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RESEARCH ARTICLE

Nonparametric tests for combined location-scale and Lehmann alternatives using adaptive approach and max-type metric

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Abstract

The paper deals with the classical two-sample problem for the combined location-scale and Lehmann alternatives, known as the versatile alternative. Recently, a combination of the square of the standardized Wilcoxon, the standardized Ansari–Bradley and the standardized Anti-Savage statistics based on the Euclidean distance has been proposed. The Anti-Savage test is the locally most powerful rank test for the right-skewed Gumbel distribution. Furthermore, the Savage test is the locally most powerful linear rank test for the left-skewed Gumbel distribution. Then, a test statistic combining the Wilcoxon, the Ansari–Bradley, and Savage statistics is proposed. The limiting distribution of the proposed statistic is derived under the null and the alternative hypotheses. In addition, the asymptotic power of the suggested statistic is investigated. Moreover, an adaptive test is proposed based on a selection rule. We compare the power performance against various fixed alternatives using Monte Carlo. The proposed test statistic displays outstanding performance in certain situations. An illustration of the proposed test statistic is presented to explain a biomedical experiment. Finally, we offer some concluding remarks.

Keywords Adaptive test · Asymptotic power · Maximum test

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1 Introduction

Classical two-sample comparisons between two populations have extensive applications in many scientific fields, including controlled experiments, biometry, psychology and industry. Many tests exist in the literature for assessing the difference in various parameters between two independent populations. Furthermore, the data used in control experiments cannot accurately estimate the population distribution because the sample size is too small, there might be a few outliers, or it could be a contaminated sample. A nonparametric test is preferable here, as we cannot assume normality or any other specific distribution.

Many researchers focus on designing a test for the difference in a single parameter, such as the location, scale, or shape parameter in a distribution-free setup. For example, the most famous tests for the two-sample location problem in the distribution-free setting are the Wilcoxon rank-sum test (Gibbons & Chakraborti, 2021), namely W, or the normal scores test (Gibbons & Chakraborti, 2021). Assuming that the population distribution is normal, the asymptotic relative efficiency (ARE) of the Wilcoxon rank-sum test and the normal scores test relative to the t test are 0.955 and 1, respectively, indicating that they are not inferior to the t test. In addition, the ARE of the Wilcoxon rank-sum test to the t test is greater than or equal to about 0.864 for all continuous distributions. Therefore, the Wilcoxon rank-sum test is still widely utilized in many applications, see. e.g. Letshedi et al. (2021), Lin et al. (2021) and Dao (2022).

The parametric two-sample test for testing the variances of the normal distribution is the F test. The Ansari–Bradley, namely AB, and Mood tests (Gibbons & Chakraborti, 2021) are well-known nonparametric tests for the two-sample scale problem. The ARE of the Ansari–Bradley test and the Mood test to the F test under the assumption of the normal distribution are 0.609 and 0.76, respectively. Although ARE of the Ansari–Bradley test is lower than that of the Mood test, the Ansari–Bradley is widely used in many applications, see, e.g. Lahmiri (2023) and Omer et al. (2023). The two location tests, based respectively on Wilcoxon and van der Waerden's normal scores for two-sample problems, assume no scale difference in the population distributions. Likewise, the two scale tests, based respectively on Ansari–Bradley and Mood scores for two-sample problems, assume no location difference in the population distributions. Actually, there are often differences in both location and scale in many practices.

Many researchers focus on designing a test for the difference in location and scale parameters between two populations simultaneously in a distribution-free setup. The most well-known tests in this circumstance, the test for the location and scale simultaneously, are the Lepage (Neuhäuser, 2012) and the Cucconi (Neuhäuser, 2012) tests. The Lepage statistic combines the square of the standardized Wilcoxon rank-sum and the standardized Ansari–Bradley statistics. The Lepage-type statistic, which is a quadratic form of location and scale statistics, has been studied by many researchers. For example, Pettitt (1976) proposed the combination of the square of standardized Wilcoxon rank-sum and the square of standardized Mood statistics. Note that the statistic of Pettitt (1976) is essentially equivalent to the statistic of Cucconi (Neuhäuser, 2012) under the continuous distributions; see Nishino and Murakami (2019). Since Murakami (2011) proposed an approximation to the distribution of the statistic of Pettitt (1976), we can apply the approximation distribution to the Cucconi statistic. For different versions of Lepage-type statistics, we refer to works of Büning and Thadewald (2000), Neuhäuser (2000), Kössler (2006), Murakami (2007), Murakami (2016) and Mukherjee and Marozzi (2019). Recently, Yamaguchi and Murakami (2023) discussed multi-aspect statistics generalizing the Lepage-type statistics in the presence of ties.

