The Keller–Segel system on bounded convex domains in critical spaces

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Dedicated to Hideo Kozono on the occasion of his 60-th Birthday

Abstract

Consider the classical Keller–Segel system on a bounded convex domain $\Omega \subset \mathbb{R}^3$. In contrast to previous works it is not assumed that the boundary of Ω is smooth. It is shown that this system admits a local, strong solution for initial data in critical spaces which extends to a global one provided the data are small enough in this critical norm. Furthermore, it is shown that this system admits for given T -periodic and sufficiently small forcing functions a unique, strong T -time periodic solution.

Keywords Keller–Segel system · Convex domains · Critical spaces · Strong periodic solutions

Mathematics Subject Classification 35Q92 · 35K59 · 35B10 · 47F05 · 92C17

1 Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, with smooth boundary, the *classical* Keller–Segel system

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$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (1.1)$$

has been investigated during the past decades by very many authors. Here u and v represent the density of a cell population and the concentration of a chemoattractant, respectively. In the situation of domains with smooth boundaries, the problem of local and global existence as well as blow-up of solutions are rather well understood, see e.g., the survey articles [9, 10, 16] and the references therein. In particular, the existence of global, strong solutions in L^p -settings has been investigated by Hideo Kozono and coworkers in [13–15]. For further results in this context, see [17, 18]. For recent results on the existence of unique, strong periodic solutions, see [8]. All these results deal with the setting of the whole space or domains with smooth boundaries.

The situation is very different when one considers the Keller–Segel system in domains with nonsmooth boundaries, as e.g. Lipschitz domains. It was recently shown by Horstmann, Meinel Schmidt and Rehberg [11] that under suitable conditions on the initial values and the geometry of the domain, one nevertheless obtains again the existence of a unique, strong, local solution to the Keller–Segel system.

In this article we study the Keller–Segel system in *bounded convex domains* and do *not* assume that the boundary of Ω is smooth. The situation of bounded convex domains with smooth boundaries was considered before e.g. in [22, 23], as in this setting, due to the Neumann boundary condition, a classical solution to (1.1) satisfies $\frac{\partial |\nabla v|^2}{\partial \nu} \leq 0$ on $\partial \Omega \times (0, \infty)$. The latter property is helpful to establish a priori estimates for the solution to (1.1) which are used to prove global existence results. Here we use the convexity of the domain to study the Keller–Segel system, similarly to [11], in domains with rough boundaries.

In contrast to [11] we investigate the Keller–Segel system within the framework of time-weighted Sobolev space allowing us to prove strong well-posedness results for initial data u_0 and v_0 lying in critical spaces. We note that the modified Keller–Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, \ t > 0, \\ v_t = \Delta v + u, & x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (1.2)$$

is scaling invariant with respect to the scaling

$$(u_\lambda, v_\lambda)(t, x) = (\lambda^2 u, v)(\lambda^2 t, \lambda x), \quad \lambda > 0$$

and that the spaces

$$B_{q,p}^{3/q-1}(\Omega) \times B_{q,p}^{3/q}(\Omega)$$

are scaling invariant spaces for the modified Keller–Segel system in $\Omega \subset \mathbb{R}^3$. For results on solutions for (1.2) in various scaling invariant spaces we refer to the work of Kozono, Sugiyama and Wachi [13–15].

The situation of bounded convex domains is, of course, less general than the framework of Lipschitz domains as considered in [11], however, in the situation of bounded convex domains, several main ingredients of the approach for smooth domains, such as knowledge of the domain of the Neumann Laplacian, its maximal L^p -regularity as well as the mixed derivative theorem carry over to the given situation. This allows us in particular to treat the case of initial data lying in critical Besov spaces as stated precisely in Theorem 1.

Given a bounded convex domain $\Omega \subset \mathbb{R}^n$, we are secondly interested in the question whether, given periodic functions f and g , there exist time periodic strong solutions to the classical Keller–Segel system. More precisely, for time periodic functions f_1 and f_2 , we consider the classical inhomogeneous Keller–Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + f_1(t), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - v + u + f_2(t), & x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0. \end{cases} \quad (1.3)$$

We prove the existence of a T -periodic solution to (1.3), which is unique in the associated maximal regularity space, provided $f = (f_1, f_2)$ is T -periodic and sufficiently small in an appropriate norm. We will perform our analysis for $f \in L^p(0, T; L^q(\Omega))$, which corresponds to the weak setting. Here $p, q \in (1, \infty)$ need to satisfy certain conditions described in detail in Section 2. These conditions are essentially due to the mixed derivative theorem (see e.g. [19, Corollary 4.5.10]), allowing to transfer time into space regularity in a very precise sense, and to Sobolev embeddings.

The strategy of our approach may be described as follows: we first rewrite the Keller–Segel systems (1.1) and (1.3) as semilinear evolution equations, respectively, within the L^p -setting. Maximal time weighted L^p -regularity estimates as well as the existence of a bounded H^∞ -calculus for the differential operators involved imply then the existence of a unique, local solution to the Keller–Segel system for initial data in critical spaces.

Secondly, the above maximal L^p -regularity estimates for the linearization of the semilinear equation allow first for maximal periodic L^p -solutions for the linear problem by the Arendt–Bu theorem, see [4]. The existence theory for maximal periodic L^p -solutions for the linear evolution equations was generalized in [8] to the semi- and quasilinear setting. More precisely, for Banach spaces X we consider time-periodic quasilinear problems of the form

$$\begin{cases} u'(t) + A(u(t))u(t) = F(t, u(t)), & t \in (0, 2\pi), \\ u(0) = u(2\pi). \end{cases} \quad (1.4)$$

Assuming natural Lipschitz conditions on A and F as well as that $A(0)$ admits maximal periodic L^p -regularity, the existence of a unique strong periodic solution to (1.4) is established provided $\|F(\cdot, 0)\|_{\mathbb{F}}$ is small enough, where $\mathbb{F} := L^p(0, 2\pi; X)$.

This article is organized as follows. In Sect. 2 we present the main results concerning the initial value problem as well as strong T -periodic solutions to the classical Keller–Segel system. Section 3 presents results on the Laplacian in the strong and weak setting for bounded convex domains. Sections 4 and 5 present the proofs of our main results.

