



Investment in time preference and long-run distribution

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Abstract

This paper presents a simple dynamic general equilibrium model in which each household can make a costly investment in patience capital at each time. We show that the interior long-run steady state is unstable, in the sense that per household, there is a one-dimensional curve lying in the two-dimensional space of its patience capital and physical capital amounts, and convergence happens only when its initial pair falls exactly on the curve. Households with the initial vectors falling in the upper side of the curve invest more in patience capital, which leads themselves to save more, and hence, the consumption level grows in the long run. Households with the initial vectors falling in the lower side opt out from investing in patience capital, leading to a decay of patience level, which leads themselves to save less and hence they perish in the long run. We also show a possibility that there is an expanding swing between the two classes.

Keywords Time preference · Patience capital · Long-run distribution

1 Introduction

1.1 The Ramsey conjecture and endogenous time preference

The standard consumption-saving model in which households maximize their discounted lifetime utilities, when put under the natural assumption that they differ in their time preferences, has an uneasy long-run implication—given a time-constant interest rate, households are generically divided into two “classes”, one such that their consumptions and wealths converge to zero or subsistence level, the other such that their consumptions and wealths diverge to infinity; moreover, in the full dynamic general equilibrium environment in which the interest rate lowers over time according to diminishing returns to capital, all but the most patient household

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economically “perish”, and only the most patient one can “survive”, actually regardless of earnings and initial distribution of capitals. Such “division of society into two classes” has been conjectured by Ramsey in his classic work Ramsey (1928), and later confirmed by Becker (1980) and Bewley (1982) in the discrete-time setting and Mitra and Sorger (2013) in the continuous-time setting.

Look at the discounted utility form, written in the continuous-time setting which the current paper adopts:

$$\int_0^{\infty} v(c(t))e^{-\beta t} dt,$$

where β denotes the discount rate. When the long-run pure interest rate r is given as fixed, we have to have either:

$$\beta > r$$

or

$$\beta < r,$$

generically. In the first case, the household saves less, or even borrows, and as this accumulates exponentially over time its consumption/wealth go to zero. In the second case, the household saves more, and as this accumulates exponentially over time its consumption/wealth go to infinity.

When the long-run interest rate is flexibly adjusted to diminishing return to capital over time, it is expected to satisfy the condition:

$$\beta = r,$$

but this can never be met when households have different β s. Thus, it has been shown that all but one with the smallest β are pushed away toward zero consumption/wealth or subsistence level in the long run.¹

There are two kinds of uneasiness here. One is that such long-run state does not look normatively right. There is nothing wrong in terms of the classical concept of welfare applied to “long-lived” households’ preferences over their consumption paths, since such equilibrium path is Pareto-efficient and even “fair” in the sense of envy-freeness if they have equal earnings and initial capital holding. According to this, it is simply that impatient households consume more in earlier periods. It is still problematic in two senses, though. First, since allocations take place over time, welfare criteria applied just to ex-ante evaluation of planned life-courses may be normatively insufficient, and it is ethically a different question whether we should accept resulting ex-post inequalities. Second, when the “long-lived” households are interpreted as families, such that nobody is responsible for in which family and in which generation and with what nature of time preference he or she is born, this extreme long-run inequality is definitely problematic.

¹ Because of this, homogeneity in time preference is vital for any macroeconomics based on the idea of representative agent, even when it allows heterogeneity in other dimensions like in Caselli and Ventura (2000).

Second is that such extreme long-run inequality looks simply unrealistic. Although the precise answer depends on how long the “long-run” is, our crude sense tells us that plural and different households are there, while there exist unignorable and perhaps severe inequalities among them due to other reasons. This motivates various positive models to describe and explain that plural and different households can economically survive.

Because the extreme long-run inequality result depends critically on the assumption of discounted utility preference, which is additively separable and stationary and seen as a knife-edge case even within the class of “rational (in the sense of dynamic consistency)” preferences, many preference models have been proposed to allow for the roles of non-separability and/or non-stationarity, and it has been investigated whether the extreme long-run inequality result is milder or still persists.

To illustrate, consider that the discount rate at each moment takes the form $\beta(x)$, where x can be either a decision variable which can be chosen at each moment, such as consumption, or a state variable such as physical capital, or accumulated habit. The idea is that in the long run, the steady-state value of x for each household, denoted x^* , can be flexibly adjusted, so that the interior steady-state condition:

$$\beta(x^*) = r$$

is met, and this can hold for different $\beta(\cdot)$ as the steady-state value is flexibly adjusted. This allows us to shift our question from existence of interior steady state, which generically fails, to its stability.

There are various existing models of such endogenous time preference. Uzawa (1968) proposes a preference model in which β depends on the current consumption, and it is axiomatized by Epstein (1983) in the discrete-time setting. Its further generalization is provided by Epstein (1987b), which allows non-linear way of discounting continuation lifetime utility, and is seen as a continuous-time counterpart of the classic discrete-time model of recursive utility by Koopmans (1960). Based on those models of recursive utility, a number of studies provide characterizations of the stability condition for economies with heterogeneous households, such as Epstein and Hynes (1983), Lucas and Stokey (1984), Epstein (1987a, b), Benhabib et al. (1988).²

These stability results assume that discount rate at each moment is increasing in present consumption. On the other hand, Das (2003, 2007), Hirose and Ikeda (2008) consider that discount rate is decreasing in present consumption. This is consistent with the empirically natural observation that the rich (poor) are more (im)patient, which was originally suggested by Fisher (1930). They show that steady state may be unstable, or that there are multiple steady states.

Note that the above-noted class is stationary, while weakening additive separability, and there preference over-consumption streams being held at each time is independent of histories of past consumptions, meaning that whatever the history is the consumer’s discounting between the present period and the next period depends

² Another possible explanation for survival of plural agents in steady state is to allow randomness of discount factors. See Becker and Zilcha (1997) for example.

only on the present consumption. This is somewhat odd as an explanation of the “Fisherian” story, because it says that regardless of wealth accumulation, a sudden change in the present consumption immediately changes the degree of patience, and also that such effect lasts only for one period. We view that patience should be modeled rather as certain kind of capital.

Shi and Epstein (1993) incorporate habit formation into time-preference determination, in which β depends on the level of consumption habit and is increasing in it, meaning that higher habit leads to more impatience. There, the habit rises as the household consumes more than the current habit level. This works as a stabilizer, as an over-consumption leads to higher habit, leading to less patience, leading to more current consumption but less in the future, leading to lower habit and more patience, and so on.

