Regularity of the m -symphonic map

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Abstract

We introduce a new energy functional of conformal invariance and consider its critical points, named the m -symphonic map. We study a Hölder continuity of m -symphonic maps from domains of $\mathbb{R}^m$ into the spheres in the higher dimension $m \geq 4$.

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1 Introduction

Symphonic maps are critical points of the energy functional, called symphonic energy,

$$E^{\text{sym}}(u) = \int_M \|u^*h\|^2 d\omega_g \quad (1.1)$$

for any smooth map u from a compact Riemannian manifold (M, g) into another compact Riemannian manifold (N, h) , where u^*h denotes the pullback of the metric h by u ,

$$(u^*h)(X, Y) = h(du(X), du(Y))$$

for any vector fields X, Y on M , and $d\omega_g$ is the volume form on (M, g) . The squared integral of the *trace* of the pullback u^*h

$$E(u) = \int_M (\text{tr}_g(u^*h))^2 d\omega_g, \quad \text{tr}_g(u^*h) = \sum_i h(du(e_i), du(e_i))$$

is the well-known 4-energy. The difference between the pullback and the trace

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$$T_u(X, Y) := h(du(X), du(Y)) - \frac{1}{m} \sum_i h(du(e_i), du(e_i))g(X, Y)$$

is a measurement of the conformality of a map from (M, g) to (N, h) , since $T_u = 0$, if a map u is conformal, where m is the dimension of the source manifold M . The maps u satisfying $T_u = 0$ is called weakly conformal, that is, $du = 0$ or u is conformal. The squared integral of T_u

$$\Psi(u) := \int_M \|T_u\|^2 d\omega_g, \quad \|T_u\|^2 = \|u^*h\|^2 - \frac{1}{m} \|du\|^4,$$

is introduced to look for maps closest to conformal ones. The first and second variation formulas, monotonicity type formula and Bochner type formula are derived and critical points of the functional are geometrically studied, relating to the conformality and homotopy of maps [15, 20, 21]. T_u is also equal to the stress energy tensor in the two dimension case $m = 2$ (refer to [3, 4, p. 392]). Both of the 4-energy and the symphonic energy (1.1) are conformally invariant in the four dimensional domain. While the symphonic energy density is the *norm* of the pullback u^*h , the 4-energy density is the *trace* of the pullback u^*h . Thus, the symphonic energy may have more geometric information than that of the 4-energy. That is our motivation of studying the symphonic maps.

The higher dimensional $m \geq 4$ analogue to (1.1) is the following functional on the m -dimensional domain, called m -symphonic energy,

$$E_m^{\text{sym}}(u) = \int_M \|u^*h\|^{\frac{m}{2}} d\omega_g \quad (1.2)$$

which is indeed invariant under any conformal changes of metrics of the m -dimensional domain. Critical points of m -symphonic energy is called m -symphonic maps. The m -symphonic maps naturally concern the usual m -harmonic maps, as explained above, and also the higher dimensional hypersurface of prescribed mean curvature, that reflects the regularity proof by use of the determinant structure of nonlinear term (see [14, 23, 26]). In this paper we prove the following regularity for m -symphonic maps, $m \geq 4$, that extends the previous result for symphonic maps in [18] to the higher dimension $m \geq 4$:

Main Theorem (Theorem 5.1) Let $m \geq 4$ and Ω be a domain of the m -dimensional Euclidean space $\mathbb{R}^m$. Then any m -symphonic map from Ω into the standard sphere is Hölder continuous.

The main theorem follows from a similar argument as in [18], of that the key ingredient is the determinant structure of the nonlinear term coming from the target constraint. The compensatory regularity of the determinant is nowadays well known and is applied to geometric equations such as harmonic maps, originally, two or higher dimensional surface of prescribed mean curvature. See [5, 12] for harmonic maps, [25, 27] for p -harmonic maps, [11] for polyharmonic maps, [22] for a new Sobolev type inequality and [16] for the H-system heat flow. However, here an algebraic property specific to the p -symphonic operator is emphasized, that is the following monotonicity inequality (Proposition 3.1):

$$\begin{aligned} & \sum_{i,j=1}^m \left(\sigma(A)^{(p-4)/4} (A_i \cdot A_j) A_j - \sigma(B)^{(p-4)/4} (B_i \cdot B_j) B_j \right) \cdot (A_i - B_j) \\ & \geq \frac{1}{4} \left(\sigma(A)^{(p-4)/4} + \sigma(B)^{(p-4)/4} \right) \sum_{i,j=1}^m |(A_i \cdot A_j) - (B_i \cdot B_j)|^2 \end{aligned}$$

for any vectors $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$, where $\sigma(\cdot)$ denotes the symphonic energy density form:

$$\sigma(A) = \sum_{i,j=1}^m (A_i \cdot A_j)^2.$$

The coercive inequality above for the p -symphonic operator is clearly different from that of the p -harmonic operator. The m -symphonic energy (1.2) is an $m/2$ powered integral of the *norm* of the pullback u^*h , on the other hand, the well-known m -harmonic maps are critical points of the m -energy, an $m/2$ powered integral of the *trace* of the pullback u^*h ,

$$E(u) = \int_M (\text{tr}_g(u^*h))^{\frac{m}{2}} d\omega_g.$$

The coerciveness of the p -symphonic operator seems to stem from the norm of pullback metric. Refer to the proof of Proposition 3.1. The proof of regularity as in the main theorem follows from the local energy growth estimate, Theorem 5.2 below. The m -symphonic maps on the m -dimensional domain is on the energy critical case in the sense that the local m -energy on a m -dimensional ball is invariant under the usual scaling transformation (see Theorem 5.2, 5.1, 5.6) in Sect. 5). Thus, the regularity criterion may naturally be given by the local energy, similarly as the m -harmonic maps on the m -dimensional domain. Such regularity argument as above have already been performed in the case that $m = 4$ by [17, 18]. Thus, the process and ingredients of the proof in the higher dimension $m \geq 4$ here are similar to that of [18]. Based on the usual rescaling argument, the classical regularity tools well-known nowadays are employed, such as Hardy and BMO estimates with the duality between the Hardy space and BMO space (see Sect. 4). Such regularity methods have been applied for harmonic and p -harmonic maps (refer to [5, 12, 25, 27]). Here, we shall show how to make use of the algebraic structure of the principal part in the p -symphonic maps, presented above, in the regularity proof.

2 Preliminaries

We gather the notations, used in the followings.

