schrödinger equations. Among them, we refer [1, 2, 27, 28, 31]. These papers are devoted to study fundamental questions of local existence and uniqueness of solutions in Sobolev space H^s , $s > 0$, by applying classical tools such as Strichartz and energy estimates. The global well-posedness was also discussed in H^1 space. The main idea of the proof for the local well-posedness was to derive a boundary integral operator for the corresponding inhomogeneous boundary condition and show the Strichartz estimates for this operator.

We now introduce some results concerning behavior of solutions for large time which is of interest in the present paper. Uniform decay rates for the energy of weakly damped defocusing semilinear Schrödinger equations with inhomogeneous Dirichlet boundary control were studied in [28]. It was shown that the decay rate of the solutions up to an exponential one holds and some regularity and stabilization properties were obtained for the strong solutions. The proof was based on the direct multiplier method combined with monotonicity and compactness techniques. In [23], asymptotic behavior of solutions of inhomogeneous Dirichlet initial-boundary value problem for nonlinear Schrödinger equations on the upper-right quarter plane was studied by using the method based on the Riemann–Hilbert approach and theory of the Cauchy type integral equations. In [8], inhomogeneous Dirichlet-boundary value problem in the half line has been considered and sufficient conditions which show asymptotic behavior of solutions have been presented. In the previous papers [7, 18] we showed that the operator $J_{x_n} = x_n + it\partial_{x_n}$ works well to inhomogeneous cases in one or two space dimensions. In these papers, we studied (1.1) with a critical power nonlinearity on the upper half-line by introducing the auxiliary function. We showed the optimal time decay estimates of solutions in L^∞ space. In [3], global existence in the time of small solutions for (1.1) was shown. Also a-priori estimate of solutions u such that

$$\begin{aligned} & \|u\|_{L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^{n-1}))} \\ & \leq C \left(\|u_0\|_{H^1(\mathbb{R}_+^n)} + \|h\|_{L^2(\mathbb{R}_+; H^{3/2}(\mathbb{R}^{n-1})) \cap W^{3/4, 2}(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))} \right), \end{aligned}$$

was obtained when

$$1 + \frac{4}{n} \leq p < 1 + \frac{4}{n-2}.$$

and

$$\|u_0\|_{H^1(\mathbb{R}_+^n)} + \|h\|_{L^2(\mathbb{R}_+; H^{3/2}(\mathbb{R}^{n-1})) \cap W^{3/4,2}(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))}$$

is small enough, where the fractional derivatives in space and time are defined by

$$H^{3/2}(\mathbb{R}^{n-1}) = \left\{ v \in L^2(\mathbb{R}^{n-1}); \|\langle \xi \rangle^{3/2} \mathcal{F}v\|_{L^2(\mathbb{R}_\xi^{n-1})} < \infty \right\}$$

and

$$\begin{aligned} & W^{3/4,2}(\mathbb{R}_+; L^2(\mathbb{R}^{n-1})) \\ &= \left\{ v \in L^2(\mathbb{R}_+; L^2(\mathbb{R}^{n-1})); \|\langle \delta \rangle^{3/4} \mathcal{F}_t v\|_{L^2(\mathbb{R}_\delta; L^2(\mathbb{R}^{n-1}))} < \infty \right\}. \end{aligned}$$

Therefore we are interested in the range of p such that $p < 1 + \frac{n}{4}$.

In papers, [7,18,22] we studied inhomogeneous Neumann boundary value problem with a critical power nonlinearity $p = 1 + \frac{2}{n}$ on $\mathbb{R}_+^n$ for $n = 1, 2$. In [13,25,26], inhomogeneous Dirichlet boundary value problem (1.1) was considered when $n = 1, 2$. In these papers sufficient conditions to show asymptotic behavior of solutions and the optimal time decay estimates of solutions in L^∞ space were presented.

As far as we know, there are no results of time decay of solutions to (1.1) in general space dimensions $n \geq 3$. We are interested in higher space dimensions.

2 Notations and main result

We define weighted Sobolev space

$$H^{m,s}(\mathbb{D}) = \left\{ v \in L^2(\mathbb{D}); \|v\|_{H^{m,s}(\mathbb{D})} = \sum_{|\alpha| \leq m, |\beta| \leq s} \|\partial^\alpha x^\beta v\|_{L^2(\mathbb{D})} < \infty \right\},$$

usual Sobolev space of order m

$$W^{m,p}(\mathbb{D}) = \left\{ v \in L^p(\mathbb{D}); \|v\|_{W^{m,p}(\mathbb{D})} = \sum_{|\alpha| \leq m} \|\partial^\alpha v\|_{L^p(\mathbb{D})} < \infty \right\},$$

where $\mathbb{D} = \mathbb{R}_+^n$ or $\mathbb{R}^{n-1}$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$. For simplicity, we write $H^m(\mathbb{D}) = H^{m,0}(\mathbb{D})$. For time derivatives, we denote

$$W_l^{s,p}(\mathbb{R}_+; X) = \left\{ v \in L^p(\mathbb{R}_+; X); \|v\|_{W_l^{s,p}(\mathbb{R}_+; X)} = \sum_{j=0}^s \left\| \langle \cdot \rangle^l \partial_t^j v \right\|_{L^p(\mathbb{R}_+; X)} < \infty \right\},$$

where X is a Banach space and $1 \leq p \leq \infty$. We call a pair of exponents (q, r) an admissible pair for Strichartz estimates if

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}, 2 \leq q, r \leq \infty, (q, r, n) \neq (2, \infty, 2).$$

To state our main result, we introduce the function space

$$X_{q,r} = \left\{ v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}_+^n)); \|v\|_{X_{q,r}} < \infty \right\},$$

where

$$\begin{aligned} \|v\|_{X_{q,r}} &= \sum_{|\alpha| \leq 1} \left(\|J^\alpha v\|_{L^q(\mathbb{R}_+; L^r(\mathbb{R}_+^n))} + \|J^\alpha v\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{R}_+^n))} \right) \\ &\quad + \|v\|_{L^q(\mathbb{R}_+; W^{1,r}(\mathbb{R}_+^n))} + \|v\|_{L^\infty(\mathbb{R}_+; H^1(\mathbb{R}_+^n))}. \end{aligned}$$

In what follows, we write $L^q L^r = L^q(\mathbb{R}_+; L^r(\mathbb{R}_+^n))$, $L^q W^{1,r} = L^q(\mathbb{R}_+; W^{1,r}(\mathbb{R}_+^n))$ for simplicity. For the conditions of boundary data, we prepare the function space

$$\begin{aligned} Z_\gamma &= \cap_{k=0}^2 W_\gamma^{k,\infty}(\mathbb{R}_+; H^{4-2k}(\mathbb{R}^{n-1})) \\ &\quad \cap_{l=0}^1 W_\gamma^{l,\infty}(\mathbb{R}_+; H^{2-2l,1}(\mathbb{R}^{n-1})) \cap L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^{n-1})), \end{aligned}$$

with $\gamma \geq 1$. We also define some notations :

$$\begin{aligned} \tilde{\Delta} &= \sum_{j=1}^{n-1} \partial_{x_j}^2, \Delta = \sum_{j=1}^n \partial_{x_j}^2, J = (J_{x_1}, \dots, J_{x_n}), \tilde{J} = (J_{x_1}, \dots, J_{x_{n-1}}) \\ \partial &= (\partial_{x_1}, \dots, \partial_{x_n}), \tilde{\partial} = (\partial_{x_1}, \dots, \partial_{x_{n-1}}) \\ J^\alpha &= J_{x_1}^{\alpha_1} \dots J_{x_n}^{\alpha_n}, \tilde{J}^\alpha = J_{x_1}^{\alpha_1} \dots J_{x_{n-1}}^{\alpha_{n-1}}, |\alpha| = \sum_{j=1}^n \alpha_j \text{ or } |\alpha| = \sum_{j=1}^{n-1} \alpha_j, \\ J_{x_j} &= x_j + it \partial_{x_j}, \partial_{x_j} = \partial / \partial x_j, \partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}, \tilde{\partial}^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_{n-1}}^{\alpha_{n-1}}. \end{aligned}$$

We now state the main result in this paper.

Theorem 2.1 *We assume that*

$$\begin{aligned} 1 + \frac{4}{n+2} < p < 1 + \frac{4}{n-2}, n \geq 3, \\ (u_0, h) \in Y \times Z_\gamma = H^1(\mathbb{R}_+^n) \cap H^{0,1}(\mathbb{R}_+^n) \times Z_\gamma, \gamma > 3 \end{aligned} \quad (2.1)$$

with

$$0 < \|u_0\|_Y \leq \varepsilon, 0 < \|h\|_{Z_\gamma} \leq \rho$$

and the compatibility condition $h(0, x') = u_0(x', 0)$ holds. Furthermore we assume that $h(t) \in C^1(\mathbb{R}^{n-1})$ and

$$|\partial_{x_j} h(t, x')| \leq C |h(t, x')|$$

for all $(t, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$ if $p < 2$. Then there exist $\rho, \varepsilon > 0$ such that the integral equation (1.2) has a unique global solution $u \in X_{q,r}$ where (q, r) is an admissible pair of the Strichartz estimate such that

$$\frac{n}{r} + \frac{2}{q} = \frac{n}{2}, \frac{2}{q} = \frac{p-1}{4} (n-2), n \geq 3.$$

Moreover time decay estimates of solutions

$$\|u(t)\|_{L^\sigma(\mathbb{R}_+^n)} \leq C \langle t \rangle^{-\frac{n}{2}(1-\frac{2}{\sigma})}$$

are valid for $2 \leq \sigma \leq 2 + \frac{4}{n-2}$.

2.1 Comments of Theorem 2.1

We define

$$U(t)\phi = c_n t^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{2it}} \phi(y) dy.$$

Then

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-\tau) |u|^{p-1} u(\tau) d\tau \quad (2.2)$$

is the integral equation of the Cauchy problem

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{p-1} u, & x \in \mathbb{R}^n, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$

In [6], it was shown that (2.2) has a unique global solution u with the time decay

$$\|u\|_{L^\sigma(\mathbb{R}^n)} \leq C \langle t \rangle^{-\frac{n}{2}\left(1-\frac{2}{\sigma}\right)}$$

if the data are small in $H^1(\mathbb{R}^n) \cap H^{0,1}(\mathbb{R}^n)$ and

$$1 + \frac{4}{n+2} < p < 1 + \frac{4}{n-2},$$

where $2 \leq \sigma \leq 2 + \frac{4}{n-2}$. Therefore our result is considered as an extension of the result of the Cauchy problem in [6] to the inhomogeneous boundary value problem.