In the context of growth problem, Schumacher (2009) and Strulik (2012) consider that discount rate at each moment depends on physical capital amount, which goes along the line that rich are more patient, and in general work in the direction to destabilize. They show that there are multiple steady states, so that the lower one is interpreted as a poverty trap.

1.2 Investment in time preference

This paper studies a model of endogenous time-preference formation, following Becker and Mulligan (1997), in which households can pay for costly investment to establish and maintain patience, where the degree of patience is now understood to be a capital. Following Doepke and Zilibotti (2005, 2008), we call such capital patience capital. A prominent example of the form of investment in patience capital is education, within or outside family.³

We view that investment in patience capital is a natural channel of endogeneity in time-preference determination, and it tells causal effect more directly than the above-noted elements. It captures better the reverse causality that the rich (poor) are more (im)patient, because they are rich (poor), not just the single direction that the rich (poor) are more (im)patient, because they are (im)patient.

Effect of education on time-preference determination is documented by experimental studies such as Alan and Ertac (2018) and Perez-Arce (2017). Also, numbers of studies document that time preference is related to socioeconomic status which naturally require inputting wealth for acquisition and maintenance, or to wealth itself (see, for example, Lawrance 1991; Barsky 1997; Tanaka et al. 2010; Dohmen 2015). Although the precise causal effects are yet unclear, it will be fair to say that acquiring and maintaining patience are at least costly.

Patience capital may be accumulated over time, while it can depreciate, as well. In the intergenerational interpretation, this means that patience capital is inherited to descendants. Although acquired characteristics are not inherited biologically, in the

³ An alternative interpretation of patience capital is that it is a kind of health capital and the resulting endogenous discount rate is associated with survival probability, where investment in it takes the form of medical expense.

literature of sociology, numbers of studies have documented that “cultural capital” is reproduced across generations within a family or a class, through cultural inheritance (see, for example, Bourdieu 1973; DiMaggio 1982).

After the works by Doepke and Zilibotti (2005, 2008), there is a growing attention to investment in patience capital in the context of endogenous formation of “spirit of capitalism” and its effect on economic development (see, for example, Kawagishi 2014; Haruyama and Park 2017). To our knowledge, though, the implication of investment in patience capital to long-run distribution of consumption and wealth in relation to the rather classic problem as explained above is not known. Does investment in time preference lead to survival and coexistence of heterogeneous types, or opposite?

Denote the amount of patience capital by π , then the endogenous discount rate is given by $\beta(\pi)$, which is decreasing in π , meaning that higher patience leads to less discounting of the future. Unlike physical capital, such patience capital is not transferrable across households after acquired. Evolution of patience capital π at each moment is determined by the amount of investment-like consumption. Precisely, it is given by:

$$\dot{\pi} = -\delta\pi + g(z),$$

where z denotes the amount of investment-like capital, g is the production function, and δ denotes the depreciation rate. Investing z requires material cost, and hence, it is in a direct material trade-off with pure consumption. The preference model is seen as a multi-period extension of the model by Becker and Mulligan (1997), where they assume determination of time preference occurs just once in the initial period. It parallels with the discrete-time dynastic model by Doepke and Zilibotti (2005, 2008), while there is a conceptual difference (or trade-off, perhaps) that they consider that each generation can invest on the next generation’s patience level for the latter’s lifetime, but the degree of altruism toward succeeding generation remains constant across all generations, and we consider that both evolve endogenously, but they are described altogether by one variable π .

First, we will consider linear technology, where interest rate r is time-constant, to isolate the problem of whether investment in time preference has a stabilizing effect for each household or not. In contrary to the previous literature on endogenous time preference, we show that the interior steady-state condition:

$$\beta(\pi^*) = r$$

is unstable in the following sense: per one household, there is a one-dimensional curve lying in the space of the two-dimensional space of state variables (π, k) , where k denotes the amount of physical capital, which pass through the interior steady state (π^*, k^*) as the only possible stable path leading to it, such that convergence to the steady state occurs only when the vector of initial values $(\pi(0), k(0))$ falls exactly on the curve.

Thus, we again see a division of society into two “classes”, where the above-described curve plays the role of border. Households with initial vectors falling in the upper side of the border are to be in the “upper class”, invest more in patience

capital, which leads themselves to save more, and hence, the consumption level grows in the long-run. Households with initial vectors falling in the lower side of the border are to be in the “lower class”, opt out from investing in patience capital, leading to a decay of patience level, which leads themselves to save less, and hence, they perish in the long-run. In this sense, the extreme long-run inequality as obtained in the basic discounted utility model is shown to be rather robust and self-confirming. We also show a possibility that there is an expanding swing between the two classes.

We then show that this result extends to economies with technologies with diminishing returns to capital, as far as the diminishment is sufficiently small. This shows that the instability result is not a mere knife-edge case.

2 The household model

Consider an infinite horizon continuous-time setting. There are two consumption goods: one is for pure consumption and the other is for “investment-consumption.” Denote the amount of pure consumption at time t by $c(t)$, and that of investment-consumption by $z(t)$. Denote a path of such pair by $(c, z) = (c(t), z(t))_{t \geq 0}$, which is the object of preference.

We consider that a household has preference represented in the form:

$$U(c, z) = \int_0^{\infty} v(c(t)) e^{-\int_0^t \beta(\pi(\tau)) d\tau} dt$$

$$\dot{\pi}(t) = -\delta\pi(t) + g(z(t))$$

$$\pi(0) = \text{given},$$

where β is decreasing in π . The preference is indirectly represented through variable π , which is interpreted as “patience capital”. Moment by moment, it is accumulated through having the investment-consumption $z(t)$, while it depreciates with rate $\delta \geq 0$.

In a market, a household has to pay for purchasing the investment-consumption good. Thus, there is a trade-off between enjoying pure consumption and establishing patience through having the investment-consumption good. Also there is a trade-off between saving and establishing patience, in addition to the standard trade-off between pure consumption and saving.

We make the following regularity assumptions on v , β , and g .

