On the uniqueness of a suitable weak solution to the Navier–Stokes Cauchy problem

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Abstract

The paper is concerned with the Navier–Stokes Cauchy problem. We investigate on some results of regularity and uniqueness related to suitable weak solutions corresponding to a special set of initial data. The suitable weak solution notion is meant in the sense introduced by Caffarelli–Kohn–Nirenberg. As further result we discuss the uniqueness of a set of suitable weak solutions (wider than the previous one) enjoying a “Prodi–Serrin” condition which is “relaxed” in space.

Keywords Navier–Stokes equations · Weak solutions · Uniqueness of solutions

Mathematics Subject Classification 35Q30 · 35B65 · 76D03

1 Introduction

We deal with the Navier–Stokes Cauchy problem

$$\begin{aligned} u_t + u \cdot \nabla u + \nabla \pi_u &= \Delta u, \quad \nabla \cdot u = 0, \quad \text{in } (0, T) \times \mathbb{R}^3, \\ u &= u_0(x) \text{ on } \{0\} \times \mathbb{R}^3. \end{aligned} \quad (1)$$

In system (1) u is the kinetic field, π_u is the pressure field, $u_t := \frac{\partial}{\partial t} u$ and $u \cdot \nabla u := u_k \frac{\partial}{\partial x_k} u$. The aim of the note is to investigate on some questions of regularity and uniqueness of weak solutions to the Cauchy problem (1). We believe that for this topic two different approaches are possible. One looks for establishing results by comparing a weak solution in the sense of

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Leray–Hopf with another weak solution enjoying extra conditions. We believe that this is the major question investigated in a very wide literature originated by Prodi and Serrin papers [15, 16], whose results employ for the uniqueness the technique of the famous Leray’s paper [12]. However the goal is slightly different in [12], because the uniqueness result has a precise role in the proof of the structure theorem. This approach conceals a possible compatibility between the *extra conditions* assumed for the weak solution and the initial datum. Hence, as consequence of the further hypothesis burdening the initial datum, the simple character of Leray–Hopf weak solution is lost. In this regard, some authors look for a characterization between the initial datum and the Prodi–Serrin conditions, see the recent contribution given by Kozono et al. [8] and see [5] for a review of results on this topic. In their approach there is no discrepancy between the extra conditions fit to obtain the uniqueness and the ones related to the regularity of solutions. Actually, the assumptions fit to ensure the uniqueness imply the regularity of a weak solution. However in [13] relaxed Prodi–Serrin conditions are considered, valid on any time interval properly contained in the one of existence, implying the regularity of the solutions for $t > 0$. Since no equivalence between the Prodi–Serrin conditions and the uniqueness is known, as well as no example of non-uniqueness is known for weak solutions corresponding to an initial datum only in L^2 , it is an open question if a weak solution satisfying a relaxed Prodi–Serrin condition is also unique.

Another possible approach looks for the coincidence between a weak solution and a solution in the existence class related to the initial datum. This second topic, which is more classical and posed by Leray in [12], is matter of study of a more recent literature. We refer to the interesting and clarifying paper [11]. As a consequence of the weaker assumptions on the initial datum with respect to the $J^{1,2}$ assumption considered by Leray, we believe that they represent the real generalization of Leray’s result [12], that is, “*Comparison d’une solution régulière et d’une solution turbulente*”.

In this note we follow this task. In particular, our existence class is in some sense close to the one of weak solutions of the L^2 -theory. So that the results could achieve a further interest.

In order to better discuss the results of this note, we need some definitions and notations.

Definition 1 Let $u_0 \in J^2(\mathbb{R}^3)$. A pair (u, π_u) , such that $u : (0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $\pi_u : (0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$, is said a weak solution to problem (1) if

- (i) for all $T > 0$, $u \in L^2(0, T; J^{1,2}(\mathbb{R}^3))$ and $\pi_u \in L^{\frac{5}{3}}((0, T) \times \mathbb{R}^3)$, for all $\psi \in J^2(\mathbb{R}^3)$, $(u(t), \psi) \in C((0, T))$
- (ii) $\lim_{t \rightarrow 0} (u(t), \psi) = (u_0, \psi)$, for all $\psi \in \mathcal{C}_0(\mathbb{R}^3)$
- (iii) for all $t, s \in (0, T)$, the pair (u, π_u) satisfies the equation:

$$\int_s^t \left[(u, \varphi_\tau) - (\nabla u, \nabla \varphi) + (u \cdot \nabla \varphi, u) + (\pi_u, \nabla \cdot \varphi) \right] d\tau + (u(s), \varphi(s)), \\ = (u(t), \varphi(t)) \text{ for all } \varphi \in C_0^1([0, T) \times \mathbb{R}^3).$$

In [1] an energy relation having also a local character is introduced to investigate on the regularity of a weak solution:

Definition 2 A pair (u, π_u) is said a suitable weak solution if it is a weak solution in the sense of the Definition 1 and, moreover,

$$\int_{\mathbb{R}^3} |u(t)|^2 \phi(t) dx + 2 \int_\sigma^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi dx d\tau \leq \int_{\mathbb{R}^3} |u(\sigma)|^2 \phi(\sigma) dx$$

$$+ \int_{\sigma}^t \int_{\mathbb{R}^3} |u|^2 (\phi_{\tau} + \Delta \phi) dx d\tau + \int_{\sigma}^t \int_{\mathbb{R}^3} (|u|^2 + 2\pi_u) u \cdot \nabla \phi dx d\tau, \quad (2)$$

for all $t \geq \sigma$, for $\sigma = 0$ and a.e. in $\sigma \geq 0$, and for all nonnegative $\phi \in C_0^{\infty}(\mathbb{R} \times \mathbb{R}^3)$.

We introduce the *acronimus* sws to mean a suitable weak solution in the sense of Definition 2. In [1], the following existence result is proved:

Theorem 1 For all $u_0 \in J^2(\mathbb{R}^3)$ there exists a sws.

As a consequence of inequality (2) and of the existence theorem one gets

Corollary 1 A sws enjoys the strong energy inequality:

$$\|u(t)\|_2^2 + 2 \int_s^t \|\nabla u(\tau)\|_2^2 d\tau \leq \|u(s)\|_2^2, \\ \text{for all } t \geq s, \text{ for } s = 0 \text{ and a.e. in } s > 0. \quad (3)$$

Moreover for all s such that (3) holds we get

$$\lim_{t \rightarrow s^+} \|u(t) - u(s)\|_2 = 0. \quad (4)$$

Following [1], in the sequel by the symbol $Q_r(t, x)$ we mean the parabolic cylinder

$$Q_r(t, x) := \{(\tau, y) : t - r^2 < \tau < t \text{ and } |y - x| < r\}.$$

In paper [4], as natural continuation of the results proved in [3] (actually some hints are already given in [3]) we proved two results of partial regularity that we resume in the following statement:

Theorem 2 Let u be a sws. Then there exists a set $\mathbb{E} \subseteq \mathbb{R}^3$, with $\mathbb{R}^3 - \mathbb{E}$ having zero H^a -Hausdorff measure, with $a = \frac{3}{2} + \varepsilon$ and arbitrary $\varepsilon > 0$, enjoying the property: for all $x \in \mathbb{E}$ there exist $\delta \in [0, 1)$ and $t > 0$ such that

$$u \in L^{\infty} \left(Q_{\left(\frac{(1-\delta)s}{4}\right)^{\frac{1}{2}}} \left(\frac{7}{6}s, x \right) \right), \text{ for all } s \in (0, t). \quad (5)$$

In particular, if (τ, y) is a Lebesgue's point in $Q_{\left(\frac{(1-\delta)s}{4}\right)^{\frac{1}{2}}} \left(\frac{7}{6}s, x \right)$, then we get

$$|u(\tau, y)| \leq \bar{c} \tau^{-\frac{1}{2}}. \quad (6)$$

Moreover, for all $R > 0$ and $\sigma > 0$ there exists a closed set $\mathbb{F}_{\sigma} \subset \mathbb{E} \cap B_R$ such that $|B_R - \mathbb{F}_{\sigma}| < \sigma$ and (5) holds for all $(s, x) \in (0, T_{\mathbb{F}_{\sigma}}) \times \mathbb{F}_{\sigma}$ and (6) holds for the Lebesgue's point (τ, y) belonging to $Q_{\left(\frac{(1-\delta)s}{4}\right)^{\frac{1}{2}}} \left(\frac{7}{6}s, x \right)$, provided that $x \in \mathbb{F}_{\sigma}$.

Remark 1

- The set $\mathbb{E}$ depends on the initial datum u_0 . Actually, the set $\mathbb{E}$ is the set of those x , ensured by the Hardy–Littlewood–Sobolev inequality, for which

$$\int_{\mathbb{R}^3} \frac{|u_0(y)|^2}{|x - y|} dy < \infty. \quad (7)$$

From (7) the claim on the Hausdorff measure of $\mathbb{R}^3 - \mathbb{E}$ is immediate.

- The set $\mathbb{E}$ is independent of the sws corresponding to the initial datum u_0 . In particular, if we assume the existence of two sws corresponding to u_0 , ignoring their possible coincidence, we can select not only the same set $\mathbb{E}$, but also the same set $\mathbb{F}_\sigma$. Different is the claim related to the existence of $\delta(x)$ and $t(x)$. They can depend on the solution.

The aim of this note is to prove some uniqueness results. For this task we have to improve the results given in Theorem 2. In particular we cannot limit ourselves to prove (5) and (6). More precisely, by means of the initial datum, we have to give a upper bound which is useful for the uniqueness.

We introduce also the notion of $J^{1,2}$ -regular solution:

Definition 3 A pair (v, π) is said a $J^{1,2}$ -regular solution if (v, π_v) , for some $T \leq \infty$, solves problem (1) almost everywhere in $(t, x) \in (0, T) \times \mathbb{R}^3$, and

$$v \in C([0, T]; J^{1,2}(\mathbb{R}^3)) \cap L^2(0, T; W^{2,2}(\mathbb{R}^3)), \quad v_t, \nabla \pi \in L^2(0, T; L^2(\mathbb{R}^3)). \quad (8)$$

It is well known that for all $v_0 \in J^{1,2}(\mathbb{R}^3)$ there exists on some interval $(0, T(v_0))$ a unique $J^{1,2}$ -regular solution (cf. e.g. [12]).

The following theorem is an improvement of Theorem 2. To obtain the improvement we consider a suitable subset of $J^2(\mathbb{R}^3)$ as set of the initial data for problem (1).

Theorem 3 Let $u(t, x)$ be a sws. Assume the existence of $v_0 \in J^{1,2}(\mathbb{R}^3)$ such that

$$\sup_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_0(y) - v_0(y)|^2}{|x - y|} dy < \frac{1}{(4c)^2}, \quad (9)$$

where c is an absolute constant. Denote by v the kinetic field of the $J^{1,2}$ -regular solution corresponding to v_0 , and by $(0, T_0)$ the related existence interval. For all $x \in \mathbb{R}^3$, there exists a $\delta \in [0, 1)$ such that

$$u \in L^\infty \left(Q_{\left(\frac{(1-\delta)s}{4}\right)^{\frac{1}{2}}}(s, x) \right), \text{ for all } s \in (0, T_0), \quad (10)$$

and for all Lebesgue's points $(\tau, y) \in Q_{\left(\frac{(1-\delta)s}{4}\right)^{\frac{1}{2}}}(s, x)$

$$|u(\tau, y)| \leq c[(1 - \delta)\tau]^{-\frac{1}{2}} + |v(\tau, y)|. \quad (11)$$

If $v_0 \equiv 0$, then we get $T_0 = \infty$.

Remark 2 The assumption (9), which detects a subset of $J^2(\mathbb{R}^3)$, allow us to consider $\mathbb{E} \equiv \mathbb{R}^3$ and $\inf_{x \in \mathbb{R}^3} t(x) \geq T_0$. This is the meaning of assumption (9) and the improvement with respect Theorem 2. Assumption (9) with $v_0 \equiv 0$ and, as a consequence, $T_0 = \infty$, corresponds to a suitable small data. Actually, Theorem 3 is either a new existence theorem of regular solutions (local for large data and global for small data) and a theorem of regularity for a sws. Although the local character of regularity, we are not able to prove the uniqueness for the sws of Theorem 3. This motives the further following results.

In the following theorems the symbols c_0 and ε_1 denote two absolute constants whose meaning is the same of the one given in Proposition 1 by Caffarelli-Kohn-Nirenberg in paper [1]. We employ these constants in the proof of Lemma 6 sect. 4.

Theorem 4 Assume that $u(t, x)$ is a sws. There is an absolute constant c_1 such that if for some $R_0 > 0$

$$\int_{\mathbb{R}^3} \frac{|u_0(y)|^2}{|x - y|} dy < \frac{\varepsilon_1}{2c_1}, \text{ for } |x| > R_0, \quad (12)$$

then

$$t^{\frac{1}{2}} |u(t, x)| < c_0 \varepsilon_1^{\frac{2}{3}}, \text{ for all } (t, x) \in (0, \infty) \times \{x : |x| > R_0\}. \quad (13)$$

Moreover, assume $v_0, v_0^* \in J^{1,2}(\mathbb{R}^3)$ such that

$$\sup_{|x| \leq R_0} \int_{\mathbb{R}^3} \frac{|u_0(y) - v_0(y)|^2}{|x - y|} dy < \frac{1}{(4c)^2}, \quad \sup_{|x| \leq R_0} \int_{\mathbb{R}^3} \frac{|u_0(y) - v_0^*(y)|^2}{|x - y|} dy < \frac{\varepsilon_1}{2c_1}, \quad (14)$$

where the constant c is as in (9). Set v and v^* the kinetic fields of the $J^{1,2}$ -regular solutions to problem (1) with initial datum v_0 and v_0^* , and with existence intervals $(0, T_0)$ and $(0, T_0^*)$, respectively. Then, for some $T^* \leq T_0^*$, we get

$$t^{\frac{1}{2}} |u(t, x)| < c_0 \varepsilon_1^{\frac{2}{3}} + t^{\frac{1}{2}} |v^*(t, x)|, \text{ for all } (t, x) \in (0, T^*) \times \mathbb{R}^3, \quad (15)$$

and, if $T^* < T_0$, the following estimate holds:

$$|u(t, x)| \leq \bar{c} T^{*-\frac{1}{2}} + |v(t, x)|, \text{ for all } (t, x) \in [T^*, T_0] \times \mathbb{R}^3, \quad (16)$$

for some constant $\bar{c}$.

Remark 3 If we denote by $(0, T_0^*)$ the interval of existence of the solution v^* , then a priori we have $T^* \leq T_0^*$. Since we do not compare the numerical value of $\varepsilon_1/2c_1$ and $1/(4c)^2$, a priori we consider $v_0 \neq v_0^*$ and as a consequence $T_0 \neq T_0^*$. Actually, the “perturbations v_0 and v_0^* to the initial datum u_0 ” fulfill a different role.

The following theorem is a weak version of the above one

Theorem 5 Assume that $u(t, x)$ is a sws. There is an absolute constant c_1 such that if for some $R'_0 > 0$

$$\int_{\mathbb{R}^3} \frac{|u_0(y)|^2}{|x - y|} dy < \frac{\varepsilon_1}{2c_1}, \text{ for all } |x| > R'_0, \quad (17)$$

then (13) holds on $(0, \infty) \times \{|x| > R'_0\}$. Moreover, for all $\sigma > 0$ there exists a closed set $\mathbb{F}_\sigma$ such that $|B_{R'_0} - \mathbb{F}_\sigma| < \sigma$ and there exist functions $v^*(t, x)$ and $v(t, x)$ enjoying (8) on $(0, T^*)$ and on $(0, T_0)$, respectively, independent of $u(t, x)$, such that

$$t^{\frac{1}{2}} |u(t, x)| < c_0 \varepsilon_1^{\frac{2}{3}} + t^{\frac{1}{2}} |v^*(t, x)|, \text{ for all } (t, x) \in (0, T^*) \times \mathbb{F}_\sigma, \quad (18)$$

and if $T^* < T_0$, the following estimate holds:

$$|u(t, x)| \leq \bar{c} T^{*-\frac{1}{2}} + |v(t, x)|, \text{ for all } (t, x) \in [T^*, T_0] \times \mathbb{F}_\sigma, \quad (19)$$

for some constant $\bar{c}$.