The pullback u^*h on the standard metric $h = (h_{\alpha\beta}(u)) = (\delta_{\alpha\beta})$ of $\mathbb{R}^n$ is

$$(u^*h)_{ij}(x) := \sum_{\alpha, \beta=1}^n h_{\alpha\beta}(u(x)) D_i u^\alpha(x) D_j u^\beta(x) = \sum_{\alpha=1}^n D_i u^\alpha(x) D_j u^\alpha(x)$$

and

$$(u^*h)^{ij}(x) := \sum_{\alpha, \beta=1}^n h_{\alpha\beta}(u(x)) D^i u^\alpha(x) D^j u^\beta(x) = \sum_{\alpha=1}^n D^i u^\alpha(x) D^j u^\alpha(x),$$

where $g = (g_{ij}) = (\delta_{ij})$ denotes the standard metric on $\mathbb{R}^m$, $g^{-1} = (g^{ij})$ and thus,

$$D^i u(x) := \sum_{j=1}^m g^{ij} D_j u(x) = D_i u(x).$$

Let Ω be a domain in $\mathbb{R}^m$, $m \geq 1$, and $p > 1$. The Sobolev space of $\mathbb{R}^n$ -valued L^p -functions on Ω is denoted by $L^{1,p}(\Omega, \mathbb{R}^n)$, $n \geq 1$,

$$L^{1,p}(\Omega, \mathbb{R}^n) = \left\{ u(x) = (u^1(x), \dots, u^n(x)) \mid \int_{\Omega} (|u|^p + |Du|^p) dx < \infty \right\},$$

where dx is the Lebesgue measure, and $D_i u^\alpha$, $i = 1, \dots, m$; $\alpha = 1, \dots, n$, are weak derivatives and $Du(x) = (D_i u^\alpha(x))$ is the gradient. $L_0^{1,p}(\Omega, \mathbb{R}^n)$ is the completion on $L^{1,p}$ -norm of $\mathbb{R}^n$ -valued smooth maps with compact support on Ω . The $\mathbb{S}^{n-1}$ -valued Sobolev maps on Ω , $L^{1,p}(\Omega, \mathbb{S}^{n-1})$, is defined as be

$$L^{1,p}(\Omega, \mathbb{S}^{n-1}) = \{ u \in L^{1,p}(\Omega, \mathbb{R}^n) \mid u(x) \in \mathbb{S}^{n-1} \text{ a.e. } x \in \Omega \}.$$

The p -symphonic energy is introduced as follows:

Definition 2.1 (p -symphonic energy) **Let $p > 1$. The p -symphonic energy is defined as**

$$E_p^{\text{sym}}(u) \stackrel{\text{def}}{=} \int_{\Omega} |u^* h|^{p/2} dx,$$

for $u \in L^{1,p}(\Omega, \mathbb{R}^n)$, where

$$|u^* h|^2 = \sum_{i,j=1}^m (u^* h)_{ij} (u^* h)^{ij} = \sum_{i,j=1}^m (D_i u \cdot D_j u)^2$$

is the symphonic energy density.

Remark 2.1 The p -symphonic energy density is of p -th powered gradients. In fact, there exists a positive constant $C > 1$ such that

$$\frac{1}{C} |Du|^p \leq |u^* h|^{p/2} \leq C |Du|^p. \quad (2.1)$$

A weak solution of p -symphonic maps from Ω into $\mathbb{S}^{n-1}$ is given as follows:

Definition 2.2 Let $p > 1$. A weakly p -symphonic map u is defined as a weak solution $u \in L^{1,p}(\Omega, \mathbb{S}^{n-1})$ of the Euler-Lagrange equation for the p -symphonic energy $E_p^{\text{sym}}(u)$

$$-\sum_{i,j=1}^m D_i \left((|u^* h|^2)^{(p-4)/4} (u^* h)^{ij} D_j u \right) = |u^* h|^{p/2} u, \quad (2.2)$$

that is, for any $\phi \in L_0^{1,p} \cap L^\infty(\Omega, \mathbb{R}^n)$,

$$\int_{\Omega} \left\{ \sum_{i,j=1}^m (|u^* h|^2)^{(p-4)/4} (u^* h)^{ij} D_j u \cdot D_i \phi - \phi \cdot u |u^* h|^{p/2} \right\} dx = 0. \quad (2.3)$$

Now we consider a weak solution of

$$\sum_{i,j=1}^m D_i \left((|v^* h|^2)^{(p-4)/4} (v^* h)^{ij} D_j v \right) = 0. \quad (2.4)$$

The operator in left hand side of (2.4) is called as p -symphonic operator. We shall present the Hölder regularity of a weak solution of (2.4), for later use.

Lemma 2.1 *Let $v \in L^{1,m}(B(1), \mathbb{R}^n)$ be a weak solution to (2.4). Then there exist positive constants $\alpha < 1$ and C depending only on m such that*

$$\int_{B(r)} |Dv|^m dx \leq C r^{m\alpha} \quad (2.5)$$

for any positive number $r < 1$ and, in particular, v is Hölder continuous in $B(1/2)$ with an exponent α .

Proof We argue as in the proof of Theorem 1 in [17] (also refer to [10, 19]). The growth estimate of local energy in (2.5) is shown to hold for a local minimizer. If the target manifold $(\mathbb{R}^n, h)$ has the Euclidean metric $h = (\delta_{\alpha\beta})$, then the Euler-Lagrange equation is (5.14) and a weak solution of (5.14) is a local minimizer. Putting $F(P) = \sigma(P)^{m/4} = \left\{ \sum_{i,j=1}^m (P_i \cdot P_j)^2 \right\}^{m/4}$ for any $P = (P_i^\alpha)$, its second derivative in componentwise is

$$\begin{aligned} \frac{\partial^2 F}{\partial P_j^\beta \partial P_i^\alpha} &= m \sigma(P)^{(m-4)/4} \left(P_i^\beta P_j^\alpha + \delta_{ij} P^\beta \cdot P^\alpha + \delta^{\alpha\beta} P_i \cdot P_j \right) \\ &\quad + \frac{m(m-4)}{4} \sigma(P)^{(m-8)/4} \sum_{k,\ell=1}^m (P_i \cdot P_\ell) (P_k \cdot P_j) P_k^\beta P_\ell^\alpha \end{aligned}$$

and thus, its quadratic form is

$$\begin{aligned} &\sum_{i,j=1}^m \sum_{\alpha,\beta=1}^n \frac{\partial^2 F}{\partial P_j^\beta \partial P_i^\alpha} \zeta_i^\alpha \zeta_j^\beta \\ &= m \sigma(P)^{(m-4)/4} \left\{ \sum_{i,j=1}^m \left((\zeta_i \cdot P_j)^2 + (\zeta_i \cdot P_j) (\zeta_j \cdot P_i) \right) + \sum_{\alpha,\beta=1}^n (\zeta^\alpha \cdot P^\beta)^2 \right\} \\ &\quad + \frac{m(m-4)}{4} \sigma(P)^{(m-8)/4} \left\{ \sum_{i,j=1}^m (\zeta_i \cdot P_j) (P_i \cdot P_j) \right\}^2 \\ &\geq m \sigma(P)^{(m-4)/4} \sum_{\alpha,\beta=1}^n (\zeta^\alpha \cdot P^\beta)^2, \end{aligned}$$

where the Cauchy inequality is used for the second term of the right hand side in the first equality. So, the local minimality of a weak solution follows from the Taylor expansion. The Hölder regularity follows from Morrey's Dirichlet growth theorem with (2.5), as usual (see [9, 10, 19]). $\square$

3 A monotonicity inequality for p -symphonic operator

Here, an algebraic inequality associated with the p -symphonic maps is presented, that will be crucially used in the regularity proof later.