In the previous paper [15], we considered the integral equation of multi-dimensional inhomogeneous Neumann boundary value problem

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{p-1} u, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}_+^n, \\ \partial_{x_n} u(t, x', 0) = h(t, x'), & \text{a.e. } (t, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1} \end{cases} \quad (2.3)$$

which is represented as

$$u(t) = U_N(t)u_0 + z_N(t) - i\lambda \int_0^t U_N(t-\tau) |u|^{p-1} u(\tau) d\tau, \quad (2.4)$$

where $U_N(t) = U_{nN}(t)\tilde{U}(t)$ with

$$U_{nN}(t)\phi = c_1 t^{-1/2} \int_0^\infty \left(e^{-\frac{(x_n-y_n)^2}{2it}} + e^{-\frac{(x_n+y_n)^2}{2it}} \right) \phi(y_n) dy_n$$

and

$$z_N(t, x) = -ic_1 \int_0^t \tau^{-\frac{1}{2}} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) h(t-\tau) d\tau.$$

In [15], we apply the operator $J_{x_n} = J_{x_n}(t)$ to both sides of (2.4) and use the commutation relation

$$J_{x_n}(t)U_N(t-\tau) = U_D(t-\tau)J_{x_n}(\tau).$$

Then by using the Strichartz estimates, we get the desired results. However in the Dirichlet case we do not have

$$J_{x_n}(t) U_D(t - \tau) \phi(\tau) = U_N(t - \tau) J_{x_n}(\tau) \phi(\tau)$$

if we do not assume $\phi(\tau, x', 0) = 0$. Indeed, we have the commutation relation

$$\begin{aligned} J_{x_n}(t) U_D(t - \tau) \phi(\tau, x) &= U_N(t - \tau) J_{x_n}(\tau) \phi(\tau, x) \\ &+ \frac{2ic_1}{(t - \tau)^{1/2}} e^{-\frac{x_n^2}{2i(t-\tau)}} \tilde{U}(t - \tau) \tau \phi(\tau, x', 0). \end{aligned} \quad (2.5)$$

The identity is obtained by a direct calculation

$$\begin{aligned} &J_{x_n}(t) U_{nD}(t - \tau) g \\ &= \frac{c_1}{(t - \tau)^{1/2}} J_{x_n}(t - \tau) \int_0^\infty \left(e^{-\frac{|x_n - y_n|^2}{2i(t-\tau)}} - e^{-\frac{|x_n + y_n|^2}{2i(t-\tau)}} \right) g(\tau, y_n) dy_n \\ &\quad - \frac{c_1}{(t - \tau)^{1/2}} \int_0^\infty \left((i\tau \partial_{x_n}) \left(e^{-\frac{|x_n - y_n|^2}{2i(t-\tau)}} - e^{-\frac{|x_n + y_n|^2}{2i(t-\tau)}} \right) \right) g(\tau, y_n) dy_n \\ &= \frac{c_1}{(t - \tau)^{1/2}} \int_0^\infty \left(e^{-\frac{|x_n - y_n|^2}{2i(t-\tau)}} + e^{-\frac{|x_n + y_n|^2}{2i(t-\tau)}} \right) J_{x_n}(\tau) g(\tau, y_n) dy_n \\ &\quad + \frac{2c_1}{(t - \tau)^{1/2}} i\tau e^{-\frac{x_n^2}{2i(t-\tau)}} g(\tau, 0), \end{aligned}$$

since

$$\begin{aligned} &J_{x_n}(t - \tau) \left(e^{-\frac{|x_n - y_n|^2}{2i(t-\tau)}} - e^{-\frac{|x_n + y_n|^2}{2i(t-\tau)}} \right) g(\tau, y_n) \\ &= x_n \left(e^{-\frac{|x_n - y_n|^2}{2i(t-\tau)}} - e^{-\frac{|x_n + y_n|^2}{2i(t-\tau)}} \right) g(\tau, y_n) \\ &\quad + \left(-(x_n - y_n) e^{-\frac{|x_n - y_n|^2}{2i(t-\tau)}} + (x_n + y_n) e^{-\frac{|x_n + y_n|^2}{2i(t-\tau)}} \right) g(\tau, y_n) \\ &= \left(e^{-\frac{|x_n - y_n|^2}{2i(t-\tau)}} + e^{-\frac{|x_n + y_n|^2}{2i(t-\tau)}} \right) y_n g(\tau, y_n). \end{aligned}$$

Hence we have

$$\begin{aligned} &J_{x_n}(t) \int_0^t U_D(t - \tau) \phi(\tau, x) d\tau = \int_0^t U_N(t - \tau) J_{x_n}(\tau) \phi(\tau, x) d\tau \\ &\quad + 2ic_1 \int_0^t \frac{1}{(t - \tau)^{1/2}} e^{-\frac{x_n^2}{2i(t-\tau)}} \tilde{U}(t - \tau) \tau \phi(\tau, x', 0) d\tau. \end{aligned} \quad (2.6)$$

It seems that it is difficult to control the second term of the right hand side of (2.6).

In order to prove the desired a-priori estimates, we make use of a different integral representation of solutions compared with (1.2) which will be shown in Lemma 3.1 below. In order to avoid the commutation relation between $J_{x_n}(t)$ and $U_D(t)$, we use a new integral form and prove a-priori estimate of

$$\left\| J_{x_n}(t) \int_0^t U_D(t - \tau) \phi(\tau, x) d\tau \right\|_{L^q(\mathbb{R}_+; L^r(\mathbb{R}_+^n))}$$

directly. More precisely, we use the equation of $I\phi = \int_0^t U_D(t-\tau)\phi(\tau, x) d\tau$ which is written as

$$i\partial_t I\phi + \frac{1}{2}\Delta I\phi = ie^{-x_n}\phi,$$

and the commutation relation between $J_{x_n}(t)$ and $i\partial_t + \frac{1}{2}\Delta$ to get the desired estimates.

We organize our paper as follows. In Sect. 2, we state notations used in this paper and main result. In Sect. 3 we present the integral equation of the different form of the original problem and give some estimates of

$$\int_0^t U_D(t-\tau)e^{-x_n}\left(i\partial_\tau + \frac{1}{2}\tilde{\Delta} + \frac{1}{2}\right)h(\tau)d\tau$$

and

$$\int_0^t U_D(t-\tau)e^{-x_n}|h(\tau)|^{p-1}h(\tau)d\tau$$

which are used in the integral representation. We also state Strichartz estimates and Sobolev inequality. In Sect. 4 we prove our main result.

3 Preliminaries

In order to prove our result, we translate (1.2) into another representation.

Lemma 3.1 *Let u be the solution of (1.2). Then we have*

$$\begin{aligned} u(t) &= U_D(t)(u_0(x', x_n) - e^{-x_n}h(0, x')) + e^{-x_n}h(t, x') \\ &\quad - i\lambda \int_0^t U_D(t-\tau)(|u|^{p-1}u(\tau) - e^{-x_n}|h(\tau)|^{p-1}h(\tau))d\tau \\ &\quad + \int_0^t U_D(t-\tau)e^{-x_n}\left(i\partial_\tau + \frac{1}{2}\tilde{\Delta} + \frac{1}{2}\right)h(\tau, x')d\tau \\ &\quad - i\lambda \int_0^t U_D(t-\tau)e^{-x_n}|h(\tau)|^{p-1}h(\tau)d\tau \end{aligned} \quad (3.1)$$

Proof We have by a direct calculation

$$\begin{aligned} \partial_{x_n}^2 U_{nD}(t)e^{-x_n} &= c_1 t^{-1/2} \int_0^\infty \left(\partial_{y_n}^2 \left(e^{-\frac{(x_n-y_n)^2}{2it}} - e^{-\frac{(x_n+y_n)^2}{2it}} \right) \right) e^{-y_n} dy_n \\ &= c_1 t^{-1/2} \int_0^\infty \partial_{y_n} \left(\left(\frac{x_n - y_n}{it} e^{-\frac{(x_n-y_n)^2}{2it}} + \frac{x_n + y_n}{it} e^{-\frac{(x_n+y_n)^2}{2it}} \right) e^{-y_n} \right) dy_n \\ &\quad + c_1 t^{-1/2} \int_0^\infty \left(\left(\partial_{y_n} \left(e^{-\frac{(x_n-y_n)^2}{2it}} - e^{-\frac{(x_n+y_n)^2}{2it}} \right) \right) e^{-y_n} \right) dy_n \\ &= -c_1 t^{-1/2} \frac{2x_n}{it} e^{-\frac{x_n^2}{2it}} + c_1 t^{-1/2} \int_0^\infty \partial_{y_n} \left(\left(e^{-\frac{(x_n-y_n)^2}{2it}} - e^{-\frac{(x_n+y_n)^2}{2it}} \right) e^{-y_n} \right) dy_n \\ &\quad + c_1 t^{-1/2} \int_0^\infty \left(e^{-\frac{(x_n-y_n)^2}{2it}} - e^{-\frac{(x_n+y_n)^2}{2it}} \right) e^{-y_n} dy_n \\ &= -c_1 t^{-1/2} \frac{2x_n}{it} e^{-\frac{x_n^2}{2it}} + U_{nD}(t)e^{-x_n}. \end{aligned}$$

Denote $U_D(t) = U_{nD}\tilde{U}(t)$, then

$$z(t, x) = c_1 \int_0^t x_n \tau^{-\frac{3}{2}} e^{-\frac{x_n^2}{2\tau}} \tilde{U}(\tau) h(t - \tau, \tilde{x}) d\tau.$$