Remark 4 We believe that the assumptions (12) and (17) are of some interest. Actually they ensure the regularity of a *sws* in the exterior of a ball. This result is hailing from [1] where, being $\nabla u_0 \in L^2(|x| > R)$, the assumption on the initial datum is slightly stronger. In the prospective to look for sufficient conditions for the well-posedness of the Navier–Stokes, at least in the case of the Cauchy problem (or more in general for some unbounded domains), they appear useful, because these assumptions ensure regularity in the exterior of the ball for all $t > 0$. So that the regularity question of a solution (or equivalently of a possible blow-up) is in the ball. We stress that $\mathbb{F}_\sigma$ depends on the initial datum. That is, the set is independent of the particular *sws* u . Roughly speaking, the theorem represents the spatial counterpart of the structure theorem (in time) give in [12]. This also is the meaning of Theorem 5.

We get the following uniqueness results:

Theorem 6 *A *sws* of Theorem 4 is unique on $(0, T_0)$ provided that $\varepsilon_1^{\frac{2}{3}} < 1/(\sqrt{2\pi}c_0)$.*

The second uniqueness result is concerned with *sws* of Theorem 5 for which a sort of (actually weaker) “Prodi–Serrin” condition is employed [6, 15, 16].

We set $\mathbb{F}_\sigma^c := B_{R_0'} - \mathbb{F}_\sigma$. For arbitrary $\rho > 0$, the set $I_\rho := \bigcup_{x \in \mathbb{F}_\sigma^c} B_\rho(x)$ is a neighborhood of $\mathbb{F}_\sigma^c$. The symbol χ_ρ is the characteristic function of I_ρ . The symbol $J_\delta[\cdot]$ denotes the mollifier, and, for $\delta \in (0, \rho)$, we set

$$h(x) := J_\delta[\chi_\rho](x) \in [0, 1], \text{ with } h(x) = 1 \text{ for all } x \in \mathbb{F}_\sigma^c. \quad (20)$$

We denote by $\mathbb{H} := \text{supp } h$.

Theorem 7 *Let u be a *sws* of Theorem 5. If, for some $\sigma > 0$ and $s > 3$, u belongs to $L^r(0, T_0; L^s(\mathbb{H}))$, with $\frac{2}{r} + \frac{3}{s} = 1$, then, for all $\eta > 0$, u is a $J^{1,2}$ -regular solution on $(\eta, T_0) \times \mathbb{R}^3$, and it is the unique corresponding to u_0 in the set of solutions detected in Theorem 5, provided that $\varepsilon_1^{\frac{2}{3}} < 1/(\sqrt{2\pi}c_0)$.*

Remark 5

- Theorem 6 states the uniqueness of the solutions furnished by Theorem 4. In the framework of the above settings, we consider Theorem 6 belonging to the second kind of approaches to the uniqueness, and like a generalization of the Leray one (actually by means of Hardy’s inequality all $u_0 \in J^{1,2}$ satisfies (12) and (14)).
- An initial datum satisfying (12) and (14) is an element of $BM O_T^{-1}$. In this setting we have the results of existence and uniqueness (both local) proved in [7] (see also [10]) and the result of uniqueness proved in [9]. However, in our local uniqueness theorem the hypotheses require no condition of smallness to the initial datum in $BM O_T^{-1}$, or else the $\lim_{t \rightarrow 0} t^{\frac{1}{2}} \|u(t)\|_\infty = 0$, u weak solution. Further comparison is due for the assumptions (12) and (14) with the ones made for other uniqueness results reproduced in [11] by Lemarié–Rieusset. Here we prefer to refer [11]. In [11] data u_0 are assumed in $L^2 \cap BM O_0^{-1} \cap \dot{B}_{q,\infty}^{-s}$, $q \in (3, \infty)$ and $-s > -1 + \frac{2}{q}$. Here

$$\dot{B}_{q,\infty}^{-s} \text{ is normed with } \sup_{t>0} t^{\frac{s}{2}} \|U(t)\|_q < \infty, \quad (21)$$

where U is the heat solution corresponding to u_0 . The uniqueness result concerns the coincidence of the Leray weak solution with a mild solution both corresponding to u_0 . This result, that we set in the second kind of uniqueness results, that is uniqueness in the

class of existence, are a generalization of the conditions on the initial data in order to obtain uniqueness with respect to the one by Leray [12].

Our result of uniqueness, that we consider also as belonging to the second kind, has a special character ascribable to two different facts. The first is that no comparison is possible between the space $L^2 \cap BMO_0^{-1} \cap \dot{B}_{q,\infty}^{-s}$ and the one of the assumptions (12) and (14). Actually, the solution to the heat equation with $u_0 \in L^2(\mathbb{R}^3)$ satisfying (12) and (14) does not satisfy (21) for any $q > 2$. In this connection see Theorem A' in [14] for the characterization of convolutions in weighted spaces (in Sect. 2, for the reader convenience we reproduce the statement of Theorem A'). The second is the fact that we can consider uniqueness in the case of $\sup_{t \in (0, T_0)} t^{\frac{1}{2}} \|u(t)\|_\infty < \infty$, (u weak solution to problem (1)) which is out of the results furnished in [11]. This is due to their approach based on the energy inequality. Instead, thanks the assumption of smallness on the “perturbation” $u_0 - v_0$, and the fact that we argue by duality in the way suggested in [6], we can avoid the condition.

- Theorem 7 is stated for $s > 3$. This restriction can be removed, hence the result also holds for $s = 3$. For the sake of brevity we omit the proof.
- Theorem 7 is meant in the first kind of uniqueness results. The theorem is based on a relaxed space Prodi–Serrin condition furnishing regularity and uniqueness. Actually, it represents a different relaxed condition with regard to the one introduced in [13], that is a relaxed condition just with respect to the time variable. In fact, for arbitrary $\sigma > 0$, on set $\mathbb{F}_\sigma^c \subset \mathbb{R}^3$, related to the “unknown” behavior of a weak solution, we have a Prodi–Serrin condition in an arbitrary neighborhood $\mathbb{H} \supset \mathbb{F}_\sigma^c$, and on $\mathbb{F}_\sigma$ the solution $u \in L_{weak}^2(0, T; L^\infty(\mathbb{F}_\sigma))$. As far as we are able to recognize, in the light of the regularity expressed in Theorem 5, these *extra* assumptions appear as the most congenial for a sws.

The plan of the paper is the following. In Sect. 2 we recall some well known results preliminary for the next sections. In Sect. 3 we formulate a weighted energy relation, which is a key tool in order to apply the criterium of regularity established in [1]. In Sect. 4 we extend to the solutions of a “perturbed Navier–Stokes system” the criterium of regularity stated in [1] for the solutions to the ordinary Navier–Stokes system. In Sect. 5 we furnish the proof of Theorems 3–5. Finally, in Sect. 6, 7 we give the proof of the uniqueness results.

2 Some preliminary results

We start recalling the following result concerning a (relaxed) Prodi–Serrin condition for regularity:

Theorem 8 *Assume that a weak solution v satisfies the condition:*

$$\text{for all } \eta > 0, v \in L^\rho(\eta, T; L^\sigma(\Omega)) \text{ with } \frac{3}{\sigma} + \frac{2}{\rho} = 1, \rho < \infty, \quad (22)$$

then v is a $J^{1,2}$ -regular solution on (η, T) .

Proof For the proof see [13] Theorem 2.

Lemma 1 *Assume that (u, π_u) is a sws. Then the pressure field admits the following representation formula*

$$\pi_u(t, x) = -D_{x_i} D_{x_j} \int_{\mathbb{R}^3} \mathcal{E}(x - y) u^i(y) u^j(y) dy, \text{ a.e. in } (t, x) \in (0, \infty) \times \mathbb{R}^3, \quad (23)$$

where $\mathcal{E}(z)$ denotes the fundamental solution of the Laplace operator, and the following estimates hold:

$$\pi_u(t, x) \in L^{\frac{5}{3}}(0, T; L^{\frac{5}{3}}(\mathbb{R}^3)), \quad (24)$$

$$\|\pi_u(1 + |x|)^{-\frac{4}{3}}\|_{\frac{3}{2}} \leq c\|u(1 + |x|)^{-\frac{2}{3}}\|_3^2, \quad (25)$$

where c is a constant independent of (u, π_u) .

Indicate by (u, π_u) and $(\bar{u}, \pi_{\bar{u}})$ two weak solutions in the sense of Definition 1, set $w = \bar{u} - u$ and $\pi_w := \pi_{\bar{u}} - \pi_u$. The pair (w, π_w) satisfies the following integral equation:

$$\begin{aligned} & \int_s^t \left[(w, \varphi_\tau) - (\nabla w, \nabla \varphi) + (w \cdot \nabla \varphi, \bar{u}) + (u \cdot \nabla \varphi, w) + (\pi_w, \nabla \cdot \varphi) \right] d\tau \\ & + (w(s), \varphi(s)) = (w(t), \varphi(t)), \text{ for all } \varphi \in C_0^1([0, T] \times \mathbb{R}^3). \end{aligned} \quad (26)$$

Lemma 2 The pressure field π_w in (26) solves the equation

$$(\pi_w, \Delta g) = -(w \otimes \bar{u} + u \otimes w, \nabla \nabla g) \text{ for all } g \in C_0^\infty(\mathbb{R}^3) \quad (27)$$

and, for $q \in (1, 3]$, π_w enjoys the following estimate:

$$\|\pi_w\|_q \leq c\|w(|u| + |\bar{u}|)\|_q, \text{ a.e. in } t > 0, \quad (28)$$

and for $q = 1$

$$\|\pi_w\|_{\mathcal{H}^1} \leq c\|w(|u| + |\bar{u}|)\|_1. \quad (29)$$

Moreover, for all smooth function h , the function $h\pi_w$ solves the equation

$$\begin{aligned} (h\pi_w, \Delta g) &= -(\pi_w, g\Delta h) - 2(\pi_w, \nabla h \cdot \nabla g) - (w \otimes \bar{u} + u \otimes w, h\nabla \nabla g) \\ &\quad - (w \otimes \bar{u} + u \otimes w, g\nabla \nabla h) - (w \otimes \bar{u} + u \otimes w, \nabla h \otimes \nabla g + \nabla g \otimes \nabla h) \end{aligned} \quad (30)$$

provided that $g \in C_0^\infty(\mathbb{R}^3)$. Moreover, if function h and its derivatives are bounded functions, we get $h\pi_w = \pi_1 + \pi_2$ with

$$\begin{aligned} \pi_1(x) &:= - \int_{\mathbb{R}^3} \mathcal{E}(x) \pi_w \Delta h - (w \otimes \bar{u} + u \otimes w) \cdot \nabla \nabla h dy, \\ \pi_2(x) &:= - \int_{\mathbb{R}^3} 2\pi_w \nabla \mathcal{E}(x) \cdot \nabla h + [w \otimes \bar{u} + u \otimes w] \cdot D(x) dy \\ &\quad - \int_{\mathbb{R}^3} \nabla \nabla \mathcal{E}(x) \cdot h(w \otimes \bar{u} + u \otimes w) dy \end{aligned} \quad (31)$$

where $\mathcal{E}(x)$ is the fundamental solution of Poisson's equation and $D(x) := [\nabla \mathcal{E}(x) \otimes \nabla h + \nabla h \otimes \nabla \mathcal{E}(x)]$. Finally, the following estimates hold:

$$\begin{aligned} \|\pi_2\|_2 &\leq c(\|hu\|_\infty + \|h\bar{u}\|_\infty)\|w\|_2 + c\|w\|_2[\|u\|_3 + \|\bar{u}\|_3], \text{ a.e. in } t > 0, \\ \|\nabla \pi_1\|_2 &\leq c\|w\|_2[\|u\|_3 + \|\bar{u}\|_3], \text{ a.e. in } t > 0. \end{aligned} \quad (32)$$

Proof The results (27)–(31) of the lemma are known. Hence we limit ourselves to prove (32). Actually with obvious meaning of the symbols, we have $\pi_2 = \pi_{HLS} + \pi_{CZ}$. Estimating $\|\pi_{HLS}\|_2$ via the Hardy–Littlewood–Sobolev theorem, we get $\|\pi_{HLS}\|_2 \leq c(\|\pi_w\|_{\frac{5}{3}} + \|\bar{u} + u\|_{\frac{5}{3}})$. Hence, recalling estimate (28), Hölder’s inequality leads to $\|\pi_{HLS}\|_2 \leq c\|w\|_2(\|u\|_3 + \|\bar{u}\|_3)$. Estimating $\|\pi_{CZ}\|_2$ via the Calderon–Zigmund theorem, we get $\|\pi_{CZ}\|_2 \leq c\|(hu + h\bar{u})w\|_2$. Hence Hölder’s inequality leads to estimate $\|\pi_{CZ}\|_2 \leq c(\|hu\|_{\infty} + \|h\bar{u}\|_{\infty})\|w\|_2$. In the case of $\|\nabla\pi_1\|_2$, applying the Hardy–Littlewood–Sobolev theorem, we get $\|\nabla\pi_1\|_2 \leq c(\|\pi_w\|_{\frac{5}{3}} + \|w\|(\|\bar{u}\| + \|u\|))\|u\|_{\frac{5}{3}}$. Hence, recalling estimate (28) for $\|\pi_w\|_{\frac{5}{3}}$, Hölder’s inequality leads to $\|\nabla\pi_1\|_2 \leq c\|w\|_2(\|u\|_3 + \|\bar{u}\|_3)$, which completes the proof.

Lemma 3 Suppose that $|x|^{\beta}u \in L^2(\mathbb{R}^3)$ and $|x|^{\alpha}\nabla u \in L^2(\mathbb{R}^3)$. Also

- (i) $r \geq 2$, $\gamma + \frac{3}{r} > 0$, $\alpha + \frac{3}{2} > 0$, $\beta + \frac{3}{2} > 0$, and $a \in [\frac{1}{2}, 1]$,
- (ii) $\gamma + \frac{3}{r} = a(\alpha + \frac{1}{2}) + (1-a)(\beta + \frac{3}{2})$ (dimensional balance),
- (iii) $a(\alpha - 1) + (1-a)\beta \leq \gamma \leq a\alpha + (1-a)\beta$.

Then, with a constant c independent of u , the following inequality holds:

$$\||x|^{\gamma}u\|_r \leq c\||x|^{\alpha}\nabla u\|_2^a \||x|^{\beta}u\|_2^{1-a}. \quad (33)$$

Proof See [1] Lemma 7.1.

Let us consider the convolution of functions on $\mathbb{R}^n$:

$$K[g](x) := \int_{\mathbb{R}^n} K(x-y)g(y)dy, \quad x \in \mathbb{R}^n.$$

Set

$$p \in [1, \infty], \quad \alpha \in \mathbb{R}, \quad L_{\alpha}^p(\mathbb{R}^n) := \left\{ g : \left[\int_{\mathbb{R}^n} |g(x)|^p |x|^{\alpha p} dx \right]^{\frac{1}{p}} < \infty \right\}.$$

Theorem 9 Suppose $r, q \in (1, \infty)$ and $p \in (1, \infty]$. Then we get

$$\|K[g]\|_{L_{-\gamma}^r} \leq c\|K\|_{L_{\beta}^q} \|g\|_{L_{\alpha}^p} \text{ for all } g \in L_{\alpha}^p(\mathbb{R}^n), \quad (34)$$

if, and only if,

$$\begin{aligned} \frac{1}{r} &= \frac{1}{p} + \frac{1}{q} + \frac{\alpha + \beta + \gamma}{n} - 1 \\ \alpha &< \frac{n}{p'}, \quad \beta < \frac{n}{q'}, \quad \gamma < \frac{n}{r} \quad \text{if } p < \infty, \\ \alpha &\in (0, n), \quad \beta < \frac{n}{q'}, \quad \gamma < \frac{n}{r} \quad \text{if } p = \infty, \end{aligned} \quad (35)$$

and

$$\alpha + \gamma \geq 0, \quad \alpha + \beta \geq 0, \quad \beta + \gamma \geq 0. \quad (36)$$

Proof See [14] Theorem A'.