Kössler and Mukherjee (2020) recently noted that traditional two-sample simultaneous tests for location and scale parameters are silent about the shape of the distributions. A change in the shape of the distribution in a two-sample problem is addressed using the Lehmann alternative (Hájek et al., 1999) and is very common in many applications, see, e.g. Razzaghi (2014), Ng et al. (2021) and Chakraborty et al. (2023). For the Lehmann alternative, for example, the Anti-Savage test (Kössler & Mukherjee, 2020), namely AS, is widely used by many researchers. However, the difference in a single parameter is rare in many applications. It is more general and advisable to consider that at the same time, a shift may occur in one or more of the three parameters, namely, location, scale, and shape parameters. Therefore, we must focus on designing approaches for simultaneously testing many parameters between two populations. Note that Kössler and Mukherjee (2020) only consider the squares of Euclidean and Mahalanobis type distance between the Wilcoxon, Ansari–Bradley and Anti-Savage statistics. The Anti-Savage test is suitable for difference in location of the right-skewed data.

However, we sometimes encounter left-skewed data in practical analysis. In this case, the Savage test, namely S, is one of the preferred tests. In the theoretical background, the Savage test is the locally most powerful rank test for the left-skewed Gumbel distribution. Then, we may consider similar combinations using Wilcoxon, Ansari–Bradley and Savage statistics. Also, the null distribution may be symmetric, but the shape alternative could lead to a right or left-skewed population, and the shift direction is unknown a priori. Then, a question regarding the choice between these arises and is addressed in the current paper.

Recently, Yamaguchi and Murakami (2023) proposed the tie-adjusted version of the Euclidian and Mahalanobis distance-based statistics of some standardized linear rank statistics, that is the multi-aspect tests. However, in practical analysis, we must determine whether to use the Wilcoxon–Ansari–Bradley–Anti-Savage statistic or the Wilcoxon–Ansari–Bradley–Savage statistic before we treat the hypothesis test. To this end, we might consider a max-type test. The larger of the two statistics as the test statistic is the first and simple way to solve this problem. For example, Neuhäuser et al. (2004) and Welz et al. (2018) compared the validity of the maximum test with various nonparametric tests for the two-sample location problem. Additionally, Neuhäuser and Hothorn (2006) discussed that a maximum test is an adaptive permutation test. As another approach to solve this problem, we also consider an adaptive test, which selects the test statistic depending on the case grouping. Büning (1996) proposed an adaptive test for the multisample location problem based on selectors suggested by Hogg et al. (2018, pp. 622–623). In Büning

(2000), a selector of skewness and tail-weight using quantile points is proposed for this problem. Büning and Thadewald (2000) proposed the two-sample location-scale adaptive test using a selector introduced by Büning (2000). For the two-sample scale problem, Kössler (1994) proposed the adaptive test based on new selectors for skewness and tail-weight. Neuhäuser et al. (2004) proposed an adaptive test for the two-sample location problem by using the selectors of Hogg et al. (2018, pp. 622–623). For the one-sample location problem, recently, Kitani and Murakami (2022) proposed an adaptive test and new selectors. Although Yamaguchi and Murakami (2023) considered the multi-aspect test statistic based on some linear rank test statistics, maximum type and adaptive-type tests are not discussed. Therefore, we focus on designing one maximum-type and another adaptive-type procedure for the two-sample testing problem.

The rest of the paper is organized as follows. We discuss statistical preliminaries, introduce a test statistic and derive the limiting distribution of the proposed statistic under the null hypothesis in Sect. 2. In addition, we derive the limiting distribution of the suggested statistic under the alternative hypothesis and investigate asymptotic power in Sect. 3.1. Section 4 introduces a maximum test and an adaptive test based on a selecting rule. We present some numerical results via Monte Carlo in Sect. 5. The proposed test statistics are compared with the classical omnibus test statistics Kolmogorov–Smirnov (Gibbons & Chakraborti, 2021), Cramér–von Mises (Anderson, 1962) and Anderson–Darling (Pettitt, 1976) as well with the test statistic of Boos (1986). Section 6 is devoted to illustrations of the proposed test. We offer some concluding remarks in Sect. 7.