Proposition 3.1 *For any vectors $A = (A_i)$, $B = (B_i)$ with $A_i, B_i \in \mathbb{R}^m$, $i = 1, \dots, m$, let $\sigma(\cdot)$ denote the symphonic energy density form :*

$$\sigma(A) \stackrel{\text{def}}{=} \sum_{i,j=1}^m (A_i \cdot A_j)^2. \quad (3.1)$$

Suppose that $p \geq 4$. Then, it hold true that

$$\begin{aligned} & \sum_{i,j=1}^m \left(\sigma(A)^{(p-4)/4} (A_i \cdot A_j) A_j - \sigma(B)^{(p-4)/4} (B_i \cdot B_j) B_j \right) \cdot (A_i - B_i) \\ & \geq \frac{1}{4} \left(\sigma(A)^{(p-4)/4} + \sigma(B)^{(p-4)/4} \right) \sum_{i,j=1}^m |(A_i \cdot A_j) - (B_i \cdot B_j)|^2 \end{aligned} \quad (3.2)$$

and that

$$\begin{aligned} & \sum_{i,j=1}^m \left(\sigma(A)^{(p-4)/4} (A_i \cdot A_j) - \sigma(B)^{(p-4)/4} (B_i \cdot B_j) \right)^2 \\ & \leq \left(\frac{p}{2} - 1 \right)^2 \left(\sigma(A)^{(p-4)/4} + \sigma(B)^{(p-4)/4} \right)^2 \sum_{i,j=1}^m |(A_i \cdot A_j) - (B_i \cdot B_j)|^2. \end{aligned} \quad (3.3)$$

Proof Put the notation

$$a_{ij} = A_i \cdot A_j, \quad b_{ij} = B_i \cdot B_j.$$

Noting that a_{ij} and b_{ij} are positive semi-definite, we have

$$\begin{aligned} \sum_{i,j=1}^m (A_i \cdot A_j) (B_i \cdot A_j) &= \sum_{i,j=1}^m a_{ij} (B_i \cdot A_j) \\ &\leq \frac{1}{2} \sum_{i,j=1}^m \{ a_{ij} (B_i \cdot B_j) + a_{ij} (A_i \cdot A_j) \} \\ &= \frac{1}{2} \sum_{i,j=1}^m \{ (A_i \cdot A_j) \cdot (B_i \cdot B_j) + (A_i \cdot A_j)^2 \} \end{aligned} \quad (3.4)$$

Replacing A by B we get

$$\sum_{i,j=1}^m (B_i \cdot B_j) (A_i \cdot B_j) \leq \frac{1}{2} \sum_{i,j=1}^m \left\{ (B_i \cdot B_j) \cdot (A_i \cdot A_j) + (B_i \cdot B_j)^2 \right\}. \quad (3.5)$$

A direct computation yields

$$\begin{aligned} & \sum_{i,j=1}^m \left(\sigma(A)^{(p-4)/4} (A_i \cdot A_j) A_j - \sigma(B)^{(p-4)/4} (B_i \cdot B_j) B_j \right) \cdot (A_i - B_i) \\ &= \sigma(A)^{(p-4)/4} \sum_{i,j=1}^m (A_i \cdot A_j)^2 - \sigma(A)^{(p-4)/4} \sum_{i,j=1}^m (A_i \cdot A_j) (B_i \cdot A_j) \\ & \quad - \sigma(B)^{(p-4)/4} \sum_{i,j=1}^m (B_i \cdot B_j) (A_i \cdot B_j) + \sigma(B)^{(p-4)/4} \sum_{i,j=1}^m (B_i \cdot B_j)^2, \end{aligned}$$

which is furthermore estimated below by (3.4, 3.5)

$$\begin{aligned} & \sigma(A)^{(p-4)/4} \left\{ \sum_{i,j=1}^m a_{ij}^2 - \frac{1}{2} \left(\sum_{i,j=1}^m a_{ij} b_{ij} + \sum_{i,j=1}^m a_{ij}^2 \right) \right\} \\ & \quad + \sigma(B)^{(p-4)/4} \left\{ -\frac{1}{2} \left(\sum_{i,j=1}^m b_{ij} a_{ij} + \sum_{i,j=1}^m b_{ij}^2 \right) + \sum_{i,j=1}^m b_{ij}^2 \right\} \\ &= \frac{1}{2} \sigma(A)^{(p-4)/4} \sum_{i,j=1}^m a_{ij}^2 \\ & \quad - \frac{1}{2} \left(\sigma(A)^{(p-4)/4} + \sigma(B)^{(p-4)/4} \right) \sum_{i,j=1}^m a_{ij} b_{ij} \\ & \quad + \frac{1}{2} \sigma(B)^{(p-4)/4} \sum_{i,j=1}^m b_{ij}^2 \end{aligned}$$

The above quantities are arranged and computed as

$$\begin{aligned} & \frac{1}{4} \left(\sigma(A)^{(p-4)/4} + \sigma(B)^{(p-4)/4} \right) \\ & \quad \times \left(\sum_{i,j=1}^m a_{ij}^2 - 2 \sum_{i,j=1}^m a_{ij} b_{ij} + \sum_{i,j=1}^m b_{ij}^2 \right) \\ & \quad + \frac{1}{4} \left(\sigma(A)^{(p-4)/4} - \sigma(B)^{(p-4)/4} \right) \left(\sum_{i,j=1}^m a_{ij}^2 - \sum_{i,j=1}^m b_{ij}^2 \right) \\ &= \frac{1}{4} \left(\sigma(A)^{(p-4)/4} + \sigma(B)^{(p-4)/4} \right) \sum_{i,j=1}^m |a_{ij} - b_{ij}|^2 \\ & \quad + \frac{1}{4} \left(\sigma(A)^{(p-4)/4} - \sigma(B)^{(p-4)/4} \right) (\sigma(A) - \sigma(B)) \\ & \geq \frac{1}{4} \left(\sigma(A)^{(p-4)/4} + \sigma(B)^{(p-4)/4} \right) \sum_{i,j=1}^m |(A_i \cdot A_j) - (B_i \cdot B_j)|^2. \end{aligned}$$

The last inequality follows from the non-negativity of the second term in the previous

quantity, since $p \geq 4$. The inequality (3.2) is obtained.

For the proof of (3.3), put the notation

$$V_{ij} = \sum_{i,j=1}^m \sigma(A)^{(p-4)/4} a_{ij}, \quad W_{ij} = \sum_{i,j=1}^m \sigma(B)^{(p-4)/4} b_{ij},$$

$$X(t)_{ij} = ta_{ij} + (1-t)b_{ij}, \quad 0 \leq t \leq 1; \quad |X(t)|^2 = \sum_{i=1}^m (ta_{ij} + (1-t)b_{ij})^2.$$

By direct computation we have

$$V_{ij} - W_{ij} = \left(\frac{p}{2} - 2\right) \int_0^1 |X(t)|^{\frac{p}{2}-4} X(t)_{ij} \sum_{k,l=1}^m X(t)_{kl} (a_{kl} - b_{kl}) dt$$

$$+ \int_0^1 |X(t)|^{\frac{p}{2}-2} dt (a_{ij} - b_{ij})$$

and thus,

$$\sum_{i,j=1}^m |V_{ij} - W_{ij}|^2 = \left(\frac{p}{2} - 2\right)^2 \sum_{i,j=1}^m \left(\int_0^1 |X(t)|^{\frac{p}{2}-4} X(t)_{ij} \sum_{k,l=1}^m X(t)_{kl} (a_{kl} - b_{kl}) dt \right)^2$$

$$+ (p-4) \int_0^1 |X(t)|^{\frac{p}{2}-2} dt \int_0^1 |X(t)|^{\frac{p}{2}-4} \left(\sum_{k,l=1}^m X(t)_{kl} (a_{kl} - b_{kl}) \right)^2 dt$$

$$+ \left(\int_0^1 |X(t)|^{\frac{p}{2}-2} dt \right)^2 \sum_{i,j=1}^m (a_{ij} - b_{ij})^2$$

$$\leq \left(\frac{p}{2} - 1\right)^2 \left(\int_0^1 |X(t)|^{\frac{p}{2}-2} dt \right)^2 \sum_{i,j=1}^m (a_{ij} - b_{ij})^2,$$

where, in the last inequality, we use the Schwarz inequality in $\mathbb{R}^{m^2}$ to estimate

$$\left| \sum_{k,l=1}^m X(t)_{kl} (a_{kl} - b_{kl}) \right|^2 \leq |X(t)|^2 \sum_{k,l=1}^m (a_{kl} - b_{kl})^2;$$

$$\sum_{i,j=1}^m \left(\int_0^1 |X(t)|^{\frac{p}{2}-4} \sum_{k,l=1}^m X(t)_{kl} (a_{kl} - b_{kl}) dt \right)^2$$

$$\leq \left(\int_0^1 |X(t)|^{\frac{p}{2}-2} dt \right)^2 \sum_{i,j=1}^m (a_{ij} - b_{ij})^2.$$