By virtue of this result, we deduce that if $u_0 \in L_{-\frac{1}{2}}^2(\mathbb{R}^3)$, then the corresponding heat solution does not belongs to $L^r(\mathbb{R}^3)$ (that is L_0^r) since (36)₁ does not hold (that is $-\frac{1}{2} + 0 \not\geq 0$). It is not difficult to verify that $L^3(\mathbb{R}^3)$ is not included in $L_{-\frac{1}{2}}^2(\mathbb{R}^3)$. As a consequence the $L_{-\frac{1}{2}}^2(\mathbb{R}^3)$ cannot be compared with Besov spaces detected with (21).

3 A weighted energy relation for suitable weak solutions

We reproduce, in a suitable way, Lemma 3.1 contained in [3]. Actually the proof of Lemma 3.1 of [3] consists of 5 steps. The first four steps realize a statement that we consider in the following Lemma 5.

We set $p(x, y, \mu) := (|x - y|^2 + \mu^2)^{-\frac{1}{2}}$, $\mu > 0$, and

$$\mathcal{E}(v, t, x, \mu) := \int_{\mathbb{R}^3} |v(\tau, y)|^2 p(x, y) dy, \quad \mathcal{D}(v, \tau, x, \mu) := \int_{\mathbb{R}^3} |\nabla v(\tau, y)|^2 p(x, y) dy. \quad (37)$$

We put before a lemma

Lemma 4 *Let (v, π_v) be a $J^{1,2}$ -regular solution. Then the following weighted energy inequality hold:*

$$\begin{aligned} \mathcal{E}(v, t, x, \mu) + \int_0^t \mathcal{D}(v, \tau, x, \mu) d\tau + 3\mu^2 \int_0^t \int_{\mathbb{R}^3} \frac{v^2(\tau, y)}{(|x - y|^2 + \mu^2)^{\frac{5}{2}}} dy d\tau \\ \leq \mathcal{E}(v, 0, x, \mu) + c \int_0^t \|v(\tau)\|_2 \|\nabla v(\tau)\|_2^5 d\tau, \end{aligned} \quad (38)$$

for all $\mu > 0$, $x \in \mathbb{R}^3$ and $t \in [0, T)$.

Proof In Lemma 2.5 of [3] it is proved the following:

$$\begin{aligned} \mathcal{E}(v, t, x, \mu) + \int_0^t \mathcal{D}(v, \tau, x, \mu) d\tau + 3\mu^2 \int_0^t \int_{\mathbb{R}^3} \frac{v^2(\tau, y)}{(|x - y|^2 + \mu^2)^{\frac{5}{2}}} dy d\tau \\ \leq \mathcal{E}(0, x, \mu) + c \int_0^t \mathcal{E}(v, \tau, x, \mu) \|\nabla v(\tau)\|_2^4 d\tau. \end{aligned} \quad (39)$$

Taking the definition of $\mathcal{E}(v, t, x, \mu)$ into account, and Lemma 3, then we get $\mathcal{E}(v, t, x, \mu) \leq c \|v(t)\|_2 \|\nabla v(t)\|_2$. Hence (38) follows from (39).

By the symbols $\mathcal{E}(w, t, x)$ and $\mathcal{D}(w, t, x)$ we mean (37) with $\mu = 0$.

Lemma 5 *Let (u, π_u) be a suitable weak solution. Assume that there exists $v_0 \in J^{1,2}(\mathbb{R}^3)$ such that for some $\varepsilon_0 \leq \frac{1}{4c^2}$ and $\mathbb{E} \subseteq \mathbb{R}^3$*

$$\sup_{x \in \mathbb{E}} \int_{\mathbb{R}^3} \frac{|u_0(y) - v_0(y)|^2}{|x - y|} dy < \varepsilon_0. \quad (40)$$

Let (v, π_v) be the $J^{1,2}$ -regular solution corresponding to v_0 and set $w := u - v$. Then there exists a $t^ > 0$ such that*

$$\mathcal{E}(w, t, x) + \frac{1}{2} \int_0^t \mathcal{D}(w, \tau, x) d\tau \leq \varepsilon_0, \text{ for all } t \in [0, t^*) \text{ and for all } x \in \mathbb{E}. \quad (41)$$

If assumption (40) holds with $v_0 = 0$, then we get (41) with $t^ = \infty$.*

Proof The proof of estimate (41) reproduces in a suitable way an idea employed in [2]. This idea follows the Leray arguments employed for the proof of the energy inequality in strong form. The proof is achieved by means of four steps.

Step 1. We start proving that for all $t > 0$, $x \in \mathbb{R}^3$ and $\mu > 0$

$$\begin{aligned} \mathcal{E}(u, t, x, \mu) + 2 \int_0^t \mathcal{D}(u, \tau, x, \mu) d\tau + \int_0^t \int_{\mathbb{R}^3} |u(\tau)|^2 \Delta_y p dy d\tau \leq \mathcal{E}(u, 0, x, \mu) \\ + \int_0^t \int_{\mathbb{R}^3} \frac{|u(\tau)|^2 u \cdot (x-y)}{(|x-y|^2 + \mu^2)^{\frac{3}{2}}} dy d\tau + 2 \int_0^t \int_{\mathbb{R}^3} \frac{\pi_u(\tau) u(\tau) \cdot (x-y)}{(|x-y|^2 + \mu^2)^{\frac{3}{2}}} dy d\tau. \end{aligned} \quad (42)$$

In the energy inequality (2) we set $\phi(\tau, y) := (|x-y|^2 + \mu^2)^{-\frac{1}{2}} h_R(y) k(\tau) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$, with h_R and k such that

$$h_R(y) := \begin{cases} 1 & \text{if } |y| \leq R \\ \in (0, 1) & \text{if } |y| \in (R, 2R) \\ 0 & \text{for } |y| \geq 2R, \end{cases} \quad \text{and } k(\tau) := \begin{cases} 1 & \text{if } |\tau| \leq t \\ \in (0, 1) & \text{if } |\tau| \in (t, 2t) \\ 0 & \text{for } |\tau| \geq 2t. \end{cases}$$

We get

$$\begin{aligned} \int_{\mathbb{R}^3} |u(t)|^2 h_R p dy + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u(\tau)|^2 h_R p dy d\tau + 3\mu^2 \int_0^t \int_{\mathbb{R}^3} \frac{|u(\tau)|^2 h_R}{(|x-y|^2 + \mu^2)^{\frac{5}{2}}} dy d\tau \\ \leq \int_{\mathbb{R}^3} |u_0|^2 h_R p dy + \int_0^t \int_{\mathbb{R}^3} \frac{|u(\tau)|^2 h_R u \cdot (x-y)}{(|x-y|^2 + \mu^2)^{\frac{3}{2}}} dy d\tau \\ + 2 \int_0^t \int_{\mathbb{R}^3} \frac{\pi_u(\tau) h_R u(\tau) \cdot (x-y)}{(|x-y|^2 + \mu^2)^{\frac{3}{2}}} dy d\tau + F(t, R) \\ := \int_{\mathbb{R}^3} |u_0|^2 h_R p dy + I_1(t, x) + I_2(t, x) + F(t, R), \end{aligned} \quad (43)$$

where we set

$$F(t, R) := \int_0^t \int_{\mathbb{R}^3} |u|^2 \left[2 \nabla h_R \cdot \nabla p + \Delta h_R p + u \cdot \nabla h_R p \right] dy d\tau + \int_0^t \int_{\mathbb{R}^3} \pi_u u \cdot \nabla h_R p dy d\tau.$$

Since $\pi_u, u^2 \in L^{\frac{5}{3}}(0, T; L^{\frac{5}{3}}(\mathbb{R}^3))$, applying Hölder's inequality and employing the decay of $\nabla h_R, \Delta h_R$, for all $t > 0$, we get $F(t, R) = o(R)$. We estimate the terms I_i , $i = 1, 2$. Since $\mu > 0$, by virtue of the integrability properties of a suitable weak solution, applying Lemma 3 we get

$$|I_1(t, x)| \leq \int_0^t \|u p^{-\frac{2}{3}}\|_3^3 d\tau \leq c \int_0^t \|u p^{-\frac{1}{2}}\|_2 \|(\nabla u) p^{-\frac{1}{2}}\|_2^2 d\tau.$$

For I_2 , applying Hölder's inequality and inequality (25), we obtain

$$|I_2(t, x)| \leq c \int_0^t \|up^{-\frac{2}{3}}\|_3 \|\pi_u p^{-\frac{4}{3}}\|_{\frac{3}{2}} d\tau \leq c \int_0^t \|up^{-\frac{2}{3}}\|_3^3 d\tau.$$

Hence, as in the previous case, applying Lemma 3, we get

$$|I_2(t, x)| \leq c \int_0^t \|up^{-\frac{1}{2}}\|_2 \|(\nabla u) p^{-\frac{1}{2}}\|_2^2 d\tau.$$

Recalling that $F(R, t)$ is an $o(R)$, employing the estimates obtained for I_i , $i = 1, 2$, via the Lebesgue dominated convergence theorem, in the limit as $R \rightarrow \infty$, for all $t > 0$ we deduce the inequality (42).

Step 2. In this step we derive a sort of Green's identity between solutions (u, π_u) , which is a suitable weak solution, and (v, π_v) , where (v, π_v) is the regular solution considered in Lemma 4, corresponding to the initial data $v_0 \in J^{1,2}(\mathbb{R}^3)$. In the following $(0, T)$ is the interval of existence of (v, π_v) . We also recall that the regular solution (v, π_v) is smooth for $t \in (0, T)$. We denote by $\lambda(\tau)$ a smooth cutoff function such that $\lambda(\tau) = 1$ for $\tau \in [s, t]$ and $\lambda(\tau) = 0$ for $\tau \in [0, \frac{s}{2}]$.

For all $t, s \in (0, T)$, we consider the weak formulation iii) of Definition 1 written with $\varphi = \lambda v p$:

$$\begin{aligned} & \int_s^t \left[(pu, v_\tau) - (p \nabla u, \nabla v) + (pu \cdot \nabla v, u) + (\pi_u, v \cdot \nabla p) \right] d\tau + (pu(s), v(s)) \\ &= (pu(t), v(t)) + \int_s^t \left[(\nabla u, v \otimes \nabla p) + (u \otimes u, v \otimes \nabla p) \right] d\tau. \end{aligned} \quad (44)$$

We multiply equation (1)₁ written for (v, π_v) by up . After integrating by parts on $(s, t) \times \mathbb{R}^3$, we get

$$\begin{aligned} & \int_s^t \left[(pu, v_\tau) + (p \nabla u, \nabla v) + (pv \cdot \nabla v, u) - (\pi_v, u \cdot \nabla p) \right] d\tau \\ &= \int_s^t \left[(\nabla u, v \otimes \nabla p) + (u \cdot v, \Delta p) \right] d\tau. \end{aligned} \quad (45)$$

making the difference between formulas (44) and (45) we get

$$\begin{aligned} & \int_s^t \left[-2(p \nabla u, \nabla v) + (pu \cdot \nabla v, u) - (pv \cdot \nabla v, u) + (\pi_u, v \cdot \nabla p) + (\pi_v, u \cdot \nabla p) \right] d\tau \\ &= (pu(t), v(t)) - (pu(s), v(s)) + \int_s^t \left[(u \otimes u, v \otimes \nabla p) - (u \cdot v, \Delta p) \right] d\tau, \end{aligned}$$

Since in a suitable neighborhood of 0 all the terms of the last integral equation are continuous on the right, letting $s \rightarrow 0^+$, we get

$$\begin{aligned} & \int_0^t \left[(pu \cdot \nabla v, u) - 2(p \nabla u, \nabla v) - (pv \cdot \nabla v, u) + (\pi_u, v \cdot \nabla p) + (\pi_v, u \cdot \nabla p) \right] d\tau \\ &= (pu(t), v(t)) - (pu(0), v_0) + \int_0^t \left[(u \otimes u, v \otimes \nabla p) - (u \cdot v, \Delta p) \right] d\tau, \end{aligned} \quad (46)$$

which furnishes the wanted Green's identity.

Step 3. Setting $w := u - v$ and $\pi_w := \pi_u - \pi_v$, let us derive the following estimate

$$\begin{aligned} & \mathcal{E}(w, t, x, \mu) + \int_0^t \mathcal{D}(w, \tau, x, \mu) d\tau \\ & \leq \mathcal{E}(w, 0, x, \mu) + c \int_0^t \mathcal{E}^{\frac{1}{2}}(w, \tau, x, \mu) \mathcal{D}(w, \tau, x, \mu) d\tau \\ & \quad + H(w, t, x, \mu), \text{ for all } t \in [0, T], x \in \mathbb{R}^3, \mu > 0, \end{aligned} \quad (47)$$

where we set

$$H(w, t, x, \mu) := c \int_0^t \|\nabla v(\tau)\|_2^4 d\tau + c \int_0^t \mathcal{E}(v, \tau, x, \mu) \mathcal{D}(v, \tau, x, \mu) d\tau.$$

We remark that from the representation formula (23) and the regularity of v we get that

$$\begin{aligned} \pi_w &= \pi^1 + \pi^2, \\ \pi^1 &:= D_{x_j} \int_{\mathbb{R}^3} D_{y_i} \mathcal{E}(x-y) w^i(y) w^j(y) dy, \\ \pi^2 &:= 2 \int_{\mathbb{R}^3} D_{y_j} \mathcal{E}(x-y) w(y) \cdot \nabla v^j(y) dy. \end{aligned} \quad (48)$$

We sum estimates (38) and (42), then we add twice formula (46), written for $s = 0$. Recalling the definition of (w, π_w) and formula (48), after a straightforward computation we get

$$\begin{aligned} & \mathcal{E}(w, t, x, \mu) + 2 \int_0^t \mathcal{D}(w, \tau, x, \mu) d\tau + 3\mu^2 \int_0^t \int_{\mathbb{R}^3} \frac{w^2(\tau, x)}{(|x-y|^2 + \mu^2)^{\frac{5}{2}}} d\tau \\ & \leq \mathcal{E}(w, 0, x, \mu) + F_1(w, t, x, \mu) + F_2(w, v, t, x, \mu), \end{aligned} \quad (49)$$

where

$$\begin{aligned} F_1 &:= F_1(w, t, x, \mu) := \int_0^t (w \otimes w, w \otimes \nabla p) d\tau + 2 \int_0^t (\pi_1, w \cdot \nabla p) d\tau \\ F_2 &:= F_2(w, v, t, x, \mu) := 2 \int_0^t (\pi_2, w \cdot \nabla p) d\tau - 2 \int_0^t (w \cdot \nabla v, wp) d\tau + \int_0^t (v \cdot \nabla p, w^2). \end{aligned}$$

The term F_1 admits the same estimate as I_1 and I_2 given in *Step 1*, hence we get

$$|F_1| \leq c \int_0^t \mathcal{E}^{\frac{1}{2}}(\tau, w, x, \mu) \mathcal{D}(\tau, w, x, \mu) d\tau \quad \text{for all } t \in (0, T), x \in \mathbb{E}, \mu > 0.$$