Hence,

$$\begin{aligned} & \frac{1}{2} \partial_{x_n}^2 \int_0^t U_D(\tau) e^{-x_n} h(t - \tau, x') d\tau \\ &= i c_1 \int_0^t x_n \tau^{-3/2} e^{-\frac{x_n^2}{2\tau}} \tilde{U}(\tau) h(t - \tau, x') d\tau \\ & \quad + \frac{1}{2} \int_0^t U_D(\tau) e^{-x_n} h(t - \tau, x') d\tau \\ &= iz(t) + \frac{1}{2} \int_0^t U_D(\tau) e^{-x_n} h(t - \tau, x') d\tau. \end{aligned} \quad (3.2)$$

We put

$$\Phi(t, x) = \int_0^t U_D(\tau) e^{-x_n} h(t - \tau, x') d\tau.$$

By a direct calculation we obtain

$$i \partial_t \Phi + \frac{1}{2} \partial_{x_n}^2 \Phi + \frac{1}{2} \tilde{\Delta} \Phi = i e^{-x_n} h(t, x'). \quad (3.3)$$

Therefore

$$\begin{aligned} \frac{1}{2} \partial_{x_n}^2 \Phi &= -i \partial_t \Phi - \frac{1}{2} \tilde{\Delta} \Phi + i e^{-x_n} h(t, x'). \\ &= -i U_D(t) e^{-x_n} h(0, x') \\ & \quad + i \int_0^t U_D(\tau) e^{-x_n} \left(i \partial_t + \frac{1}{2} \tilde{\Delta} \right) h(t - \tau, x') d\tau \\ & \quad + i e^{-x_n} h(t, x'). \end{aligned} \quad (3.4)$$

From (3.2) and (3.4) it follows that

$$\begin{aligned} z(t) - \frac{1}{2} i \int_0^t U_D(\tau) e^{-x_n} h(t - \tau, x') d\tau \\ &= -U_D(t) e^{-x_n} h(0, x') + e^{-x_n} h(t, x') \\ & \quad + \int_0^t U_D(\tau) e^{-x_n} \left(i \partial_t + \frac{1}{2} \tilde{\Delta} \right) h(t - \tau, x') d\tau. \end{aligned}$$

We substitute this identity to the formula (1.2) to get (3.1). This completes the proof of the lemma. $\square$

In what follows, we consider the integral equation of (3.1). We note that

$$(u_0(x', x_n) - e^{-x_n} h(0, x'))|_{x_n=0} = 0$$

and

$$(|u|^{p-1} u(\tau) - e^{-x_n} |h|^{p-1} h(\tau))|_{x_n=0} = 0.$$

Therefore from the commutation relation (2.6)

$$\begin{aligned}
 J_{x_n} u(t) &= U_N(t) x_n (u_0(x', x_n) - e^{-x_n} h(0, x')) \\
 &\quad - i\lambda \int_0^t U_N(t-\tau) J_{x_n} (|u|^{p-1} u(\tau) - e^{-x_n} |h|^{p-1} h(\tau)) d\tau \\
 &\quad + J_{x_n} e^{-x_n} h(t, x') - i\lambda J_{x_n} \int_0^t U_D(t-\tau) e^{-x_n} |h|^{p-1} h(\tau) d\tau \\
 &\quad + i J_{x_n} \int_0^t U_D(t-\tau) e^{-x_n} \left(i\partial_\tau + \frac{1}{2}\tilde{\Delta} + \frac{1}{2} \right) h(\tau) d\tau. \quad (3.5)
 \end{aligned}$$

We state the Strichartz estimates.

Lemma 3.2 Assume that

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}, \quad 2 \leq q, r \leq \infty, (q, r, n) \neq (2, \infty, 2).$$

Then

$$\|U(t) u_0\|_{L^q L^r} \leq C \|u_0\|_{L^2(\mathbb{R}_+^n)},$$

where $U(t) = U_N(t)$ or $U(t) = U_D(t)$. Furthermore if $\tilde{q}, \tilde{r}$ satisfy the same restriction as q, r and $\tilde{q}', \tilde{r}'$ are their dual exponents:

$$\frac{2}{\tilde{q}} + \frac{n}{\tilde{r}} = \frac{n}{2}, \quad 2 \leq \tilde{q}, \tilde{r} \leq \infty, (\tilde{q}, \tilde{r}, n) \neq (2, \infty, 2)$$

and

$$\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1, \quad \frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1,$$

then the estimate

$$\left\| \int_0^t U(t-\tau) F(\tau) d\tau \right\|_{L^q L^r} \leq C \|F\|_{L^{\tilde{q}'} L^{\tilde{r}'}} \quad (3.6)$$

is valid.

We apply Lemma 3.2 to (3.5) to get

$$\begin{aligned}
 \|J_{x_n} u\|_{L^q L^r} &\leq C \|x_n (u_0(x', x_n) - e^{-x_n} h(0, x'))\|_{L^2(\mathbb{R}_+^n)} \\
 &\quad + C \|J_{x_n} (|u|^{p-1} u - e^{-x_n} |h|^{p-1} h)\|_{L^{q'} L^{r'}} \\
 &\quad + C \left\| \langle t \rangle \|h(t)\|_{L^r(\mathbb{R}^{n-1})} \right\|_{L^q(\mathbb{R}_+)} + \left\| J_{x_n} \int_0^t U_D(t-\tau) e^{-x_n} \phi(\tau) d\tau \right\|_{L^q L^r}, \quad (3.7)
 \end{aligned}$$

where

$$\phi(\tau) = \phi(\tau, x') = i \left(-\lambda |h|^{p-1} h(\tau, x') + \left(i\partial_\tau + \frac{1}{2}\tilde{\Delta} + \frac{1}{2} \right) h(\tau, x') \right), \quad (3.8)$$

and $(q, r), (q', r')$ are an admissible pair of the Strichartz estimate and its dual respectively. Similarly we obtain

$$\begin{aligned} & \left\| \partial_{x_n} u \right\|_{L^q L^r} \leq C \left\| \partial_{x_n} (u_0(x', x_n) - e^{-x_n} h(0, x')) \right\|_{L^2(\mathbb{R}_+^n)} \\ & + C \left\| \partial_{x_n} (|u|^{p-1} u - e^{-x_n} |h|^{p-1} h) \right\|_{L^{q'} L^{r'}} \\ & + C \left\| \|h(t)\|_{L^r(\mathbb{R}^{n-1})} \right\|_{L^q(\mathbb{R}_+)} + \left\| \partial_{x_n} \int_0^t U_D(t-\tau) e^{-x_n} \phi(\tau) d\tau \right\|_{L^q L^r}. \end{aligned} \quad (3.9)$$

We put

$$(I\phi)(t, x) = \int_0^t U_D(t-\tau) e^{-x_n} \phi(\tau) d\tau \quad (3.10)$$

and consider the estimates of the last terms of the right hand sides of (3.7) and (3.9). We now prove

Lemma 3.3 *We assume*

$$\phi \in E_1 = \cap_{k=0}^1 W_\eta^{k, \infty}(\mathbb{R}_+; H^{2-2k}(\mathbb{R}^{n-1})),$$

where $\eta > 1$. Then the following estimates

$$\left\| \partial_{x_n}^2 I\phi \right\|_{L^\infty L^2} \leq C \|\phi\|_{E_1},$$

and

$$\left\| \partial_{x_j}^2 I\phi \right\|_{L^\infty L^2} \leq C \|\phi\|_{E_1}$$

for $1 \leq j \leq n-1$ are valid.

Proof By a direct calculation we have

$$\begin{cases} i \partial_t I\phi + \frac{1}{2} \Delta I\phi = i e^{-x_n} \phi, \\ (I\phi)(0, x) = 0, \\ (I\phi)(t, x', 0) = 0 \end{cases} \quad (3.11)$$

and as a result

$$\partial_{x_n}^2 I\phi = -2i \partial_t I\phi - \tilde{\Delta} I\phi + 2i e^{-x_n} \phi. \quad (3.12)$$

By a simple calculation we obtain

$$\begin{aligned} \partial_t I\phi &= \partial_t \int_0^t U_D(t-\tau) e^{-x_n} \phi(\tau) d\tau = e^{-x_n} \phi(t) \\ &\quad - \int_0^t \partial_\tau (U_D(t-\tau) e^{-x_n} \phi(\tau)) d\tau + \int_0^t U_D(t-\tau) e^{-x_n} \partial_\tau \phi(\tau) d\tau \\ &= U_D(t) e^{-x_n} \phi(0) + \int_0^t U_D(t-\tau) e^{-x_n} \partial_\tau \phi(\tau) d\tau. \end{aligned}$$

Hence, via (3.12)

$$\begin{aligned} -2i \partial_t I\phi - \tilde{\Delta} I\phi + 2i e^{-x_n} \phi &= -2i U_D(t) e^{-x_n} \phi(0) \\ &\quad - 2i I \partial_t \phi - I \tilde{\Delta} \phi + 2i e^{-x_n} \phi(t) \end{aligned}$$

from which it follows that

$$\begin{aligned}
& \left\| \partial_{x_n}^2 I \phi(t) \right\|_{L^2(\mathbb{R}_+^n)} \leq \|\phi(0)\|_{L^2(\mathbb{R}^{n-1})} + \|\phi(t)\|_{L^2(\mathbb{R}^{n-1})} \\
& \quad + \int_0^t \|\partial_\tau \phi(\tau)\|_{L^2(\mathbb{R}^{n-1})} + \|\phi(\tau)\|_{H^2(\mathbb{R}^{n-1})} d\tau \\
& \leq C \left(\|\phi\|_{W_\eta^{1,\infty}(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))} + \|\phi\|_{W_\eta^{0,\infty}(\mathbb{R}_+; H^2(\mathbb{R}^{n-1}))} \right).
\end{aligned}$$

This completes the proof of the first estimate. Analogously the last one is estimated by the identity

$$\partial_{x_j}^2 I \phi = \int_0^t U_D(t-\tau) e^{-x_n} \partial_{x_j}^2 \phi(\tau) d\tau.$$

This completes the proof of the lemma. $\square$

We also need the following lemma.