The proof of (3.3) is complete. $\square$

4 Function spaces

The Hardy space and the space of functions of bounded mean oscillation are crucially used in the proof of the main theorem. Here let us recall the definitions and properties of these function spaces (See more details in [7, 24, 28]).

Definition 4.1 (Hardy space) Let u be in $L^1(\mathbb{R}^m)$. We say that u belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^m)$ if the grand maximal function of u

$$\|u\|_{\mathcal{H}^1(\mathbb{R}^m)} = \int_{\mathbb{R}^m} \sup_{r>0} |\phi_r * u(x)| dx$$

is finite in $L^1(\mathbb{R}^m)$, where $\phi_r(x) = r^{-m}\phi(r^{-1}x)$ for $r > 0$ and ϕ is a smooth function on $\mathbb{R}^m$ with compact support in $B(0, 1) = \{x \in \mathbb{R}^m; |x| < 1\}$. The definition does not depend on choice of a function ϕ .

Definition 4.2 (Functions of bounded mean oscillation) Let u belong to $L^1_{\text{loc}}(\mathbb{R}^m)$. u is of bounded mean oscillation (abbreviated as BMO) if

$$\|u\|_{BMO(\mathbb{R}^m)} = \sup_{B \subset \mathbb{R}^m} \frac{1}{|B|} \int_B |u - (u)_B| dx < \infty,$$

where the supremum is taken over all finite ball $B \subset \mathbb{R}^m$, $|B|$ is the n -dimensional Lebesgue measure of B , and $(u)_B$ denotes the integral average of u over B , $(u)_B = \frac{1}{|B|} \int_B u(x) dx$.

The functions space BMO, modulo constants, is a Banach space with the norm $\|\cdot\|_{BMO}$ defined above.

We will need the Hardy and BMO spaces on the domain $\Omega \subset \mathbb{R}^m$.

Definition 4.3 (Hardy space on Ω) $L^1(\Omega)$ -function u belongs to $\mathcal{H}^1(\Omega)$ if and only if $\tilde{u} \in \mathcal{H}^1(\mathbb{R}^m)$, where $\tilde{u}$ is the extension of u by letting $u = 0$ outside Ω . A norm on $\mathcal{H}^1(\Omega)$ is defined as

$$\|u\|_{\mathcal{H}^1(\Omega)} = \|\tilde{u}\|_{\mathcal{H}^1(\mathbb{R}^m)}.$$

Definition 4.4 (BMO on Ω) $L^1(\Omega)$ -function u is in $BMO(\Omega)$ if and only if $\tilde{u} \in BMO(\mathbb{R}^m)$, where $\tilde{u}$ is the extension of u to all of $\mathbb{R}^m$ by letting $u = 0$ outside Ω . A norm on $BMO(\Omega)$ is defined as

$$\|u\|_{BMO(\Omega)} = \|\tilde{u}\|_{BMO(\mathbb{R}^m)}.$$

The Fefferman-Stein inequality means the duality between $\mathcal{H}^1(\mathbb{R}^m)$ and $BMO(\mathbb{R}^m)$, $(\mathcal{H}^1(\mathbb{R}^m))^* = BMO(\mathbb{R}^m)$. See the proof in [7, 24].

Lemma 4.1 (Fefferman-Stein inequality) Whenever $u \in \mathcal{H}^1(\mathbb{R}^m)$ and $v \in BMO(\mathbb{R}^m)$, there holds

$$\left| \int_{\mathbb{R}^m} uv dx \right| \leq C \|u\|_{\mathcal{H}^1(\mathbb{R}^m)} \|v\|_{BMO(\mathbb{R}^m)},$$

where the positive constant C depending only on n .

Finally, we require the Hardy space estimate in L^p -setting, due to Coifman, Lions, Meyer and Semmes (refer the proof in [2, 5]).

Lemma 4.2 *Suppose that $u \in L^{1,m}(\mathbb{R}^n, \mathbb{R})$ and $v = (v^i) \in L^{m/(m-1)}(\mathbb{R}^n, \mathbb{R}^n)$, and furthermore,*

$$\operatorname{div} v = \sum_{i=1}^n D_i v^i = 0 \quad \text{in the distribution sense.} \quad (4.1)$$

Then there holds $v \cdot Du = \sum_{i=1}^m v^i D_i u \in \mathcal{H}^1(\mathbb{R}^n, \mathbb{R})$ and

$$\|v \cdot Du\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \left(\|u\|_{L^{1,m}(\mathbb{R}^n)}^m + \|v\|_{L^{m/(m-1)}(\mathbb{R}^n)}^{m/(m-1)} \right). \quad (4.2)$$

5 Hölder continuity of m -symphonic maps

In this section we shall prove our main theorem.

Theorem 5.1 *Let Ω be a domain of the m -dimensional Euclidean space $\mathbb{R}^m$ with $m \geq 4$. Then any m -symphonic map from Ω into the standard sphere is Hölder continuous.*

The proof of Theorem 5.1 is based on a local energy growth estimate (refer to [9, 10, 13, 29]).

Let $\Omega \subset \mathbb{R}^m$ be a domain and $u \in L^{1,m}(\Omega, \mathbb{S}^n)$ be a weakly m -symphonic map as in Definition 2.2. For any ball $B(x_0, r) \subset \Omega$, the local energy is defined by

$$E(x_0, r) = \int_{B(x_0, r)} |Du|^m dx. \quad (5.1)$$

Theorem 5.2 (Energy growth estimate) *There exist positive constants $\varepsilon_0, \theta < 1$ such that the following holds true: If, for $B(x_0, r) \subset \Omega$ with $x_0 \in \Omega$ and $r > 0$,*

$$E(x_0, r) \leq \varepsilon_0, \quad (5.2)$$

then

$$E(x_0, \theta r) \leq \frac{1}{2} E(x_0, r). \quad (5.3)$$

The everywhere regularity of a weakly m -symphonic map in Theorem 5.1 follows from applying inductively Theorem 5.2 (refer to [18, Proof of Theorem 1.1, Page 7–8]) : There exists a positive number $\alpha < 1$ such that

$$E(\bar{x}, \rho) \leq 2 \left(\frac{\rho}{r} \right)^{2\alpha} E(\bar{x}, R_0/2).$$

holds for any $\bar{x} \in B(x_0, R_0/2)$ and any positive number $\rho < R_0/2$. Finally, we employ the Morrey Dirichlet growth theorem to have that the solution u is uniformly Hölder continuous in $B(x_0, R_0/2)$ and thus, locally Hölder continuous in Ω .