For term F_2 we estimate the first two terms in a different way from the last. Taking the representation formula of π_2 into account, we get

$$\left| \int_0^t (\pi_2, w \cdot \nabla p) d\tau - \int_0^t (w \cdot \nabla v, wp) d\tau \right| = \left| \int_0^t p \nabla \pi_2 \cdot w dy d\tau + \int_0^t (w \cdot \nabla v, wp) dy d\tau \right|.$$

Hence, applying the same arguments employed in Lemma 4 to estimate J_1 and J_2 , for all $t \in [0, T)$, $x \in \mathbb{E}$, $\mu > 0$, we get

$$\begin{aligned} \left| \int_0^t (\pi_2, w \cdot \nabla p) d\tau - \int_0^t (w \cdot \nabla v, wp) d\tau \right| &\leq \int_0^t \|wp^{\frac{1}{2}}\|_4^2 \|\nabla v\|_2 d\tau \\ &\leq \int_0^t \mathcal{E}^{\frac{1}{3}}(w, \tau, x, \mu) \mathcal{D}(w, \tau, x, \mu) d\tau + c \int_0^t \|\nabla v(\tau)\|_2^4 d\tau. \end{aligned}$$

For the last term in F_2 , applying Hölder's inequality, we get

$$\left| \int_0^t (v \cdot \nabla p, w^2) d\tau \right| \leq \int_0^t \|wp^{\frac{1}{2}}\|_4^2 \|vp\|_2^2 d\tau.$$

By virtue of estimate (33), applying Young's inequality with obvious meaning of the symbols we deduce:

$$\begin{aligned} \left| \int_0^t (v \cdot \nabla p, w^2) d\tau \right| &\leq c \int_0^t \mathcal{E}^{\frac{1}{4}}(w, \tau) \mathcal{D}^{\frac{3}{4}}(w, \tau) \mathcal{E}^{\frac{1}{4}}(v, \tau) \mathcal{D}^{\frac{1}{4}}(v, \tau) d\tau \\ &\leq \int_0^t \mathcal{E}^{\frac{1}{3}}(w, \tau) \mathcal{D}(w, \tau) d\tau + c \int_0^t \mathcal{E}(v, \tau) \mathcal{D}(v, \tau) d\tau. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} |F_2| &\leq 2 \int_0^t \mathcal{E}^{\frac{1}{3}}(w, \tau, x, \mu) \mathcal{D}(w, \tau, x, \mu) d\tau + c \int_0^t \|\nabla v(\tau)\|_2^4 d\tau \\ &\quad + c \int_0^t \mathcal{E}(v, \tau, x, \mu) \mathcal{D}(v, \tau, x, \mu) d\tau, \quad \text{for all } t \in [0, T), x \in \mathbb{E}, \mu > 0. \end{aligned}$$

Finally, applying Young's inequality with obvious meaning of the symbols we get

$$|F_2| \leq \int_0^t \mathcal{D}(w, \tau) d\tau + c \int_0^t \mathcal{E}^{\frac{1}{2}}(w, \tau) \mathcal{D}(w, \tau) d\tau + c \int_0^t \|\nabla v(\tau)\|_2^4 d\tau \\ + c \int_0^t \mathcal{E}(v, \tau) \mathcal{D}(v, \tau) d\tau, \text{ for all } t \in [0, T), x \in \mathbb{E}, \mu > 0.$$

Estimates for F_1 , F_2 and (49) furnish the integral inequality (47).

Step 4. Deduction of estimate (41).

Under our assumptions, a fortiori we get

$$\mathcal{E}(w, 0, x, \mu) < \varepsilon_0, \text{ for all } \mu > 0 \text{ and } x \in \mathbb{E}. \quad (50)$$

Moreover, by virtue of the $J^{1,2}$ -regularity of the solution (v, π_v) there exists a t^* such that

$$H(v, t, x, \mu) < \varepsilon_0 - \mathcal{E}(w, 0, x, \mu), \text{ for all } x \in \mathbb{E} \text{ and } \mu > 0 \text{ and } t \in (0, t^*). \quad (51)$$

Since u and v are right continuous in L^2 -norm in $t = 0$, for all $\mu > 0$ and $x \in \mathbb{E}$ the same continuity property holds for $\mathcal{E}(w, t, x, \mu)$. Therefore there exists a $\delta = \delta(x, \mu) > 0$ such that

$$\mathcal{E}(w, t, x, \mu) \leq \varepsilon_0, \text{ for all } t \in [0, \delta). \quad (52)$$

Hence the validity of estimates (47) and (51) yields for any $t \in [0, \delta)$

$$\mathcal{E}(w, t, x, \mu) + \int_0^t \mathcal{D}(w, \tau, x, \mu) d\tau < \varepsilon_0 + c \int_0^t \mathcal{E}^{\frac{1}{2}}(w, \tau, x, \mu) \mathcal{D}(w, \tau, x, \mu) d\tau, \quad (53)$$

that, thanks to (52) and our assumption on ε_0 , gives (41) on $[0, \delta)$.

Let us show that estimate (52) holds for $t \in [0, t^*)$. For all $x \in \mathbb{E}$ and $\mu > 0$, the function

$$f(t, x, \mu) := \mathcal{E}(w, 0, x, \mu) + c \int_0^t \mathcal{E}^{\frac{1}{2}}(w, \tau, x, \mu) \mathcal{D}(w, \tau, x, \mu) d\tau + H(v, t, x, \mu)$$

is uniformly continuous on $[0, t^*)$. Hence there exists $\eta = \eta(x, \mu) > 0$ such that

$$|t_1 - t_2| < \eta \Rightarrow |f(t_1) - f(t_2)| < \varepsilon \varepsilon_0.$$

We claim that estimate (52) and, consequently, estimate (41), also holds for $t \in [\delta, \delta + \eta)$. Assuming the contrary, there exists $\bar{t} \in [\delta, \delta + \eta)$ such that for some $\varepsilon > 0$ we have

$$\mathcal{E}(w, \bar{t}, x, \mu) \geq (1 + \varepsilon) \varepsilon_0. \quad (54)$$

On the other hand, the validity of (47) and assumption (51) yield

$$\mathcal{E}(w, \bar{t}, x, \mu) + \int_0^{\bar{t}} \mathcal{D}(w, \tau, x, \mu) d\tau \leq (f(\bar{t}) - f(\delta)) + f(\delta) \\ < (1 + \varepsilon) \varepsilon_0 + c \int_0^{\delta} \mathcal{E}^{\frac{1}{2}}(w, \tau, x, \mu) \mathcal{D}(w, \tau, x, \mu) d\tau.$$

Estimate (52) allows to deduce that

$$c \int_0^\delta \mathcal{E}^{\frac{1}{2}}(w, \tau, x, \mu) \mathcal{D}(w, \tau, x, \mu) d\tau < \frac{1}{2} \int_0^\delta \mathcal{D}(w, \tau, x, \mu) d\tau.$$

Hence the last two estimates imply

$$\mathcal{E}(w, \bar{t}, x, \mu) < (1 + \varepsilon)\varepsilon_0,$$

which is in contradiction with (54). Since the arguments are independent of δ , the result holds for any $t \in [0, t^*)$, which, for all $\varepsilon > 0$, proves

$$\mathcal{E}(w, t, x, \mu) < (1 + \varepsilon)\varepsilon_0, \text{ for all } t \in [0, t^*), x \in \mathbb{E} \text{ and } \mu > 0.$$

This last and the validity of estimates (47) and assumption (51), for all $\varepsilon > 0$, allow us to deduce

$$\mathcal{E}(w, t, x, \mu) + \frac{1 - \varepsilon}{2} \int_0^t \mathcal{D}(w, \tau, x, \mu) d\tau < \varepsilon_0, \text{ for all } t \in [0, t^*), x \in \mathbb{E} \text{ and } \mu > 0.$$

By virtue of dominate convergence theorem, and then employing the arbitrariness of $\varepsilon > 0$, we also get

$$\mathcal{E}(w, t, x, 0) + \frac{1}{2} \int_0^t \mathcal{D}(w, \tau, x, 0) d\tau \leq \varepsilon_0, \text{ for all } t \in [0, t^*), x \in \mathbb{E},$$

which completes the proof of (41). Assuming $v_0 = 0$, in the above computations we get $H(v, t, x, \mu) = 0$ for all $t > 0$. Starting from (53), and being t^* free from constrains, the claim is immediate on $(0, \infty)$.

4 A result of partial regularity for a suitable weak solution to the perturbed Navier–Stokes system

Let us consider the following perturbed Navier–Stokes Cauchy problem:

$$\begin{aligned} w_t - \Delta w + \nabla \pi &= -w \cdot \nabla w - v \cdot \nabla w - w \cdot \nabla v, \\ \nabla \cdot w &= 0, \text{ in } (0, T) \times \mathbb{R}^3, \\ w &= w_0, \text{ on } \{0\} \times \mathbb{R}^3. \end{aligned} \quad (55)$$

We assume that for all $T > 0$

$$v \in L^\infty(0, T; J^{1,2}(\mathbb{R}^3)) \cap L^2(0, T; W^{2,2}(\mathbb{R}^3)). \quad (56)$$

For weak solution to the Cauchy problem (55) we mean a pair (w, π) which satisfies Definition 1 with the integral equation iii) substituted by the weak formulation of (55).

For any non negative scalar function $\phi(t, x) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$, we set

$$\mathcal{E}(t, \phi) := \int_{\mathbb{R}^3} |w|^2 \phi dx, \quad \mathcal{D}(t, \phi) := \int_{\mathbb{R}^3} |\nabla w|^2 \phi dx.$$

For a suitable weak solution to problem (55) we mean a weak solution (w, π) to problem (55) such that for all $t > s$ a.e. in $s > 0$ and for $s = 0$

$$\begin{aligned} \mathcal{E}(t, \phi) + 2 \int_s^t \mathcal{D}(\tau, \phi) d\tau &\leq \mathcal{E}(s, \phi) + \int_s^t (|w|^2, \phi_\tau + \Delta \phi) d\tau \\ &\quad + \int_s^t (|w|^2, (w + v) \cdot \nabla \phi) d\tau - 2 \int_s^t (w \cdot \nabla v, w \phi) d\tau \\ &\quad + 2 \int_s^t (\pi, w \cdot \nabla \phi) d\tau. \end{aligned} \quad (57)$$

We set

$$M(w, t, x, r) := r^{-2} \int_{Q_r} (|w|^3 + |w||\pi|) + r^{-\frac{13}{4}} \int_{t-r^2}^t \left(\int_{|x-y|<r} |\pi| dy \right)^{\frac{5}{4}} d\tau, \quad (58)$$

where, we recall, we set

$$Q_r = Q_r(t, x) := \{(\tau, y) : t - r^2 < \tau < t \text{ and } |y - x| < r\}. \quad (59)$$

When the parabolic cylinder Q_r is centered in $(0, 0)$, then the above neighborhood becomes

$$Q_r = Q_r(0, 0) := \{(\tau, y) : -r^2 < \tau < 0 \text{ and } |y| < r\}. \quad (60)$$

In the computations one can indifferently argument on the parabolic cylinder $Q_r(t, x)$ and $Q_r(0, 0)$. Actually, as remarked in [1], in some steps of the proofs the adoption of the cylinder $Q_r(0, 0)$ simplifies the notation, as the inductive procedure employed to prove Lemma 6 below. The usual euclidean ball is denoted by $B_r(x)$ and by B_r , when no confusion occurs.

Our aim is to prove

Lemma 6 *Let (w, π) be a suitable weak solution to problem (55) in some parabolic cylinder $Q_r(t, x)$. There exist absolute positive constants $\varepsilon_1, c_0 > 0$ and $\varepsilon_2 > 0$ such that, assuming*

$$M(w, t, x, r) \leq \varepsilon_1, \quad (61)$$

and

$$r \iint_{Q_r} |v|^6 + r^2 \int_{t-r^2}^t \|v(\tau)\|_\infty^4 d\tau + r \iint_{Q_r} |\nabla v|^3 < \varepsilon_2, \quad (62)$$

then

$$|w(\tau, y)| \leq c_0 \varepsilon_1^{\frac{2}{3}} r^{-1}, \text{ a.e. in } (\tau, y) \in Q_{\frac{r}{2}}(t, x). \quad (63)$$

In particular, a suitable weak solution w is regular in $Q_{\frac{r}{2}}(t, x)$.

Of course the proof of the above proposition consists in an adaption to problem (55) of the well known result by Caffarelli–Kohn–Nirenberg (see [1] Proposition 1 and Corollary 1).

In order to prove Lemma 6 we need some preliminary lemmas that are related to the terms of the “unperturbed” field v .

We assume that $\{\phi_m\}$ is a sequence of cutoff functions such that

$$\begin{aligned} \frac{1}{c}r_m^{-3} \leq \phi_m \leq cr_m^{-3}, \quad |\nabla \phi_m| \leq cr_m^{-4} \text{ on } Q_m, \quad 2 \leq m, \\ \phi_m \leq cr_k^{-3}, \quad |\nabla \phi_m| \leq cr_k^{-4} \text{ on } Q_{k-1} - Q_k, \quad 1 < k \leq m, \end{aligned} \quad (64)$$

with $r_k := 2^{-k}$ and Q_k given by (59) with radius $r_k := 2^{-k}$. Moreover, for some $\varepsilon_1 > 0$ and for all k , we set

$$\iint_{Q_k} |w|^3 \leq r_k^5 \varepsilon_1^{\frac{2}{3}} \Leftrightarrow \iint_{Q_k} |w|^3 \leq \varepsilon_1^{\frac{2}{3}} \quad (65)$$

Lemma 7 *Let v and w be two fields enjoying (56) and (65), respectively. Then we get*

$$\left| \iint_{Q_1} w^2 v \cdot \nabla \phi_m \right| + \left| \iint_{Q_1} w \cdot \nabla v \cdot w \phi_m \right| \leq c \varepsilon_1^{\frac{4}{9}} \left[\left(\iint_{Q_1} |v|^6 \right)^{\frac{1}{6}} + \left(\iint_{Q_1} |\nabla v|^3 \right)^{\frac{1}{3}} \right], \quad (66)$$

where ϕ_m is defined in (64).

Proof We separate the estimates of the two integrals. Taking into account the properties (64) and employing (65), applying Hölder's inequality, we get

$$\begin{aligned} \left| \iint_{Q_1} w^2 v \cdot \nabla \phi_m \right| &\leq c \sum_{k=1}^m r_k^{-4} \iint_{Q_k} |v| w^2 \leq c \sum_{k=1}^m r_k^{-\frac{19}{6}} \left[\iint_{Q_k} |w|^3 \right]^{\frac{2}{3}} \left[\iint_{Q_k} |v|^6 \right]^{\frac{1}{6}} \\ &\leq c \sum_{k=1}^m r_k^{\frac{1}{6}} \left[\iint_{Q_k} |w|^3 \right]^{\frac{2}{3}} \left[\iint_{Q_k} |v|^6 \right]^{\frac{1}{6}} \leq c \varepsilon_1^{\frac{4}{9}} \sum_{k=1}^{\infty} r_k^{\frac{1}{6}} \left[\iint_{Q_1} |v|^6 \right]^{\frac{1}{6}}. \end{aligned}$$

Analogously, we get

$$\begin{aligned} \left| \iint_{Q_1} w \cdot \nabla v \cdot w \phi_m \right| &\leq c \sum_{k=1}^m r_k^{-3} \iint_{Q_k} |\nabla v| w^2 \leq c \sum_{k=1}^m r_k^{-3} \left[\iint_{Q_k} |w|^3 \right]^{\frac{2}{3}} \left[\iint_{Q_k} |\nabla v|^3 \right]^{\frac{1}{3}} \\ &\leq c \sum_{k=1}^m r_k^{\frac{1}{3}} \left[\iint_{Q_k} |w|^3 \right]^{\frac{2}{3}} \left[\iint_{Q_k} |\nabla v|^3 \right]^{\frac{1}{3}} \leq c \varepsilon_1^{\frac{4}{9}} \sum_{k=1}^{\infty} r_k^{\frac{1}{3}} \left[\iint_{Q_1} |\nabla v|^3 \right]^{\frac{1}{3}}. \end{aligned}$$

The above inequalities lead to the proof of (66).

