ORIGINAL RESEARCH

Finet's Law as a Special Case of the Generalised Murray's Law

Jakub Zieliński[1] · Jędrzej M. Nowosielski[1]

Abstract

A thorough knowledge of statistical properties of the coronary artery tree is very important in cardiology. We present generalised form of Murray's law—the first law that described the relationship between vessel diameters in bifurcation. We show that other frequently used laws: Huo-Kassab and Finet's rules are the special cases of its generalised form. We show theoretically that the Finet's law is met with an apparently paradoxical relationship between the lengths and diameters of the vessels. Based on the analysis of CT 3D scans, we show that in the left coronary artery the diameters and lengths of vessels are inversely proportional which explains the applicability of the Finet's law. We justify theoretically the value of coefficient defining Finet's law.

Keywords Murray's law · Finet's law · Artery bifurcation · Computed tomography of coronary arteries · Energy

Introduction

Diseases of the cardiovascular system are leading cause of death in developed countries. The main problems remain atherosclerosis and heart failure. Atherosclerosis may affect coronary, cerebral or peripheral arteries. It increases the vascular resistance and consequently increases heart energy consumption. Heart failure reduces cardiac output (the volume of blood being pumped by the heart per unit of time) or reduces energy conversion efficiency of the heart, hence reducing the vascular resistance is crucial during treatment of the cardiovascular diseases. For this reason, especially in the bypass surgery or placing vascular stents, the key is to understand the properties of optimal vascular geometry. In this publication, we focused on understanding the characteristics of the coronary circulation.

✉ Jakub Zieliński
  jziel@icm.edu.pl

  Jędrzej M. Nowosielski
  jnow@icm.edu.pl

1 Interdisciplinary Centre for Mathematical and Computational Modelling (ICM), University of Warsaw (UW), ul. Pawińskiego 5a, 02-106 Warsaw, Poland

Optimal geometry of the arterial tree should have both minimal vascular resistance and the lowest possible volume of the tree. This ensures simultaneous minimisation of the energy needed to deliver oxygen to the tissues and the necessary blood volume. Increased blood volume is also associated with an increased amount of energy necessary to produce blood or blood vessels.

The first solution to the problem was proposed by Cecil Murray [1]. Let us consider a vessel with a diameter d_0, which splits into two branches with diameters d_1 and d_2 (Fig. 1), then according to Murray's law

$$d_0^3 = d_1^3 + d_2^3. \tag{1}$$

Experimental studies have shown that Murray's law is also true for tubules of some leaves [2], but more accurate measurements showed deviations from Murray's law in the vascular system [3–7]. These tests were performed using CT (computed tomography), IVUS (intravascular ultrasound), direct *post mortem* measurements, etc.

Numerous authors have proposed modifications of Murray's law [8–12]. Depending on whether we consider a laminar, steady or pulsating flow, the resistance is described by the Poiseuille or Womersley law. Similarly, it can be assumed that the flow is transient or turbulent, and the liquid is Newtonian or not. The dependence of vessel lengths on their diameter is also unclear—Murray assumed that these quantities are directly proportional.

Fig. 1 The schematic of a single bifurcation, which consists of the mother vessel and the two daughter vessels
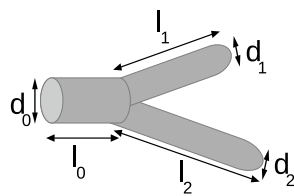
Depending on the assumptions about the flow and the length of the vessels, a slightly different form of the scaling law is obtained. A thorough knowledge of this law is important in cardiology for several reasons. First of all, it is sometimes necessary to place a stent in a vascular branching. For the success of the procedure, it is critical that all three stent diameters match the geometry of a particular patient. It is therefore important to know what diameters often occur. Secondly, currently non-invasive imaging of small vessels is not possible. Consequently, in order to carry out computer simulations of blood flow in small vessels, it is necessary to know the statistical properties of branching.

In interventional cardiology, two laws based on Murray's law are commonly used: the Huo–Kassab law (HK) [9]

$$d_0^{7/3} = d_1^{7/3} + d_2^{7/3} \tag{2}$$

and the area preserving branching (APB)

$$d_0^2 = d_1^2 + d_2^2. \tag{3}$$

However, the law proposed by Gérard Finet et al. [3] is most often used clinically [12]. This law is:

$$d_0 = 0.678(d_1 + d_2). \tag{4}$$

Finet's law seems to have surprising mathematical properties, because for a very small one of the distal vessels ($d_2 \approx 0$) there is: $d_0 \approx 0.678\,d_1$, while we expect that the departure of a small vessel will not cause a significant change in the diameter of the main vessel. So for $d_2 \approx 0$, one would expect $d_0 \approx d_1$. Nevertheless, Finet's law is the best when the needs of interventional cardiologists are considered.