Assumption 1 $v : \mathbb{R}_+ \rightarrow [0, \bar{v})$ is twice-continuously differentiable on \mathbb{R}_{++} and satisfies $v' > 0$ and $v'' < 0$. Also, it satisfies $\lim_{c \rightarrow 0} v'(c) = \infty$ and $\lim_{c \rightarrow \infty} v'(c) = 0$.

In contrast to the case of standard discounted utility, the assumption that the period utility function is positive-valued has a behavioral content, since if it is negative, the household may prefer to become more impatient. We assume v is bounded from above to guarantee that the lifetime utility function is well defined.

Assumption 2 $\beta : \mathbb{R}_{++} \rightarrow (\beta, \infty)$ is twice-continuously differentiable on \mathbb{R}_{++} , and satisfies $\beta' < 0$ and $\beta'' > 0$. Also, it holds $\lim_{\pi \rightarrow 0} \beta(\pi) = \infty$ and $\lim_{\pi \rightarrow \infty} \beta(\pi) = \underline{\beta}$.

We assume that β is bounded away from below by $\underline{\beta} > 0$. There are two reasons. One is that it is natural, in the sense that even after indefinite accumulation of patience capital, there is a fundamental minimal degree of discounting. Second, the minimal degree of discounting guarantees that the lifetime utility is well defined, because, otherwise, the household’s lifetime utility may blow up by means of making oneself indefinitely more patient.

Finally, we impose the following assumption on production of patience capital, which is quite natural.

Assumption 3 $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is twice-continuously differentiable on \mathbb{R}_{++} , and satisfies $g' > 0$, $g'' < 0$, and $g(0) = 0$. Also, it satisfies $\lim_{z \rightarrow 0} g'(z) = \infty$ and $\lim_{z \rightarrow \infty} g'(z) = 0$.

3 Dynamic competitive equilibrium and its welfare property

At each time, an output good is produced from an input good and labor. Let $F(K, L)$ denote the aggregate production function, where K denotes the amount of capital input and L denotes the labor input, and it is assumed to be three-times differentiable over \mathbb{R}_{++}^2 , weakly concave, and exhibits constant returns to scale. To suppress notation, we assume that depreciation of physical capital is already taken into account in the specification of F .

Each household has 1 unit of labor time which can be used as production input, and for simplicity, we assume that the households have no preference for leisure.

The produced output can be allocated either as the pure consumption good or the investment-consumption good or capital holding, as we implicitly assume that there is a linear technology which converts between them, and without loss of generality, we assume that the rate of conversion is 1.

Let $I = \{1, \dots, n\}$ be the set of households. Let $c_i(t)$ denote the amount of pure consumption, $z_i(t)$ denote the amount of investment-consumption, and $k_i(t)$ denote the amount of capital being held for household $i \in I$ and time t , while $k_i(0)$ is given. Let $r(t)$ denote the return rate of capital and $w(t)$ denote wage at time t . Here, an household allocation is denoted by $(c_i, z_i, k_i) = (c_i(t), z_i(t), k_i(t))_{t \geq 0}$ and a social one is denoted by $(c, z, k) = (c_i, z_i, k_i)_{i \in I}$.

Definition 1 An allocation (c, z, k) is a dynamic competitive equilibrium if there exists a path $(r, w) = (r(t), w(t))_{t \geq 0}$, such that (c_i, z_i, k_i) maximizes:

$$U_i(c_i, z_i) = \int_0^\infty v_i(c_i(t)) e^{-\int_0^t \beta_i(\pi_i(\tau)) d\tau} dt$$

under the constraints:

$$\begin{aligned} \dot{\pi}_i(t) &= -\delta_i \pi_i(t) + g_i(z_i(t)) \\ \dot{k}_i(t) &= r(t)k_i(t) + w(t) - c_i(t) - z_i(t) \\ c_i(t), z_i(t) &\geq 0 \\ \pi_i(0), k_i(0) &= \text{given} \end{aligned}$$

for every $i \in I$, so that the No-Ponzi condition:

$$\lim_{t \rightarrow \infty} \frac{k_i(t)}{e^{\int_0^t r(\tau) d\tau}} \geq 0$$

is met, and $(\sum_{i \in I} k_i(t), n)$ solves the profit-maximization problem:

$$\max_{K,L} F(K, L) - r(t)K - w(t)L$$

and the market-clearing condition:

$$\sum_{i \in I} c_i(t) + \sum_{i \in I} z_i(t) + \sum_{i \in I} \dot{k}_i(t) = F\left(\sum_{i \in I} k_i(t), n\right)$$

is met for all t .

We assume that there is no borrowing constraint or that it is not binding in finite time range, as we are focusing on the question whether the economy can converge to an interior long-run steady state.

Under the No-Ponzi condition, we can consolidate the series of sequential budget constraints into one, as:

$$\int_0^\infty \frac{c_i(t) + z_i(t)}{e^{\int_0^t r(\tau) d\tau}} dt \leq \int_0^\infty \frac{w(t)}{e^{\int_0^t r(\tau) d\tau}} dt + r(0)k_i(0).$$

Hence, the current definition of equilibrium falls in the Arrow–Debreu–McKenzie framework.

Proposition 1 *If $(c_i, z_i, k_i)_{i \in I}$ is a dynamic competitive equilibrium allocation, then $(c_i, z_i)_{i \in I}$ is an Arrow–Debreu–McKenzie equilibrium allocation.*

As far as the investment-like consumption is understood as still falling in the category of “consumption”, there is nothing wrong with the classic concept of welfare, since any equilibrium consumption path is Pareto-efficient. Moreover, when each household is responsible for the initial level of patience capital and the way of reproducing it is “a matter of taste”, the equilibrium path is even “fair” in the sense of absence of envy among households when they are endowed with equal initial amounts of physical capital.

Say that a feasible allocation $(c, z) = (c_i, z_i)_{i \in I}$ is ex-ante Pareto-efficient if there is no feasible allocation $(\tilde{c}, \tilde{z}) = (\tilde{c}_i, \tilde{z}_i)_{i \in I}$, such that $U_i(\tilde{c}_i, \tilde{z}_i) \geq U_i(c_i, z_i)$ for all $i \in I$ and $U_i(\tilde{c}_i, \tilde{z}_i) > U_i(c_i, z_i)$ for at least one $i \in I$.