Denoted by ϕ a smooth nonnegative cutoff function with support in B_ρ and such that $\phi = 1$ on $B_{\frac{3}{4}\rho}$, taking into account that from (55) we formally deduce $\Delta \pi = \nabla w \cdot (\nabla w)^T + 2 \nabla w \cdot (\nabla v)^T$, then for the pressure field π of (55) the following representation formula holds:

$$\pi \phi := \pi_w + \pi_v,$$

where we set

$$\begin{aligned}\pi_w &:= \tilde{\pi}_w + \pi_{w_2} + \pi_{w_3}, \quad \text{with } \tilde{\pi}_w := -\frac{3}{4\pi} \nabla_x \int_{\mathbb{R}^3} \nabla_y \frac{1}{|x-y|} \phi w \otimes w dy, \\ \pi_{w_2} &:= \frac{3}{2\pi} \int_{\mathbb{R}^3} w \otimes w \cdot \nabla \frac{1}{|x-y|} \otimes \nabla \phi dy + \frac{3}{4\pi} \int_{\mathbb{R}^3} w \otimes w \cdot \frac{1}{|x-y|} \cdot \nabla \nabla \phi dy, \\ \pi_{w_3} &:= \frac{3}{2\pi} \int_{\mathbb{R}^3} \pi \nabla \frac{1}{|x-y|} \cdot \nabla \phi dy + \frac{3}{4\pi} \int_{\mathbb{R}^3} \pi \frac{1}{|x-y|} \Delta \phi dy,\end{aligned}$$

and

$$\begin{aligned}\pi_v &:= \tilde{\pi}_v + \pi_{v_2}, \quad \text{with } \tilde{\pi}_v := -\frac{3}{2\pi} \nabla_x \int_{\mathbb{R}^3} \nabla_y \frac{1}{|x-y|} \phi w \otimes v dy, \\ \pi_{v_2} &:= \frac{3}{\pi} \int_{\mathbb{R}^3} w \otimes v \cdot \nabla \frac{1}{|x-y|} \otimes \nabla \phi dy + \frac{3}{2\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} w \otimes v \cdot \nabla \nabla \phi dy.\end{aligned}$$

For our aims is important to evaluate the term

$$L(r) := r^{-2} \iint_{Q_r} |w| \cdot |\pi - \bar{\pi}_r|, \quad (67)$$

where we set

$$\bar{\pi}_r := \oint_{B_r} \pi dy,$$

that we increase by

$$L(r) \leq r^{-2} \iint_{Q_r} |w| \cdot |\pi_w - \bar{\pi}_{w_r}| + r^{-2} \iint_{Q_r} |w| \cdot |\pi_v - \bar{\pi}_{v_r}| =: L_w(r) + L_v(r), \quad (68)$$

where we set

$$\bar{\pi}_{w_r} := \oint_{B_r} \pi_w dy \quad \text{and} \quad \bar{\pi}_{v_r} := \oint_{B_r} \pi_v dy.$$

We introduce the following quantities related to (w, π) :

$$\begin{aligned}A(r) &:= \sup_{-r^2 < t < 0} r^{-1} \int_{B_r} |w(t)|^2, \quad \delta(r) := r^{-1} \iint_{Q_r} |\nabla w|^2, \\ G(r) &:= r^{-2} \iint_{Q_r} |w|^3, \quad K(r) := r^{-\frac{13}{4}} \int_{-r^2}^0 \left[\int_{B_r} |\pi| \right]^{\frac{5}{4}} d\tau.\end{aligned}$$

The definition of suitable weak solution to problem (55) ensures that $\pi \in L^{\frac{5}{3}}(0, T; L^{\frac{5}{3}}(\mathbb{R}^3))$. Hence we get

$$K(r) < \infty \quad \text{for all } r > 0. \quad (69)$$

Analogously, via assumption (62) related to (v, π_v) , for all $r > 0$, the following is well posed

$$V(r) := r^{-2} \iint_{Q_r} |v|^3.$$

Lemma 8 *The following estimate holds:*

$$G(r) \leq c A^{\frac{3}{4}}(r) [A^{\frac{3}{4}}(r) + \delta^{\frac{3}{4}}(r)], \quad (70)$$

where c is a constant independent of r, w .

Proof See [1] Lemma 3.1.

In [1] the authors prove Lemma 3.2 related to $L_w(r)$:

Lemma 9 *Let $r \leq \frac{1}{2}\rho$, then*

$$\begin{aligned} L_w(r) \leq & c \left[\frac{r}{\rho} \right]^{\frac{7}{5}} A^{\frac{1}{5}}(r) G^{\frac{1}{5}}(r) K_w^{\frac{4}{5}}(\rho) + c \left[\frac{r}{\rho} \right]^{\frac{5}{3}} G^{\frac{1}{3}}(r) G^{\frac{2}{3}}(\rho) \\ & + c G^{\frac{1}{3}}(r) G^{\frac{2}{3}}(2r) + cr^3 G^{\frac{1}{3}}(r) \sup_{-r^2 < t < 0} \int_{2r < |y| < \rho} \frac{|w(t)|^2}{|y|^4} \end{aligned} \quad (71)$$

where c is a constant independent of r, w, π_w and ρ .

By the same arguments employed in [1] to obtain Lemma 9 (that is, Lemma 3.2 in [1]), we prove for the term $L_v(r)$ the following

Lemma 10 *Let $r \leq \frac{1}{2}\rho$, then*

$$\begin{aligned} L_v(r) \leq & c \left[\frac{r}{\rho} \right]^{\frac{5}{3}} G^{\frac{1}{3}}(r) G^{\frac{1}{3}}(\rho) V^{\frac{1}{3}}(\rho) + c G^{\frac{1}{3}}(r) G^{\frac{1}{3}}(2r) V^{\frac{1}{3}}(2r) \\ & + cr^2 G^{\frac{1}{3}}(r) \sup_{-r^2 < t < 0} \left[\int_{2r < |y| < \rho} \frac{|w|^2}{|y|^4} dy \right]^{\frac{1}{2}} \left[\int_{-r^2}^0 \|v(t)\|_{L^\infty(B_\rho)}^4 dt \right]^{\frac{1}{4}}, \end{aligned} \quad (72)$$

where c is a constant independent of r, v, π_v and ρ .

Proof We recall that, via the representation formula of $\pi_v = \tilde{\pi}_v + \pi_{v_2}$, we are going to discuss the following inequality:

$$\begin{aligned} L_v(r) &= r^{-2} \iint_{Q_r} |w| \cdot |\pi_v - \bar{\pi}_v| \\ &\leq r^{-2} \iint_{Q_r} |w| \cdot |\tilde{\pi}_v - \bar{\tilde{\pi}}_v| + r^{-2} \iint_{Q_r} |w| \cdot |\pi_{v_2} - \bar{\pi}_{v_2}|. \end{aligned} \quad (73)$$

For the latter term on the right hand side of (73), in a sequential way we apply Hölder's inequality:

$$\begin{aligned}
 r^{-2} \iint_{Q_r} |w| \cdot |\pi_{v2} - \bar{\pi}_{v2}| &\leq cr \int_{-r^2}^0 \left[\int_{B_r} |w(\tau, y)|^3 dy \right]^{\frac{1}{3}} \sup_{B_r} |\nabla \pi_{v2}(\tau)| d\tau \\
 &\leq c \frac{r}{\rho^4} \int_{-r^2}^0 \left[\int_{B_r} |w(\tau, y)|^3 dy \right]^{\frac{1}{3}} \int_{B_\rho} |w(\tau, y)| |v(\tau, y)| dy d\tau \\
 &\leq c \frac{r}{\rho^3} \left[\iint_{Q_r} |w|^3 \right]^{\frac{1}{3}} \left[\iint_{Q_\rho} |w|^3 \right]^{\frac{1}{3}} \left[\iint_{Q_\rho} |v|^3 \right]^{\frac{1}{3}} \\
 &\leq c \left[\frac{r}{\rho} \right]^{\frac{5}{3}} G^{\frac{1}{3}}(r) G^{\frac{1}{3}}(\rho) V^{\frac{1}{3}}(\rho),
 \end{aligned}$$

thus we have obtained the first term on the right hand side of (72). For the former term on the right-hand side of (73), we consider $\tilde{\pi}_v := \pi'_v + \pi''_v$, where we set

$$\begin{aligned}
 \pi'_v &:= -\frac{3}{4\pi} \nabla_x \int_{|x-y| < 2r} \nabla_y \frac{1}{|x-y|} \phi w \otimes v dy \\
 \pi''_v &:= -\frac{3}{4\pi} \nabla_x \int_{|x-y| > 2r} \nabla_y \frac{1}{|x-y|} \phi w \otimes v dy.
 \end{aligned}$$

The term π'_v has the kernel which is singular of Calderon–Zigmund type. Hence one easily deduce

$$r^{-2} \iint_{Q_r} |w| |\pi'_v - \bar{\pi}'_v| \leq cr^{-2} \left[\iint_{Q_r} |w|^3 \right]^{\frac{1}{3}} \left[\iint_{Q_{2r}} |w|^{\frac{3}{2}} |v|^{\frac{3}{2}} \right]^{\frac{2}{3}} \leq c G^{\frac{1}{3}}(r) G^{\frac{1}{3}}(2r) V^{\frac{1}{3}}(2r).$$

For π''_v we initially stress that

$$\|\pi''_v - \bar{\pi}''_v\|_{L^\infty(B_r)} \leq cr \|\nabla \pi''_v\|_{L^\infty(B_r)},$$

and

$$\|\nabla \pi''_v\|_{L^\infty(B_r)} \leq c \int_{2r < |y| < \rho} \frac{|w||v|}{|y|^4} dy \leq c \left[\int_{2r < |y| < \rho} \frac{|w|^2}{|y|^4} dy \right]^{\frac{1}{2}} \left[\int_{2r < |y| < \rho} \frac{|v|^2}{|y|^4} dy \right]^{\frac{1}{2}}.$$

Thus

$$\begin{aligned}
 r^{-2} \iint_{Q_r} |w| |\pi''_v - \bar{\pi}''_v| &\leq cr \left[\iint_{Q_r} |w|^3 \right]^{\frac{1}{3}} \left[\int_{-r^2}^0 \|\nabla \pi''_v(t)\|_{L^\infty(B_r)}^{\frac{3}{2}} dt \right]^{\frac{2}{3}} \\
 &\leq cr \left[\iint_{Q_r} |w|^3 \right]^{\frac{1}{3}} \left[\int_{-r^2}^0 \int_{2r < |y| < \rho} \frac{|w|^2}{|y|^4} dy \right]^{\frac{3}{4}} \left[\int_{2r < |y| < \rho} \frac{|v|^2}{|y|^4} dy \right]^{\frac{3}{4}} dt^{\frac{2}{3}} \\
 &\leq cr^{\frac{5}{3}} G^{\frac{1}{3}}(r) \left[\int_{-r^2}^0 \int_{2r < |y| < \rho} \frac{|w|^2}{|y|^4} dy \right]^{\frac{3}{4}} \left[\int_{2r < |y| < \rho} \frac{|v|^2}{|y|^4} dy \right]^{\frac{3}{4}} dt^{\frac{2}{3}}.
 \end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned}
 r^{-2} \iint_{Q_r} |w| |\pi_v'' - \bar{\pi}_v''| &\leq cr^{\frac{5}{3}} G^{\frac{1}{3}}(r) \sup_{-r^2 < t < 0} \left[\int_{2r < |y| < \rho} \frac{|w|^2}{|y|^4} dy \right]^{\frac{1}{2}} \left[\int_{-r^2}^0 \int_{2r < |y| < \rho} \frac{|v|^2}{|y|^4} dy dt \right]^{\frac{3}{4}} \\
 &\leq cr^{\frac{7}{6}} G^{\frac{1}{3}}(r) \sup_{-r^2 < t < 0} \left[\int_{2r < |y| < \rho} \frac{|w|^2}{|y|^4} dy \right]^{\frac{1}{2}} \left[\int_{-r^2}^0 \|v(t)\|_{L^\infty(B_\rho)}^{\frac{3}{2}} dt \right]^{\frac{2}{3}} \\
 &\leq cr^2 G^{\frac{1}{3}}(r) \sup_{-r^2 < t < 0} \left[\int_{2r < |y| < \rho} \frac{|w|^2}{|y|^4} dy \right]^{\frac{1}{2}} \left[\int_{-r^2}^0 \|v(t)\|_{L^\infty(B_\rho)}^4 dt \right]^{\frac{1}{4}}.
 \end{aligned}$$

We are in a position to prove Lemma 6. The proof completely follows the arguments employed in [1]. Actually we have only to show that the adjoint of the terms $v \cdot \nabla w$, $w \cdot \nabla v$ and the modified pressure field work in the inductive procedure.

Proof of Lemma 6 For the proof we adopt the parabolic cylinder $Q_1(0, 0)$. The aim is to prove that

$$\int_{|x-a| < r_m} |w(x, s)|^2 dx \leq c_0 \varepsilon_1^{\frac{2}{3}}, \quad (74)$$

for each $(s, a) \in Q_{\frac{1}{2}}(0, 0)$ and for all $m \geq 2$, where we set $r_m := 2^{-m}$.

Step 1. Setting up the induction.

The assumptions are

$$\iint_{Q_1} (|w|^3 + |w||\pi|) + \int_{-1}^0 \left[\int_{B_1} |\pi| dx \right]^{\frac{5}{4}} dt \leq \varepsilon_1, \quad (75)$$

and, for the coefficient v ,

$$\iint_{Q_1} |v|^6 + \int_{-1}^0 \|v(\tau)\|_{L^\infty}^4 d\tau + \iint_{Q_1} |\nabla v|^3 < \varepsilon_2. \quad (76)$$

Considering the generalized energy inequality (57) with function $\phi = 1$ on Q_2 and support enclosed in Q_1 , assuming (75) and (76), by virtue of Lemma 7, we obtain

$$\sup_{s - \frac{1}{4} < t < s} \int_{B(a, \frac{1}{2})} |w|^2 dx + \frac{1}{8} \iint_{Q_2} |\nabla w|^2 \leq c_0 \varepsilon_1^{\frac{2}{3}},$$

provided that ε_1 and ε_2 are suitable. Since $Q_{\frac{1}{2}}(s, a) \subset Q_1(0, 0)$ for all $(s, a) \in Q_{\frac{1}{2}}(0, 0)$, we get

$$\iint_{Q_{\frac{1}{2}}} (|w|^3 + |w||\pi|) + \int_{s - \frac{1}{4}}^s \left[\int_{B_{\frac{1}{2}}} |\pi| dx \right]^{\frac{5}{4}} dt \leq \varepsilon_1. \quad (77)$$

The inductive procedure is the following: for $k \geq 3$,

$$\iint_{Q_k} |w|^3 + r_k^{\frac{3}{2}} \iint_{Q_k} |w| |\pi - \bar{\pi}_k| \leq \varepsilon_1^{\frac{2}{3}} \quad (78)$$

and, for $k \geq 2$,

$$\sup_{s-r_k^2 < t < s} \int_{B(a, r_k)} |w|^2 dx + r_k^{-3} \iint_{Q_k} |\nabla w|^2 \leq c \varepsilon_1^{\frac{2}{3}}, \quad (79)$$

then we also have

$$\begin{aligned} (78) \text{ true for } 2 \leq k \leq m &\Rightarrow (79) \text{ for } m, \\ \text{and} & \\ (79) \text{ true for } 3 \leq k \leq m &\Rightarrow (78) \text{ for } m+1. \end{aligned} \quad (80)$$

Step 2. Implication (80)₂.