2 Preliminaries and main results

Consider the Keller–Segel system (1.1), where $\Omega \subset \mathbb{R}^3$ denotes a *bounded convex domain*, ν the outward unit normal on $\partial\Omega$ and f a given T -periodic function. We study the initial value problem (1.1) for critical spaces as well as the existence of time periodic solutions to the Keller–Segel system (1.3).

We start by rewriting the Keller–Segel system (1.1) as an evolution equation in the Banach space $X_0 := (W^{1,q'}(\Omega))' \times L^q(\Omega)$ for $1/q + 1/q' = 1$ with $q, q' \in (1, \infty)$ by introducing the operator A and for $w = (u, v)^T$ the mapping F as

$$A := \begin{pmatrix} \Delta_N^w & 0 \\ 1 & \Delta_N - 1 \end{pmatrix}, \quad F(w) := \begin{pmatrix} -\nabla \cdot (u \nabla v) \\ 0 \end{pmatrix}. \quad (2.1)$$

Here Δ_N and Δ_N^w denote the Neumann Laplacian on $L^q(\Omega)$ and $W^{-1,q}(\Omega) = W^{1,q'}(\Omega)'$ equipped with the domains $D(\Delta_N) := \{u \in W^{2,q}(\Omega) : \partial_\nu u = 0 \text{ on } \partial\Omega\}$ and $W^{1,q}(\Omega)$, respectively. For details, see Section 3. Setting $X_1 := D(\Delta_N^w) \times D(\Delta_N)$, we see that $X_1 \hookrightarrow X_0$ is densely embedded and that $A : X_1 \rightarrow X_0$ is bounded.

The Keller–Segel system (1.1) corresponds then to the equation

$$\begin{cases} w'(t) - Aw(t) = F(w(t)), & t \in (0, T), \\ w(0) = w_0. \end{cases} \quad (2.2)$$

For a Banach space X , $1 < p < \infty$, a time weight $\mu \in (1/p, 1]$, a time interval $J \subset [0, \infty)$, we set

$$\begin{aligned} L_\mu^p(J; X) &= \{u \in L_{loc}^1(J; X) : [t \mapsto t^{1-\mu} u(t)] \in L^p(J; X)\}, \\ H_\mu^{1,p}(J; X) &= \{u \in L_\mu^p(J; X) \cap H_{loc}^{1,1}(J; X) : u' \in L_\mu^p(J; X)\}. \end{aligned}$$

Here u' denotes the time derivative of u in the distributional sense.

We aim for solutions in the maximal regularity space

$$\mathbb{E}_\mu(J) := H_\mu^{1,p}(J; X_0) \cap L_\mu^p(J; X_1).$$

The associated real interpolation space $X_{\mu-1/p}$ for the initial data is given by $X_{\mu-1/p} = (X_0, X_1)_{\mu-1/p, p}$. Since in the given situation of bounded convex domains, the domains of Δ_N^w and Δ_N are explicitly known, it follows as in Theorem 5.2 of [2] that

$$(X_0, X_1)_{s,p} = B_{q,p}^{2s-1}(\Omega) \times {}_NB_{q,p}^{2s}(\Omega),$$

where

$${}_NB_{q,p}^{2s}(\Omega) = \begin{cases} \{u \in B_{q,p}^{2s}(\Omega) : \partial_\nu u = 0 \text{ on } \partial\Omega\}, & s \in (1/2 + 1/2q, 1), \\ B_{q,p}^{2s}(\Omega), & s \in (0, 1/2 + 1/2q). \end{cases}$$

We show in Section 4 that the critical value μ_c of μ for which we obtain local well-posedness for (1.1) is given by $\mu_c = 3/2q + 1/p$. It is thus natural to call this value of μ the critical weight and it is hence meaningful to name $X_{\mu_c-1/p,p}$ the critical space for (1.1).

Our first result on the Keller–Segel system on bounded convex domains reads as follows.

Theorem 1 Let $\Omega \subset \mathbb{R}^3$ be a bounded and convex domain and $p, q \in (1, \infty)$ such that $q \in (3/2, 2]$ and $3/2q + 1/p \leq 1$. Then, for all $(u_0, v_0) \in B_{q,p}^{3/q-1}(\Omega) \times {}_NB_{q,p}^{3/q}(\Omega)$ the chemotaxis system (1.1) admits a unique solution $w = (u, v)^T$ with

$$w \in H_\mu^{1,p}(0, a; X_0) \cap L_\mu^p(0, a; X_1)$$

for some $a > 0$ and $\mu = 3/2q + 1/p$.

Remark 1 It is interesting to compare Theorem 1 with the results obtained in [11] dealing with a much more complex geometrical situation. It shows that the method of time-weights combined with the special situation of bounded convex domains allows to improve the regularity index for the initial data by more than 1, meaning from $B_{q,r}^s(\Omega)$ for $s > 3/q + 1$ and $r > 2(1 - 3/q)^{-1}$ in [11] to $B_{q,p}^{3/q}(\Omega)$.

In order to extend Theorem 1 to a global existence result for small data, we consider for $1 < r < \infty$ the space $L_0^r(\Omega) = \{u \in L^r(\Omega) : \int_\Omega u = 0\}$ consisting of all L^r -functions having mean zero. We then set $X_0 := (W^{1,q'}(\Omega) \cap L_0^{q'}(\Omega))' \times L^q(\Omega)$ and consider the weak Neumann Laplacian $\Delta_{N,0}^w$ in $(W^{1,q'}(\Omega) \cap L_0^{q'}(\Omega))'$ as defined in Section 3. Set

$$A_0 := \begin{pmatrix} \Delta_{N,0}^w & 0 \\ 1 & \Delta_N - 1 \end{pmatrix}, \quad F(w) := \begin{pmatrix} -\nabla \cdot (u \nabla v) \\ 0 \end{pmatrix}.$$

on X_0 with $X_1 = D(\Delta_{N,0}^w) \times D(\Delta_N)$. Note that

$$(X_0, X_1)_{s,p} = B_{q,p}^{2s-1}(\Omega) \cap L_0^q(\Omega) \times {}_NB_{q,p}^{2s}(\Omega).$$

Then the general theory of semilinear equations in time-weighted spaces (see e.g. [20, Cor.2.2]) implies the following result.