Say that an allocation $(c, z) = (c_i, z_i)_{i \in I}$ *ex-ante envy-free* if $U_i(c_i, z_i) \geq U_i(c_j, z_j)$ for all $i, j \in I$. We put the word “ex-ante” to emphasize that it is about ex-ante evaluation of planned life-courses. Note that here, every i is taken to be responsible for its preference, including the initial amount of patience capital $\pi_i(0)$.

Proposition 2 *Arrow–Debreu–McKenzie equilibrium allocation (c, z) is Pareto-efficient. It is also ex-ante envy-free if $k_i(0) = k_j(0)$ for all $i, j \in I$.*

Despite of Pareto efficiency and fairness from the ex-ante viewpoint, we show in the next section that the long-run ex-post consequence can be an extreme inequality.

4 Analysis

4.1 Equilibrium dynamics and steady state

Let

$$\alpha_i(t) = e^{-\int_0^t \beta_i(\pi_i(\tau))d\tau}$$

which follows the differential equation:

$$\dot{\alpha}_i(t) = -\beta_i(\pi_i(t))\alpha_i(t).$$

To simply notation, hereafter, we omit t as far as no confusion arises. Then, the maximization problem for a generic price-taking household is formulated as:

$$\max \int_0^\infty \alpha_i v_i(c_i) dt$$

subject to

$$\begin{aligned} \dot{k}_i &= rk_i + w - c_i - z_i \\ \dot{\pi}_i &= -\delta_i \pi_i + g_i(z_i) \\ \dot{\alpha}_i &= -\beta_i(\pi_i)\alpha_i \\ c_i, z_i &\geq 0 \\ k_i(0) &= \text{given} \\ \pi_i(0) &= \text{given} \\ \alpha_i(0) &= 1. \end{aligned}$$

Set up the Hamiltonian, assuming an interior path, as:

$$H_i = \alpha_i v_i(c_i) + \lambda_i [rk_i + w - c_i - z_i] + \mu_i [-\delta_i \pi_i + g_i(z_i)] + \nu_i [-\beta_i(\pi_i)\alpha_i].$$

Then, the individually optimal path is characterized by:

$$\begin{aligned}
\alpha_i v'_i(c_i) - \lambda_i &= 0 \\
-\lambda_i + \mu_i g'_i(z_i) &= 0 \\
\dot{\lambda}_i &= -\lambda_i r \\
\dot{\mu}_i &= \delta_i \mu_i + \beta'_i(\pi_i) v_i \alpha_i \\
\dot{v}_i &= -v_i(c_i) + \beta_i(\pi_i) v_i \\
\dot{k}_i &= r k_i + w - c_i - z_i \\
\dot{\pi}_i &= -\delta_i \pi_i + g_i(z_i) \\
\dot{\alpha}_i &= -\beta_i(\pi_i) \alpha_i
\end{aligned}$$

plus three transversality conditions:

$$\begin{aligned}
\lim_{t \rightarrow \infty} \lambda_i(t) k_i(t) &\leq 0 \\
\lim_{t \rightarrow \infty} \mu_i(t) \pi_i(t) &\leq 0 \\
\lim_{t \rightarrow \infty} v_i(t) \alpha_i(t) &\leq 0.
\end{aligned}$$

Note that together with the No-Ponzi condition, the first transversality condition reduces to:

$$\lim_{t \rightarrow \infty} \frac{k_i(t)}{e^{\int_0^t r(\tau) d\tau}} = 0.$$

Also, because of positivity of π_i , together with the first-order condition, the second transversality condition reduces to:

$$\lim_{t \rightarrow \infty} \frac{\pi_i(t)}{g'_i(z_i(t)) e^{\int_0^t r(\tau) d\tau}} = 0,$$

and it is relevant only when π_i diverges to positive infinity. Note also that the third transversality condition is met by any path, under our assumption that utility is bounded and the discount rate is bounded away from zero.

We eliminate λ_i , μ_i , and α_i , while v_i cannot be, and obtain the dynamics:

$$\begin{aligned}
\dot{c}_i &= \frac{(\beta_i(\pi_i) - r) v'_i(c_i)}{v''_i(c_i)} \\
\dot{z}_i &= -\frac{g'_i(z_i)}{g''_i(z_i)} \left[r + \delta_i + \frac{g'_i(z_i) \beta'_i(\pi_i) v_i}{v'_i(c_i)} \right] \\
\dot{v}_i &= -v_i(c_i) + \beta_i(\pi_i) v_i \\
\dot{\pi}_i &= -\delta_i \pi_i + g_i(z_i) \\
\dot{k}_i &= r k_i + w - c_i - z_i.
\end{aligned}$$

On the other hand, from the profit-maximization condition in equilibrium, it holds:

$$r = F_1\left(\sum_i k_i, n\right)$$

$$w = F_2\left(\sum_i k_i, n\right).$$

By taking the time derivative of the above, we obtain the dynamics of equilibrium interest rate and wage:

$$\dot{r} = F_{11}\left(\sum_i k_i, n\right) \left[r \sum_i k_i + nw - \sum_i c_i - \sum_i z_i \right]$$

$$\dot{w} = F_{21}\left(\sum_i k_i, n\right) \left[r \sum_i k_i + nw - \sum_i c_i - \sum_i z_i \right].$$

Interior steady state $((c_i^*, z_i^*, v_i^*, \pi_i^*, k_i^*)_{i \in I}, r^*, w^*)$, if exists, is determined by:

$$\beta_i(\pi_i^*) = r^*$$

$$g_i(z_i^*) = \delta_i \pi_i^*$$

$$\frac{v_i'(c_i^*)}{v_i(c_i^*)} = - \frac{g_i'(z_i^*) \beta_i'(\pi_i^*)}{r^*(r^* + \delta_i)}$$

$$v_i^* = \frac{v_i(c_i^*)}{r^*}$$

$$k_i^* = \frac{c_i^* + z_i^* - w^*}{r^*}$$

$$r^* = F_1\left(\sum_i k_i^*, n\right)$$

$$w^* = F_2\left(\sum_i k_i^*, n\right).$$

As far as they all survive, the households end up with the same value of discount rate in steady state, which is equal to the given interest rate. Note that general existence of interior steady state is not obvious, though, because β_i has the natural lower bound $\underline{\beta}_i$ and it might be possible that the r^* can be above $\max_i \underline{\beta}_i$ only by violating the other steady-state conditions. In the next subsections, we present a class of technologies, such that a unique steady state exists.