We start remarking that (78) regards two terms. As proved in [1], by choosing $c\varepsilon_1 \leq \frac{1}{2}$, the following chain of inequalities

$$\iint_{Q_{m+1}} |w|^3 \leq c \iint_{Q_m} |w|^3 \leq c A^{\frac{3}{4}}(r_m) (A^{\frac{3}{4}}(r_m) + \delta^{\frac{3}{4}}(r_m)) \leq \frac{\varepsilon_1^{\frac{2}{3}}}{2} \quad (81)$$

is a consequence of Lemma 8 and the inductive assumption (79) for $k = m$. Hence we omit further comments. Now, we consider the latter term of (78) which involves the pressure field. The proof is analogous to the one furnished by Caffarelli–Kohn–Nirenberg, we have just to consider the adjoint of the terms in v . The key tools for the estimate are Lemma 9 and Lemma 10. We premise that, by the definition of the quantities $A(r)$, $G(r)$, and by the inductive hypothesis (79) for $k = m$, easily we obtain

$$\begin{aligned} G(r_{m+1}) &\leq c G(r_m) \leq c \varepsilon_1 r_m^3, \\ A(r_{m+1}) &\leq c A(r_m) \leq c \varepsilon_1^{\frac{2}{3}} r_m^2, \end{aligned} \quad (82)$$

and via (69) we also deduce

$$K \left(\frac{1}{4} \right) \leq c \varepsilon_1. \quad (83)$$

The estimate of the quoted latter term trivially admits the following

$$\begin{aligned} r_{m+1}^{\frac{3}{2}} \iint_{Q_{m+1}} |w| |\pi - \bar{\pi}_{m+1}| &= r_{m+1}^{-\frac{12}{5}} L(r_{m+1}) \\ &\leq r_{m+1}^{-\frac{12}{5}} L_w(r_{m+1}) + r_{m+1}^{-\frac{12}{5}} L_v(r_{m+1}), \end{aligned} \quad (84)$$

where we took (67) and (68) into account. By virtue of Lemma 9, following all the arguments employed in [1], one obtain

$$L_w(r_{m+1}) \leq c r_m^{\frac{12}{5}} \varepsilon_1. \quad (85)$$

Now, reproducing the arguments, we realize the same estimate for $L_v(r_{m+1})$ starting by Lemma 10. We set $\mathfrak{V} := \left[\int_{-1}^0 \|v(t)\|_{\infty}^4 dt \right]^{\frac{1}{4}}$. Recalling the definition of V , applying Hölder's

inequality, we have

$$V(r_m) \leq cr_m^{\frac{3}{2}} \left[\int_{s-r_m^2}^s \|v(t)\|_{L^\infty(B_\rho)}^4 dt \right]^{\frac{3}{4}} \leq cr_m^{\frac{3}{2}} \mathfrak{V}^3 \quad (86)$$

Being possible to set $\rho = \frac{1}{4}$, for the first two terms on the right hand side of (72) we have

$$\begin{aligned} r_{m+1}^{\frac{7}{5}} A^{\frac{1}{5}}(r_{m+1}) G^{\frac{1}{5}}(r_{m+1}) K^{\frac{4}{5}}\left(\frac{1}{4}\right) &\leq cr_m^{\frac{12}{5}} \varepsilon_1, \\ r_{m+1}^{\frac{5}{3}} G^{\frac{1}{3}}(r_{m+1}) G^{\frac{1}{3}}\left(\frac{1}{4}\right) V^{\frac{1}{3}}\left(\frac{1}{4}\right) &\leq cr_m^{\frac{8}{3}} \varepsilon_1^{\frac{2}{3}} \left[\int_{s-\frac{1}{4}}^s \|v(t)\|_{L^3(B_{\frac{1}{4}})}^3 dt \right]^{\frac{1}{3}}. \end{aligned} \quad (87)$$

For the third term on the right hand side of (72), employing (86), we get

$$G^{\frac{1}{3}}(r_{m+1}) G^{\frac{1}{3}}(r_m) V^{\frac{1}{3}}(r_m) \leq cr_m^{\frac{5}{3}} \varepsilon_1^{\frac{2}{3}} \mathfrak{V} \quad (88)$$

and, finally, by virtue of (82), we obtain for the last term of (72)

$$\begin{aligned} r_{m+1}^2 G^{\frac{1}{2}}(r_{m+1}) \sup_{\substack{-r_{m+1}^2 < t < 0 \\ r_m < |y| < \frac{1}{4}}} \left[\int \frac{|w|^2}{|y|^4} dy \right]^{\frac{1}{2}} \mathfrak{V} \\ \leq cr_m^3 \varepsilon_1^{\frac{1}{3}} \left[\sum_{k=2}^m \int_{r_k < |y| < r_{k-1}} \frac{|w|^2}{|y|^4} dy \right]^{\frac{1}{2}} \mathfrak{V} \leq cr_m^3 \varepsilon_1^{\frac{1}{3}} \left[\sum_{k=2}^m \frac{1}{r_k^3} A_k \right]^{\frac{1}{2}} \mathfrak{V} \\ \leq cr_m^3 \varepsilon_1^{\frac{2}{3}} \left[\sum_{k=2}^m \frac{1}{r_k} \right]^{\frac{1}{2}} \mathfrak{V} \leq cr_m^{\frac{5}{3}} \varepsilon_1^{\frac{2}{3}} \left[\sum_{p=0}^{\infty} \frac{1}{2^p} \right]^{\frac{1}{2}} \mathfrak{V} \end{aligned} \quad (89)$$

Considering for the $L_v(r)$ estimate (72), and increasing the right hand side by means of estimates (87) and (89), since $r < 1$ and we can consider ε_2 such that $c\varepsilon_2 < \varepsilon_1^{\frac{1}{3}}$, we get

$$L_v(r) \leq c\varepsilon_1. \quad (90)$$

Hence, recalling that for all $r < \frac{3}{4}\rho$ we have $L(r) \leq L_w(r) + L_v(r)$, via estimates (85) and (90), we complete the proof of step 2.

Step 3. Implication (80)₁.

Recalling that the center of the parabolic cylinder is the point $(0, 0)$, in the generalized energy inequality (57) we insert the scalar test function $\phi_m := \chi \psi_m$ where function χ is smooth nonnegative cutoff function with support $Q_{\frac{1}{3}}(0, 0)$ and $\chi = 1$ on $Q_{\frac{1}{4}}(0, 0)$, and, for $t < r_m^2$,

$$\psi_m := \frac{1}{(r_m^2 - t)^{\frac{3}{2}}} \exp\left(-\frac{|x|^2}{4(r_m^2 - t)}\right).$$

For all $m \in \mathbb{N}$, function ψ_m is a solution backward in time of the heat equation. The sequence $\{\phi_m\}$ enjoys the property listed in (64), and also

$$\phi_{m_t} + \Delta \phi_m = 0, \quad \text{in } Q_{\frac{1}{4}}(0, 0), \quad \text{and } |\phi_{m_t} + \Delta \phi_m| \leq c, \quad \text{everywhere.} \quad (91)$$

Substituting ϕ_m in (57), by virtue of (64), we realize the inequality

$$\sup_{-r_m^2 < t \leq 0} \int_{|x| < r_m} |w(t, x)|^2 dx + \frac{1}{r_m^3} \iint_{Q_m} |\nabla w(t, y)|^2 dy dt \leq c \sum_{i=1}^5 I_i, \quad (92)$$

where we set

$$\begin{aligned} I_1 &:= \iint_{Q_1} |w|^2 |\phi_{mt} + \Delta \phi_m|, \quad I_2 := \iint_{Q_1} |w|^3 |\nabla \phi_m|, \quad I_3 := \left| \iint_{Q_1} \pi w \cdot \nabla \phi_m \right|, \\ I_4 &:= \left| \iint_{Q_1} w \cdot \nabla v \cdot w \phi_m \right|, \quad I_5 := \left| \iint_{Q_1} v \cdot \nabla w \cdot w \phi_m \right|. \end{aligned}$$

Now we should have to estimate each term from I_1 to I_5 . Since the estimates from I_1 to I_3 are the same of the ones deduced in [1] for the terms I , II , and III , then for them we only carry the bounds calculated in [1] (p. 792–794, step 3) and we limit ourselves to find the bounds for the residual. Hence we set

$$I_1 + I_2 + I_3 \leq c\varepsilon^{\frac{2}{3}}. \quad (93)$$

For the terms I_4 and I_5 we employ (66):

$$I_4 + I_5 \leq c\varepsilon_1^{\frac{4}{9}} \left[\left(\iint_{Q_1} |v|^6 \right)^{\frac{1}{6}} + \left(\iint_{Q_1} |\nabla v|^3 \right)^{\frac{1}{3}} \right]. \quad (94)$$

Hence, choosing ε_2 in such a way that $c\varepsilon_2 < \varepsilon_1^{\frac{2}{9}}$, via (93) and (94), we complete the proof of step 3.

The lemma is completely proved.

Lemma 11 *Let w be a suitable weak solution to problem (55) with assumptions (56) for coefficient v . Let $S > 0$ and $x \in \mathbb{R}^3$. Assume the existence of a $t(x) > 0$ such that*

$$\mathcal{E}(w, t, x) + \frac{1}{2} \int_0^t \mathcal{D}(w, \tau, x) d\tau \leq S, \text{ for all } t \in [0, t(x)). \quad (95)$$

Then there exists a $\delta(x) \in [0, 1)$ such that for all $s \in (0, t(x))$, $w(t, y) \in L^\infty(Q_{(\frac{(1-\delta)s}{4})}^{\frac{1}{2}}(s, x))$. In particular, we get

$$|w(t, y)| \leq c[(1-\delta)s]^{-\frac{1}{2}}, \quad (96)$$

for all Lebesgue's points $(t, y) \in Q_{(\frac{(1-\delta)s}{4})}^{\frac{1}{2}}(s, x)$.

Proof The function

$$f(t, x) := \int_t^{t(x)} \mathcal{D}(w, \tau, x) d\tau \quad (97)$$

is uniformly continuous on $[0, t(x)]$. Hence for $\varepsilon > 0$ there exists a $\tilde{\delta}(x) \in (0, 1]$ such that

$$\int_{t(1-\tilde{\delta})}^t \mathcal{D}(w, \tau, x) d\tau < \varepsilon, \text{ for all } t \in [0, t(x)]. \quad (98)$$

Now for a suitable ε , we can satisfy the assumptions of Lemma 6. Actually, recalling the definition of $M(w, t, x, r)$ given in (58) and applying the Hölder's inequality, for the first term of (58) we get

$$\begin{aligned} r^{-2} \int \int_{Q_r} (|w|^3 + |w||\pi|) &\leq \iint_{Q_r} \frac{|w(\tau, y)|^3}{|x-y|^2} + \frac{|w(\tau, y)||\pi(\tau, y)|}{|x-y|^2} \\ &\leq \iint_{Q_r} \frac{|w(\tau, y)|^3}{|x-y|^2} + \left[\iint_{Q_r} \frac{|w(\tau, y)|^3}{|x-y|^2} \right]^{\frac{1}{3}} \left[\iint_{Q_r} \frac{|\pi(\tau, y)|^{\frac{3}{2}}}{|x-y|^2} \right]^{\frac{2}{3}}. \end{aligned}$$

Hence, by virtue of inequalities (25) for the pressure term and via the weighted interpolating inequality (33), we arrive at

$$r^{-2} \int \int_{Q_r} (|w|^3 + |w||\pi|) \leq c \int_{t-r^2}^t \mathcal{E}^{\frac{1}{2}}(w, \tau, x) \mathcal{D}(w, \tau, x) d\tau. \quad (99)$$

Analogously, applying Hölder's inequality

$$\begin{aligned} r^{-\frac{13}{4}} \int_{t-r^2}^t \left(\int_{|x-y|<r} |\pi| dy \right)^{\frac{5}{4}} d\tau &\leq r^{-2} \int_{t-r^2}^t \left(\int_{|x-y|<r} |\pi|^{\frac{3}{2}} dy \right)^{\frac{5}{6}} d\tau \\ &\leq \left[\int_{t-r^2}^t \int_{|x-y|<r} \frac{|\pi|^{\frac{3}{2}}}{|x-y|^2} dy d\tau \right]^{\frac{5}{6}}. \end{aligned}$$

Hence, by virtue of inequality (25) and weighted interpolating inequality (33), we arrive at

$$r^{-\frac{13}{4}} \int_{t-r^2}^t \left(\int_{|x-y|<r} |\pi| dy \right)^{\frac{5}{4}} d\tau \leq c \int_{t-r^2}^t \mathcal{E}^{\frac{1}{2}}(w, \tau, x) \mathcal{D}(w, \tau, x) d\tau. \quad (100)$$

By virtue of hypothesis (95), via estimates (98)–(100), setting $r^2 = \tilde{\delta}t$, we arrive at

$$M(w, t, x, r) \leq c S^{\frac{1}{2}} \int_{(1-\tilde{\delta})t}^t \mathcal{D}(w, \tau, x) d\tau < c S^{\frac{1}{2}} \varepsilon, \text{ for all } t \in (0, t(x)). \quad (101)$$

Hence, choosing $\varepsilon := \varepsilon_1 / c S^{\frac{1}{2}}$, we realize condition (61) for w with $r^2 = \tilde{\delta}t$. Condition (62) is immediate from the assumptions. Hence, setting $\delta := 1 - \tilde{\delta}$, the claim of the lemma is a consequence of Lemma 6.

Corollary 2 Let w be a suitable weak solution to problem (55) with assumptions (56) for v . Let $S > 0$ and $\mathbb{E} \subseteq \mathbb{R}^3$. Assume the existence of a $\tilde{T} > 0$ such that

$$\mathcal{E}(w, t, x) + \frac{1}{2} \int_0^t \mathcal{D}(w, \tau, x) d\tau \leq S, \text{ for all } (t, x) \in (0, \tilde{T}) \times \mathbb{E}. \quad (102)$$

Then for all $x \in \mathbb{E}$ there exists a $\delta(x) \in [0, 1)$ such that, for all $s \in (0, \tilde{T})$, $w(t, y) \in L^\infty(Q_{\left(\frac{(1-\delta)s}{4}\right)^{\frac{1}{2}}}(s, x))$. In particular, for all $(s, x) \in (0, \tilde{T}) \times \mathbb{E}$, and for all Lebesgue's points $(t, y) \in Q_{\left(\frac{(1-\delta)s}{4}\right)^{\frac{1}{2}}}(s, x)$, we get

$$|w(t, y)| \leq c[(1 - \delta)s]^{-\frac{1}{2}}. \quad (103)$$

Finally, if $S \leq \frac{\varepsilon_1}{2c_1}$, with ε_1 as in Lemma 6, then (103) holds with $\delta = 0$ too.

Proof The proof is an immediate consequence of the previous lemma. Actually, the assumption ensures that function (97) can be defined in the following way:

$$f(t, x) := \int_t^{\tilde{T}} \mathcal{D}(\tau, x, w) d\tau.$$

Hence, for all x , the time interval of uniform continuity of this $f(t, x)$ is $[0, \tilde{T}]$, hence independent of x . This, via estimate (101) and Lemma 6, ensures that (103) holds on $(0, \tilde{T})$. If $S^{\frac{3}{2}} \leq \frac{\varepsilon_1}{2c}$, then, further the above considerations, by summing estimates (99) and (100), we arrive at

$$M(w, t, x, r) \leq 2cS^{\frac{1}{2}} \int_{(1-\delta)t}^t \mathcal{D}(w, \tau, x) d\tau < 2cS^{\frac{3}{2}} < \varepsilon_1, \text{ for all } t \in (0, \tilde{T}),$$

which is verified independently of the uniform continuity, that is in the end independently of δ too.