Derivation of the Generalised Murray's Law

In order to show weaknesses of Murray's law, the derivation of its generalised form is briefly described below. For the sake of simplicity, the single bifurcation, which consists of three vessels (the mother vessel and the two daughter vessels), is considered here for optimisation (Fig. 1). Let us assume that ith vessel is a cylinder with diameter d_i and length l_i, where $i = 0$ denotes the mother vessel and $i = 1$, $i = 2$ denote the daughter vessels. Murray assumed that

length of vessel segments is directly proportional to their diameters. We use more general formula

$$l_i = A \cdot d_i^{\beta}, \tag{5}$$

where A is some constant. The volume of the considered chunk of the vessel tree is given by

$$V = \frac{\pi}{4}\left(l_0 \cdot d_0^2 + l_1 \cdot d_1^2 + l_2 \cdot d_2^2\right) = \frac{\pi}{4} \cdot A\left(d_0^{2+\beta} + d_1^{2+\beta} + d_2^{2+\beta}\right). \tag{6}$$

Our goal is to find relation between diameters of the three vessels which minimises vessel resistance of the whole considered tree chunk for a fixed total volume of the tree chunk. The lowest vessel resistance results in the lowest energy expense. Vascular resistance Z is defined as the quotient of pressure drop Δp and volume of blood flowing per unit of time Q:

$$Z = \frac{\Delta p}{Q}. \tag{7}$$

The considered three-vessel chunk of the tree is in fact a simple series-parallel connection. It seems to be justified to write

$$Z = Z_0 + \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}}. \tag{8}$$

We still need to calculate the resistance of the single cylindrical vessel. Murray utilised the classic Hagen–Poiseuille law

$$Z = \frac{128\mu l}{\pi d^4}, \tag{9}$$

where μ is viscosity of blood (dynamic viscosity). The above-mentioned formula is valid provided following conditions are satisfied:

– the considered fluid is Newtonian and incompressible,
– the flow is laminar and steady,
– the vessel is a cylindrical pipe of constant cross-section,
– the length of the pipe is much larger than its diameter.

However, these assumptions are not met in the human circulatory system. Hence, one often assumes that resistance is inversely proportional not to fourth power but to some different power of the vessel diameter

$$Z = B\frac{l}{d^{\alpha}}, \tag{10}$$

where B is a constant dependent, among other factors, on the viscosity and density of the fluid. In the special case $\alpha = 4$ we get the Hagen–Poiseuille flow.

The pulsating flow is characterised by different velocity profile in cross-section and that is why there is different impedance dependence on the vessel diameter (due to the negligible effect of the phase difference between pressure and flow changes and the relatively low frequency of blood pulsing, we use the term *resistance* instead of *impedance*). It can be described by Womersley's law. Exponent α lies between 2 and 4 depending on the oscillation frequency. The turbulent flow in turn depends even stronger on the vessel diameter because the diameter change influences the Reynolds number, i.e. for a fixed flow the Reynolds number drops with increasing diameter due to the drop of the average velocity. Hence, one usually assumes that exponent α lies within interval $4-5$. Similarly, assuming non-Newtonian fluid results in change of velocity profile. Consequently, the non-Newtonian fluid assumption changes exponent α. The exponent values for various types of flow and corresponding characteristics of physiological settings are listed in Table 1.

In various parts of the vessel tree the fluid flow is characterised by distinct values of exponent α. It is therefore sensible to use more general formula (10) rather than just Hagen–Poiseuille law (9) in order to derive the generalised Murray's law. The formula describing three-vessel chunk of the tree is then following

$$Z = B\left(\frac{l_0}{d_0^{\alpha}} + \frac{1}{\frac{d_1^{\alpha}}{l_1} + \frac{d_2^{\alpha}}{l_2}}\right). \tag{11}$$

When we substitute relation (5) between vessel length l_i and vessel diameter d_i into the above formula, we obtain

$$Z = A \cdot B\left(d_0^{\beta-\alpha} + \frac{1}{d_1^{\alpha-\beta} + d_2^{\alpha-\beta}}\right). \tag{12}$$