2 Simultaneous statistic for the location-scale-shape parameters

Let $X_1 = (X_{11}, \dots, X_{1n_1})$ and $X_2 = (X_{21}, \dots, X_{2n_2})$ be two random samples of size n_1 and n_2 from absolutely continuous populations with the cumulative distribution functions (cdf) F_1 and F_2, respectively. Consider the pooled sample of size $N = n_1 + n_2$ and let V_i, $i = 1, \dots, N$ be 1 if the i^{th} smallest of N observations is from X_1, and otherwise 0. Then, a two-sample linear rank statistic is given by

$$\mathrm{LRT} = \sum_{i=1}^{N} a_i V_i,$$

where the a_i are appropriate scores. As noted before, we test the location μ, scale e^{σ} and shape e^{δ} parameters at the same time. Then we are interested in testing the hypothesis

$$H_0 : F_2(x) = F_1(x)$$

against

$$H_1 : F_2(x) = \left[F_1\left(\frac{x-\mu}{e^{\sigma}}\right)\right]^{e^{\delta}}, \quad \text{at least one of } \mu \neq 0,\ \sigma \neq 0,\ \delta \neq 0. \tag{1}$$

Under this setup, we propose a new statistic for (1) in Sect. 2.1. Furthermore, we derive the limiting null distribution of the proposed test statistic in Sect. 2.2.

2.1 A test statistic for the versatile alternative

Kössler and Mukherjee (2020) recently noted that traditional two-sample Lepage-type statistics are silent about the shape of the distributions. However, in various applications, a change in the shape of the distribution along with the location and scale is also widespread. Then, Kössler and Mukherjee (2020) proposed the Euclidean-type statistic for the test problem (1) as follows:

$$\mathrm{T}_1 = \left(\frac{\mathrm{W} - \mathrm{E[W]}}{\sqrt{\mathrm{V[W]}}}\right)^2 + \left(\frac{\mathrm{AB} - \mathrm{E[AB]}}{\sqrt{\mathrm{V[AB]}}}\right)^2 + \left(\frac{\mathrm{AS} - \mathrm{E[AS]}}{\sqrt{\mathrm{V[AS]}}}\right)^2,$$

where

$$\mathrm{W} = \sum_{i=1}^{N} iV_i, \quad \mathrm{E[W]} = \frac{n_1(N+1)}{2}, \quad \mathrm{V[W]} = \frac{n_1 n_2(N+1)}{12},$$

$$\mathrm{AB} = \frac{n_1(N+1)}{2} - \sum_{i=1}^{N} \left| i - \frac{N+1}{2} \right| V_i,$$

$$\mathrm{E[AB]} = \begin{cases} \frac{n_1(N+2)}{4} & \text{if } N \text{ is even}, \\ \frac{n_1(N+1)^2}{4N} & \text{if } N \text{ is odd}, \end{cases}$$

$$\mathrm{V[AB]} = \begin{cases} \frac{n_1 n_2(N^2-4)}{48(N-1)} & \text{if } N \text{ is even}, \\ \frac{n_1 n_1(N+1)(N^2+3)}{48N^2} & \text{if } N \text{ is odd}, \end{cases}$$

$$\mathrm{AS} = \sum_{i=1}^{N} \left(1 - \sum_{j=i}^{N} \frac{1}{j}\right) V_i, \quad \mathrm{E[AS]} = 0, \quad \mathrm{V[AS]} = \frac{n_1 n_2}{N-1}\left(1 - \frac{H_N}{N}\right),$$

$$H_N = \sum_{j=1}^{N} \frac{1}{j}.$$

Remark that, see, e.g. Kössler and Mukherjee (2020), the Anti-Savage test (AS) is the locally most powerful rank test for location under the right-skewed Gumbel distribution with cdf

$$F_R(x) = \exp\{-\exp(-x)\}, \quad f_R(x) = \exp\{-\exp(-x) - x\}, \quad x \in \mathbb{R}.$$

However, there exists the left-skewed Gumbel distribution given by

$$F_L(x) = 1 - \exp\{-\exp(x)\}, \quad f_L(x) = \exp\{-\exp(x) + x\}, \quad x \in \mathbb{R}.$$