Lemma 3.4 *Let*

$$\phi \in E_2 = W_\gamma^{1,\infty}(\mathbb{R}_+; L^2(\mathbb{R}^{n-1})) \cap W_\gamma^{0,\infty}(\mathbb{R}_+; H^2(\mathbb{R}^{n-1})),$$

where $\gamma > 3$. Then the following estimates

$$\|J_{x_n}^2 I \phi\|_{L^\infty L^2} \leq C \|\phi\|_{E_2}$$

and

$$\|\partial_{x_n} J_{x_n} I \phi\|_{L^\infty L^2} \leq C \|\phi\|_{E_2}$$

are valid.

Proof From the energy estimate we obtain

$$\begin{aligned}
& \frac{d}{dt} \|J_{x_n}^2 I \phi(t)\|_{L^2(\mathbb{R}_+^n)}^2 + \operatorname{Im} \int_{\mathbb{R}_+^n} \partial_{x_n} \left(\partial_{x_n} J_{x_n}^2 I \phi \cdot \overline{J_{x_n}^2 I \phi} \right) dx \\
& = 2 \operatorname{Im} \left(J_{x_n}^2 e^{-x_n} \phi, \overline{J_{x_n}^2 I \phi} \right).
\end{aligned} \tag{3.13}$$

We consider the second terms of left hand sides of (3.13). By (2.5)

$$\begin{aligned}
& J_{x_n} U_D(t-\tau) e^{-x_n} g(\tau) = U_N(t-\tau) J_{x_n}(\tau) e^{-x_n} g \\
& \quad + 2ic_1(t-\tau)^{-1/2} e^{-\frac{x_n^2}{2i(t-\tau)}} \widetilde{U}(t-\tau) \tau g(\tau).
\end{aligned} \tag{3.14}$$

Via (3.14), the commutation relation

$$J_{x_n}(t) U_N(t-\tau) = U_D(t-\tau) J_{x_n}(\tau)$$

and the identity

$$\begin{aligned}
& J_{x_n}(t) (t-\tau)^{-1/2} e^{-\frac{x_n^2}{2i(t-\tau)}} = i\tau \partial_{x_n} (t-\tau)^{-1/2} e^{-\frac{x_n^2}{2i(t-\tau)}} \\
& \quad + (t-\tau)^{-1/2} e^{-\frac{x_n^2}{2i(t-\tau)}} J_{x_n}(t-\tau)
\end{aligned}$$

we obtain

$$\begin{aligned}
& J_{x_n}(t)^2 U_D(t-\tau) e^{-x_n} g = U_D(t-\tau) J_{x_n}(\tau)^2 e^{-x_n} g \\
& \quad - 2c_1 \partial_{x_n} (t-\tau)^{-1/2} e^{-\frac{x_n^2}{2i(t-\tau)}} \widetilde{U}(t-\tau) \tau^2 g(\tau, x').
\end{aligned}$$

As a consequence,

$$\begin{aligned}
 J_{x_n}^2 I \phi &= \int_0^t U_D(t-\tau) J_{x_n}(\tau)^2 e^{-x_n} \phi(\tau) d\tau \\
 &\quad - 2c_1 \partial_{x_n} \int_0^t (t-\tau)^{-1/2} e^{-\frac{x_n^2}{2i(t-\tau)}} \tilde{U}(t-\tau) \tau^2 \phi(\tau, x') d\tau \\
 &= \int_0^t U_D(t-\tau) J_{x_n}^2 e^{-x_n} \phi(\tau) d\tau \\
 &\quad - 2c_1 \partial_{x_n} \int_0^t \tau^{-1/2} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) (t^2 \phi)(t-\tau, x') d\tau,
 \end{aligned} \quad (3.15)$$

from which it follows that

$$J_{x_n}^2 I \phi|_{x_n=0} = -2t^2 \phi(t, x').$$

Hence we obtain

$$\begin{aligned}
 \|J_{x_n}^2 I \phi|_{x_n=0}\|_{L^2(\mathbb{R}^{n-1})} &\leq C t^2 \|\phi(t)\|_{L^2(\mathbb{R}^{n-1})} \\
 &\leq C \langle t \rangle^{-(\gamma-2)} \|\phi\|_{W_\gamma^{0,\infty}(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))}.
 \end{aligned} \quad (3.16)$$

By using the identity

$$\begin{aligned}
 &\partial_{x_n} U_D(t-\tau) J_{x_n}(\tau)^2 e^{-x_n} \phi(\tau) \\
 &= c_1 (t-\tau)^{-1/2} \partial_{x_n} \int_0^\infty \left(e^{-\frac{(x_n-y_n)^2}{2i(t-\tau)}} - e^{-\frac{(x_n+y_n)^2}{2i(t-\tau)}} \right) \tilde{U}(t-\tau) J_{y_n}(\tau)^2 e^{-y_n} \phi(\tau) dy_n \\
 &= -c_1 (t-\tau)^{-1/2} \int_0^\infty \partial_{y_n} \left(\left(e^{-\frac{(x_n-y_n)^2}{2i(t-\tau)}} + e^{-\frac{(x_n+y_n)^2}{2i(t-\tau)}} \right) \tilde{U}(t-\tau) J_{y_n}(\tau)^2 e^{-y_n} \phi(\tau) \right) dy_n \\
 &\quad + c_1 (t-\tau)^{-1/2} \int_0^\infty \left(e^{-\frac{(x_n-y_n)^2}{2i(t-\tau)}} + e^{-\frac{(x_n+y_n)^2}{2i(t-\tau)}} \right) \tilde{U}(t-\tau) \partial_{x_n} J_{x_n}(\tau)^2 e^{-x_n} \phi(\tau) dy_n \\
 &= 2c_1 (t-\tau)^{-1/2} e^{-\frac{x_n^2}{2i(t-\tau)}} \tilde{U}(t-\tau) \tau^2 \phi(\tau) - U_N(t-\tau) \partial_{x_n} J_{x_n}^2 e^{-x_n} \phi(\tau)
 \end{aligned}$$

we obtain

$$\begin{aligned}
 &\partial_{x_n} \int_0^t U_D(t-\tau) J_{x_n}^2 e^{-x_n} \phi(\tau) d\tau \\
 &= 2 \int_0^t c_1 (t-\tau)^{-1/2} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(t-\tau) \tau^2 \phi(\tau) d\tau - \int_0^t U_N(t-\tau) \partial_{x_n} J_{x_n}^2 e^{-x_n} \phi(\tau) d\tau \\
 &= 2 \int_0^t c_1 \tau^{-1/2} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) (t^2 \phi)(t-\tau) d\tau - \int_0^t U_N(t-\tau) \partial_{x_n} J_{x_n}^2 e^{-x_n} \phi(\tau) d\tau.
 \end{aligned}$$

Hence it follows

$$\begin{aligned}
 \partial_{x_n} J_{x_n}^2 I \phi &= 2 \int_0^t c_1 \tau^{-1/2} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) (t^2 \phi)(t-\tau) d\tau \\
 &\quad - \int_0^t U_N(t-\tau) \partial_{x_n} J_{x_n}^2 e^{-x_n} \phi(\tau) d\tau \\
 &\quad - \partial_{x_n}^2 \int_0^t c_1 \tau^{-1/2} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) (t^2 \phi)(t-\tau, x') d\tau.
 \end{aligned}$$

The last term of the right hand side of the above is represented as

$$\begin{aligned}
 & -\partial_{x_n}^2 \int_0^t c_1 \tau^{-1/2} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) (t^2 \phi)(t - \tau, x') d\tau \\
 & = (2i\partial_t + \tilde{\Delta}) \int_0^t c_1 \tau^{-1/2} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) (t^2 \phi)(t - \tau, x') d\tau \\
 & = c_1 t^{-1/2} e^{-\frac{x_n^2}{2it}} \tilde{U}(t) (t^2 \phi)(0, x') \\
 & \quad + \int_0^t c_1 \tau^{-1/2} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) ((2i\partial_t + \tilde{\Delta}) t^2 \phi)(t - \tau, x') d\tau \\
 & = \int_0^t c_1 \tau^{-1/2} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) ((2i\partial_t + \tilde{\Delta}) t^2 \phi)(t - \tau, x') d\tau.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \partial_{x_n} J_{x_n}^2 I\phi|_{x_n=0} & = 2 \int_0^t c_1 \tau^{-1/2} \tilde{U}(\tau) (t^2 \phi)(t - \tau) d\tau \\
 & \quad - 2 \int_0^t c_1 \tau^{-1/2} e^{-\frac{y_n^2}{2i\tau}} \tilde{U}(\tau) (\partial_{y_n} J_{y_n}^2 e^{-y_n} \phi)(t - \tau) d\tau dy_n \\
 & \quad + \int_0^t c_1 \tau^{-1/2} \tilde{U}(\tau) ((2i\partial_t + \tilde{\Delta}) t^2 \phi)(t - \tau, x') d\tau
 \end{aligned}$$

from which we get

$$\begin{aligned}
 \left\| \partial_{x_n} J_{x_n}^2 I\phi|_{x_n=0} \right\|_{L^2(\mathbb{R}^{n-1})} & \leq C \int_0^t \frac{1}{\sqrt{t-\tau}} \langle \tau \rangle^2 \left(\|\partial_\tau \phi(\tau)\|_{L^2(\mathbb{R}_+^{n-1})} + \|\phi(\tau)\|_{H^2(\mathbb{R}_+^{n-1})} \right) d\tau \\
 & \leq C \langle t \rangle^{-\gamma_0} \left(\|\phi\|_{W_{\gamma-1/4}^{1,\infty}(\mathbb{R}_+; L^2(\mathbb{R}_+^{n-1}))} + \|\phi\|_{W_{\gamma-1/4}^{0,\infty}(\mathbb{R}_+; H^2(\mathbb{R}_+^{n-1}))} \right),
 \end{aligned} \tag{3.17}$$

where $\gamma_0 = \min\{(\gamma - 2 - 3/2), 1/2\}$. By (3.16) and (3.17)

$$\begin{aligned}
 & \left| \operatorname{Im} \int_{\mathbb{R}_+^n} \partial_{x_n} \left(\partial_{x_n} J_{x_n}^2 I\phi \cdot \overline{J_{x_n}^2 I\phi} \right) dx \right| \\
 & \leq C \langle t \rangle^{-\gamma_1} \left(\|\phi\|_{W_{\gamma-1/4}^{1,\infty}(\mathbb{R}_+; L^2(\mathbb{R}_+^{n-1}))} + \|\phi\|_{W_{\gamma-1/4}^{0,\infty}(\mathbb{R}_+; H^2(\mathbb{R}_+^{n-1}))} \right)^2,
 \end{aligned}$$