Corollary 1 Let $\Omega \subset \mathbb{R}^3$ be a bounded and convex domain and $p, q \in (1, \infty)$ such that $q \in (3/2, 2]$ and $3/2q + 1/p \leq 1$. Then there exists $r_0 > 0$ such that the local solution w given in Theorem 1 exists globally and converges exponentially to zero in $(X_0, X_1)_{1-1/p,p}$ provided $\|w_0\|_{(X_0, X_1)_{3/2q,p}} \leq r_0$.

Our second main result concerns the existence of strong T -periodic solutions to the Keller–Segel model in bounded convex domains. For recent results on periodic solutions in the situation of bounded domains with smooth boundaries we refer to [8]. Similarly as above, we rewrite the Eq. (1.3) as an evolution equation on $X_0 := (W^{1,q'}(\Omega) \cap L_0^{q'}(\Omega))' \times L^q(\Omega)$, where $L_0^r(\Omega) = \{u \in L^r(\Omega) : \int_\Omega u = 0\}$ for $1 < r < \infty$. To this end, we recall A_0 and define for $w = (u, v)^T$ the mapping H as

$$A_0 = \begin{pmatrix} \Delta_{N,0}^w & 0 \\ 1 & \Delta_N - 1 \end{pmatrix}, \quad H(t, w) := \begin{pmatrix} -\nabla \cdot (u \nabla v) + f_1(t) \\ f_2(t) \end{pmatrix}. \quad (2.3)$$

Here Δ_N and $\Delta_{N,0}^w$ denote the Neumann Laplacian on $L^q(\Omega)$ and $(W^{1,q'}(\Omega) \cap L_0^{q'}(\Omega))'$ equipped with the domains $D(\Delta_N) := \{u \in W^{2,q}(\Omega) : \partial_\nu u = 0 \text{ on } \partial\Omega\}$ and

$W^{1,q}(\Omega) \cap L_0^q(\Omega)$, respectively. For details, see Section 3. Setting $X_1 := D(\Delta_{N,0}^w) \times D(\Delta_N)$, we see that $X_1 \hookrightarrow X_0$ is densely embedded and that $A_0 : X_1 \rightarrow X_0$ is bounded.

The Keller–Segel system in the periodic setting corresponds then to the equation

$$\begin{cases} w'(t) - A_0 w(t) = H(t, w(t)), & t \in (0, T), \\ w(0) = w(T). \end{cases} \quad (2.4)$$

Being interested again in strong solutions, we define

$$\begin{aligned} \mathbb{E}_1 &:= L^p(0, T; W^{1,q}(\Omega) \cap L_0^q(\Omega)) \cap W^{1,p}(0, T; (W^{1,q'}(\Omega) \cap L_0^{q'}(\Omega))'), \\ \mathbb{E}_2 &:= L^p(0, T; D(\Delta_N)) \cap W^{1,p}(0, T; L^q(\Omega)), \end{aligned} \quad (2.5)$$

as well as

$$\mathbb{F} := L^p(0, T; X_0) \text{ and } \mathbb{E} := \mathbb{E}_1 \times \mathbb{E}_2. \quad (2.6)$$

Our result on periodic solutions to (1.3) reads as follows.

Theorem 2 *Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain, $T > 0$, $p, q \in (1, \infty)$ such that $3/2 < q \leq 2$ and $3/2q + 1/p < 1$ and let $f = (f_1, f_2) \in L^p(0, T; X_0)$ be a T -periodic function.*

Then there exists $r_0 > 0$ such that for any $r \in (0, r_0)$ there is $\delta = \delta(r) > 0$ such that if $\|f\|_{\mathbb{F}} < \delta$, then there exists a T -periodic solution $w = (u, v)^T \in \mathbb{E}$ to (1.3), which is unique in $\overline{B_{\mathbb{E}}}(0, r)$.

Remark 2 (Nonnegative solutions). We remark that the solutions obtained in Corollary 1 and Theorem 2 are not nonnegative, as $u(t) \in L_0^q(\Omega)$ for all $t \geq 0$. Since u and v represent the density of a cell population and the concentration of a chemoattractant, respectively, nonnegative solutions of Keller–Segel systems are of particular interest. Nonnegative solutions can be obtained from Corollary 1 and Theorem 2 by using the approach from [8, Theorem 2.4 and Corollary 2.5]. Given a nonnegative solution to (1.1) or (1.3) (for $f_1 \equiv 0$) with $M := \int_{\Omega} u_0 \geq 0$, $(U, V) := (u - M, v - M)$ is a solution to a modified Keller–Segel system, namely where the first equation in (1.1) or (1.3) is replaced by $u_t = \Delta u - \nabla \cdot ((u + M)\nabla v)$. The existence proof for the modified system is completely similar to the reasoning in Corollary 1 and Theorem 2, respectively. Then $(u, v) = (U + M, V + M)$ is a nonnegative solution to (1.1) or (1.3), respectively.

3 The Laplacian on bounded and convex domains

For bounded domains $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ it is well known that the Laplacian Δ with domain $D(\Delta) = \{u \in W^{2,q}(\Omega) : \partial_\nu u = 0 \text{ on } \partial\Omega\}$ generates an analytic semigroup of positive contractions on $L^q(\Omega)$ for all $q \in (1, \infty)$ and that it satisfies the maximal L^p -regularity property, see [5].

The results by Wood [24] show that this is no longer the case for arbitrary Lipschitz domains. However, under suitable assumptions on the Lipschitz domain Ω and the exponent q , the above operator still generates a positive, analytic and contractive semigroup on $L^q(\Omega)$.

In the following, we consider bounded, convex domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and define the Neumann–Laplacian Δ_N on $L^q(\Omega)$ for $1 < q < \infty$ by

$$\begin{aligned}\Delta_N u &:= \Delta u, \\ D(\Delta_N) &:= \{u \in W^{2,q}(\Omega) : \partial_\nu u = 0 \text{ on } \partial\Omega\}.\end{aligned}$$

For the particular situation of $n = 3$ and $1 < q \leq 2$ it was proved by Wood [24] that Δ_N generates an analytic semigroup of positive contractions on $L^q(\Omega)$ provided Ω is bounded and convex. He proved in addition that $-\Delta_N$ admits the property of $L^p[0, T]$ -regularity on $L^q(\Omega)$ provided $1 < q \leq 2$ and $0 < T < \infty$.