Remark 6 We stress that the above lemma and corollary are criteria for the regularity which furnish estimates (96) and (103) respectively. In the sequel we understand that in order to satisfy the assumption of the lemma and of the corollary some suitable conditions of smallness are required for w_0 . However we stress that w_0 is the initial data of a “perturbation” of a suitable weak solution u , thus for u a different result of regularity holds.

5 Proof of Theorems 3–5

Proof of Theorem 3 Setting $w := u - v$, where v is the $J^{1,2}$ -regular solution corresponding to v_0 , we are in the hypotheses of Lemma 5, hence the weighted energy inequality (41) holds. As a consequence we can apply to w Corollary 2 in order to deduce $w(t, y) \in L^\infty(Q_{\left(\frac{(1-\delta)s}{4}\right)^{\frac{1}{2}}}(s, x))$ as well estimate (103). Hence, by means of the triangular inequality, we easily arrive at estimate (11). The case of $v_0 = 0$ should be discussed in the same way. For the sake of the brevity we omit the details.

Proof of Theorem 4 Our hypothesis (12) allows us to deduce (41) with $w = u$, $t^* = \infty$, $\varepsilon_0 \leq \frac{\varepsilon_1}{2c_1}$ and $\mathbb{E} = \{x : |x| > R_0\}$. Hence, by virtue of the last claim of Corollary 2, taking into account that $\tilde{T} = \infty$, via a suitable c_1 , we obtain (103) with $w = u$, $\delta = 0$ and for all $(s, x) \in (0, \infty) \times \mathbb{E}$. Considering the latter assumption of (14), v_0^* belongs to $J^{1,2}(\mathbb{R}^3)$, then on some interval $(0, T_0^*)$ there exists the $J^{1,2}$ -regular solution (v^*, π_{v^*}) to problem (1). On cylinder $(0, T_0^*) \times \mathbb{R}^3$ we can consider the difference (w^*, π^*) with $w^* := u - v^*$ and $\pi^* := \pi_u - \pi_{v^*}$. Set $\varepsilon_0 := \frac{\varepsilon_1}{2c_1}$ in Lemma 5, then the following weighted energy inequality holds ($t^* \equiv T^*$ below):

$$\mathcal{E}(w^*, t, x) + \frac{1}{2} \int_0^t \mathcal{D}(w^*, \tau, x) d\tau \leq \frac{\varepsilon_1}{2c_1}, \text{ for all } (t, x) \in [0, T^*) \times \mathbb{E}. \quad (104)$$

We set

$$v(t, x) := \begin{cases} v^*(t, x), & \text{if } (t, x) \in (0, T_0^*) \times \mathbb{R}^3, \\ 0 & \text{if } t \in [T_0^*, \infty) \times \mathbb{R}^3. \end{cases}$$

The above position allow us to consider (w^*, π^*) as a suitable weak solution to problem (55) on the cylinder $(0, T_0^*) \times \mathbb{R}^3$. By virtue of (104) we satisfy the assumptions of Corollary 2 for all $(t, x) \in (0, T^*) \times \{|x| \leq R_0\}$. The last claim of Corollary 2 ensures (103) with $\delta = 0$. This last, via the triangular inequality, allows us to arrive at (15). In order to complete the proof of the theorem we have to prove estimate (16). So that we have to argue for $t \geq T^*$. As already claimed, estimate (103) holds for u , thus estimate (16) holds for all $t > T^*$ and $|x| > R_0$. In particular they hold for $t \in [T^*, \max\{T_0, T^*\})$. Hence (16) is true for $t \in [T^*, \max\{T_0, T^*\})$ and $|x| > R_0$. Taking into account the former assumption of (14), by virtue of the first claim of Corollary 2, we get (103) with $\delta(x)$. This last allows us to arrive at (11). Hence we can consider a covering of $[T^*, \max\{T_0, T^*\}] \times \{|x| \leq R_0\}$ by means of a family of open parabolic neighborhood $\mathcal{Q}_{\left(\frac{(1-\delta(x))\delta}{4}\right)^{\frac{1}{2}}}(s, x)$, $(s, x) \in [T^*, \max\{T_0, T^*\}] \times \{|x| \leq R_0\}$. Since the set $[T^*, \max\{T_0, T^*\}] \times \{|x| \leq R_0\}$ is a compact, then from the covering we realize a finite family of parabolic cylinder $\tilde{\mathcal{Q}}_h$, $h = 1, \dots, M$, which covers the compact set. Hence any $(t, x) \in [T^*, \max\{T_0, T^*\}] \times \{|x| \leq R_0\}$ belongs to some $\tilde{\mathcal{Q}}_h$, thus, with a constant $\bar{c}$ independent of (t, x) , we get

$$|u(t, x)| \leq \bar{c}r_h^{-\frac{1}{2}} + |v(t, x)| \leq \bar{c}T^{*- \frac{1}{2}} + |v(t, x)|,$$

which proves (16).

The difference between Theorems 4 and 5 is in the fact that in the latter the hypotheses do not provide for the existence of v_0 and v_0^* satisfying estimates (14). In the following lemma we show that, under the only assumption $u_0 \in J^2(\mathbb{R}^3)$, there exist suitable v_0 and v_0^* such that (14) holds on a subset $\mathbb{F}_\sigma \subseteq B_{R'_0}$ whose complement set in $B_{R'_0}$ has Lebesgue measure less than σ , where $\sigma > 0$ is arbitrarily chosen.

Lemma 12 *Let u_0 be in $J^2(\mathbb{R}^3)$ and let $\varepsilon > 0$. For all $R > 0$ and $\sigma > 0$ there exist a closed set $\mathbb{F} \subseteq B_R$ and $\mathfrak{g} \in J^{1,2}(\mathbb{R}^3)$ such that*

$$|B_R - \mathbb{F}| < \sigma, \text{ and } \sup_{\mathbb{F}} \int_{\mathbb{R}^3} \frac{|u_0 - \mathfrak{g}|^2}{|x - y|} dy < \varepsilon. \quad (105)$$

Proof Let us consider the mollified sequence $\{J_m[u_0]\}$ converging to u_0 in $J^2(\mathbb{R}^3)$. As consequence, by virtue of the Hardy–Littlewood–Sobolev theorem, the sequence

$$\{\psi_m(x)\}, \text{ with } \psi_m(x) := \int_{\mathbb{R}^3} \frac{|u_0 - J_m[u_0]|^2}{|x - y|} dy, \text{ converges to 0 a.e. in } B_R.$$

By virtue of the Severini–Egorov theorem, for all $\sigma > 0$ there exists a closed set $\mathbb{F} \subseteq B_R$ such that

$$|B_R - \mathbb{F}| < \sigma \text{ and } \psi_m \rightarrow 0 \text{ uniformly on } \mathbb{F}.$$

The lemma is proved.

Now, we are in a position to prove Theorem 5.

Proof of Theorem 5 The first claim, that is (13) on $(0, \infty) \times \{x : |x| > R'_0\}$, holds as proved in Theorem 4. By virtue of Lemma 12 we realize estimate (105) on some closed set $\mathbb{F}_\sigma$, that we twice consider. A first time we choose $\varepsilon \leq \frac{\varepsilon_1}{2c_1}$. In such case we denote by v_0^* the function g in (105). By means of v_0^* we deduce (41) with $\varepsilon \leq \varepsilon_1/2c_1$. Thus, via Corollary 2, we obtain (103) with $\delta = 0$ on some interval $(0, T^*) \times \mathbb{F}_\sigma$. Since $w = u - v^*$, where v^* is the $J^{1,2}$ -regular solution corresponding to v_0^* , we deduce the validity of (18). A second time we choose $\varepsilon \leq 1/(4c)^2$. In such case we denote by v_0 the function g in (105). By means of v_0 we deduce (41) with $\varepsilon_0 = 1/(4c)^2$. Thus, via Corollary 2, we obtain (103) for all $(s, x) \in (0, T_0) \times \mathbb{F}_\sigma$. *A priori* we have to consider $T_0 \neq T^*$. If $T^* < T_0$, since $w = u - v$, where v is the $J^{1,2}$ -regular solution corresponding to v_0 , we deduce the validity of (19). Actually, we have a family of open set of the kind $Q_{[(1-\delta)t/4]^{-1/2}}(t, x)$ covering the compact set $[T^*, T_0] \times \mathbb{F}_\sigma$. Hence, we can extract a finite family $\tilde{Q}_1, \dots, \tilde{Q}_m$ in order to deduce (19) with $T^* \leq \min_{h \in \{1, \dots, m\}} r_h^2$.

6 Proof of Theorem 6

The proof is based on a duality argument classical in PDEs that goes back to the Holmgren uniqueness theorem for the system of the Cauchy–Kowalewskaya type and considered by Foias in [6] for the Navier–Stokes equations using as adjoint the linear Stokes Cauchy problem. We are able to repeat the Foias arguments, but without extra assumptions on the solutions.

We denote by ψ the solution to the Cauchy problem:

$$\begin{aligned} \psi_t - \Delta \psi &= -\nabla \pi \psi, \quad \nabla \cdot \psi = 0 \text{ in } (0, T) \times \mathbb{R}^3, \\ \psi &= \psi_0 \in \mathcal{C}_0(\mathbb{R}^3) \text{ on } \{0\} \times \mathbb{R}^3. \end{aligned} \quad (106)$$

It is well known that ψ is a smooth solution with $\psi \in C([0, T]; J^p(\mathbb{R}^3))$, for all $p \in (1, \infty)$. Moreover, the following estimates hold:

$$\begin{aligned} q \geq p, \quad \|\psi(t)\|_q &\leq ct^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\psi_0\|_p, \quad \text{for all } t > 0, \\ \|\nabla \psi(t)\|_q &\leq c_1 t^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\psi_0\|_p, \quad \text{for all } t > 0. \end{aligned} \quad (107)$$

In the case of $p = q = 2$, in (107) we have $c = 1$ and $c_1 = 1/\sqrt{2}$.

For $t > 0$, we set $\hat{\psi}(\tau, x) := \psi(t - \tau, x)$ provided that $(\tau, x) \in (0, t) \times \mathbb{R}^3$. It is well known that $\hat{\psi}$ is a solution backward in time with $\hat{\psi}(t, x) = \psi_0(x)$.

Lemma 13 Let u be a sws. Then relation iii. of Definition 2, firstly written for $\varphi \in C_0^1([0, T] \times \mathbb{R}^3)$, can be extended to all $\varphi \in C([0, T]; J^{1,2}(\Omega))$ with $\varphi_t \in L^2(0, T; L^2(\Omega))$.

Proof This is a classical result essentially due to Prodi [15]. It is based on the integrability properties of a sws and a classical density argument.

Proof of Theorem 6 We denote by (w, π_w) the difference of two solutions (u, π_u) and $(\bar{u}, \pi_{\bar{u}})$ furnished by Theorem 4. Hence, for all $t > s \geq 0$, (w, π_w) solves the integral equation:

$$\int_s^t \left[(w, \varphi_\tau) - (\nabla w, \nabla \varphi) + (u \cdot \nabla \varphi, w) + (w \cdot \nabla \varphi, \bar{u}) \right] d\tau + (w(s), \varphi(s)) = (w(t), \varphi(t)), \quad (108)$$

for all $\varphi \in C([0, T]; J^{1,2}(\Omega))$ with $\varphi_t \in L^2(0, T; L^2(\Omega))$. For given $t > 0$ and $s > 0$, we set $\widehat{\psi}$ in place of φ . Integrating by parts, we get

$$\int_s^t \left[(u \cdot \nabla \widehat{\psi}, w) + (w \cdot \nabla \widehat{\psi}, \bar{u}) \right] d\tau + (w(s), \widehat{\psi}(s)) = (w(t), \psi_0). \quad (109)$$

Taking estimate (15) into account for both the solutions, applying Hölder's inequality and estimates (107) for $p = q = 2$, for all $t \in (0, T^*)$, we deduce the estimate

$$\begin{aligned} |(w(t), \psi_0)| &\leq \|w(s)\|_2 \|\psi(t-s)\|_2 + \int_s^t \left[\frac{\sqrt{2}c_0\varepsilon_1^{\frac{2}{3}}}{\sqrt{\tau}} + \|v^*(\tau)\|_\infty \right] \|w(\tau)\|_2 \|\nabla \psi(t-\tau)\|_2 d\tau \\ &\leq \|w(s)\|_2 \|\psi_0\|_2 + \|\psi_0\|_2 \int_s^t \left[\frac{\sqrt{2}c_0\varepsilon_1^{\frac{2}{3}}}{\sqrt{\tau}} + \|v^*(\tau)\|_\infty \right] \|w(\tau)\|_2 (t-\tau)^{-\frac{1}{2}} d\tau, \end{aligned}$$

for all $\psi_0 \in \mathcal{C}_0(\mathbb{R}^3)$. Letting $s \rightarrow 0$, since $\|w(0)\|_2 = 0$, we get

$$\|w(t)\|_2 \leq \int_s^t \left[\frac{\sqrt{2}c_0\varepsilon_1^{\frac{2}{3}}}{\sqrt{\tau}} + \|v^*(\tau)\|_\infty \right] \|w(\tau)\|_2 (t-\tau)^{-\frac{1}{2}} d\tau.$$

Recalling that $\int_0^t (t-\tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} d\tau = \pi$, and observing that v , via the embedding theorem, belongs to $L^4(0, T^*; L^\infty(\mathbb{R}^3))$, we can deduce the uniqueness on $(0, t^*) \subseteq (0, T^*)$ provided that $\sqrt{2}c_0\varepsilon_1^{\frac{2}{3}}\pi < 1$ and t^* is sufficiently small. Now, we have to discuss the uniqueness for $t \in [t^*, T_0)$. On the other hand by virtue of (16) and Theorem 8, both the solutions are local regular on $(0, T_0)$, therefore from some $t_0 \in (0, t^*)$ we have to discuss the uniqueness of two regular solutions. Thus the claim of the uniqueness trivially follows.

7 Proof of Theorem 7

In the hypotheses of Theorem 7 and by virtue of estimates (18) and (19), we state that, for all $\eta > 0$, the sws $u \in L^r(\eta, T_0; L^s(\mathbb{R}^3))$. Hence, by virtue of Theorem 8, u is a local regular solution for all $t \in (0, T_0)$. This proves the first part of the theorem.