where $\gamma_1 = (\gamma - 2) + \min\{(\gamma - 2 - 3/4), 1/2\}$. We apply the estimate to (3.13) to find that

$$\begin{aligned}
 \|J_{x_n}^2 I\phi(t)\|_{L^2(\mathbb{R}_+^n)}^2 & \leq \|J_{x_n}^2 I\phi(0)\|_{L^2(\mathbb{R}_+^n)}^2 \\
 & \quad + C \left(\|\phi\|_{W_{\gamma-1/4}^{1,\infty}(\mathbb{R}_+; L^2(\mathbb{R}_+^{n-1}))} + \|\phi\|_{W_{\gamma-1/4}^{0,\infty}(\mathbb{R}_+; H^2(\mathbb{R}_+^{n-1}))} \right)^2 \\
 & \quad + C \int_0^t \langle \tau \rangle^2 \|\phi(\tau)\|_{L^2(\mathbb{R}_+^{n-1})} \|J_{x_n}^2 I\phi(\tau)\|_{L^2(\mathbb{R}_+^n)} d\tau \\
 & \leq \|J_{x_n}^2 I\phi(0)\|_{L^2(\mathbb{R}_+^n)}^2
 \end{aligned}$$

$$\begin{aligned}
& + C \left(\|\phi\|_{W_{\gamma-1/4}^{1,\infty}(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))} + \|\phi\|_{W_{\gamma-1/4}^{0,\infty}(\mathbb{R}_+; H^2(\mathbb{R}^{n-1}))} \right)^2 \\
& + C \|\phi\|_{W_{\gamma}^{0,\infty}(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))} \int_0^t \langle \tau \rangle^{-(\gamma-2)} \|J_{x_n}^2 I\phi(\tau)\|_{L^2(\mathbb{R}_+^n)} d\tau. \quad (3.18)
\end{aligned}$$

We now prove $\|J_{x_n}^2 I\phi(0)\|_{L^2(\mathbb{R}_+^n)} = 0$. Indeed, from (3.15)

$$\begin{aligned}
\|J_{x_n}^2 I\phi(t)\|_{L^2(\mathbb{R}_+^n)} & \leq \left\| \int_0^t U_D(t-\tau) J_{x_n}^2 e^{-x_n} \phi(\tau) d\tau \right\|_{L^2(\mathbb{R}_+^n)} \\
& + \left\| \partial_{x_n} \int_0^t \frac{1}{\sqrt{2\pi i \tau}} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) (t^2 \phi)(t-\tau, x') d\tau \right\|_{L^2(\mathbb{R}_+^n)}
\end{aligned}$$

and by integration by parts

$$\begin{aligned}
& \partial_{x_n} \int_0^t \tau^{-1/2} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) (t^2 \phi)(t-\tau, x') d\tau \\
& = i \int_0^t \tau^{-3/2} x_n e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) (t^2 \phi)(t-\tau, x') d\tau \\
& = i \int_0^t \tau^{-1/2} x_n \frac{2}{2i\tau + x_n^2} \left(\partial_\tau \tau e^{-\frac{x_n^2}{2i\tau}} \right) \tilde{U}(\tau) (t^2 \phi)(t-\tau, x') d\tau \\
& = i \int_0^t \partial_\tau \left(\tau^{-1/2} x_n \frac{2}{2i\tau + x_n^2} \tau e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) (t^2 \phi)(t-\tau, x') \right) d\tau \\
& \quad - i \int_0^t \tau e^{-\frac{x_n^2}{2i\tau}} \partial_\tau \left(\frac{x_n}{\sqrt{\tau}} \frac{2}{2i\tau + x_n^2} \tilde{U}(\tau) (t^2 \phi)(t-\tau, x') \right) d\tau.
\end{aligned}$$

Hence it follows $\|J_{x_n}^2 I\phi(0)\|_{L^2(\mathbb{R}_+^n)} = 0$ and as a consequence we have the first estimate of the lemma by (3.18). The second one is obtained in the same way and so, we omit it. This completes the proof of the lemma. $\square$

Now we prove some useful estimates.

Lemma 3.5 *Let*

$$\phi \in E_3 = W_{\gamma}^{0,\infty}(\mathbb{R}_+; H^1(\mathbb{R}^{n-1})) \cap W_{\gamma-1}^{0,\infty}(\mathbb{R}_+; H^{0,1}(\mathbb{R}^{n-1})),$$

$\gamma > 3$, then the following estimates

$$\|J_{x_n} J_{x_j} I\phi\|_{L^\infty L^2} \leq C \|\phi\|_{E_3}$$

and

$$\|J_{x_n} \partial_{x_j} I\phi\|_{L^\infty L^2} \leq C \|\phi\|_{E_3}$$

for $1 \leq j \leq n-1$ are valid.

Proof From the energy estimate we obtain

$$\begin{aligned}
& \frac{d}{dt} \|J_{x_n} J_{x_j} I\phi(t)\|_{L^2(\mathbb{R}_+^n)}^2 + \operatorname{Im} \int_{\mathbb{R}_+^n} \partial_{x_n} (\partial_{x_n} J_{x_n} J_{x_j} I\phi \cdot \overline{J_{x_n} J_{x_j} I\phi}) dx \\
& = 2\operatorname{Im} (J_{x_n} J_{x_j} e^{-x_n} \phi, \overline{J_{x_n} J_{x_j} I\phi}). \quad (3.19)
\end{aligned}$$

By the direct calculations we have

$$\begin{aligned} J_{x_n} J_{x_j} I \phi(t) &= J_{x_n} \int_0^t U_D(t-\tau) e^{-x_n} J_{x_j} \phi(\tau) d\tau \\ &= \int_0^t U_N(t-\tau) (J_{x_n} e^{-x_n} J_{x_j} \phi)(\tau) d\tau \\ &\quad + 2c_1 \int_0^t (t-\tau)^{-1/2} e^{-\frac{x_n^2}{2(t-\tau)}} \tilde{U}(t-\tau) (i\tau J_{x_j} \phi)(\tau) d\tau. \end{aligned}$$

Therefore from (3.14) with g replaced by $J_{x_j} \phi$ it follows

$$\begin{aligned} \partial_{x_n} J_{x_n} J_{x_j} I \phi(t) &= \partial_{x_n} J_{x_n} \int_0^t U_D(t-\tau) e^{-x_n} J_{x_j} \phi(\tau) d\tau \\ &= \int_0^t U_D(t-\tau) \partial_{x_n} J_{x_n} e^{-x_n} J_{x_j} \phi(\tau) d\tau \\ &\quad + 2c_1 \partial_{x_n} \int_0^t (t-\tau)^{-1/2} e^{-\frac{x_n^2}{2(t-\tau)}} \tilde{U}(t-\tau) i\tau J_{x_j} \phi(\tau) d\tau. \end{aligned}$$

Hence,

$$\begin{aligned} &\left| \operatorname{Im} \int_{\mathbb{R}_+^n} \partial_{x_n} (\partial_{x_n} J_{x_n} J_{x_j} I \phi \cdot \overline{J_{x_n} J_{x_j} I \phi}) dx \right| \\ &\leq C \|t J_{x_j} \phi(t)\|_{L^2(\mathbb{R}^{n-1})} \int_0^t (t-\tau)^{-1/2} \left\| \int_0^\infty (J_{y_n} e^{-y_n} J_{x_j} \phi)(\tau) dy_n \right\|_{L^2(\mathbb{R}^{n-1})} d\tau \\ &\leq C \langle t \rangle \left(\|\phi(t)\|_{H^{0,1}(\mathbb{R}^{n-1})} + \langle t \rangle \|\phi(t)\|_{H^1(\mathbb{R}^{n-1})} \right) \\ &\quad \times \int_0^t (t-\tau)^{-1/2} \langle \tau \rangle \left(\|\phi(\tau)\|_{H^{0,1}(\mathbb{R}^{n-1})} + \langle \tau \rangle \|\phi(\tau)\|_{H^1(\mathbb{R}^{n-1})} \right) d\tau \\ &\leq C \left(\|\phi\|_{W_{\gamma-5/4}^{0,\infty}(\mathbb{R}_+; H^{0,1}(\mathbb{R}^{n-1}))} + \|\phi\|_{W_{\gamma-1/4}^{0,\infty}(\mathbb{R}_+; H^1(\mathbb{R}^{n-1}))} \right)^2 \langle t \rangle^{-(\gamma-9/4)} \\ &\quad \times \int_0^t (t-\tau)^{-1/2} \langle \tau \rangle^{-(\gamma-9/4)} d\tau \\ &\leq C \left(\|\phi\|_{W_{\gamma-5/4}^{0,\infty}(\mathbb{R}_+; H^{0,1}(\mathbb{R}^{n-1}))} + \|\phi\|_{W_{\gamma-1/4}^{0,\infty}(\mathbb{R}_+; H^1(\mathbb{R}^{n-1}))} \right)^2 \langle t \rangle^{-\gamma_2}. \end{aligned}$$

$\gamma_2 = (\gamma - 9/4) + \min\{\gamma - 11/4, 1/2\}$. We apply the estimate to (3.19) to get

$$\begin{aligned} \|J_{x_n} J_{x_j} I \phi(t)\|_{L^2(\mathbb{R}_+^n)}^2 &\leq C \left(\|\phi\|_{W_{\gamma-5/4}^{0,\infty}(\mathbb{R}_+; H^{0,1}(\mathbb{R}^{n-1}))} + \|\phi\|_{W_{\gamma-1/4}^{0,\infty}(\mathbb{R}_+; H^1(\mathbb{R}^{n-1}))} \right)^2 \\ &\quad + \int_0^t \|J_{x_n} J_{x_j} e^{-x_n} \phi(\tau)\|_{L^2(\mathbb{R}_+^n)} \|J_{x_n} J_{x_j} I \phi(\tau)\|_{L^2(\mathbb{R}_+^n)} d\tau \\ &\leq C \left(\|\phi\|_{W_{\gamma-5/4}^{0,\infty}(\mathbb{R}_+; H^{0,1}(\mathbb{R}^{n-1}))} + \|\phi\|_{W_{\gamma-1/4}^{0,\infty}(\mathbb{R}_+; H^1(\mathbb{R}^{n-1}))} \right)^2 \\ &\quad + C \left(\|\phi\|_{W_{\gamma}^{0,\infty}(\mathbb{R}_+; H^1(\mathbb{R}^{n-1}))} + \|\phi\|_{W_{\gamma-1}^{0,\infty}(\mathbb{R}_+; H^{0,1}(\mathbb{R}^{n-1}))} \right) \\ &\quad \times \int_0^t \langle \tau \rangle^{-(\gamma-2)} \|J_{x_n} J_{x_j} I \phi(\tau)\|_{L^2(\mathbb{R}_+^n)} d\tau \end{aligned}$$

from which we get the first result. The second estimate is obtained similarly, and so we omit it. This completes the proof of the lemma. $\square$

Denote

$$\|I\phi\|_{\tilde{X}} = \sum_{|\alpha| \leq 1} (\|J^\alpha J_{x_n} I\phi\|_{L^\infty L^2} + \|\partial^\alpha J_{x_n} I\phi\|_{L^\infty L^2}).$$

We have the following result by Lemmas 3.3–3.5.