Then, the Jacobian matrix evaluated at the interior steady state takes the form of $(5n + 2) \times (5n + 2)$ matrix:

$$\begin{pmatrix} A_1 & O & \cdots & O & P_1 \\ O & A_2 & \cdots & O & P_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & A_n & P_n \\ E & E & \cdots & E & Q \end{pmatrix},$$

where

$$A_i = \begin{pmatrix} 0 & 0 & 0 & \frac{\beta'_i v'_i}{v''_i} & 0 \\ -(r + \delta_i) \frac{g'_i v''_i}{g''_i v'_i} & r + \delta_i & (r + \delta_i) r \frac{g'_i}{g''_i v'_i} & (r + \delta_i) \frac{g'_i \beta''_i}{g''_i \beta'_i} & 0 \\ -v'_i & 0 & r & \frac{\beta'_i v_i}{r} & 0 \\ 0 & g'_i & 0 & -\delta_i & 0 \\ -1 & -1 & 0 & 0 & r \end{pmatrix}, \quad P_i = \begin{pmatrix} -\frac{v'_i}{v''_i} & 0 \\ -\frac{g'_i}{g''_i} & 0 \\ 0 & 0 \\ 0 & 0 \\ k_i & 1 \end{pmatrix}$$

for each i and

$$E = \begin{pmatrix} -F_{11} & -F_{11} & 0 & 0 & F_{11}r \\ -F_{21} & -F_{21} & 0 & 0 & F_{21}r \end{pmatrix}$$

$$Q = \begin{pmatrix} F_{11} \sum_i k_i & F_{11}n \\ F_{21} \sum_i k_i & F_{21}n \end{pmatrix}.$$

4.2 The case of linear technology

Here, we restrict attention to consumption and investment under linear technology, $F(K, L) = rK + wL$, to isolate the problem of whether investment in patience capital has a stabilizing or rather destabilizing effect at the pure household level. Assume that $r > \max_i \beta_i$, and then, a unique interior steady state exists. Otherwise, there is no steady state with all households surviving, and some households automatically perish in the long run.

When technology is linear, there is no interaction between households, the aggregate outcome is simply the sum of individual ones, and it suffices to look at each household problem separately. Since $F_{11} = F_{21} = 0$ here and E and Q are zero matrices, we only need to look at the stability property of each diagonal block A_i separately.

To see if the system described by A_i is stable, note that since π_i and k_i are the state variables, the number of stable roots required for stability with a unique optimal path is exactly two.

Proposition 3 *The number of stable roots for each A_i is one. Let θ_{11} denote the only stable root for A_i , and $\theta_{12}, \theta_{13}, \theta_{14}, \theta_{15}$ denote the unstable roots, ordered in the ascending manner according to their real parts, and then, at least $\theta_{11}, \theta_{14}, \theta_{15}$ are real and it holds:*

$$\theta_{i1} < -\delta_i < 0 < \operatorname{Re}\theta_{i2} \leq \operatorname{Re}\theta_{i3} < \theta_{i4} = r < r + \delta_i < \theta_{i5}.$$

Proof Pick any i . Since it is clear that one eigenvalue of A_i is r , we can restrict attention to its submatrix:

$$\begin{pmatrix} 0 & 0 & 0 & \frac{\beta'_i v'_i}{v''_i} \\ -(r + \delta_i) \frac{g'_i v''_i}{g''_i v'_i} & r + \delta_i & (r + \delta_i) r \frac{g'_i}{g''_i v'_i} & (r + \delta_i) \frac{g'_i \beta''_i}{g''_i \beta'_i} \\ -v'_i & 0 & r & \frac{\beta'_i v_i}{r} \\ 0 & g'_i & 0 & -\delta_i \end{pmatrix}.$$

Then, its characteristic polynomial is:

$$\begin{aligned} p_i(\theta_i) = & (\theta_i + \delta_i)\theta_i(\theta_i - r)(\theta_i - (r + \delta_i)) \\ & - (r + \delta_i) \frac{(g'_i)^2 \beta''_i}{g''_i \beta'_i} \theta_i(\theta_i - r) \\ & - (r + \delta_i) r \frac{(g'_i)^2}{g''_i} \beta'_i \left(1 - \frac{(v'_i)^2}{v''_i v_i} \right), \end{aligned}$$

which is an even function around $r / 2$. Note that one can verify: $-(r + \delta_i) r \frac{(g'_i)^2}{g''_i} \beta'_i \left(1 - \frac{(v'_i)^2}{v''_i v_i} \right) < 0$.

Since the first line above, a fourth-order function being even around $r / 2$ equals to zero at $\theta_i = -\delta_i, 0, r, r + \delta_i$, and the second line above, a second-order function being even around $r / 2$, equals to zero at $\theta_i = 0, r$, their sum equals to zero at $-\delta_i - \omega_i, 0, r, r + \delta_i + \omega_i$ with some $\omega_i > 0$.

Since the third line above is negative, $p_i(\theta_i) = 0$ has at least two real roots $-\delta_i - \rho_i$ and $r + \delta_i + \rho_i$ with $\rho_i > 0$. When $p_i(r/2) \geq 0$, the other two are real and they are $r/2 - \sigma_i$ and $r/2 + \sigma_i$ with $0 \leq \sigma_i < r/2$. The case of $p_i(r/2) < 0$ cannot be ruled out, because we can always make $v'' < 0$ arbitrarily close to zero without changing v_i, v'_i and the other values and the steady state, so that $-(r + \delta_i) r \frac{(g'_i)^2}{g''_i} \beta'_i \left(1 - \frac{(v'_i)^2}{v''_i v_i} \right)$ is a large negative number. Then, the other two roots are complex and they are $r/2 - \sigma_i \sqrt{-1}$ and $r/2 + \sigma_i \sqrt{-1}$ with σ_i being some real number. \square

Since the number of stable roots for each household’s linearized system is one, the interior steady state is unstable, while there is a stable manifold with dimension one, a curve.