Let $\bar{u}$ be another SWS ensured by Theorem 5 and corresponding to u_0 . We set $w = \bar{u} - u$. Let h be the smooth function defined in (20). We write $w = hw + (1 - h)w$. Hence we get $\|w\|_2 \leq \|hw\|_2 + \|(1 - h)w\|_2$. We separately estimate hw and $(1 - h)w$. We set $\bar{h} := 1 - h$. The term $\bar{h}w$ is estimated by duality starting from the integral equation

$$\begin{aligned} (w(t), \varphi(t)) &= (w(s), \varphi(s)) \\ &+ \int_s^t \left[(w, \varphi_\tau) - (\nabla w, \nabla \varphi) + (\bar{u} \cdot \nabla \varphi, w) + (w \cdot \nabla \varphi, u) + (\pi_w, \nabla \cdot \varphi) \right] d\tau, \end{aligned} \quad (110)$$

that we consider for all $\varphi \in C([0, T]; W^{1,2}(\Omega))$ with $\varphi_t \in L^2(0, T; L^2(\Omega))$ and $\varphi = 0$ in a neighborhood of T . Let ψ be the solution to the heat equation with initial data $\psi_0 \in C_0^\infty(\mathbb{R}^3)$, and $\hat{\psi}$ the solution ψ written backward in time on $(0, t) \times \mathbb{R}^3$. We substitute φ with $\bar{h}\hat{\psi}$. Integrating by parts, formula (110) becomes

$$\begin{aligned} (w(t)\bar{h}, \psi_0) &= \int_0^t \left[(\bar{u} \cdot \nabla \psi, \bar{h}w) + (w \cdot \nabla \psi, \bar{h}u) - (\nabla \pi^1, \bar{h}\psi) + (\bar{h}\pi^2, \nabla \cdot \psi) \right. \\ &\quad \left. + (\bar{u} \cdot \nabla \bar{h}, \psi \cdot w) + (w \cdot \nabla \bar{h}, \psi \cdot u) + (\pi^2 \nabla \bar{h}, \psi) \right] d\tau =: \sum_{i=1}^7 I_i, \end{aligned} \quad (111)$$

where, via the representation (31), we have mean π_w decomposed in $\pi^1 + \pi^2$. Recalling that Theorem 5 ensures estimates (13) and (18) and (19) for u and $\bar{u}$ corresponding to the same data u_0 , united with the properties related to function h given in (20), for all $\tau \in (0, T_0)$, we have

$$\|\bar{h}u(\tau)\|_\infty + \|\bar{h}\bar{u}(\tau)\|_\infty \leq \frac{2c_0\varepsilon_1^{\frac{2}{3}}}{\sqrt{\tau}} + A(\tau) \quad (112)$$

where we set

$$A(\tau) := \left[\chi_{[0, T^*]}(\tau) \|v^*(\tau)\|_\infty + \chi_{[T^*, t]}(\tau) \left(\frac{c}{(T^*)^{\frac{1}{2}}} + \|v(\tau)\|_\infty \right) \right].$$

Hence, applying Hölder's inequality, we get

$$\begin{aligned} |I_1 + I_2| &\leq \int_0^t (\|\bar{h}u\|_\infty + \|\bar{h}\bar{u}\|_\infty) \|\nabla \psi(t - \tau)\|_2 \|w\|_2 d\tau \\ &\leq \|\psi_0\|_2 \int_0^t \left[\frac{\sqrt{2}c_0\varepsilon_1'}{\sqrt{\tau}} + A(\tau) \right] (t - \tau)^{-\frac{1}{2}} \|w(\tau)\|_2 d\tau. \end{aligned}$$

Via estimates (32), we obtain the following estimates:

$$\begin{aligned}
 |I_3(t)| &= \left| \int_0^t (\nabla \pi_1, \bar{h} \widehat{\psi}) d\tau \right| \leq \|\psi_0\|_2 \int_0^t \|\nabla \pi_1\|_2 d\tau \\
 &\leq c \|\psi_0\|_2 \int_0^t \|w\|_2 [\|u\|_3 + \|\bar{u}\|_3] B(t, \tau) d\tau \\
 |I_4(t)| &= \left| \int_0^t (\pi_2, \bar{h} \nabla \cdot \psi) d\tau \right| \leq c \|\psi_0\|_2 \int_0^t \|\pi_2\|_2 (t - \tau)^{-\frac{1}{2}} d\tau \\
 &\leq c \|\psi_0\|_2 \int_0^t \left[\frac{\sqrt{2} c_0 \varepsilon'_1}{\sqrt{\tau}} + A(\tau) \right] (t - \tau)^{-\frac{1}{2}} \|w\|_2 + \|w\|_2 [\|u\|_3 + \|\bar{u}\|_3] B(t, \tau) d\tau
 \end{aligned}$$

where we set $B(t, \tau) := 1 + (t - \tau)^{-\frac{1}{2}}$ and we took estimate (112) into account. Finally, recalling that $\nabla \bar{h}$ has compact support, applying Hölder's inequality, and employing (29) for the pressure field, we arrive at

$$\begin{aligned}
 |I_5 + I_6 + I_7| &\leq c \int_0^t \|\psi\|_\infty [(\|u\|_2 + \|\bar{u}\|_2) \|w\|_2 + \|\pi_w\|_{\mathcal{H}^1}] d\tau \\
 &\leq c \|\psi_0\|_2 \|u_0\|_2 \int_0^t (t - \tau)^{-\frac{3}{4}} \|w(\tau)\|_2 d\tau.
 \end{aligned}$$

The above estimates and formula (111), uniformly in $t \in (0, T_0)$, ensure

$$\|\bar{h}w(t)\|_2 \leq \sqrt{2} \pi c_0 \varepsilon_1^{\frac{2}{3}} \sup_{(0, t)} \|w(t)\|_2 + \int_0^t \left[\frac{A(\tau)}{(t - \tau)^{\frac{1}{2}}} + \frac{\|u_0\|_2}{(t - \tau)^{\frac{3}{4}}} \right] \|w(\tau)\|_2 d\tau. \quad (113)$$

Now we estimate $\|hw(t)\|_2$. For this task we employ Leray's technique adapted to *sws*. We consider $\phi = h^2$ in the energy relation of each weighted energy relations related to the *sws* u and $\bar{u}$. So that from (2) we get

$$\begin{aligned}
 \int_{\mathbb{R}^3} |hu(t)|^2 dx + 2 \int_\sigma^t \int_{\mathbb{R}^3} |h \nabla u|^2 dx d\tau &\leq \int_{\mathbb{R}^3} |u(\sigma)h|^2 dx \\
 + \int_\sigma^t \int_{\mathbb{R}^3} |u|^2 \Delta h^2 dx d\tau + \int_\sigma^t \int_{\mathbb{R}^3} (|u|^2 + 2\pi_u) u \cdot \nabla h^2 dx d\tau, \quad (114)
 \end{aligned}$$

for all $t \geq \sigma$, for $\sigma = 0$ and a.e. in $\sigma \geq 0$, and analogously for $\bar{u}$

$$\begin{aligned} \int_{\mathbb{R}^3} |h\bar{u}(t)|^2 dx + 2 \int_{\sigma}^t \int_{\mathbb{R}^3} |h\nabla\bar{u}|^2 dx d\tau &\leq \int_{\mathbb{R}^3} |h\bar{u}(\sigma)|^2 dx \\ &+ \int_{\sigma}^t \int_{\mathbb{R}^3} |\bar{u}|^2 \Delta h^2 dx d\tau + \int_{\sigma}^t \int_{\mathbb{R}^3} (|\bar{u}|^2 + 2\pi_{\bar{u}}\bar{u} \cdot \nabla h^2) dx d\tau, \end{aligned} \quad (115)$$

for all $t \geq \sigma$, for $\sigma = 0$ and a.e. in $\sigma \geq 0$. Since u is $J^{1,2}$ -regular, we substitute the test function φ with $h^2 u$ in the weak formulation of $\bar{u}$

$$\begin{aligned} \int_s^t \left[(\bar{u}, u_{\tau} h^2) - (\nabla\bar{u}, h^2 \nabla u) - (\bar{u} \cdot \nabla\bar{u}, h^2 u) \right] d\tau \\ + (\bar{u}(s), h^2 u(s)) = (\bar{u}(t), h^2 u(t)) + G(s, t), \end{aligned}$$

where we set

$$G(s, t) := \int_s^t \left[(\nabla\bar{u}, \nabla h^2 \otimes u) - (\pi_{\bar{u}} \nabla h^2 \cdot u) \right] d\tau.$$

Analogously, employing the regularity of u , we set $\bar{u}h^2 \in L^2(0, T; J^{1,2}(\mathbb{R}^3))$ as test function for the equation of (u, π_u) . Recalling that, for all $\psi \in L^2(\mathbb{R}^3)$, $(u(t), \psi)$ is continuous function of t , we get

$$\int_s^t \left[(u_{\tau}, \bar{u}h^2) + (h^2 \nabla\bar{u}, \nabla u) + (u \cdot \nabla u, h^2 \bar{u}) \right] d\tau = G_1(s, t),$$

where we set

$$G_1(s, t) := \int_s^t \left[-(\nabla u, \nabla h^2 \otimes \bar{u}) + (\pi_u \nabla h^2 \cdot \bar{u}) \right] d\tau.$$

Making the difference between the two weak formulations we get

$$\begin{aligned} -2 \int_s^t (\nabla\bar{u}, h^2 \nabla u) &= \int_s^t \left[(\bar{u} \cdot \nabla\bar{u}, h^2 u) + (u \cdot \nabla u, h^2 \bar{u}) + G(s, t) - G_1(s, t) \right] d\tau \\ &- (\bar{u}(s), h^2 u(s)) + (\bar{u}(t), h^2 u(t)). \end{aligned}$$

Now, multiplying for 2 the last relation and summing it with the weighted energy relations (114) and (115), we get

$$\begin{aligned} \|hw(t)\|_2^2 + 2 \int_s^t \|h\nabla w(\tau)\|_2^2 d\tau &\leq \|hw(s)\|_2^2 \\ &+ \int_s^t \left[(|\bar{u}|^2 \bar{u} + |u|^2 u, \nabla h^2) + 2(\bar{u} \cdot \nabla\bar{u}, h^2 u) + 2(u \cdot \nabla u, h^2 \bar{u}) \right] d\tau \end{aligned}$$

$$+ \int_s^t (|w(\tau)|^2, \Delta h^2) d\tau - 4 \int_s^t (\pi_{\bar{u}} - \pi_u, \nabla h \cdot wh) d\tau =: \|w(s)h\|_2^2 + \sum_{i=1}^3 I_i.$$

We estimate separately the integrals I_i , $i = 1, 2, 3$. Considering the mollified $\bar{u}_\varepsilon$, set $w_\varepsilon := \bar{u}_\varepsilon - u$, via an integrating by parts, we get

$$\begin{aligned} 2(\bar{u} \cdot \nabla \bar{u}, uh^2) + (|\bar{u}|^2, \bar{u} \cdot \nabla h^2) &= 2 \lim_{\varepsilon} [(\bar{u}_\varepsilon \cdot \nabla \bar{u}, (u - \bar{u})h^2)] \\ &= -2 \lim_{\varepsilon} (\bar{u}_\varepsilon \cdot \nabla \bar{u}, wh^2), \end{aligned} \quad (116)$$

$$2(u \cdot \nabla u, \bar{u}h^2) - 2(u \cdot \nabla u, uh^2) = 2(u \cdot \nabla u, wh^2). \quad (117)$$

Summing the right hand sides of (116) and (117), we arrive at

$$\begin{aligned} -2 \lim_{\varepsilon} (\bar{u}_\varepsilon \cdot \nabla \bar{u} - u \cdot \nabla u, wh^2) &= -2 \lim_{\varepsilon} ((\bar{u}_\varepsilon - u) \cdot \nabla \bar{u} + u \cdot \nabla (\bar{u} - u), wh^2) \\ &= -2 \lim_{\varepsilon} (w_\varepsilon \cdot \nabla \bar{u} + u \cdot \nabla w, wh^2). \end{aligned} \quad (118)$$

Hence, via (116)–(118), for I_1 we obtain

$$\begin{aligned} I_1 &= 2 \int_s^t \left[-\lim_{\varepsilon} [(w_\varepsilon \cdot \nabla w, wh^2) + (w_\varepsilon \cdot \nabla u, wh^2)] - (u \cdot \nabla w, wh^2) \right] d\tau \\ &= \int_s^t \left[\lim_{\varepsilon} [(w_\varepsilon, \nabla h^2 |w|^2) + 2(w_\varepsilon \cdot \nabla h^2, u \cdot w)] \right. \\ &\quad \left. + (u \cdot \nabla h^2, |w|^2) + 2(w_\varepsilon \cdot \nabla w, uh^2) \right] d\tau \\ &= \int_s^t \left[(w \cdot \nabla h^2, |w|^2) + 2(w \cdot \nabla h^2, u \cdot w) \right. \\ &\quad \left. + (u \cdot \nabla h^2, |w|^2) + 2(w \cdot \nabla w, uh^2) \right] d\tau, \end{aligned}$$

where we taken into account that u enjoys the $J^{1,2}$ -regularity. Recalling the assumption on u related to the support $\mathbb{H}$ of h , by applying Holder's inequality, we get

$$\begin{aligned} |I_1| &\leq c \int_s^t \|w\|_2 \|w\|_3 \|hw\|_6 d\tau + c \int_s^t \|u\|_3 \|w\|_2 \|hw\|_6 d\tau \\ &\quad + c \int_s^t \|u\|_{L^s(\mathbb{H})} \|hw\|_{\frac{2s}{s-2}} \|h \nabla w\|_2 d\tau. \end{aligned}$$

Thus via the Gagliardo–Nirenberg inequality, we get

$$\begin{aligned} |I_1| &\leq c \int_s^t (\|w\|_3 + \|u\|_3) \|w\|_2 [\|h \nabla w\|_2 + \|\nabla hw\|_2] d\tau \\ &\quad + c \int_s^t \|u\|_{L^s(\mathbb{H})} \|wh\|_2^{\frac{s-3}{s}} \left[\|h \nabla w\|_2^{\frac{3}{2}} + \|\nabla hw\|_2^{\frac{3}{2}} \right] \|h \nabla w\|_2 d\tau \end{aligned}$$

Since $\Delta h^2 \in L^\infty(\mathbb{R}^3)$, we get

$$|I_2| \leq c \int_s^t \|w(\tau)\|_2^2 d\tau.$$

Finally, recalling (27), we write

$$(\pi_{\bar{u}} - \pi_u, \Delta g) = -(w \otimes w + u \otimes w + w \otimes u, \nabla \nabla g), \text{ for all } g \in C_0^\infty(\mathbb{R}^3),$$

via (28), by applying Hölder's inequality, we easily arrive at

$$\begin{aligned} |I_3| &\leq c \int_s^t \|\pi_{\bar{u}} - \pi_u\|_{\frac{5}{5}} \|hw\|_6 d\tau \leq c \int_s^t [2\|u\|_3 \|w\|_2 + \|w\|_3 \|w\|_2] \|hw\|_6 d\tau \\ &\leq c \int_s^t [\|u\|_3 + \|w\|_3] \|w\|_2 [\|h \nabla w\|_2 + 2\|w \nabla h\|_2] d\tau. \end{aligned}$$

Collecting the estimates related to I_i , $i = 1, 2, 3$, and substituting on the right-hand side of the energy relation related to wh , applying Young's inequality, we get

$$\begin{aligned} \|hw(t)\|_2^2 + \int_s^t \|h \nabla w(\tau)\|_2^2 d\tau &\leq \|hw(s)\|_2^2 \\ &+ c \int_s^t [\|u\|_{L^s(\mathbb{H})}^r \|w\|_2^2 + \|u\|_{L^s(\mathbb{H})}^2 \|w\|_2^2 + \|w\|_2^2 \\ &+ [\|u\|_3 + \|w\|_3]^2 \|w\|_2^2 + [\|w\|_3 + \|u\|_3] \|w\|_2^2] d\tau. \end{aligned}$$

Letting $s \rightarrow 0$ in the above inequality, and recalling the assumption on u and employing the energy inequality, we get the estimate

$$\|hw(t)\|_2 \leq k(t) \sup_{(0,t)} \|w(\tau)\|_2, \quad (119)$$

where $k(t)$ is a continuous function with the limit property $\lim_{t \rightarrow 0} k(t) = 0$. Estimate (113) and estimate (119) furnish

$$\|w(t)\|_2 \leq [\sqrt{2\pi} c_0 \varepsilon_1^{\frac{2}{3}} + k(t)] \sup_{(0,t)} \|w(t)\|_2 + \int_0^t \left[\frac{A(\tau)}{(t-\tau)^{\frac{1}{2}}} + \frac{\|u_0\|_2}{(t-\tau)^{\frac{3}{4}}} \right] \|w(\tau)\|_2 d\tau.$$

Since $\sqrt{2\pi} c_0 \varepsilon_1^{\frac{2}{3}} < 1$, for a suitable $t^* \in (0, T_0)$ we get $w(t) = 0$ for all $t \in [0, t^*]$. Then, recalling that in the beginning of the section we claimed that u is regular, for $t > 0$, the Leray arguments furnish the coincidence of u and $\bar{u}$ on $[t^*, T_0)$.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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