Lemma 3.6 *We assume that*

$$\phi \in \tilde{Z}_\gamma = \cap_{k=0}^1 W_\gamma^{k,\infty}(\mathbb{R}_+; H^{2-2k}(\mathbb{R}^{n-1})) \cap W_{\gamma-1}^{0,\infty}(\mathbb{R}_+; H^{0,1}(\mathbb{R}^{n-1})),$$

where $\gamma > 3$. Then

$$\|I\phi\|_{\tilde{X}} \leq C \|\phi\|_{\tilde{Z}_\gamma}.$$

By Lemma 3.6 we have

Lemma 3.7 *Let $\phi_1(t) = (\partial_t + \frac{1}{2}\tilde{\Delta} + \frac{1}{2})h(t)$ and*

$$h \in E_4 = \cap_{k=0}^2 W_\gamma^{k,\infty}(\mathbb{R}_+; H^{4-2k}(\mathbb{R}^{n-1})) \cap_{l=0}^1 W_{\gamma-1}^{l,\infty}(\mathbb{R}_+; H^{2-2l,1}(\mathbb{R}^{n-1})),$$

where $\gamma > 3$. Then the following estimate

$$\|I\phi_1\|_{\tilde{X}} \leq C \|h\|_{E_4}$$

holds.

Also using Lemma 3.6 we have the following result.

Lemma 3.8 *Let $\phi_2(t) = |h|^{p-1}h(t)$, $p > 1$ and*

$$h \in E_5 = \cap_{k=0}^1 W_\gamma^{k,\infty}(\mathbb{R}_+; H^{2-2k}(\mathbb{R}^{n-1})) \cap W_{\gamma-1}^{0,\infty}(\mathbb{R}_+; H^{0,1}(\mathbb{R}^{n-1})) \cap L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^{n-1})),$$

where $\gamma > 3$. Furthermore we assume that $h(t) \in C^1(\mathbb{R}^{n-1})$

$$\sum_{j=1}^{n-1} |\partial_{x_j} h(t, x')| \leq C |h(t, x')|$$

for any $(t, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$. Then the following estimate

$$\|I\phi_2\|_{\tilde{X}} \leq C \|h\|_{E_5}^p$$

holds.

Proof By Lemma 3.6

$$\|I\phi_2\|_{\tilde{X}} \leq C \sum_{k=0}^1 \|\phi_2\|_{W_\gamma^{k,\infty}(\mathbb{R}_+; H^{2-2k}(\mathbb{R}^{n-1}))} + C \|\phi_2\|_{W_{\gamma-1}^{0,\infty}(\mathbb{R}_+; H^{0,1}(\mathbb{R}^{n-1}))}. \quad (3.20)$$

Since

$$\begin{aligned}
& \| |h|^{p-1} h \|_{W_{\gamma-1}^{0,\infty}(\mathbb{R}_+; H^{0,1}(\mathbb{R}^{n-1}))} \\
& \leq C \|h\|_{L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^{n-1}))}^{p-1} \|h\|_{W_{\gamma-1}^{0,\infty}(\mathbb{R}_+; H^{0,1}(\mathbb{R}^{n-1}))} \leq C \|h\|_{E_5}^p \\
& \| |h|^{p-1} h \|_{W_\gamma^{1,\infty}(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))} \\
& \leq C \|h\|_{L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^{n-1}))}^{p-1} \|h\|_{W_\gamma^{1,\infty}(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))} \\
& \leq C \|h\|_{E_5}^p
\end{aligned}$$

it is enough to consider

$$\|\phi_2\|_{W_\gamma^{0,\infty}(\mathbb{R}_+; H^2(\mathbb{R}^{n-1}))}.$$

If $p \geq 2$, we have by Sobolev

$$\| |h|^{p-1} h(t) \|_{H^2(\mathbb{R}^{n-1})} \leq C \|h(t)\|_{L^\infty(\mathbb{R}^{n-1})}^{p-1} \|h(t)\|_{H^2(\mathbb{R}^{n-1})}.$$

For $p < 2$

$$\begin{aligned}
& \| |h|^{p-1} h(t) \|_{H^2(\mathbb{R}^{n-1})} \leq C \|h(t)\|_{L^\infty(\mathbb{R}^{n-1})}^{p-1} \|h(t)\|_{H^2(\mathbb{R}^{n-1})} \\
& + C \sum_{j,k=1}^{n-1} \left\| \frac{|\partial_{x_j} h(t)|}{|h(t)|} |h(t)|^{p-1} \right\|_{L^\infty(\mathbb{R}^{n-1})} \|\partial_{x_k} h(t)\|_{L^2(\mathbb{R}^{n-1})}.
\end{aligned}$$

Then by the assumption $|\partial_{x_j} h(t, x')| \leq C |h(t, x')|$, we find

$$\| |h|^{p-1} h(t) \|_{H^2(\mathbb{R}^{n-1})} \leq C \|h(t)\|_{L^\infty(\mathbb{R}^{n-1})}^{p-1} \|h(t)\|_{H^2(\mathbb{R}^{n-1})}.$$

Therefore it follows that

$$\| |h|^{p-1} h \|_{W_\gamma^{0,\infty}(\mathbb{R}_+; H^2(\mathbb{R}^{n-1}))} \leq C \|h\|_{L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^{n-1}))}^{p-1} \|h\|_{W_{\gamma\delta}^{0,\infty}(\mathbb{R}_+; H^2(\mathbb{R}^{n-1}))}.$$

The lemma is proved. $\square$

Now we state the Sobolev inequality on the domain $\mathbb{R}_+^n$.

Lemma 3.9 Assume that g is extended to $\mathbb{R}_+^n$ as an even or odd function with respect to x_n smoothly and

$$2 \leq \sigma \begin{cases} \leq 2 + \frac{4}{n-2}, n \geq 3 \\ < \infty, n = 2 \\ \leq \infty, n = 1 \end{cases}.$$

Then the following estimates

$$\|g\|_{L^\sigma(\mathbb{R}_+^n)} \leq C \|g\|_{L^2(\mathbb{R}_+^n)}^{1 - \left(\frac{1}{2} - \frac{1}{\sigma}\right)n} \|\partial g\|_{L^2(\mathbb{R}_+^n)}^{\left(\frac{1}{2} - \frac{1}{\sigma}\right)n} \quad (3.21)$$

and

$$\|g\|_{L^\sigma(\mathbb{R}_+^n)} \leq C t^{-\frac{n}{2}\left(1 - \frac{2}{\sigma}\right)} \|g\|_{L^2(\mathbb{R}_+^n)}^{1 - \left(\frac{1}{2} - \frac{1}{\sigma}\right)n} \|Jg\|_{L^2(\mathbb{R}_+^n)}^{\left(\frac{1}{2} - \frac{1}{\sigma}\right)n} \quad (3.22)$$

are valid, provided that the right hand sides are finite.

Now we prove

Lemma 3.10 *Let h and (q, r) satisfy the conditions on Theorem 2.1. Then the following estimates*

$$\|J_{x_n} I \phi\|_{L^q L^r} + \|J_{x_n} I \phi\|_{L^\infty L^2} \leq C \|h\|_{E_4}$$

and

$$\|J_{x_n} I \phi_1\|_{L^q L^r} + \|J_{x_n} I \phi_1\|_{L^\infty L^2} \leq C \|h\|_{E_5}^p$$

hold.