Then, the Savage test is the locally most powerful linear rank test for location if F_L is the underlying cdf. see, e.g. Hájek et al. (1999, pp. 105–106). Therefore, in this paper, we consider another type of tri-aspect statistic for test problem (1) as follows:

$$\mathrm{T}_2 = \left(\frac{\mathrm{W}-\mathrm{E[W]}}{\sqrt{\mathrm{V[W]}}}\right)^2 + \left(\frac{\mathrm{AB}-\mathrm{E[AB]}}{\sqrt{\mathrm{V[AB]}}}\right)^2 + \left(\frac{\mathrm{S}-\mathrm{E[S]}}{\sqrt{\mathrm{V[S]}}}\right)^2, \tag{2}$$

where

$$\mathrm{S} = \sum_{i=1}^{N}\left(\sum_{j=N+1-i}^{N}\frac{1}{j}\right)V_i, \quad \mathrm{E[S]} = n_1, \quad \mathrm{V[S]} = \frac{n_1 n_2}{N-1}\left(1-\frac{H_N}{N}\right).$$

In addition, Kössler and Mukherjee (2020) proposed the Mahalanobis-type statistic T_3 as follows:

$$\mathrm{T}_3 = \boldsymbol{T}_{\mathrm{M}_1}\boldsymbol{\Sigma}_{\mathrm{M}_1}^{-1}\boldsymbol{T}'_{\mathrm{M}_1}, \tag{3}$$

where

$$\boldsymbol{T}_{\mathrm{M}_1} = \left(\frac{\mathrm{W}-\mathrm{E[W]}}{\sqrt{\mathrm{V[W]}}}, \frac{\mathrm{AB}-\mathrm{E[AB]}}{\sqrt{\mathrm{V[AB]}}}, \frac{\mathrm{AS}-\mathrm{E[AS]}}{\sqrt{\mathrm{V[AS]}}}\right),$$

$$\boldsymbol{\Sigma}_{\mathrm{M}_1} = \begin{pmatrix} 1 & 0 & \rho_{\mathrm{W,AS}} \\ 0 & 1 & \rho_{\mathrm{AB,AS}} \\ \rho_{\mathrm{W,AS}} & \rho_{\mathrm{AB,AS}} & 1 \end{pmatrix},$$

$$\rho_{\mathrm{W,AS}} = \frac{\sqrt{3}}{2}\sqrt{\frac{N-1}{N+1}}\left(1-\frac{H_N}{N}\right)^{-\frac{1}{2}},$$

$$\rho_{\mathrm{AB,AS}} = \begin{cases} -\sqrt{\dfrac{3N^2}{N^2-4}\left(1-\dfrac{H_N}{N}\right)^{-1}}\left(\dfrac{1}{2}-H_N+\displaystyle\sum_{j=1}^{\frac{N}{2}}\dfrac{1}{j}\right) & \text{if } N \text{ is even}, \\ -\sqrt{\dfrac{3(N^2-1)}{N^2+3}\left(1-\dfrac{H_N}{N}\right)^{-1}}\left[\dfrac{N+3}{2(N+1)}-\left\{H_N-\displaystyle\sum_{j=1}^{\frac{N-1}{2}}\dfrac{1}{j}\right\}\right] & \text{if } N \text{ is odd}, \end{cases}$$

See, Remark of 3.1 of Mukherjee et al. (2021) for the details of the expression of $\rho_{\mathrm{AB,AS}}$.

Similarly to T_2, we consider the another-type of statistic based on the Mahalanobis distance as follows:

$$\mathrm{T}_4 = \boldsymbol{T}_{\mathrm{M}_2}\boldsymbol{\Sigma}_{\mathrm{M}_2}^{-1}\boldsymbol{T}'_{\mathrm{M}_2}, \tag{4}$$

where

$$\boldsymbol{T}_{\mathrm{M}_2} = \left(\frac{\mathrm{W} - \mathrm{E[W]}}{\sqrt{\mathrm{V[W]}}}, \frac{\mathrm{AB} - \mathrm{E[AB]}}{\sqrt{\mathrm{V[AB]}}}, \frac{\mathrm{S} - \mathrm{E[S]}}{\sqrt{\mathrm{V[S]}}} \right),$$

$$\boldsymbol{\Sigma}_{\mathrm{M}_2} = \begin{pmatrix} 1 & 0 & \rho_{\mathrm{W,S}} \\ 0 & 1 & \rho_{\mathrm{AB,S}} \\ \rho_{\mathrm{W,S}} & \rho_{\mathrm{AB,S}} & 1 \end{pmatrix},$$

$$\rho_{\mathrm{W,S}} = \rho_{\mathrm{W,AS}},$$

$$\rho_{\mathrm{AB,S}} = -\rho_{\mathrm{AB,AS}}.$$

Note that T_1, T_2, T_3 and T_4 are special cases of Yamaguchi and Murakami (2023).