Since Δ_N generates a positive semigroup on $L^q(\Omega)$, it follows from the results in [6] (see also Section 10.7 of [12]) that $-\Delta_N$ admits a bounded H^∞ -calculus on $L^q(\Omega)$ as well as on $L^q_0(\Omega)$ with $\phi^\infty_{-\Delta_N} \leq \pi/2$ provided Ω is bounded, convex and $n = 3$ and $1 < q \leq 2$. Since the semigroup generated by $\Delta_N - \omega$ is bounded analytic on $L^q(\Omega)$ for suitable $\omega \geq 0$, we conclude by Theorem 10.7.13 of [12] that $\phi^\infty_{-\Delta_N + \omega} < \pi/2$. This reproves in particular the fact that $-\Delta_N$ admits maximal L^p -regularity on $L^q(\Omega)$ provided Ω is bounded, convex and $1 < q \leq 2$.

Since $-\Delta_N$ admits a bounded H^∞ -calculus on $L^q(\Omega)$ it follows that

$D((-\Delta_N)^{1/2}) = W^{1,q}(\Omega) =: Y_1$ and that $(-\Delta_N)^{1/2}$ admits a bounded H^∞ -calculus on $Y_0 := L^q(\Omega)$. Setting $B_0 := (-\Delta_N + 1)^{1/2}$, Theorem V.1.5.1 of [3] implies that (Y_0, B_0) defines an interpolation-extrapolation scale (Y_α, B_α) for $\alpha \in \mathbb{R}$. In particular, the operator $B_{-1} : Y_0 \rightarrow Y_{-1} = W^{-1,q}(\Omega) = (W^{1,q'}(\Omega))'$ defines a linear isomorphism provided Ω is bounded convex and $1 < q \leq 2$.

We next introduce the complex interpolation–extrapolation scale (X_α, A_α) , $\alpha \in \mathbb{R}$, generated by $(X_0, A_0) = (L^q(\Omega), \Delta_N)$. Consider the weak Neumann–Laplacian Δ^w_N on $W^{-1,q}(\Omega)$ defined by

$$\Delta^w_N := A_{-1/2} : W^{1,q}(\Omega) \rightarrow W^{-1,q}(\Omega),$$

which has the explicit representation

$$\langle \Delta^w_N \phi, \psi \rangle = \langle \nabla u, \nabla \psi \rangle_{L^2}$$

for $(u, \phi) \in W^{1,q}(\Omega) \times W^{1,q'}(\Omega)$ with $1/q + 1/q' = 1$. Since

$$\|f(-\Delta^w_N)\|_{\mathcal{L}(W^{-1,q})} = \|B_{-1} f(-\Delta^w_N) (B_{-1})^{-1}\|_{\mathcal{L}(W^{-1,q})} \leq C \|f(-\Delta_N)\|_{\mathcal{L}(L^q)}$$

for $f \in H^\infty(\Sigma_\varphi)$ it follows that $-\Delta^w_N$ has a bounded H^∞ -calculus on $W^{-1,q}(\Omega)$ with the same angle as $-\Delta_N$. Here $H^\infty_0(\Sigma_\varphi)$ and $f(-\Delta^w_N)$ are defined as in [5]. In particular, Δ^w_N admits maximal L^p -regularity on $W^{-1,q}(\Omega)$.

Concerning bounded imaginary powers of Δ_N note that the transference principle of Coifman–Weiss implies that $\|(-\Delta_N)^{it}\|_{L^q} \leq M(1+t^2)e^{\pi|t|/2}$ for $t \in \mathbb{R}$. Since Δ_N is self-adjoint on $L^2(\Omega)$ it follows by interpolation that $-\Delta_N$ has bounded imaginary powers on $L^q(\Omega)$ of angle θ , where $\theta > \pi|1/q - 1/2|$. As before, here $n = 3$ and $q \in (1, 2]$.

We summarize our calculations in the following proposition.

Proposition 1 *Let $p \in (1, \infty)$, $1 < q \leq 2$ and $\Omega \subset \mathbb{R}^3$ be bounded convex domain.*

- (a) The operator $-\Delta_N$ admits a bounded H^∞ -calculus on $L^q(\Omega)$ and there exists $\omega \geq 0$ such that $-\Delta_N + \omega$ admits a bounded H^∞ -calculus on $L^q(\Omega)$ of angle $\phi_{-\Delta_N + \omega}^\infty < \pi/2$.
- (b) The operator $-\Delta_N^w$ admits a bounded H^∞ -calculus on $W^{-1,q}(\Omega)$ and there exists $\omega \geq 0$ such that $-(\Delta_N + \omega)^w$ admits a bounded H^∞ -calculus on $W^{-1,q}(\Omega)$ of angle $\phi_{-(\Delta_N + \omega)^w}^\infty < \pi/2$.
- (c) The operators $-\Delta_N$ and $-\Delta_N^w$ admit bounded imaginary powers on $L^q(\Omega)$ and $W^{-1,q}(\Omega)$, respectively, of angle $\varphi > \pi|1/q - 1/2|$.
- (d) The operators $-\Delta_N^w$ and $-\Delta_N$ admit maximal $L^p([0, T])$ -regularity on $W^{-1,q}(\Omega)$ and $L^q(\Omega)$, respectively.

We also consider the Neumann Laplacian $\Delta_{N,0}$ on $L_0^q(\Omega)$ as well as the weak Neumann Laplacian on $(W^{1,q'}(\Omega) \cap L_0^{q'}(\Omega))'$, where as above $L_0^r(\Omega) = \{u \in L^r(\Omega) : \int_\Omega u = 0\}$ for $1 < r < \infty$ and set

$$\begin{aligned}\Delta_{N,0}^w u &:= \Delta_N^w u, \\ D(\Delta_{N,0}^w) &:= W^{1,q}(\Omega) \cap L_0^q(\Omega).\end{aligned}$$

Corollary 2 Let $p \in (1, \infty)$, $1 < q \leq 2$ and $\Omega \subset \mathbb{R}^3$ be a bounded and convex domain.