Proof of Theorem 5.2 We proceed the proof by reduction to absurdity. Suppose that, for any positive number $\theta < 1$, there exist families of weak solutions $\{u_k\}$ and balls $B(x_k, r_k) \subset \Omega$, $k = 1, 2, \dots$, such that

$$E(x_k, r_k) = \int_{B(x_k, r_k)} |Du_k|^m dx \leq 2^{-k} \quad (5.4)$$

and

$$E(x_k, \theta r_k) > 2^{-1} E(x_k, r_k). \quad (5.5)$$

Now we make a scaling transformation as follows: For $x \in B(1) = B(0, 1)$ let

$$\begin{aligned} v_k(x) &= \frac{u_k(x_k + r_k x) - a_k}{E(x_k, r_k)^{1/m}}, \\ w_k(x) &= u_k(x_k + r_k x) \\ &= a_k + E(x_k, r_k)^{1/m} v_k(x), \end{aligned} \quad (5.6)$$

$$a_k = \frac{1}{|B(x_k, r_k)|} \int_{B(x_k, r_k)} u_k(x) dx. \quad (5.7)$$

Then, we find by (5.1), (5.6) and Poincaré inequality that

$$\int_{B(1)} |Dv_k|^m dx = 1; \quad \int_{B(1)} |v_k|^m dx \leq C \quad (5.8)$$

On the other hand, (5.5) is equivalent to

$$\begin{aligned} \int_{B(\theta)} |Dv_k|^m dx &= E(x_k, r_k)^{-1} \int_{B(x_k, \theta r_k)} |Du_k(y)|^m dy \\ &> 1/2. \end{aligned} \quad (5.9)$$

By (5.8), the sequence $\{v_k\}$ is bounded in $L^{1,m}(B(1), \mathbb{R}^n)$ and thus, there exist a subsequence $\{v_k\}$, denoted by the same notation as above, and the limit map $v_\infty \in L^{1,m}(B(1), \mathbb{R}^n)$ such that

$$v_k \rightarrow v_\infty \quad \text{strongly in } L^m(B(1), \mathbb{R}^n), \quad (5.10)$$

$$Dv_k \rightharpoonup Dv_\infty \quad \text{weakly in } L^m(B(1), \mathbb{R}^{4n}). \quad (5.11)$$

Each scaled solution v_k also satisfies the scaled equation of the symphonic maps on $B(1)$

$$\begin{aligned} \int_{B(1)} \sigma(Dv_k)^{(m-4)/4} \sum_{i,j=1}^m (D_i v_k \cdot D_j v_k) D_j v_k \cdot D_i \phi dx \\ = E(x_k, r_k)^{1/m} \int_{B(1)} \sigma(Dv_k)^{m/4} w_k \cdot \phi dx \end{aligned} \quad (5.12)$$

for any smooth map ϕ defined on $B(1)$ into $\mathbb{R}^n$, with compact support. By (5.6) it is clear that

$$\begin{aligned} |w_k(x)| &= |u_k(x_k + r_k x)| \\ &= 1 \quad \text{almost every } x \in B(1). \end{aligned} \quad (5.13)$$

Now we assert that the limit map v_∞ satisfies

$$\int_{B(1)} \sigma(Dv_\infty)^{(m-4)/4} \sum_{i,j=1}^m (D_i v_\infty \cdot D_j v_\infty) D_j v_\infty \cdot D_i \phi \, dx = 0 \quad (5.14)$$

for any $\mathbb{R}^n$ -valued smooth map ϕ on $B(1)$ with compact support. The proof will be given in Sect. 6.

By (5.8) and (2.5), there holds

$$\begin{aligned} \int_{B(\theta)} |Dv_k|^{m-4} |Dv_\infty|^4 \, dx &\leq \left(\int_{B(1)} |Dv_k|^m \, dx \right)^{(m-4)/m} \left(\int_{B(\theta)} |Dv_\infty|^m \, dx \right)^{4/m} \\ &\leq 4^{-1}, \end{aligned} \quad (5.15)$$

provided θ is chosen small such that

$$C^{4/m} \theta^{4\alpha} \leq 4^{-1}.$$

Here we shall claim the strong convergence of gradients, that is proved in Sect. 7,

$$\begin{aligned} &\left(\sigma(Dv_k)^{(m-4)/4} + \sigma(Dv_\infty)^{(m-4)/4} \right) \sum_{i,j=1}^m (D_i v_k \cdot D_j v_k - D_i v_\infty \cdot D_j v_\infty)^2 \\ &\longrightarrow 0 \quad \text{strongly in } L^1(B(1/2), \mathbb{R}) \end{aligned} \quad (5.16)$$

and, in particular,

$$|Dv_k|^{m-4} \sum_{i,j=1}^m ||Dv_k|^2 - |Dv_\infty|^2|^2 \longrightarrow 0 \quad \text{strongly in } L^1(B(1/2), \mathbb{R}).$$

Then, taking the limit $k \rightarrow \infty$ in (5.9) forces, for $\theta < 1/2$,

$$\lim_{k \rightarrow \infty} \int_{B(\theta)} |Dv_\infty|^{m-4} |Dv_\infty|^4 \, dx \geq 2^{-1},$$

contradicting to the limit taken in (5.15) as $k \rightarrow \infty$. $\square$

6 Weak compactness

In this section we show the validity of (5.14), that is, the limit map v_∞ obtained in (5.10) and (5.11) verifies (5.14).

Theorem 6.1 *Let $v = v_\infty \in L^{1,m}(B(1), \mathbb{R}^n)$ be the limit map obtained by (5.10) and (5.11). Then it holds that*

$$\int_{B(1)} \sigma(Dv)^{(m-4)/4} \sum_{i,j=1}^m (D_i v \cdot D_j v) D_j v \cdot D_i \phi \, dx = 0 \quad (6.1)$$

for any $\mathbb{R}^n$ -valued smooth map ϕ with compact support in $B(1)$.

The proof relies on a weak compactness, that is the convergence in the distributional sense of our m -symphonic operator

$$\sum_{i,j=1}^m D_i \left(\sigma(Dv)^{(m-4)/4} (D_i v \cdot D_j v) D_j v \right), \quad (6.2)$$

under that of sequence $\{v_k\}$ in (5.11) and (5.10) (refer to [6] for harmonic case and [1, 8, 30] for p -Laplacian case).

Lemma 6.1 (Strong convergence of the gradient) *There holds, for any μ , $1 \leq \mu < 2$, and any K compactly contained in $B(1)$,*

$$\int_K \left\{ \left(\sigma(Dv_k)^{(m-4)/4} + \sigma(Dv_\infty)^{(m-4)/4} \right) \sum_{i,j=1}^m (D_i v_k \cdot D_j v_k - D_i v_\infty \cdot D_j v_\infty)^2 \right\}^{\mu/2} dx \quad (6.3)$$

$$\longrightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Proof For the symphonic maps with $p = 4$ the similar assertion as above is obtained in [18], following the argument in [1, Theorem 2.1, pp. 31–33]. Here, the proof is almost same as that in the symphonic maps case $p = 4$, except for the use of algebraic inequalities. Thus, we indicate only how to use the algebraic inequalities Proposition 3.1 peculiar to the p -symphonic operator.