Proof It is clear that by Lemma 3.7 and Lemma 3.8

$$\|J_{x_n} I \phi\|_{L^\infty L^2} \leq C \|h\|_{E_4}, \quad \|J_{x_n} I \phi_1\|_{L^\infty L^2} \leq C \|h\|_{E_5}^p.$$

Since $J_{x_n} I \phi$ can be extended to whole space as an even function with respect to x_n , by Lemma 3.9,

$$\begin{aligned} \|J_{x_n} I \phi\|_{L^r(\mathbb{R}_+^n)} &\leq C \langle t \rangle^{-\frac{n}{2}(1-\frac{2}{r})} \sum_{|\alpha| \leq 1} \left(\|J^\alpha J_{x_n} I \phi(t)\|_{L^2(\mathbb{R}_+^n)} + \|\partial^\alpha J_{x_n} I \phi(t)\|_{L^2(\mathbb{R}_+^n)} \right) \\ &\leq C \|h\|_{E_4} \langle t \rangle^{-\frac{n}{2}(1-\frac{2}{r})} \end{aligned}$$

and similarly,

$$\|J_{x_n} I \phi_1\|_{L^r(\mathbb{R}_+^n)} \leq C \|h\|_{E_5}^p \langle t \rangle^{-\frac{n}{2}(1-\frac{2}{r})}.$$

Therefore we have the desired estimates. The lemma is proved. $\square$

4 Proof of Theorem 2.1

We define

$$X_{q,r} = \left\{ v \in C(\mathbb{R}_+; L^2(\mathbb{R}_+^n)); \|v\|_{X_{q,r}} < \infty \right\},$$

endowed with a norm

$$\|v\|_{X_{q,r}} = \sum_{|\alpha| \leq 1} \|J^\alpha v\|_{L^q L^r \cap L^\infty L^2} + \sum_{|\alpha| \leq 1} \|\partial^\alpha v\|_{L^q L^r \cap L^\infty L^2},$$

where (q, r) satisfies the condition of Theorem 2.1. Via Lemma 3.1 the solution of (1.1) can be rewritten in the following form

$$\begin{aligned} u(t) &= Mu(t) \\ &= U_D(t) (u_0(x', x_n) - e^{-x_n} h(0, x')) + e^{-x_n} h(t, x') \\ &\quad - i\lambda \int_0^t U_D(t-\tau) (|u|^{p-1} u(\tau) - e^{-x_n} |h|^{p-1} h(\tau)) d\tau \\ &\quad + \Phi(t, x), \end{aligned} \tag{4.1}$$

where $U_D(t) = U_{nD}(t) \tilde{U}(t)$ and

$$\begin{aligned}\Phi(t) &= -i\lambda \int_0^t U_D(t-\tau) e^{-x_n} |h|^{p-1} h(\tau) d\tau \\ &\quad + i \int_0^t U_D(t-\tau) e^{-x_n} \left(i\partial_\tau + \frac{1}{2}\tilde{\Delta} + \frac{1}{2} \right) h(\tau, x') d\tau.\end{aligned}$$

Let us consider the linearized equation of (4.1) such that

$$u(t) = Mv(t), \quad (4.2)$$

where $\|v\|_{X_{q,r}} \leq \sqrt{\varepsilon}$. If $\phi|_{x_n=0} = 0$, then we have by a direct calculation

$$J_{x_n}(t) U_D(t-\tau) \phi = U_N(t-\tau) J_{y_n}(\tau) \phi.$$

We find that

$$J_{x_j}(t) U_D(t-\tau) = U_{nD}(t-\tau) J_{x_j}(t) \tilde{U}(t-\tau) = U_D(t-\tau) J_{x_j}(\tau)$$

for $1 \leq j \leq n-1$. We apply the operator J_{x_n} to both sides of (4.2) to get

$$\begin{aligned}J_{x_n} u(t) &= U_D(t) x_n(u_0(x', x_n) - e^{-x_n} h(0, x')) - i J_{x_n} e^{-x_n} h(t, x') \\ &\quad - i\lambda \int_0^t U_D(t-\tau) J_{x_n}(\tau) (|v|^{p-1} v(\tau) - e^{-x_n} |h|^{p-1} h(\tau)) d\tau \\ &\quad + J_{x_n} \Phi(t).\end{aligned} \quad (4.3)$$

By Lemma 3.2,

$$\begin{aligned}&\left\| \int_0^t U_D(t-\tau) J_{x_n}(\tau) (|v|^{p-1} v(\tau) - e^{-x_n} |h|^{p-1} h(\tau)) d\tau \right\|_Y \\ &\leq C \| |v|^{p-1} J_{x_n} v \|_{L^{q'} L^{r'}} + C \| J_{x_n} e^{-x_n} |h|^{p-1} h \|_{L^{q'} L^{r'}},\end{aligned} \quad (4.4)$$

$$\begin{aligned}&\| U_D(t) x_n(u_0(x', x_n) - e^{-x_n} h(0, x')) \|_{L^q L^r \cap L^\infty L^2} \\ &\leq C \| x_n u_0 - e^{-x_n} h(0, x') \|_{L^2(\mathbb{R}_+^n)} \leq C \| x_n u_0 \|_{L^2(\mathbb{R}_+^n)} + C \| h(0) \|_{L^2(\mathbb{R}^{n-1})}\end{aligned} \quad (4.5)$$

and by Lemma 3.10,

$$\| J_{x_n} \Phi \|_{L^q L^r \cap L^\infty L^2} \leq C \left(\| h \|_{E_4} + \| h \|_{E_5}^p \right). \quad (4.6)$$

We also have

$$\begin{aligned}&\| J_{x_n} e^{-x_n} h(t, x') \|_{L^q L^r \cap L^\infty L^2} \leq C \left\| \langle t \rangle \| h(t) \|_{L^r(\mathbb{R}_+^n)} \right\|_{L^q(\mathbb{R}_+)} \\ &\quad + C \left\| \langle t \rangle \| h(t) \|_{L^2(\mathbb{R}_+^n)} \right\|_{L^\infty(\mathbb{R}_+)} \leq C \| h \|_{E_4}.\end{aligned} \quad (4.7)$$

Let us consider the right hand side of (4.4). We have by the Hölder inequality

$$\| |v|^{p-1} J_{x_n} v \|_{L^{q'} L^{r'}} \leq C \| v \|_{L^\beta L^\sigma}^{p-1} \| J_{x_n} v \|_{L^q L^r}, \quad (4.8)$$

where

$$\beta = \frac{p-1}{1 - \frac{n}{2} \frac{p-1}{\sigma}}, \quad \sigma = \frac{2n}{n-2}$$

By Lemma 3.9 we find that for $\tilde{v} = v - e^{-x_n} h(t, x')$

$$\begin{aligned}\|\tilde{v}\|_{L^\sigma(\mathbb{R}_+^n)} &\leq C \langle t \rangle^{-\frac{n}{2}\left(1-\frac{2}{\sigma}\right)} \sum_{|\alpha|\leq 1} \left(\|J^\alpha \tilde{v}\|_{L^2(\mathbb{R}_+^n)} + \|\partial^\alpha \tilde{v}\|_{L^2(\mathbb{R}_+^n)} \right) \\ &\leq C \langle t \rangle^{-\frac{n}{2}\left(1-\frac{2}{\sigma}\right)} \sum_{|\alpha|\leq 1} \left(\|J^\alpha v\|_{L^2(\mathbb{R}_+^n)} + \|\partial^\alpha v\|_{L^2(\mathbb{R}_+^n)} + \rho \right)\end{aligned}$$

for $2 \leq \sigma \leq 2 + \frac{4}{n-2}$, since $\tilde{v}(t, x', 0) = v(t, x', 0) - h(t, x') = 0$ by the boundary condition and so $\tilde{v}$ can be extended to an odd function with respect to x_n . Hence,

$$\begin{aligned}\|v\|_{L^\sigma(\mathbb{R}_+^n)} &\leq \|\tilde{v}\|_{L^\sigma(\mathbb{R}_+^n)} + \|h\|_{L^\sigma(\mathbb{R}^{n-1})} \\ &\leq C \langle t \rangle^{-\frac{n}{2}\left(1-\frac{2}{\sigma}\right)} \sum_{|\alpha|\leq 1} \left(\|J^\alpha v\|_{L^2(\mathbb{R}_+^n)} + \|\partial^\alpha v\|_{L^2(\mathbb{R}_+^n)} + \|h\|_{Y_4} \right),\end{aligned}$$

which implies

$$\|v\|_{L^\beta(\mathbb{R}_+; L^\sigma(\mathbb{R}_+^n))}^\beta \leq C \left(\varepsilon^{\frac{\beta}{2}} + \rho^\beta \right) \int \langle t \rangle^{-\frac{n}{2}\left(1-\frac{2}{\sigma}\right)\beta} dt \leq C \left(\varepsilon^{\frac{\beta}{2}} + \|h\|_{E_4}^\beta \right) \quad (4.9)$$

with

$$1 + \frac{4}{n+2} < p < 1 + \frac{4}{n-2}. \quad (4.10)$$

From (4.8) and (4.9) it follows

$$\| |v|^{p-1} J_{x_n} v \|_{L^{q'} L^{r'}} \leq C \left(\varepsilon^{\frac{p}{2}} + \|h\|_{E_4}^p \right). \quad (4.11)$$

We also have

$$\begin{aligned}\| J_{x_n} e^{-x_n} |h|^{p-1} h \|_{L^{q'} L^{r'}} &\leq C \| \langle t \rangle \langle x_n \rangle e^{-x_n} |h|^{p-1} h \|_{L^{q'} L^{r'}} \\ &\leq C \left\| e^{-\frac{x_n}{2p}} h \right\|_{L^\beta L^\sigma}^{p-1} \| \langle t \rangle e^{-\frac{x_n}{2p}} h \|_{L^q L^{r'}} \\ &\leq C \left\| e^{-\frac{x_n}{2p}} h \right\|_{L^\beta H^1}^{p-1} \| \langle t \rangle e^{-\frac{x_n}{2p}} h \|_{L^q H^1} \\ &\leq C \|h\|_{L^\beta H^1}^{p-1} \| \langle t \rangle h \|_{L^q H^1} \leq C \|h\|_{E_5}^p.\end{aligned} \quad (4.12)$$

By (4.3)-(4.7), (4.11), (4.12)

$$\| J_{x_n} u \|_{L^q L^{r'} \cap L^\infty L^2} \leq C \left(\varepsilon + \|h\|_{E_4} + \varepsilon^{\frac{p}{2}} + \|h\|_{E_5}^p \right). \quad (4.13)$$

Similarly, we obtain

$$\| \partial_{x_n} u \|_{L^q L^{r'} \cap L^\infty L^2} \leq C \left(\varepsilon + \|h\|_{E_4} + \varepsilon^{\frac{p}{2}} + \|h\|_{E_5}^p \right). \quad (4.14)$$

We apply the operator J_{x_j} , $1 \leq j \leq n-1$ to both sides of (4.2) and get

$$\begin{aligned}J_{x_j} u(t) &= U_D(t) x_j \left(u_0(x', x_n) - e^{-x_n} h(0, x') \right) - i e^{-x_n} J_{x_j} h(t, x') \\ &\quad - i \lambda \int_0^t U_D(t-\tau) J_{x_j} |v|^{p-1} v(\tau) d\tau \\ &\quad + J_{x_j} \Phi_1(t, x),\end{aligned} \quad (4.15)$$

where

$$J_{x_j} \Phi_1(t) = i \int_0^t U_D(t-\tau) e^{-x_n} \left(i \partial_\tau + \frac{1}{2} \tilde{\Delta} + \frac{1}{2} \right) J_{x_j} h(\tau, x') d\tau.$$