- (a) Then there exists $w \geq 0$ such that $-\Delta_{N,0} + \omega$ admits a bounded H^∞ -calculus on $L_0^q(\Omega)$ of angle $\phi_{-\Delta_{N,0} + \omega}^\infty < \pi/2$ and $-(\Delta_{N,0} + \omega)^w$ admits a bounded H^∞ -calculus on $(W^{1,q'}(\Omega) \cap L_0^{q'}(\Omega))'$ of angle $\phi_{-(\Delta_{N,0} + \omega)^w}^\infty < \pi/2$.
- (b) The operator $-\Delta_{N,0}^w$ admits bounded imaginary powers on $(W^{1,q'}(\Omega) \cap L_0^{q'}(\Omega))'$ of angle $\varphi > \pi|1/q - 1/2|$. In particular, the operator $-\Delta_{N,0}^w$ admits maximal $L^p([0, T])$ -regularity on $(W^{1,q'}(\Omega) \cap L_0^{q'}(\Omega))'$.

Remark 3 Consider the operators A and A_0 defined as in (2.1) and (2.3) on the associated spaces X_0 with domains X_1 . Then X_0 is an UMD space and Proposition 1 and Corollary 2 imply that A and A_0 admit a bounded H^∞ -calculus on X_0 . The mixed derivative theorem [19, Cor.4.5.10] implies then for $\beta \in (0, 1)$, $p \in (1, \infty)$ the embedding

$$H^{1,p}(\mathbb{R}; X_0) \cap L^p(\mathbb{R}; X_1) \hookrightarrow H^{1-\beta,q}(\mathbb{R}; X_\beta),$$

where $\Omega \subset \mathbb{R}^3$ is bounded and convex.

4 The initial value problem on convex domains

We start this section by recalling from [21] and [20] (see also Chapter I of [7]) some results on semilinear evolution equations, on which we will base the proof of Theorem 1.

Let X_0, X_1 be Banach spaces such that $X_1 \hookrightarrow X_0$ is densely embedded, and let $A : X_1 \rightarrow X_0$ be bounded. For $0 < T \leq \infty$ consider the semi-linear problem

$$u' + Au = F(u) + f \quad \text{on } (0, T) \quad \text{with} \quad u(0) = u_0. \quad (4.1)$$

For a Banach space X , a time weight $\mu \in (1/p, 1]$, a time interval $J \subset [0, \infty)$ and $k \in \mathbb{N}$, we set

$$\begin{aligned} L_\mu^p(J; X) &= \{u \in L_{loc}^1(J; X) : [t \mapsto t^{1-\mu}u(t)] \in L^p(J; X)\}, \\ H_\mu^{1,p}(J; X) &= \{u \in L_\mu^p(J; X) \cap H_{loc}^{1,1}(J; X) : u' \in L_\mu^p(J; X)\}, \end{aligned} \quad (4.2)$$

Here u' denotes the time derivative of u in the distributional sense. We aim for solutions in the maximal regularity space

$$\mathbb{E}_\mu(J) := H_\mu^{1,p}(J; X_0) \cap L_\mu^p(J; X_1).$$

As space for the initial data u_0 we introduce the real interpolation space

$$u_0 \in X_{\gamma,\mu} = (X_0, X_1)_{\mu-1/p,p} \quad \text{and for } f \in \mathbb{F}_\mu(J) := L_\mu^p(J; X_0),$$

where $p \in (1, \infty)$. We define for $\beta \in [0, 1]$ the space X_β as the complex interpolation space $[X_0, X_1]_\beta$.

The following existence and uniqueness results are based on the following assumptions:

(H1) A has maximal L^p -regularity for $p \in (1, \infty)$.

(H2) $F : X_\beta \rightarrow X_0$ satisfied the estimate

$$\|F(u_1) - F(u_2)\|_{X_0} \leq C(\|u_1\|_{X_\beta} + \|u_2\|_{X_\beta})(\|u_1 - u_2\|_{X_\beta})$$

for some $C > 0$ independent of u_1, u_2 .

(H3) $2\beta - 1 + 1/p \leq \mu$.

(S) X_0 is of class UMD, and the embedding

$$H^{1,p}(\mathbb{R}; X_0) \cap L^p(\mathbb{R}; X_1) \hookrightarrow H^{1-\beta,q}(\mathbb{R}; X_\beta)$$

is valid for each $\beta \in (0, 1)$ and $p \in (1, \infty)$.

For the definition and properties of UMD spaces see e.g. [12].

Proposition 2 [21, Theorem 1.2] Assume that the assumptions (H1), (H2), (H3) and (S) hold and let

$$u_0 \in X_{\gamma,\mu} \quad \text{and} \quad f \in L_\mu^p(0, T; X_0).$$

Then there exists a time $T' = T'(u_0, f)$ with $0 < T' \leq T$ such that problem (4.1) admits a unique solution

$$u \in H_\mu^{1,p}(0, T'; X_0) \cap L_\mu^p(0, T'; X_1).$$

Furthermore, the solution u depends continuously on the data.

Remark 4

- (a) We note that condition (S) holds true whenever X_0 is of class UMD and there is an operator $A_\# \in \mathcal{H}^\infty(X_0)$ with domain $D(A_\#) = X_1$ satisfying $\phi_{A_\#}^\infty < \pi/2$, see Remark 1.1 of [21].
- (b) Due to the embeddings

$$\mathbb{E}_\mu(0, T') \hookrightarrow C([0, T']; X_{\gamma, \mu}) \quad \text{and} \quad \mathbb{E}_\mu(\delta, T') \hookrightarrow C([\delta, T']; X_\gamma), \quad \delta > 0,$$

there is an instantaneous smoothing effect typical for parabolic equations, compare e.g. [19, Section 3.5.2].