Let, for any positive $\delta < 1$ and $k \geq 1$,

$$S(\delta, k) = \{x \in B(1) : |v_k(x) - v_\infty(x)| > \delta\} \quad (6.4)$$

The estimation in [18, (5.7), Page 12] is changed in the following: By use of the algebraic inequality (3.2) in Proposition 3.1 we compute as

$$\begin{aligned}
& \frac{1}{4} \int_{K \setminus S(\delta, k)} \left(\sigma(Dv_k)^{(m-4)/4} + \sigma(Dv_\infty)^{(m-4)/4} \right) \\
& \quad \times \sum_{i,j=1}^m (D_i v_k \cdot D_j v_k - D_i v_\infty \cdot D_j v_\infty)^2 dx \\
& \leq \frac{1}{4} \int_{B(1) \setminus S(\delta, k)} \left(\sigma(Dv_k)^{(m-4)/4} + \sigma(Dv_\infty)^{(m-4)/4} \right) \\
& \quad \times \sum_{i,j=1}^m (D_i v_k \cdot D_j v_k - D_i v_\infty \cdot D_j v_\infty)^2 \eta dx \\
& \leq \int_{B(1) \setminus S(\delta, k)} \sum_{i,j=1}^m \left(\sigma(Dv_k)^{(m-4)/4} (D_i v_k \cdot D_j v_k) D_j v_k \right. \\
& \quad \left. - \sigma(Dv_\infty)^{(m-4)/4} (D_i v_\infty \cdot D_j v_\infty) D_j v_\infty \right) \cdot (D_i v_k - D_i v_\infty) \eta dx \\
& = \int_{B(1) \setminus S(\delta, k)} \sum_{i,j=1}^m \sigma(Dv_k)^{(m-4)/4} (D_i v_k \cdot D_j v_k) D_j v_k \cdot (D_i v_k - D_i v_\infty) \eta dx \\
& \quad - \int_{B(1) \setminus S(\delta, k)} \sum_{i,j=1}^m \sigma(Dv_\infty)^{(m-4)/4} (D_i v_\infty \cdot D_j v_\infty) D_j v_\infty \cdot (D_i v_k - D_i v_\infty) \eta dx.
\end{aligned} \tag{6.5}$$

The estimation of each term in the right hand side of (6.5) is performed similarly as in [18, Page 11–14]. $\square$

Now the proof of Theorem 6.1 follows from Lemma 6.1.

Proof of Theorem 6.1 Let ϕ be a $\mathbb{R}^n$ -valued smooth map with compact support in $B(1)$. We compute as

$$\begin{aligned}
& \left| \int_{B(1)} \sigma(Dv_k)^{(m-4)/4} \sum_{i,j=1}^m (D_i v_k \cdot D_j v_k) D_j v_k \cdot D_i \phi dx \right. \\
& \quad \left. - \int_{B(1)} \sigma(Dv_\infty)^{(m-4)/4} \sum_{i,j=1}^m (D_i v_\infty \cdot D_j v_\infty) D_j v_\infty \cdot D_i \phi dx \right| \\
& \leq \left| \int_{B(1)} \sum_{i,j=1}^m \left(\sigma(Dv_k)^{(m-4)/4} D_i v_k \cdot D_j v_k \right. \right. \\
& \quad \left. \left. - \sigma(Dv_\infty)^{(m-4)/4} D_i v_\infty \cdot D_j v_\infty \right) D_j v_k \cdot D_i \phi dx \right| \\
& \quad + \left| \int_{B(1)} \sigma(Dv_\infty)^{(m-4)/4} \sum_{i,j=1}^m D_i v_\infty \cdot D_j v_\infty (D_j v_k - D_j v_\infty) \cdot D_i \phi dx \right|.
\end{aligned} \tag{6.6}$$

The second term in the right hand side of (6.6) simply converges to zero as $k \rightarrow \infty$, by the weak convergence (5.11), since, with $m' = \frac{m}{m-1}$,

$$\sigma(Dv_\infty)^{(m-4)/4} \sum_{i=1}^m (D_i v_\infty \cdot Dv_\infty) \cdot D_i \phi \in L^{m'/(m-2)}(B(1), \mathbb{R}^{4n}) \subset L^{m'}(B(1), \mathbb{R}^{4n}).$$

By virtue of (6.3) in Lemma 6.1, the first term in the right hand side of (6.6) is estimated above as

$$\begin{aligned} & \left| \int_{B(1)} \sum_{i,j=1}^m \left(\sigma(Dv_k)^{(m-4)/4} D_i v_k \cdot D_j v_k \right. \right. \\ & \quad \left. \left. - \sigma(Dv_\infty)^{(m-4)/4} D_i v_\infty \cdot D_j v_\infty \right) D_j v_k \cdot D_i \phi \, dx \right| \\ & \leq \int_{B(1) \cap \text{supp} \phi} \left(\sum_{i,j=1}^m \left(\sigma(Dv_k)^{(m-4)/4} D_i v_k \cdot D_j v_k \right. \right. \\ & \quad \left. \left. - \sigma(Dv_\infty)^{(m-4)/4} D_i v_\infty \cdot D_j v_\infty \right)^2 \right)^{1/2} \left(\sum_{i,j=1}^m (D_j v_k \cdot D_i \phi)^2 \right)^{1/2} dx \\ & \leq C \int_{B(1) \cap \text{supp} \phi} \left(\sigma(Dv_k)^{(m-4)/4} + \sigma(Dv_\infty)^{(m-4)/4} \right) \\ & \quad \times \left(\sum_{i,j=1}^m (D_i v_k \cdot D_j v_k - D_i v_\infty \cdot D_j v_\infty)^2 \right)^{1/2} \left(\sum_{i,j=1}^m (D_j v_k \cdot D_i \phi)^2 \right)^{1/2} dx \\ & \leq \left\{ \int_{B(1) \cap \text{supp} \phi} \left((\sigma(Dv_k)^{(m-4)/4} + \sigma(Dv_\infty)^{(m-4)/4}) \right. \right. \\ & \quad \times \left. \left. \sum_{i,j=1}^m |D_i v_k \cdot D_j v_k - D_i v_\infty \cdot D_j v_\infty|^2 \right)^{\mu/2} dx \right\}^{1/\mu} \\ & \quad \times \left\{ \int_{B(1)} \left((\sigma(Dv_k)^{(m-4)/4} + \sigma(Dv_\infty)^{(m-4)/4}) \sum_{i,j=1}^m (D_j v_k \cdot D_i \phi)^2 \right)^{\mu/2(\mu-1)} dx \right\}^{1-1/\mu}. \end{aligned} \quad (6.7)$$

Here, the first inequality of (6.7) is by the Schwarz inequality for Euclidean inner product on $\mathbb{R}^{m^2}$. The second inequality follows from the algebraic inequality (3.3) in Proposition 3.1. In the last inequality of (6.7), the Hölder inequality is used with exponent μ as

$$\frac{m}{m-1} \leq \mu < 2 \implies \frac{\mu}{\mu-1} \leq m$$

and thus, the second term in the right hand side of (6.7) is bounded by (5.8) and, by (6.3) in Lemma 6.1, the first term goes to zero as $k \rightarrow \infty$.

