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EDITORIAL

Preface to the Focused Issue on High-Order Numerical Methods for Evolutionary PDEs

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The underlying fields of this special issue of CAMC include mathematical modeling through evolutionary partial differential equations (PDEs), advanced high-order non-linear numerical methods for their approximation and applications in various engineering branches, in physics, biology and medicine, to name but a few. Of special interest are mathematical models based on systems of hyperbolic balance laws, including stiff source terms. From the numerical point of view, such systems of PDEs are very challenging. This is particularly so on the assumption that numerical algorithms will be of high order of accuracy in both space and time for smooth solutions. This is because high-order methods are able to compute solutions with small errors much more efficiently (lower CPU cost) than low-order methods on very fine meshes, by orders of magnitude in fact. However, in the presence of discontinuities, or even large gradients, high-order methods must be non-linear, in the sense of Godunov's theorem. Such non-linear methods must reconcile two contradictory requirements, namely high order of accuracy for both smooth solutions and absence of, or much reduced, spurious oscillations for discontinuous solutions. Moreover, the presence of source terms in the PDEs poses additional challenges to the design of non-linear high-order methods. If such source terms are stiff, then reconciling high order of accuracy and stiffness becomes even more challenging.

Two major classes of high-order, non-linear numerical methods exist, namely semi-discrete and fully discrete methods. Semi-discrete methods separate space and time discretizations. The spatial discretization makes use of non-linear spatial reconstruction, examples of which are ENO and WENO methods; see Jiang and Shu [12]. The time discretization tackles the solution of systems of ordinary differential equations (ODEs) in time, for which non-linear ODE solvers such as TVD Runge-Kutta schemes are often used; see Harten

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et al. [11] and Jiang and Shu [12]. The class of fully discrete methods discretizes space and time simultaneously. A representative class of very high-order non-linear fully discrete methods is the ADER methodology introduced by Toro and collaborators in [18]. ADER methods require a non-linear spatial reconstruction procedure and the solution of the generalized Riemann problem to compute a high-order intercell numerical flux. If source terms are present, the corresponding numerical source is computed from a space-time volume integral evaluated on a time-evolved reconstruction polynomial within the volume. Various versions of the ADER approach exist, depending on the method employed to solve the GRP. Major advances of the ADER methodology are due to Titarev et al. [17, 19], and Dumbser et al. [6–8]. Both semi-discrete and fully discrete methods have finite-volume and discontinuous Galerkin finite-element versions. The above-described approaches are the main numerical methodologies at the bases of the European Workshop on High Order NOnlinear numerical Methods for evolutionary PDEs: theory and applications (HONOM), since its beginning back in 2005.

This focused issue of CAMC contains works that were presented in the European Workshop on HONOM 2019 that was held at Escuela Técnica Superior de Ingenieros de Minas y Energía, Universidad Politécnica de Madrid, Spain in April 1–5, 2019. The aim of this conference was to present new research in the field of high-order numerical methods applied to mathematical models based on PDEs to simulate a wide range of physical phenomena. Most of the papers included in this focused issue cover different topics in the field of numerical methods for evolutionary PDEs.

- In [2], continuous Galerkin methods for hyperbolic problems are studied.
- Neural networks applied to shock detection are presented in [1].
- A semi-Lagrangian solver for 3D free surface flows is introduced in [4].
- Multidimensional hyperbolic conservation laws are treated in [3].
- The paper [16] deals with the entropy split method.
- Advances in high-order discontinuous Galerkin schemes are presented in [15].
- Low Mach number IMEX schemes are described in [21].
- In [14], high-order finite-volume schemes with adaptive stencil construction are introduced.
- The paper [20] deals with the high-order ADER-AENO reconstruction.
- In [5], the modeling of phase-change problems is presented.
- In [9], well-balanced schemes for sediment transport are introduced.
- Second-order ALE schemes are described in [10].
- Two-phase oil-water movement is studied in [13].

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