When investigating the question of a global solution, we consider

$$t_+(u_0) := \sup\{T' > 0 : \text{equation (4.1) admits a solution } v \in \mathbb{E}_\mu(0, T')\}.$$

By the above Proposition 2, this set is non-empty, and we say that (4.1) has a *global solution* if for $f \in L^q_\mu(0, T; X_0)$, $0 < T < \infty$, which can be extended trivially to $(0, \infty)$, one has $t_+(u_0) = \infty$. Global existence results can be derived from suitable a priori bounds following [19, Theorem 5.7.1].

In the particular case of bilinear nonlinearities, i.e. $F(u) = G(u, u)$ with $G : X_\beta \times X_\beta \rightarrow X_0$ bilinear and bounded one obtains a global solution to (4.1) provided the data are small enough in the $(X_0, X_1)_{1-1/p, p}$ -norm. More precisely, the following corollary to Proposition 2 holds.

Corollary 3 [20, Cor. 2.2] *Let the assumptions of Proposition 2 hold and assume that $f = 0$ and that $F(u) = G(u, u)$, where $G : X_\beta \times X_\beta \rightarrow X_0$ is bilinear and bounded. If $0 \in \varrho(A)$ and $1/p < 1 - \beta$, then the trivial solution to (4.1) is exponentially stable in $(X_0, X_1)_{1-1/p, p}$.*

Let now $\Omega \subset \mathbb{R}^3$ be a bounded and convex domain and suppose that $1 < q \leq 2$. We rewrite the chemotaxis system (1.1) as a semilinear equation for $w = (u, v)^T$ of the form

$$\begin{cases} w'(t) + \mathcal{A}w(t) = F(w(t)), & t \in (0, T), \\ w(0) = w_0 \end{cases} \quad (4.3)$$

in $X_0 := W^{1,q}(\Omega) \times L^q(\Omega)$. Here $\mathcal{A}$ on X_0 is defined by

$$\mathcal{A} := - \begin{pmatrix} \Delta_N^w & 0 \\ 1 & \Delta_N - 1 \end{pmatrix}, \quad (4.4)$$

where Δ_N^w with domain $D(\Delta_N^w) = W^{1,q}(\Omega)$ denotes the Neumann Laplacian on $(W^{1,q}(\Omega))'$ and Δ_N with domain $D(\Delta_N) = W^{2,q}_N(\Omega) = \{u \in W^{2,q}(\Omega) : \partial_\nu u = 0 \text{ on } \partial\Omega\}$ denotes the Neumann Laplacian on $L^q(\Omega)$. Hence $X_1 = D(\Delta_N^w) \times D(\Delta_N)$. Furthermore F is given by

$$F(w) := \begin{pmatrix} -\nabla \cdot (u \nabla v) \\ 0 \end{pmatrix}, \quad (4.5)$$

In order to estimate the nonlinearity F we use Hölder's inequality to obtain

$$\begin{aligned} \|F(w_1) - F(w_2)\|_{X_0} &\leq \| \operatorname{div} (u_1 \nabla v_1) - \operatorname{div} (u_2 \nabla v_2) \|_{X_0} \\ &\leq C \| (u_1 - u_2) \nabla v_1 \|_{L^q} + \| u_2 \nabla (v_1 - v_2) \|_{L^q} \\ &\leq C \| u_1 - u_2 \|_{L^{r,q}} \| v_1 \|_{H^{1,r,q}} + \| u_2 \|_{L^{r',q}} \| v_1 - v_2 \|_{H^{1,r,q}} \end{aligned}$$

with $1/r + 1/r' = 1$ and $r', r > 1$. We note that $X_\beta = X_\beta^1 \times X_\beta^2$, where $X_\beta^1 = H^{2\beta-1,q}(\Omega)$ and $X_\beta^2 = \{u \in H^{2\beta,q}(\Omega) : \partial_\nu u = 0 \text{ on } \Omega\}$ if $\beta \in (1/2 + 1/2q, 1]$ and $X_\beta^2 = H^{2\beta,q}(\Omega)$ if

$\beta \in [0, 1/2 + 1/2q)$. Hence, employing Sobolev's inequalities (see Theorem 4.12 of [1] for non-smooth domains) we obtain

$$X_\beta^1 \hookrightarrow H^{2\beta-1,q}(\Omega) \hookrightarrow L^{r'q}(\Omega) \text{ and } X_\beta^2 \hookrightarrow H^{2\beta,q}(\Omega) \hookrightarrow H^{1,rq}(\Omega)$$

provided $2\beta - 1 - 3/q = -3/qr'$ and $2\beta - 3/q = 1 - 3/qr$. This yields $\beta = (2 + 3/q)/4$. The condition $\beta < 1$ requires thus the condition $q > 3/2$. Condition (H3) requires $2\beta \leq 1 + \mu - 1/p$ and the optimal choice for $\mu = \mu_c$ is $\mu_c = 3/2q + 1/p$. The condition $\mu \leq 1$ requires then the condition $3/2q + 1/p \leq 1$. We then obtain

$$\begin{aligned} \|F(w_1) - F(w_2)\|_{X_0} &\leq C(\|(u_1 - u_2)\|_{X_\beta^1} \|v_1\|_{X_\beta^2} + \|u_2\|_{X_\beta^1} \|(v_1 - v_2)\|_{X_\beta^2}) \\ &\leq C(\|w_1\|_{X_\beta} + \|w_2\|_{X_\beta}) \|w_1 - w_2\|_{X_\beta}. \end{aligned}$$

Summarizing, note that assumptions (H1) and (S) of Proposition 2 are satisfied due to assertions (a), (b) and (d) of Proposition 1 and Remark 3. The above calculations show that also the assumptions (H2) and (H3) are satisfied. We thus conclude the assertion of Theorem 1 by Proposition 2.

The assertion of Corollary 1 follows from Corollary 3.