In the same way as in the proofs of (4.11) and (4.12), by (4.15) we have

$$\begin{aligned} \|J_{x_j} u\|_{L^q L^r \cap L^\infty L^2} &\leq C\varepsilon + C \|e^{-x_n} J_{x_j} h\|_{L^q L^r \cap L^\infty L^2} \\ &\quad + C\varepsilon^{\frac{p}{2}} + C \left\| e^{-x_n} \left(-\partial_\tau + \frac{1}{2} i \tilde{\Delta} + \frac{1}{2} i \right) J_{x_j} h(\tau, x') \right\|_{L^1 L^2} \\ &\leq C \left(\varepsilon + \|h\|_{E_4} + \varepsilon^{\frac{p}{2}} \right) \end{aligned} \quad (4.16)$$

for $1 \leq j \leq n-1$. Similarly,

$$\|\partial_{x_j} u\|_{L^q L^r \cap L^\infty L^2} \leq C \left(\varepsilon + \|h\|_{E_4} + \varepsilon^{\frac{p}{2}} + \|h\|_{E_5}^p \right). \quad (4.17)$$

By (4.13), (4.14), (4.16), (4.17)

$$\begin{aligned} \|u\|_{X_{q,r}} &= \|Mv\|_{X_{q,r}} \leq C \left(\varepsilon + \|h\|_{E_4} + \varepsilon^{\frac{p}{2}} + \|h\|_{E_5}^p \right) \\ &\leq C \left(\varepsilon + \|h\|_{Z_\gamma} + \varepsilon^{\frac{p}{2}} + \|h\|_{Z_\gamma}^p \right) \end{aligned} \quad (4.18)$$

since $Z_\gamma = E_4 \cap E_5$ with $\gamma > 3$. We put $u_1 = Mv_1$, $u_2 = Mv_2$, where $\|v_j\|_X \leq \sqrt{\varepsilon}$, $j = 1, 2$, then we have

$$(u_1 - u_2)(t) = -i\lambda \int_0^t U_D(t-\tau) (|v_1|^{p-1} v_1(\tau) - |v_2|^{p-1} v_2(\tau)) d\tau.$$

In the same way as in the proof of (4.17) we get

$$\|u_1 - u_2\|_{L^q L^r \cap L^\infty L^2} \leq C\varepsilon^{\frac{p-1}{2}} \|v_1 - v_2\|_{L^q L^r \cap L^\infty L^2}. \quad (4.19)$$

Therefore (4.18) and (4.19) say that there exist $\varepsilon, \rho > 0$ such that M is a contraction mapping from $X_{q,r,\sqrt{\varepsilon}} = \{v; \|v\|_{X_{q,r}} \leq \sqrt{\varepsilon}\}$ into itself. From 3.9 there exist time decay estimates of solutions such that

$$\|u\|_{L^\sigma(\mathbb{R}_+^n)} \leq C \langle t \rangle^{-\frac{n}{2} \left(1 - \frac{2}{\sigma}\right)},$$

where $2 \leq \sigma \leq 2 + \frac{4}{n-1}$. This completes the proof of Theorem 2.1.

Acknowledgements We would like to thank referees for useful comments. The work of NH is partially supported by JSPS KAKENHI Grant numbers JP20K03680, JP19H05597. The work of EIK is partially supported by CONACYT 252053-F. The work of TO is partially supported by JSPS KAKENHI Grant number JP19H05597.

Data availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Audiard, C.: On the non-homogeneous boundary value problem for Schrödinger equations. *Discrete Contin. Dyn. Syst.* **33**, 3861–3884 (2013)
2. Audiard, C.: On the boundary value problem for the Schrödinger equation: compatibility conditions and global existence. *Anal. PDE* **8**, 1113–1143 (2015)
3. Audiard, C.: Global Strichartz estimates for the Schrödinger equation with non zero boundary conditions and applications. *Ann. Inst. Fourier (Grenoble)* **69**(1), 31–80 (2019)
4. Bona, J.L., Sun, S.-M., Zhang, B.-Y.: Nonhomogeneous boundary-value problems for one dimensional nonlinear Schrödinger equations. *J. Math. Pures Appl.* **9**(109), 1–66 (2018)
5. Brezis, H., Gallouet, T.: Nonlinear Schrödinger evolution equation. *Nonlinear Anal.* **4**, 677–681 (1980)
6. Cazenave, T., Weissler, F.B.: Rapidly decaying solutions of the nonlinear Schrödinger equation. *Commun. Math. Phys.* **147**, 75–100 (1992)
7. Esquivel, L., Hayashi, N., Kaikina, E.I.: Inhomogeneous Neumann-boundary value problem for one dimensional nonlinear Schrödinger equations via factorization techniques. *J. Math. Phys.* **60** 091507(9), 29 pp (2019)
8. Esquivel, L., Hayashi, N., Kaikina, E.: Inhomogeneous Dirichlet-boundary value problem for one dimensional nonlinear Schrödinger equations via factorization techniques. *J. Differ. Equ.* **266**(2–3), 1121–1152 (2019)
9. Fokas, A.S., Himonas, A., Mantzavinos, D.: The nonlinear Schrödinger equation on the half-line. *Trans. Am. Math. Soc.* **369**(1), 681–709 (2017)
10. Fokas, A.S., Lenells, J.: An integrable generalization of the nonlinear Schrödinger equation on the half-line and solitons. *Inverse Probl.* **25**(11), 115006 (2009)
11. Hayashi, N.: Time decay of solutions to the Schrödinger equation in exterior domains. I. *Ann. Inst. Henri. Poincaré, Physique Théorique* **50**, 71–81 (1989)
12. Hayashi, N.: Time decay of solutions to the Schrödinger equation in exterior domains. II. *Ann. Inst. Henri. Poincaré Physique Théorique* **50**, 83–93 (1989)
13. Hayashi, N.: Smoothing effect for nonlinear Schrödinger equations in exterior domains. *J. Funct. Anal.* **89**, 444–458 (1990)
14. Hayashi, N.: Global existence of small radially symmetric solutions to quadratic nonlinear evolution equations in an exterior domain. *Math. Z.* **215**, 381–419 (1994)
15. Hayashi, N., Kaikina, E., Ogawa, T.: Inhomogeneous Neumann boundary value problem for nonlinear Schrödinger equations in the upper half-space. To appear in *Differential Integral Equations* (2021)
16. Hayashi, N., Kaikina, E.: *Nonlinear Theory of Pseudodifferential Equations on a Half-line*, North-Holland Mathematics Studies, vol. 194, p. 319. Elsevier Science B.V., Amsterdam (2004)
17. Hayashi, N., Kaikina, E.: Inhomogeneous Dirichlet-boundary value problem for nonlinear Schrödinger equations with a power nonlinearity on the upper half-plane. *Nonlinear Anal.* **187**, 279–306 (2019)
18. Hayashi, N., Kaikina, E.: Neumann inhomogeneous initial-boundary value problem for the 2D nonlinear Schrödinger equation. *Nonlinear Differ. Equ. Appl.* **27**(2), 32 pp (2020). <https://doi.org/10.1007/s00030-019-0605-3>
19. Holmer, J.: The initial-boundary-value problem for the 1D nonlinear Schrödinger equation on the half-line. *Differ. Integral Equ.* **18**(6), 647–668 (2005)
20. Kaikina, E.: A new unified approach to study fractional PDE equations on a half-line. *Complex Var. Elliptic Equ.* **58**(1), 55–77 (2013)
21. Kaikina, E.: Asymptotics for inhomogeneous Dirichlet initial-boundary value problem for the nonlinear Schrödinger equation. *J. Math. Phys.* **54**(11), 111504 (2013)
22. Kaikina, E.: Inhomogeneous Neumann initial boundary value problem for the nonlinear Schrödinger equation. *J. Differ. Equ.* **255**, 3338–3356 (2013)
23. Kaikina, E.: Inhomogeneous initial-boundary value problem for the 2D nonlinear Schrödinger equation. *J. Math. Phys.* **59**(6), 061506 (2018)
24. Ivan Naumkin, P.: Cubic nonlinear Dirac equation in a quarter plane. *J. Math. Anal. Appl.* **434**(2), 1633–1664 (2016)
25. Naumkin, P.I.: Klein-Gordon equation with critical nonlinearity and inhomogeneous Dirichlet boundary conditions. *Differ. Integral Equ.* **29**(1–2), 55–92 (2016)
26. Naumkin, P.I.: Initial-boundary value problem for the one dimensional Thirring model. *J. Differ. Equ.* **261**(8), 4486–4523 (2016)
27. Özsarı, T.: Well-posedness for nonlinear Schrödinger equations with boundary forces in low dimensions by Strichartz estimates. *J. Math. Anal. Appl.* **424**(1), 487–508 (2015)

28. Özşarı, T., Kalantarov, K.V., Lasiecka, I.: Uniform decay rates for the energy of weakly damped defocusing semilinear Schrödinger equations with inhomogeneous Dirichlet boundary control. *J. Differ. Equ.* **251**(7), 1841–1863 (2011)
29. Ogawa, T.: A proof of Trudinger's inequality and its application to nonlinear Schrödinger equations. *Nonlinear Anal.* **14**(9), 765–769 (1990)
30. Ogawa, T., Ozawa, T.: Trudinger type inequalities and uniqueness of weak solutions for the nonlinear Schrödinger mixed problem. *J. Math. Anal. Appl.* **155**(2), 531–540 (1991)
31. Ran, Y., Sun, S.-S., Zhang, B.-Y.: Nonhomogeneous boundary value problems of nonlinear Schrödinger equations in a half plane. *SIAM J. Math. Anal.* **50**(3), 2773–2806 (2018)
32. Strauss, W., Bu, C.: An inhomogeneous boundary value problem for nonlinear Schrödinger equations. *J. Differ. Equ.* **173**, 79–91 (2001)
33. Tsutsumi, Y.: Global solutions of the nonlinear Schrödinger equations in exterior domains. *Commun. Partial Differ. Equ.* **8**, 1337–1374 (1983)