In the vessel system there are both symmetrical ramifications, i.e. with $d_1 = d_2$, and strongly asymmetrical where a small vessel departs from a large one and $d_0 \approx d_1 \gg d_2$. We introduce asymmetry coefficient

$$\gamma = \frac{d_2}{d_1}. \tag{13}$$

Table 1 Exponent α for various types of flow given with characteristics of physiological settings

α	Flow	Human circulatory system
$2 < \alpha < 4$	Womersley	Large and medium arteries
$\alpha = 4$	Hagen–Poiseuille	Small arteries and veins
$4 < \alpha < 5$	Turbulent	Ascending aorta

For the sake of clarity, let us assume that $\gamma \leq 1$. Using asymmetry coefficient γ we obtain following formulas describing the volume and resistance of the considered three-vessel tree chunk:

$$V = \frac{\pi}{4} \cdot A\left(d_0^{2+\beta} + d_1^{2+\beta}\left(1 + \gamma^{2+\beta}\right)\right) \tag{14}$$

and

$$Z = A \cdot B\left(d_0^{\beta-\alpha} + \frac{1}{d_1^{\alpha-\beta}\left(1 + \gamma^{\alpha-\beta}\right)}\right). \tag{15}$$

Next, d_1 may be expressed as a function of V, d_0, β, γ and A using formula (14). The formula describing d_1 is then substituted into formula (15) and we obtain

$$Z = A \cdot B\left(d_0^{\beta-\alpha} + \frac{\left(\frac{V/\pi/A - d_0^{2+\beta}}{1+\gamma^{2+\beta}}\right)^{\frac{\beta-\alpha}{2+\beta}}}{1 + \gamma^{\alpha-\beta}}\right). \tag{16}$$

We assume that γ is fixed, i.e there is some fixed asymmetry. We consider also V, α, β and A fixed model parameters. In contrast, d_0 is considered optimised variable. From a condition for derivative equal to 0 we obtain

$$d_0^{2+\beta} = \frac{\left(1 + \gamma^{\alpha-\beta}\right)^{\frac{2+\beta}{2+\alpha}}}{\left(1 + \gamma^{2+\beta}\right)^{\frac{\alpha-\beta}{2+\alpha}}}\left(d_1^{2+\beta} + d_2^{2+\beta}\right). \tag{17}$$

The above equation expresses the generalised Murray law.

In the special case when the Hagen–Poiseuille law (9) is satisfied, i.e. $\alpha = 4$ and additionally $\beta = 1$, the generalised Murray's law (17) reduces to Murray's law (1).

On the other hand, if $\alpha = \frac{8}{3}$ and $\beta = \frac{1}{3}$, the generalised Murray's law reduces to the Huo–Kassab law (2). The value α of $\frac{8}{3}$ corresponds to Womersley's flow, which seems to be the type of the flow occurring in the coronary arteries.

The discrepancy between values of β in Murray's law and the Huo–Kassab law is somewhat problematic. Moreover, for $\beta = -1$ the generalised Murray's law reduces approximately to Finet's law (4). After substitution of $\beta = -1$ into (17), we obtain

$$d_0 = \frac{\left(1 + \gamma^{\alpha+1}\right)^{\frac{1}{2+\alpha}}}{\left(1 + \gamma\right)^{\frac{\alpha+1}{2+\alpha}}}\left(d_1 + d_2\right), \tag{18}$$

which can be expressed as

$$d_0 = X \cdot \left(d_1 + d_2\right), \tag{19}$$

where

$$X = \frac{\left(1 + \gamma^{\alpha+1}\right)^{\frac{1}{2+\alpha}}}{(1 + \gamma)^{\frac{\alpha+1}{2+\alpha}}}. \tag{20}$$

For $\gamma > 0.5$ factor X weakly depends on α and γ. The dependence is shown in Fig. 2. Factor X lies in interval 0.6–0.76 provided $\gamma > 0.5$ and value of α is sensible, i.e it is between 2 and 5. It is consistent with coefficient 0.678 defining Finet's law.

We showed that the general form of Murray's law is reduced to Finet's law in the case of an inversely proportional relationship between the vessels length and diameter and relatively small asymmetry of daughters' vessels ($\gamma > 0.5$). It means that Finet's law can be considered a special case of the generalised Murray's law with $\beta = -1$. However, $\beta = -1$ signifies a surprising relationship (inversely proportional) between the vessel length and diameter, which needs to be verified experimentally. Moreover, we need to check if asymmetry coefficient γ is high enough.