5 Strong time periodic solutions

We start by recalling from [4] that time periodic solutions to linear Cauchy problems within the maximal L^p -regularity class are well understood. Indeed, let X_0 be a Banach space and $A : X_1 \rightarrow X_0$ be a linear operator. For $p \in (1, \infty)$ and setting

$$\mathbb{F} := L^p(0, 2\pi; X_0), \quad \mathbb{E} := H_p^1(0, 2\pi; X_0) \cap L^p(0, 2\pi; X_1), \quad (5.1)$$

we say that A admits *maximal periodic L^p -regularity* if for each $f \in \mathbb{F}$ there is a unique solution $u \in \mathbb{E}$ to

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in (0, 2\pi), \\ u(0) = u(2\pi). \end{cases}$$

In this case, by the closed graph theorem, there exists a constant $M > 0$ satisfying $\|u\|_{\mathbb{E}} \leq M\|f\|_{\mathbb{F}}$. Maximal periodic L^p -regularity has been characterized by Arendt and Bu [4] in terms of the $\mathcal{R}$ -boundedness of the resolvent of A on a suitable subset of $\mathbb{C}$. Their result reads as follows.

Proposition 3 [4, Theorem 2.3] *Let $1 < p < \infty$, X be a UMD-space and A be a closed operator in X . Then the following assertions are equivalent.*

- a) *A admits maximal periodic L^p -regularity.*
- b) *$i\mathbb{Z} \subset \varrho(-A)$ and $(k(ik + A)^{-1})_{k \in \mathbb{Z}}$ is $\mathcal{R}$ -bounded.*

For definitions of UMD-spaces and $\mathcal{R}$ -bounded families of operators as well as their properties, we refer to [5, 12, 19]. The relationship between maximal periodic L^p -regularity and maximal L^p -regularity for the Cauchy problem can be described, following again the results of Arendt and Bu [4], as follows.

Proposition 4 [4, Thm. 5.1] *Let X be a Banach space and $-A : D(A) \rightarrow X$ be the generator of a C_0 -semigroup on X . Then A admits maximal periodic L^p -regularity if and only if $1 \in \rho(e^{-2\pi A})$ and A admits maximal L^p -regularity, i.e., if for each $f \in \mathbb{F}$ there is a unique solution $u \in \mathbb{E}$ to*

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in (0, 2\pi), \\ u(0) = 0. \end{cases}$$

An extension of the above result to the quasilinear and semilinear situation was given in [8]. To formulate this result, let X_0, X_1 be Banach spaces with X_1 densely embedded in X_0 and for $1 < p < \infty$ set $X_\gamma = (X_0, X_1)_{1-1/p, p}$. We introduce the following Lipschitz condition on the nonlinearity F .

(L) *Let $F : [0, 2\pi] \times X_\gamma \rightarrow X_0$ satisfy $F(\cdot, v(\cdot)) \in \mathbb{F}$ for all $v \in \mathbb{E}$ and suppose that for each $R > 0$ there exists $C(R) > 0$ such that*

$$\|F(\cdot, v(\cdot)) - F(\cdot, w(\cdot))\|_{\mathbb{F}} \leq C(R)\|v - w\|_{\mathbb{E}}$$

for all $v, w \in \overline{B_{\mathbb{E}}}(0, R)$.

Proposition 5 [8, Cor. 3.5] *Let X_0, X_1 be Banach spaces with X_1 densely embedded in X_0 and let $A : X_1 \rightarrow X_0$ be a closed linear operator satisfying maximal periodic L^p -regularity for $p \in (1, \infty)$. Assume furthermore that (L) is satisfied. There is $\delta_1 > 0$ such that, if $C(R) < \delta_1$ for some $R > 0$, then there are $\delta_2 > 0$ and $r > 0$ such that if $\|F(\cdot, 0)\|_{\mathbb{F}} < \delta_2$ there is a unique solution $u \in \overline{B_{\mathbb{E}}}(0, r)$ to*

$$\begin{cases} u'(t) + Au(t) = F(t, u(t)), & t \in (0, 2\pi), \\ u(0) = u(2\pi). \end{cases} \quad (5.2)$$

To prove Theorem 2 we note first that $-A_0$ on X_0 defined as in (2.3) admits maximal periodic L^p -regularity on X_0 . In fact, we have $0 \in \rho(A_{N,0}^w)$ and $0 \in \rho(A_N - 1)$ and hence $0 \in \rho(-A_0)$. Due to Corollary 2, $-A_{N,0}^w$ on $(W^{1,q'}(\Omega) \cap L_0^{q'}(\Omega))'$ as well as $-(A_N - 1)$ on $L^q(\Omega)$ admit maximal L^p -regularity. Hence, the triangular structure of A_0 implies that $-A_0$ admits maximal periodic L^p -regularity on X_0 .

Next, we show that $H(\cdot, w(\cdot)) \in \mathbb{F}$ is satisfied for $w \in \mathbb{E}$. By Hölder's inequality

$$\begin{aligned} \|H(\cdot, w(\cdot))\|_{\mathbb{F}} &\leq c\|u \nabla v\|_{L^p(0,T;L^q(\Omega))} + \|f\|_{\mathbb{F}} \\ &\leq c\|u\|_{L^{2p}(0,T;L^{2q}(\Omega))}\|v\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} + \|f\|_{\mathbb{F}}. \end{aligned} \quad (5.3)$$

In view of the assumption $\frac{3}{2} < q \leq 2$ and $\frac{1}{p} + \frac{3}{2q} < 1$ we may choose $\theta \in (0, 1)$ satisfying $\frac{1}{2p} < \theta < \frac{1}{2} - \frac{3}{4q}$. The mixed derivate theorem and Sobolev estimates for convex domains (see e.g. [1] Thm. 4.12) yield

$$\begin{aligned} \|H(\cdot, w(\cdot))\|_{\mathbb{F}} &\leq c \|u\|_{H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega))} \|v\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} + \|f\|_{\mathbb{F}} \\ &\leq c \|w\|_{\mathbb{E}}^2 + \|f\|_{\mathbb{F}}. \end{aligned}$$