Finally, we apply the convergence in (6.6) for the scaled equation (5.12) with (5.4), (5.8) and (5.13), to have (6.1). $\square$

7 Strong compactness

In this section we will show the strong convergence of $\{Dv_k\}$ in (6.3). Here the maps v_k and v_∞ are extended into all of the space $\mathbb{R}^m$ by letting $v_k = 0$ and $v_\infty = 0$ outside $B(1) = B(0, 1)$. Let η be a real-valued smooth function on $\mathbb{R}^m$ such that $0 \leq \eta \leq 1$, $\text{supp } \eta \subset B(5/8)$ and $\eta = 1$ in $B(1/2)$. In the following we often abbreviate as $B(r)$ a ball $B(0, r)$ with center of the origin 0 and radius $r > 0$. Now we present the BMO estimate, that is similar to [18, Lemma 6.1, Page 16] (also see [5]).

Lemma 7.1 *The sequence $\{\eta v_k\}$ is bounded in $BMO(\mathbb{R}^m, \mathbb{R}^n)$.*

Now we define the vector field on $B(1)$ as

$$\begin{aligned} F_k^{\alpha\beta} &= (F_{k,i}^{\alpha\beta}), \quad i = 1, \dots, m, \\ F_{k,i}^{\alpha\beta} &:= \sigma(Dv_k)^{(m-4)/4} \left(w_k^\alpha \sum_{j=1}^m (D_i v_k \cdot D_j v_k) D_j v_k^\beta \right. \\ &\quad \left. - w_k^\beta \sum_{j=1}^m (D_i v_k \cdot D_j v_k) D_j v_k^\alpha \right) \end{aligned} \quad (7.1)$$

for $\alpha, \beta = 1, \dots, n$ and $k = 1, 2, \dots$

Lemma 7.2 *The sequence $F_k^{\alpha\beta}$ of $\mathbb{R}^m$ -valued vector fields is bounded in $L^{m/m-1}(B(1), \mathbb{R}^m)$ for $\alpha, \beta = 1, \dots, n$, and satisfies*

$$\int_{B(1)} D\phi \cdot F_k^{\alpha\beta} dx = 0, \quad \alpha, \beta = 1, \dots, n, \quad (7.2)$$

for any $k = 1, 2, \dots$ and, for any real-valued function $\phi \in L_0^{1,m} \cap L^\infty(B(1), \mathbb{R})$.

The proof is done similarly as in [18, Lemma 6.2, Page 17–18].

Lemma 7.3 *The sequence $\{D(\eta v_k^\alpha) \cdot F_k^{\alpha\beta}\}$ is bounded in $\mathcal{H}^1(\mathbb{R}^m, \mathbb{R})$ for any $\alpha, \beta = 1, \dots, n$.*

Proof Since $v_k \in L^{1,m}(B(1), \mathbb{R}^n)$ by (5.8), the assertion is obtained from Lemma 7.2 and (4.2) in Lemma 4.2. $\square$

Proposition 7.1 *The sequence $\{Dv_k\}$ of gradients of the scaled solutions is strongly convergent in the following sense: For any $i, j = 1, \dots, m$,*

$$\begin{aligned} & \left(\sigma(Dv_k)^{(m-4)/4} + \sigma(Dv_\infty)^{(m-4)/4} \right) \sum_{i,j=1}^m |D_i v_k \cdot D_j v_k - D_i v \cdot D_j v|^2 \\ & \longrightarrow 0 \quad \text{strongly in } L^1(B(1/2), \mathbb{R}). \end{aligned} \quad (7.3)$$

Proof Subtracting (5.14) for the limit map $v = v_\infty$ from (5.12) for v_k we have

$$\begin{aligned} & \int_{B(1)} \sum_{i,j=1}^m \left(\sigma(Dv_k)^{(m-4)/4} (D_i v_k \cdot D_j v_k) D_j v_k \right. \\ & \quad \left. - \sigma(Dv)^{(m-4)/4} (D_i v \cdot D_j v) D_j v \right) \cdot D_i \phi \, dx \\ & = E(x_k, r_k)^{1/m} \int_{B(1)} \left\{ \sum_{i,j=1}^m (D_i v_k \cdot D_j v_k)^2 \right\}^{m/4} w_k \cdot \phi \, dx \end{aligned} \quad (7.4)$$

for any $\phi \in L_0^{1,m} \cap L^\infty(B(1), \mathbb{R}^n)$. Choosing $\phi = \eta^2 (v_k - v)$ in (7.4), the left hand side of the resulting equation is estimated below as

$$\begin{aligned} & \int_{B(1)} \sum_{i,j=1}^m \left(\sigma(Dv_k)^{(m-4)/4} (D_i v_k \cdot D_j v_k) D_j v_k \right. \\ & \quad \left. - \sigma(Dv)^{(m-4)/4} (D_i v \cdot D_j v) D_j v \right) ((D_i v_k - D_i v) \eta^2 + 2\eta D_i \eta (v_k - v)) \, dx \\ & \geq \int_{B(1/2)} \left(\sigma(Dv_k)^{(m-4)/4} + \sigma(Dv_\infty)^{(m-4)/4} \right) \sum_{i,j=1}^m |D_i v_k \cdot D_j v_k - D_i v \cdot D_j v|^2 \, dx \\ & \quad - C \left\| \left(\sigma(Dv_k)^{(m-4)/4} (D_j v_k \cdot Dv_k) \cdot Dv_k \right. \right. \\ & \quad \left. \left. - \sigma(Dv_\infty)^{(m-4)/4} (D_j v \cdot Dv) \cdot Dv \right) \right\|_{L^{m/(m-1)}(B(1))} \|v_k - v\|_{L^m(B(1))}, \end{aligned} \quad (7.5)$$

where the algebraic inequality (3.2) in Proposition 3.1 is applied for the first term, and the Hölder inequality for the second term. The right hand side of (7.4) is computed by use of the so-called Hélein technique

$$\begin{aligned}
& \int_{B(1)} \left\{ \sum_{i,j=1}^m (D_i v_k \cdot D_j v_k)^2 \right\}^{m/4} w_k \cdot (v_k - v) \eta^2 dx \\
&= \int_{B(1)} \sum_{i,j=1}^m \sum_{\alpha, \beta=1}^n \eta^2 D_i v_k^\alpha \sigma(D v_k)^{(m-4)/4} \\
&\quad \times \left(D_j v_k^\alpha (D_i v_k \cdot D_j v_k) w_k^\beta - D_j v_k^\beta (D_i v_k \cdot D_j v_k) w_k^\alpha \right) (v_k^\beta - v^\beta) dx \\
&= - \int_{B(1)} \sum_{\alpha, \beta=1}^n \eta^2 D v_k^\alpha \cdot F_k^{\alpha\beta} (v_k^\beta - v^\beta) dx \\
&= - \int_{B(1)} \sum_{\alpha, \beta=1}^n D(\eta v_k^\alpha) \cdot F_k^{\alpha\beta} \eta (v_k^\beta - v^\beta) dx \\
&\quad + \int_{B(1)} \sum_{\alpha, \beta=1}^n v_k^\alpha D \eta \cdot F_k^{\alpha\beta} \eta (v_k^\beta - v^\beta) dx,
\end{aligned} \tag{7.6}$$

where we use by (5.6) and (5.13) that

$$\begin{aligned}
D w_k &= E(x_k, r_k)^{1/m} D v_k, \\
D_i v_k \cdot w_k &= E(x_k, r_k)^{-1/m} \sum_{\alpha=1}^n D_i w_k^\alpha w_k^\alpha \\
&= E(x_k, r_k)^{-1/m} 2^{-1} D_i |w_k|^2 \\
&= 0.
\end{aligned} \tag{7.7}$$