To get an idea about the dynamics, we can verify that the eigenvector for $\theta_{i4} = r$ is $(0, 0, 0, 1)^T$, and the eigenvector for each $\theta_{ik}, k = 1, 2, 3, 5$, is:

$$\left(\frac{1}{\theta_{ik}} \frac{\beta'_i v'_i}{v''_i}, \frac{\theta_{ik} + \delta_i}{g'_i}, \frac{1}{r - \theta_{ik}} \left[\frac{1}{\theta_{ik}} \frac{\beta'_i (v'_i)^2}{v''_i} - \frac{\beta'_i v_i}{r} \right], 1, \frac{1}{r - \theta_{ik}} \left[\frac{1}{\theta_{ik}} \frac{\beta'_i v'_i}{v''_i} + \frac{\theta_{ik} + \delta_i}{g'_i} \right] \right)^T,$$

which implies that the local dynamics is given by:

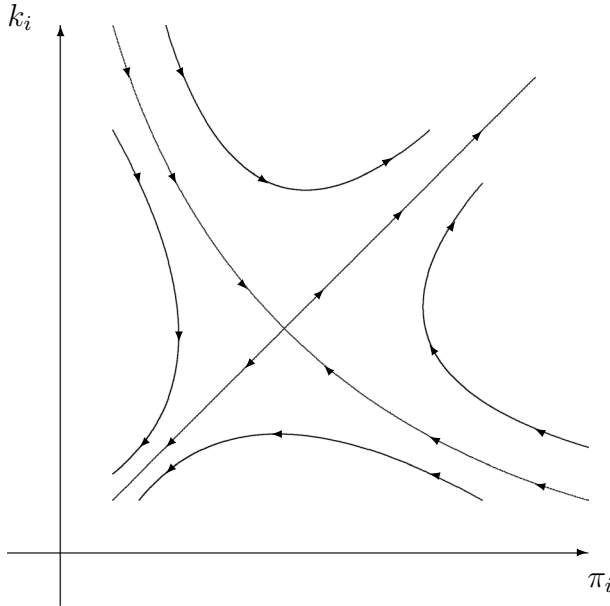


Fig. 1 Flows projected on the space of state variables: the case of θ_{i2}, θ_{i3} being real

$$\begin{pmatrix} c_i - c_i^* \\ z_i - z_i^* \\ v_i - v_i^* \\ \pi_i - \pi_i^* \\ k_i - k_i^* \end{pmatrix} = s_{i4} e^{rt} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \sum_{k=1,2,3,5} s_{ik} e^{\theta_{ik}t} \begin{pmatrix} \frac{1}{\theta_k} \frac{\beta'_i v'_i}{v''_i} \\ \frac{g'_i}{\theta_k + \delta} \\ \frac{1}{r - \theta_k} \left[\frac{1}{\theta_k} \frac{\beta'_i (v'_i)^2}{v''_i} - \frac{\beta'_i v_i}{r} \right] \\ 1 \\ \frac{1}{r - \theta_k} \left[\frac{1}{\theta_k} \frac{\beta'_i v'_i}{v''_i} + \frac{\theta_k + \delta_i}{s'_i} \right] \end{pmatrix},$$

where the constants $\{s_{ik}\}_{k=1,2,3,4,5}$ are determined according to the initial values. Note that the eigenvectors projected on the space of state variables (π_i, k_i) are downward for θ_{i1} and θ_{i5} , upward for θ_{i2} and θ_{i3} when they are real, and vertical for $\theta_{i4} = r$. Hence, the curve of stable manifold projected on the space of state variables is downward-sloping.

Since $\theta_{i4} = r$ and $\theta_{i5} > r + \delta_i$ are inconsistent with the conjunction of No-Ponzi condition and transversality condition, however, the associated upper left/lower right directions will not play a role in the global dynamics.

Thus, when θ_{i2} and θ_{i3} are real, the flows projected on the space of state variables look like Fig. 1. Here, the space is partitioned by the projected curve, and if the initial $(\pi_i(0), k_i(0))$ falls in one side, the household will invest more on patience capital, which leads to higher saving in the future and more accumulation of capital, resulting in more consumption in further future, and so on, and if it falls in the other side, the household will opt out from investing on patience capital, which leads to lower saving in the future and eating up the capital, resulting in less consumption in

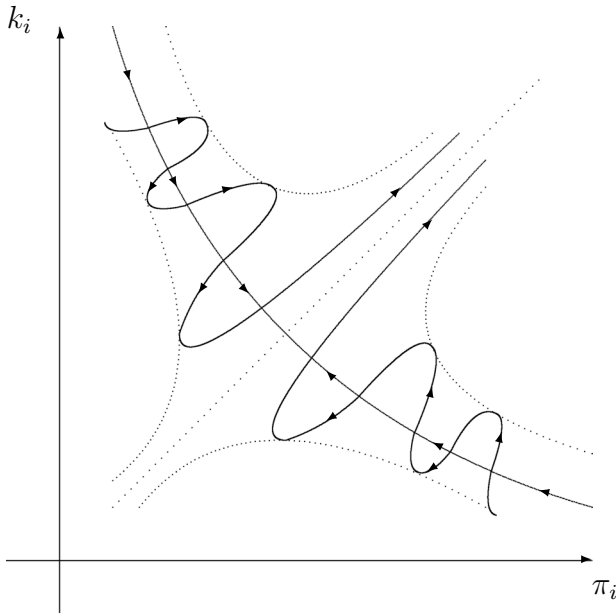


Fig. 2 Flows projected on the space of state variables: the case of θ_{i2}, θ_{i3} being complex

further future, and so on, while the speed of divergence to infinity/convergence to zero must be sufficiently modest to meet the transversality conditions. Thus, we see a division of society into two “classes”, even among ex-ante identical households, depending on their initial values of physical capital and patience capital.

In the literature of (analytical) Marxian economics, they make a seemingly ad hoc assumption that capitalists are interested only in accumulation and workers are totally myopic, make no saving, and always live at the subsistence level (see, for example, Okisio 1963; Morishima 1977; Roemer 1988). The instability result may serve as an explanation that such extreme division can be rather self-confirming.

Note that this saddle-point instability is about steady state projected onto the two-dimensional space of state variables. The familiar saddle-point nature which we observe for example in the standard optimal growth model is about steady state in the space of one state variable (such as physical capital) and one jump variable (such as consumption). There, the household or planner can choose the initial value of the jump variable, and the fact that a steady state being a saddle point means that it is stable with regard to a unique optimal path. Here, neither a household or a planner can choose the initial pair of physical capital and patience capital, and therefore, a steady state being a saddle-point means that it is unstable in the above-stated sense.