Furthermore, since $(W^{1,q'}(\Omega))' \subset (W^{1,q'}(\Omega) \cap L_0^{q'}(\Omega))'$, we see that the first component of H belongs to $L^p(0, T; (W^{1,q'}(\Omega) \cap L_0^{q'}(\Omega))')$. Hence, $H(\cdot, w(\cdot)) \in \mathbb{F}$. In order to verify condition (L), let $w, \tilde{w} \in \overline{B}_{\mathbb{E}}(0, R)$. By Hölder's inequality, the mixed derivative theorem and Sobolev's inequality we obtain

$$\begin{aligned} \|H(\cdot, w(\cdot)) - H(\cdot, \tilde{w}(\cdot))\|_{\mathbb{F}} &\leq c (\|u - \tilde{u}\|_{L^{2p}(0,T;L^{2q}(\Omega))} \|v\|_{L^{2p}(0,T;W^{1,2q}(\Omega))} \\ &\quad + \|\tilde{u}\|_{L^{2p}(0,T;L^{2q}(\Omega))} \|v - \tilde{v}\|_{L^{2p}(0,T;W^{1,2q}(\Omega))}) \\ &\leq c (\|u - \tilde{u}\|_{H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega))} \|v\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))} \\ &\quad + \|\tilde{u}\|_{H^{\theta,p}(0,T;H^{-1+2(1-\theta),q}(\Omega))} \|v - \tilde{v}\|_{H^{\theta,p}(0,T;H^{2(1-\theta),q}(\Omega))}) \\ &\leq c (\|u - \tilde{u}\|_{\mathbb{E}_1} \|v\|_{\mathbb{E}_2} + \|\tilde{u}\|_{\mathbb{E}_1} \|v - \tilde{v}\|_{\mathbb{E}_2}) \\ &\leq c (\|w\|_{\mathbb{E}} + \|\tilde{w}\|_{\mathbb{E}}) \|w - \tilde{w}\|_{\mathbb{E}} \\ &\leq cR \|w - \tilde{w}\|_{\mathbb{E}}. \end{aligned}$$

The proof of Theorem 2 is complete.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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References

- Adams, R.A., Fournier, J.J.F.: Sobolev Spaces. Academic Press, Amsterdam (2003)
- Amann, H.: Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In: H.J. Schmeisser, H. Triebel (eds.), Function Spaces, Differential Operators and Nonlinear Analysis, Teubner, pp. 9–126 (1993)
- Amann, H.: Linear and Quasilinear Parabolic Problems, Monographs in Mathematics, Vol. 89, Birkhäuser (1995)
- Arendt, W., Bu, S.: The operator-valued Marcinkiewicz multiplier theorem and maximal regularity. Math. Z. **240**, 311–343 (2002)
- Denk, R., Hieber, M., Prüss, J.: $\mathcal{R}$ -Boundedness, Fourier multipliers and problems of elliptic and parabolic type. Mem Am. Math. Soc. **788** (2003)
- Hieber, M., Prüss, J.: Functional calculi for linear operators in vector-valued L^p -spaces via the transference principle. Adv. Differ. Equ. **3**, 847–872 (1998)

7. Hieber, M., Robinson, J., Shibata, Y.: Mathematical Analysis of the Navier–Stokes Equations. Lecture Notes in Mathematics, vol. 2254. CIME Found. Subseries, Springer, Cham (2020)
8. Hieber, M., Stinner, C.: Strong time periodic solutions to Keller–Segel systems: an approach by the quasilinear Arendt–Bu theorem. *J. Differ. Equ.* **269**, 1636–1655 (2020)
9. Hillen, T., Painter, K.J.: A user’s guide to PDE models for chemotaxis. *J. Math. Biol.* **58**, 183–217 (2009)
10. Horstmann, D.: From 1970 until present: the Keller–Segel model in chemotaxis and its consequences I. *Jahresber. Deutsch. Math.-Verein.* **105**(2003), 103–165
11. Horstmann, D., Meinlschmidt, H., Rehberg, J.: The full Keller–Segel model is well-posed on nonsmooth domains. *Nonlinearity* **31**, 1560–1592 (2018)
12. Hytönen, T., van Neerven, J., Veraar, M., Weis, L.: Analysis in Banach Spaces, Vol. II. Springer, New York (2017)
13. Kozono, H., Sugiyama, Y.: Global strong solution to the semi-linear Keller–Segel system of parabolic–parabolic type with small data in scale invariant spaces. *J. Differ. Equ.* **247**, 1–32 (2009)
14. Kozono, H., Sugiyama, Y.: Strong solutions to the Keller–Segel system with the weak $L^{n/2}$ initial data and its application to the blow-up rate. *Math. Nachr.* **283**, 732–751 (2010)
15. Kozono, H., Sugiyama, Y., Wachi, T.: Existence and uniqueness theorem on mild solutions to the Keller–Segel system in the scaling invariant space. *J. Differ. Equ.* **252**, 1213–1228 (2012)
16. Langkeit, J., Winkler, M.: Facing low regularity in chemotaxis systems. *Jahresber. Deutsch. Math.-Verein.* **122**, 35–64 (2020)
17. Nakaguchi, E., Osaki, K.: L_p -estimates of solutions to n -dimensional parabolic–parabolic system for chemotaxis with subquadratic degradation. *Funkcial. Ekvac.* **59**, 51–66 (2016)
18. Nakaguchi, E., Osaki, K.: Global existence of solutions to an n -dimensional parabolic–parabolic system for chemotaxis with logistic-type growth and superlinear production. *Osaka J. Math.* **55**, 51–70 (2018)
19. Prüss, J., Simonett, G.: Moving Interfaces and Quasilinear Parabolic Evolution Equations, Monographs in Mathematics, Vol. 105, Birkhäuser (2016)
20. Prüss, J., Simonett, G., Wilke, M.: Critical spaces for quasilinear parabolic evolution equations and applications. *J. Differ. Equ.* **264**, 2028–2074 (2018)
21. Prüss, J., Wilke, M.: Addendum to the paper “On quasilinear parabolic evolution equations in weighted L_p -spaces II”. *J. Evol. Equ.* **17**, 1381–1388 (2017)
22. Tao, Y.: Global dynamics in a higher dimensional repulsion chemotaxis model with nonlinear sensitivity. *Discrete Contin. Dyn. Syst. Ser. B* **18**, 2705–2722 (2013)
23. Tao, Y., Winkler, M.: Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with subcritical sensitivity. *J. Differ. Equ.* **252**, 692–715 (2012)
24. Wood, I.: Maximal L^p -regularity for the Laplacian on Lipschitz domains. *Math. Z.* **255**, 855–875 (2007)