By the Fefferman-Stein inequality, Lemma 4.1, the first term in the right hand side of (7.6) is bounded for each $\alpha, \beta = 1, \dots, n$ by

$$\begin{aligned}
& \left| \int_{B(1)} D(\eta v_k^\alpha) \cdot F_k^{\alpha\beta} \eta (v_k^\beta - v^\beta) dx \right| \\
&\leq \|D(\eta v_k^\alpha) \cdot F_k^{\alpha\beta}\|_{\mathcal{H}^1(\mathbb{R}^m)} \|\eta (v_k^\beta - v^\beta)\|_{BMO(\mathbb{R}^m)} \\
&\leq C,
\end{aligned} \tag{7.8}$$

where Lemmas 7.1 and 7.3 are crucially used and, ηv is in $BMO(\mathbb{R}^m, \mathbb{R}^n)$, by the Poincaré inequality, (5.8) and (5.11). The second term in the right hand side of (7.6) is estimated above for each $\alpha, \beta = 1, \dots, n$ as

$$\begin{aligned}
& \left| \int_{B(1)} D \eta v_k^\alpha \cdot F_k^{\alpha\beta} \eta (v_k^\beta - v^\beta) dx \right| \\
&\leq C \|F_k^{\alpha\beta}\|_{L^{m/(m-1)}(B(1))} \|v_k^\alpha (v_k^\beta - v^\beta)\|_{L^m(B(1))} \\
&\leq C,
\end{aligned} \tag{7.9}$$

where, again, by (5.8), the sequence $\{F_k^{\alpha\beta}\}$ is bounded in $L^{m/(m-1)}(B(1), \mathbb{R}^m)$ and, the sequence $\{v_k^\alpha (v_k^\beta - v^\beta)\}$ is also bounded in $L^m(B(1), \mathbb{R}^{n^2})$ by the Sobolev inequality in the m -dimensional domain $B(1)$, $L^{1,m}(B(1)) \hookrightarrow L^q(B(1))$ for any $q \geq 1$ and (5.8)

$$\begin{aligned}
\|v_k^\alpha (v_k^\beta - v^\beta)\|_{L^m(B(1))} &\leq \|v_k^\alpha\|_{L^{2m}(B(1))} \|v_k^\beta - v^\beta\|_{L^{2m}(B(1))} \\
&\leq C \|v_k^\alpha\|_{L^{1,m}(B(1))} \left(\|v_k^\beta\|_{L^{1,m}(B(1))} + \|v^\beta\|_{L^{1,m}(B(1))} \right) \\
&\leq C.
\end{aligned}$$

Finally, the desired convergence (7.3) follows from (7.4), (7.5), (7.6), (7.8) and (7.9). $\square$

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References

- Chen, Y., Hong, M.-C., Hungerbühler, N.: Heat flow of p -harmonic maps with values into spheres. *Math. Z.* **215**, 25–35 (1994)
- Coifman, R., Lions, P.L., Meyer, Y., Semmes, S.: Compensated compactness and Hardy spaces. *J. Math. Pures Appl.* **72**, 247–286 (1993)
- Eells, J., Lemaire, L.: A report on harmonic maps. *Bull. Lond. Math. Soc.* **10**, 1–68 (1978)
- Eells, J., Lemaire, L.: Another report on harmonic maps. *Bull. Lond. Math. Soc.* **20**, 385–524 (1988)
- Evans, L.C.: Partial regularity for stationary harmonic maps into spheres, *Arch. Ration. Mech. Anal.* **20**, 385–524 (1988)
- Evans, L.C.: Weak convergence methods of nonlinear partial differential equations. CBMS Regional Conference Series in Mathematics, 74. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI (1990)
- Fefferman, C., Stein, E.M.: H^p spaces of several variables. *Acta Math.* **192**(3–4), 137–193 (1972)
- Fuchs, M.: The blow-up of p -harmonic maps. *Manuscr. math.* **81**, 89–94 (1993)
- Giaquinta, M.: Introduction to Regularity Theory for Nonlinear Elliptic Systems. Birkhäuser Verlag, Basel (1993)
- Giusti, E.: Direct Methods in the Calculus of Variations. World Scientific Publishing Co., London (2003)
- Goldstein, P., Strzelecki, P., Zatorska-Goldstein, A.: On polyharmonic maps into spheres in the critical dimension. *Ann. Inst. H. Poincaré Anal. Non. Linéaire* **26**, 1387–1405 (2009)
- Hélein, F.: Harmonic maps, conservation laws and moving frames, *Cambridge Tracts in Mathematics*, vol. 150. Cambridge University Press, Cambridge (2002)
- Hildebrandt, S., Widman, K.-O.: Some regularity results for quasilinear elliptic systems of second order. *Math. Z.* **142**, 67–86 (1975)
- Iwaniec, T., Martin, G.: Geometric function theory and non-linear analysis, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, (2001). Zbl 1045.30011 MR 1859913
- Kawai, S., Nakauchi, N.: Some results for stationary maps of a functional related to pullback metrics. *Nonlinear Anal.* **74**, 2284–2295 (2011)
- Leone, C., Misawa, M., Verde, A.: A global existence result for the heat flow of higher dimensional H -systems. *J. Math. Pures Appl.* **97**(3), 282–294 (2012)

17. Misawa, M., Nakauchi, N.: A Hölder continuity of minimizing symphonic maps. *Nonlinear Anal.* **75**, 5971–5974 (2012)
18. Misawa, M., Nakauchi, N.: A Hölder continuity of symphonic maps into the spheres. *Calc. Var. PDE* **55**, 1–20 (2016)
19. Morrey, C.B.: *Multiple Integrals in the Calculus of Variations*. Springer, Berlin (1966)
20. Nakauchi, N.: A variational problem related to conformal maps. *Osaka J. Math.* **48**, 717–739 (2011)
21. Nakauchi, N., Takenaka, Y.: A variational problem for pullback metrics. *Ricerche di Mat.* **60**, 219–235 (2011)
22. Rivière, T., Strzelecki, P.: A sharp nonlinear Gagliardo-Nirenberg-type estimates and applications to the regularity of elliptic systems. *Comm. Partial Differ. Equ.* **30**(4–6), 589–604 (2005)
23. Schikorra, A., Strzelecki, P.: Invitation to H-systems in higher dimensions: known results, new facts, and related open problems. *EMS Surv. Math. Sci.* **4**(1), 21–42 (2017)
24. Stein, E.M.: *Harmonic Analysis, Real-Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton University Press, Princeton (1993)
25. Strzelecki, P.: Regularity of p-harmonic maps from p-dimensional ball into a sphere. *Manuscr. Math.* **82**(3–4), 407–415 (1994)
26. Strzelecki, P., Zatorska-Goldstein, A.: A compactness theorem for weak solutions of higher-dimensional H-systems. *Duke Math. J.* **121**(2), 269–284 (2004)
27. Takeuchi, H.: Some conformal properties of p-harmonic maps and a regularity for sphere-valued p-harmonic maps. *J. Math. Soc. Jpn.* **46**(2), 217–234 (1994)
28. Torchinsky, A.: *Real Variable Methods in Harmonic Analysis*. Academic Press, New York (1986)
29. Widman, K.-O.: Hölder continuity of solutions of elliptic systems. *Manuscr. Math.* **5**, 299–308 (1971)
30. Zeidler, E.: *Nonlinear Functional Analysis and its Applications II / B (Nonlinear Monotonic operators)*. Springer, New York (1990)