Still we cannot rule out the case that θ_{i2} and θ_{i3} are complex. Then, the unstable flows form expanding cycles spinning around the curve of stable manifold, which look like Fig. 2 when projected on the space of state variables. This is a puzzling case, since when such an unstably cyclical orbit passes above and below the curve of stable manifold (i.e., when their projections cross on the space of state variables), the household

can switch between the stable path and the unstable orbit, by restarting the life with the current state variable (π_i, k_i) . This can be viewed as a kind of indeterminacy, as the initial value $(\pi_i(0), k_i(0))$ may not determine an unique optimal path even for a single household.

4.3 Diminishing returns to capital and capital/labor complementarity

Just to explain that the instability result as above is not a mere knife-edge case due to linear technology, we show that the above argument extends to economies with diminishing returns to capital and capital/labor complementarity, while the degrees of diminishment and complementarity are sufficiently small.

We can take for example the class of CES production functions with the form $F(K, L) = A[aK^\rho + (1 - a)L^\rho]^\frac{1}{\rho} - \eta K$, where η denotes the rate of depreciation, such that the corresponding production set coincides with that of the linear production function $rK + wL$ along the ray connecting the origin and $(\sum_i k_i^*, n)$. By varying A, a, ρ, η , we can make F_{11} and F_{21} arbitrarily close to zero at $(\sum_i k_i^*, n)$.

Then, the characteristic polynomial for the Jacobian matrix evaluated at the steady state is given by:

$$\begin{vmatrix} \theta I - A_1 & O & \dots & O & & -P_1 \\ O & \theta I - A_2 & \dots & O & & -P_2 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ O & O & \dots & \theta I - A_n & & -P_n \\ -E & -E & \dots & -E & \theta - F_{11}\sum_i k_i^* & -F_{11}n \\ & & & & -F_{21}\sum_i k_i^* & \theta - F_{21}n \end{vmatrix},$$

where I denotes the 5×5 identity matrix.

Consider doing cofactor expansion of the above determinant formula first along the $5n + 2$ th row and then along the $5n + 1$ th row. Because all the entries of the second row of E are constant multiples of F_{21} and all the entries of the first row of E are constant multiples of F_{11} , all the terms in the expanded determinant formula other than the one obtained by going along the diagonal are constant multiples of either of F_{11}, F_{21} , or $F_{11}F_{21}$. Thus, the characteristic equation takes the form:

$$\left(\theta - F_{11} \sum_i k_i^*\right) (\theta - F_{21}n) \prod_i |\theta I - A_i| + F_{11}G_1(\theta) + F_{21}G_2(\theta) + F_{11}F_{21}G_{12}(\theta) = 0,$$

where $G_1(\theta), G_2(\theta)$, and $G_{12}(\theta)$ are $5n$ th order polynomials.

Now, consider that F_{11} and F_{21} are arbitrarily close to zero. Then, the collection of roots in the characteristic equation is arbitrarily close to the one for:

$$\theta^2 \prod_i |\theta I - A_i| = 0.$$

Because the term $\prod_i |\theta I - A_i|$ is independent of F_{11} and F_{21} , the signs of $5n$ roots being close to the $5n$ ones for $\prod_i |\theta I - A_i| = 0$, which are all non-zero, do not change when F_{11} and F_{21} are sufficiently small. Out of those $5n$ roots, from the

previous result we see that the number of stable roots is n . The signs of the remaining two are ambiguous as they may switch across zero under small perturbation.

Summing up, we obtain the following claim.

Proposition 4 *Pick any $r > \max_i \beta_{-i}$ and w for which an interior steady state $(c_i^*, z_i^*, v_i^*, \pi_i^*, k_i^*)_{i \in I}$ exists in the corresponding linear technology economy. Then, there is a range of technology F with constant returns to scale and $F_{11}(\sum_i k_i^*, n) < 0$ and $F_{21}(\sum_i k_i^*, n) > 0$, which results in the same interior steady state with $F_1(\sum_i k_i^*, n) = r$ and $F_2(\sum_i k_i^*, n) = w$, such that the number of stable roots in the linearized system is either n or $n + 1$ or $n + 2$, while stability requires; it is $2n$.*

Thus, while there is a possibility of stability and even indeterminacy when $n = 1$ or $n = 2$, the similar kind of instability shows up again when $n \geq 3$, and its nature gets closer to the one under linear technology as n tends to be large.

5 Conclusion

We conclude by listing the remaining problems and suggesting future directions for the research.

A complete and global characterization of equilibrium path is obviously desired, especially for the case of stronger diminishing returns to capital and capital/labor complementarity, while we believe that we have spelled out the critical nature of it. This will require a more involved technical treatment of the system of differential equations obtained here.

There will be two effects of lowering interest rate over time adjusting to diminishing returns to capital, while the total effect is ambiguous. One is that as r tends to go below β , the bar for remaining in the “upper class” tends to be higher. This will discourage households from investing in patience capital and they will opt out. The other is that lowering interest rate makes patience capital relatively cheaper compared to physical capital, and this encourages investment in the former.

A numerical approach may help to get a picture. For example, in the case of time-constant interest rate, we can obtain the Hamilton–Jacobi–Bellman equation:

$$\beta(\pi)V(\pi, k) = \max_{c, z} \left\{ v(c) + V_1(\pi, k)[- \delta \pi + g(z)] + V_2(\pi, k)[rk + w - c - z] \right\},$$

and there are substantial numbers of studies which allow us to solve such class of functional equation.

Also, its discrete-time counterpart is solvable by the recursive method (this confirms that the household model is dynamically consistent). The Bellman equation

$$V(\pi, k) = \max \left\{ v(c) + \frac{1}{1 + \beta(\pi)} V(\pi', k') \right\},$$

where

$$\begin{aligned}\pi' &= (1 - \delta)\pi + g(z) \\ k' &= (1 + r)k - c - z\end{aligned}$$

is solvable by the simple contraction-mapping method.

Also, one can think of a more general form of production of patience capital, like

$$\dot{\pi} = h(z, \pi),$$

while it will require handling complementarity between patience capital and investment.

From the standpoint of classic concept of welfare, there is nothing wrong with having severe ex-post inequalities in the long run, as time preference and the way of reproducing it are understood as “a matter of taste”. This is even “fair” from the ex-ante viewpoint. Yet, we will need to ask whether we should accept resulting ex-post inequalities and how we should reconcile between ex-ante equity and ex-post equity.