As stated above, Finet's law cannot be applied to a strong asymmetry ($\gamma \approx 0$). The generalised Murray's law suffers from a similar drawback despite its generality. It arises from an implicit assumption underpinning formula describing resistance of the series-parallel connection (8). In order to consider daughter vessels connected in parallel and to utilise formula (8) one needs to assume that pressure on the ends of both daughter vessels (vessel 1 and vessel 2) is the same. Such an assumption is not satisfied if coefficient γ is close to 0. In the case of small γ, i.e. $d_1 \gg d_2$, the daughter vessel 2 is in fact the vessel with micro-circulation and one should not expect it to have the same pressure as the large daughter vessel 1. The formula (17) should be regarded unsuitable when coefficient γ is close to 0. The same limitation also applies to original Murray's law.

Statistical Analysis of Vessel Lengths and Diameters Based on CT Angiography

In order to experimentally verify the relationship between the length and diameter of coronary arteries and asymmetry coefficient CT studies of 19 subjects were analysed. The used data was a result of the "Novel Method for Functional Assessment of Coronary Artery Stenosis with In-Silico Flow Modelling based on Multirow Computed Tomography Imaging (TRAFIC)" project, co-funded by The National Centre for Research and Development (Poland) within PBS programme (PBS1/A9/18/2013).

Vessel lengths and diameters were derived from computed tomography (CT) angiography. The length of a given coronary vessel is understood in this paper to be the length between two consecutive bifurcations. Strictly speaking, the diameter varies even along a single vessel and therefore we use the average diameter weighted with point-to-point distance. The analysis was performed separately for the left coronary artery (LCA) and the right coronary artery (RCA).

In the case of the left coronary artery the mean value of asymmetry coefficient γ equals 0.73 and its standard deviation is 0.15. Moreover, condition $\gamma > 0.5$ is satisfied for 91.5% bifurcations. When the right coronary artery is concerned, the mean value of asymmetry coefficient γ equals 0.67 and its standard deviation is 0.20. Condition $\gamma > 0.5$ is satisfied for 74.7% bifurcations. This means that the low asymmetry condition ($\gamma > 0.5$) is well satisfied in the left coronary artery (LCA) and is not met in the right coronary artery (RCA).

Fig. 2 Dependence of factor X on exponent α from vessel resistance formula (10) and asymmetry coefficient γ
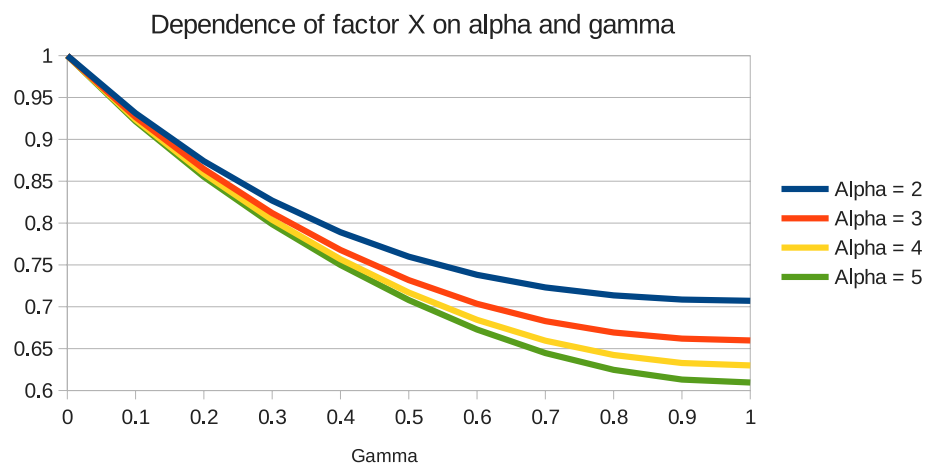
To investigate the dependence of length on the diameter of coronary vessels, we will transform Eq. (5). After applying logarithm to both sides this equation we get

$$\ln(l_i) = \ln(A) + \beta \ln(d_i). \tag{21}$$

Hence, parameter β may be determined with a simple linear regression, which results in $\beta = -1.02 \pm 0.11$ for LCA and $\beta = -0.69 \pm 0.14$ for RCA. Data points and the fitted linear function for LCA are shown in Fig. 3. It worth noticing that both values of β obtained with the regression are negative. In the case of RCA, the result is not statistically significant, whereas in the case of LCA the value -1 is within a 95% confidence interval. In the case of LCA the experimentally obtained value of β is close to -1 and it justifies the reduction of the generalised Murray's law to Finet's law.

The obtained negative value of β is neither a numerical artefact nor a statistical fluctuation. It stems from two reasons. Firstly, the left coronary artery starts with the short and thick vessel named left main coronary artery, which bifurcates into two large arteries, i.e. left circumflex artery and left anterior descending artery. One can see in Fig. 4 that there are other large vessels (left marginal artery and diagonal branch) departing from these two arteries (left circumflex artery and left anterior descending artery). That is why thick vessels forming LCA are relatively short. Secondly, one may see in Fig. 4 that distal narrower vessels are longer. It is due to imaging method of the CT angiography—in reality there are numerous tiny vessels, invisible in the CT scan, departing from the visible larger vessels.
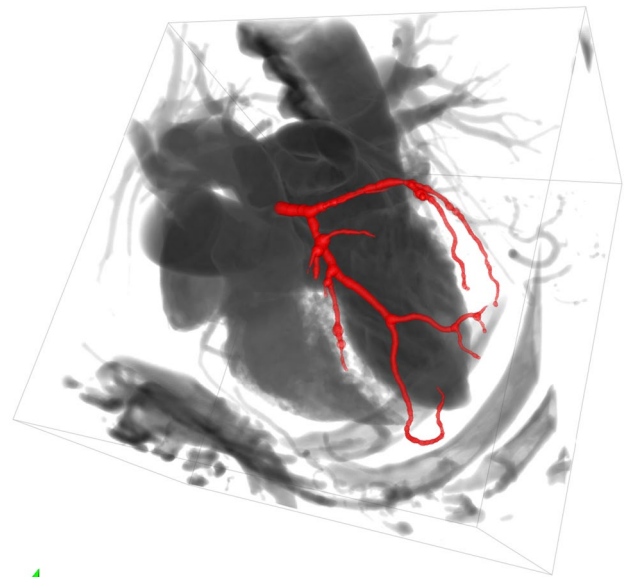

Fig. 4 Left coronary vessel artery reconstructed from CT angiography

Conclusions

Analysis of coronary vessel trees based on CT angiography indicates that vessel length is inversely proportional to its diameter in the case of the left coronary artery (LCA). It translates to $\beta = -1$ in the generalised Murray's law. In such a case the generalised Murray's law is in turn reduced to Finet's law, which is preferred by cardiologists.

Moreover, we have showed that if vessel length is inversely proportional to its diameter ($\beta = -1$) and

Fig. 3 Relation between length and diameter for vessels belonging to the left coronary artery

f(x) = - 1.02x + 3.60
R² = 0.17
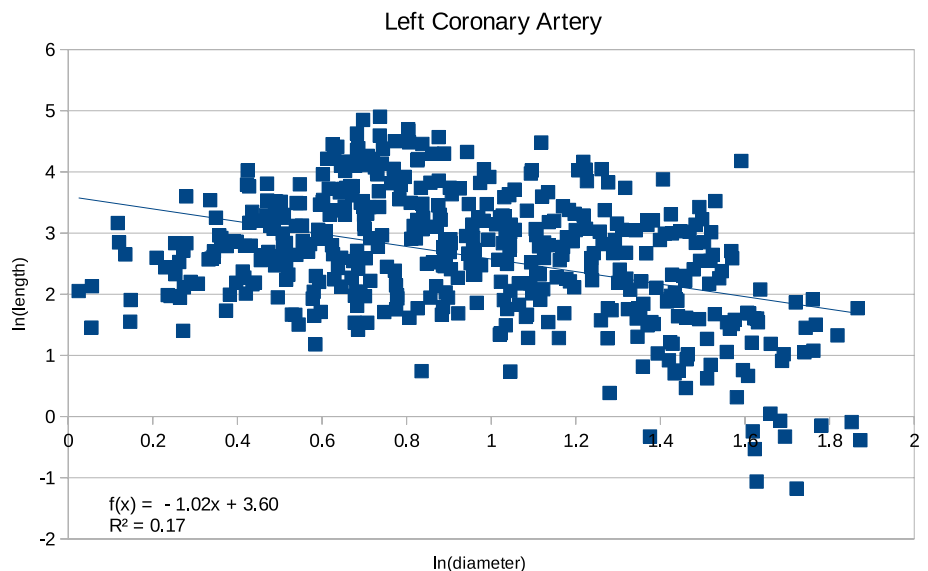
asymmetry of distal vessels is relatively small ($\gamma > 0.5$), the coefficient 0.678 occurring in the Finet's formula should be in the range 0.6–0.76. This coefficient is very insensitive to flow model assumptions. Until now, to the authors' best knowledge, it was only experimentally determined. This work is therefore the theoretical justification for the Finet's formula.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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