There is a difficulty, however, that even when we accept ex-post redistribution of physical capital, the same kind of inequality may emerge again, because it is impossible to redistribute patience capital. Then, the question becomes whether we should continuously redistribute physical capital to compensate for the infeasibility of redistribution of patience capital (or cultural capital/human capital in general), but this is even more ethically demanding.

One may think of a governmental intervention to promote investment in patience capital, but this leads to inefficiency according to the classic concept of welfare, since there is no externality between households here.

Finally, note that we considered an entirely frictionless economy with perfect foresights. This leaves a question whether any market friction or bounded foresight/consistency strengthens instability or rather stabilizes long-run distribution.⁴

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⁴ In the standard case of fixed discount rates, Sorger (2008) argues that the most patient household cannot be price-taking as it collects large portion of wealth in the society, which, in turn, results in allowing all households to have positive consumption/wealth in the long run.

References

- Alan, S., & Ertac, S. (2018). Fostering patience in the classroom: Results from a randomized educational intervention. *Journal of Political Economy*.
- Barsky, R. B., et al. (1997). Preference parameters and behavioral heterogeneity: An experimental approach in the health and retirement study. *The Quarterly Journal of Economics*, 112(2), 537–579.
- Becker, G. S., & Mulligan, C. B. (1997). The endogenous determination of time preference. *The Quarterly Journal of Economics*, 112(3), 729–758.
- Becker, R. A. (1980). On the long-run steady state in a simple dynamic model of equilibrium with heterogeneous households. *The Quarterly Journal of Economics*, 95(2), 375–382.
- Becker, R., & Zilcha, I. (1997). Stationary Ramsey equilibria under uncertainty. *Journal of Economic Theory*, 75(1), 122–140.
- Benhabib, J., Jafarey, S., & Nishimura, K. (1988). The dynamics of efficient intertemporal allocations with many agents, recursive preferences, and production. *Journal of Economic Theory*, 44(2), 301–320.
- Bewley, T. (1982). An integration of equilibrium theory and turnpike theory. *Journal of mathematical economics*, 10(2–3), 233–267.
- Bourdieu, P. (1973). *Cultural reproduction and social reproduction* (p. 178). London: Tavistock.
- Caselli, F., & Ventura, J. (2000). A representative consumer theory of distribution. *American Economic Review*, 90(4), 909–926.
- Das, M. (2003). Optimal growth with decreasing marginal impatience. *Journal of Economic Dynamics and Control*, 27(10), 1881–1898.
- Das, M. (2007). Persistent inequality: An explanation based on limited parental altruism. *Journal of Development Economics*, 84(1), 251–270.
- Doepke, M., & Zilibotti, F. (2005). Social class and the spirit of capitalism. *Journal of the European Economic Association*, 3(23), 516–524.
- Doepke, M., & Zilibotti, F. (2008). Occupational choice and the spirit of capitalism. *The Quarterly Journal of Economics*, 123(2), 747–793.
- Dohmen, T., et al. (2015). Patience and the wealth of nations. Unpublished manuscript. University of Bonn.
- DiMaggio, P. (1982). Cultural capital and school success: The impact of status culture participation on the grades of US high school students. *American Sociological Review*, 47, 189–201.
- Epstein, L. G. (1983). Stationary cardinal utility and optimal growth under uncertainty. *Journal of Economic Theory*, 31(1), 133–152.
- Epstein, L. G. (1987). A simple dynamic general equilibrium model. *Journal of Economic theory*, 41(1), 68–95.
- Epstein, L. G. (1987). The global stability of efficient intertemporal allocations. *Econometrica*, 55, 329–355.
- Epstein, L. G., & Hynes, J. A. (1983). The rate of time preference and dynamic economic analysis. *Journal of Political Economy*, 91, 611–635.
- Fisher, I. (1930). *The theory of interest*. New York 43.
- Hirose, K., & Ikeda, S. (2008). On decreasing marginal impatience. *The Japanese Economic Review*, 59, 259–274.
- Haruyama, T., & Park, H. (2017). A simple dynastic economy with parental time investment in children's patience. *Economic Modelling*, 61, 235–247.
- Kawagishi, T. (2014). Investment for patience in an endogenous growth model. *Economic Modelling*, 42, 508–515.
- Koopmans, T. C. (1960). Stationary ordinal utility and impatience. *Econometrica*, 28, 287–309.
- Lawrance, E. C. (1991). Poverty and the rate of time preference: evidence from panel data. *Journal of Political economy*, 99(1), 54–77.
- Lucas, R. E., & Stokey, N. L. (1984). Optimal growth with many consumers. *Journal of Economic Theory*, 32(1), 139–171.
- Mitra, T., & Sorger, G. (2013). On Ramsey's conjecture. *Journal of Economic Theory*, 148(5), 1953–1976.
- Morishima, M. (1977). *Marx's economics a dual theory of value and growth*. CUP Archive.
- Okisio, N. (1963). A mathematical note on Marxian theorems. *Weltwirtschaftliches Archiv*, 91, 287–299.

- Perez-Arce, F. (2017). The effect of education on time preferences. *Economics of Education Review*, 56, 52–64.
- Ramsey, F. P. (1928). A mathematical theory of saving. *The Economic Journal*, 38(152), 543–559.
- Roemer, J. E. (1988). *Analytical foundations of Marxian economic theory*. Cambridge: Cambridge University Press.
- Schumacher, I. (2009). Endogenous discounting via wealth, twin-peaks and the role of technology. *Economics Letters*, 103(2), 78–80.
- Shi, S., & Epstein, L. G. (1993). Habits and time preference. *International Economic Review*, 34, 61–84.
- Sorger, G. (2008). Strategic saving decisions in the infinite-horizon model. *Economic Theory*, 36(3), 353–377.
- Strulik, H. (2012). Patience and prosperity. *Journal of Economic Theory*, 147(1), 336–352.
- Tanaka, T., Camerer, C. F., & Nguyen, Q. (2010). Risk and time preferences: Linking experimental and household survey data from Vietnam. *American Economic Review*, 100(1), 557–71.
- Uzawa, H. (1968). *Time preference, the consumption function, and optimum asset holdings*. *Value, capital and growth: papers in honor of Sir John Hicks* (pp. 485–504). Edinburgh: The University of Edinburgh Press.

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