2.2 The limiting null distributions of T_2 and T_4

The test statistic's distribution plays a vital role in testing the hypothesis. Using the exact permutation method, we can derive the exact distribution of a test statistic for small sample sizes. However, deriving the exact distribution is often difficult when the sample sizes are moderate to large. Then, in this section, we derive the limiting distributions of T_2 and T_4 under the null hypothesis.

Let $\lambda_i, i = 1, 2, 3$ be the eigenvalues of the asymptotic correlation matrix $\boldsymbol{\Sigma}_{\mathrm{S}}$

$$\boldsymbol{\Sigma}_{\mathrm{S}} = \lim_{\min\{n_1,n_2\}\to\infty} \boldsymbol{\Sigma}_{\mathrm{M}_2} = \lim_{\min\{n_1,n_2\}\to\infty} \begin{pmatrix} 1 & 0 & \rho_{\mathrm{W,S}} \\ 0 & 1 & \rho_{\mathrm{AB,S}} \\ \rho_{\mathrm{W,S}} & \rho_{\mathrm{AB,S}} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & \frac{\sqrt{3}}{2}(1 - 2\log 2) \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2}(1 - 2\log 2) & 1 \end{pmatrix}. \tag{5}$$

Since the eigenvalues of matrix (5) are equal to that of the matrix (17) in Kössler and Mukherjee (2020), we immediately obtain the following two theorems by a similar procedure to that of Kössler and Mukherjee (2020).

Theorem 1 *Assume that* $n_1/N \in (0, 1)$ *as* $\min(n_1, n_2) \to \infty$. *The limiting null distribution of* T_2 *is approximately equivalent to* $1.7299Z + 0.2692$, *where* $Z \sim \chi^2_{df}$ *is a chi square random variable with* $df = 1.5786$ *degrees of freedom.*

Proof See the Appendix 1. □

Then, as a consequence of Theorem 1, we get the following corollary.

Corollary 1 *Assume that* $n_1/N \in (0, 1)$ *as* $\min(n_1, n_2) \to \infty$. *The level* α *critical point of* T_2 *can be approximated by*

$$t_{Q,1-\alpha} = 1.7299 t_{Z,1-\alpha} + 0.2692,$$

where $t_{Z,1-\alpha}$ *is the* $1-\alpha$ *quantile of the distribution of the random variable Z.*

By replacing AS in Kössler and Mukherjee (2020) with S, we immediately obtain Lemma 1.

Lemma 1 *Assume that* $n_1/N \in (0,1)$ *as* $\min(n_1,n_2) \to \infty$. *The asymptotic null joint distribution of* $\boldsymbol{T}_{\mathrm{M}_2}$ *is a trivariate normal with mean vector* (0, 0, 0) *and variance-covariance matrix given by* (5).

Theorem 2 *Assume that* $n_1/N \in (0,1)$ *as* $\min(n_1,n_2) \to \infty$. *The limiting null distribution of* T_4 *converges to a Chi-square distribution with three degrees of freedom.*

Proof See Appendix 2. □

3 The distribution of test statistics under the alternative hypothesis

The score generating functions of W, AB, AS and S are respectively given by

$$\begin{aligned}
\phi_{\mathrm{W}}(u) &= 2u-1,\\
\phi_{\mathrm{AB}}(u) &= 1-2|2u-1|,\\
\phi_{\mathrm{AS}}(u) &= 1+\log(u),\\
\phi_{\mathrm{S}}(u) &= -1-\log(1-u),\\
u &\in (0,1).
\end{aligned}$$

Remark that the score generating function of AB in Kössler and Mukherjee (2020) should have an opposite sign, see Kössler (2006). However, it does not affect the eigenvalues of the correlation matrix and asymptotic null distributions. Let the parameter vector $\boldsymbol{\Theta}'_N = (\bar{\mu},\bar{\sigma},\bar{\delta})/\sqrt{N}$, $\lambda = n_1/N \in (0,1)$ as $\min(n_1,n_2) \to \infty$ and the parameter vector $\boldsymbol{\Theta}' = \lim \boldsymbol{\Theta}'_N = (\mu,\sigma,\delta)$. Let be $\mathrm{Score} \in \{\mathrm{W},\mathrm{AB},\mathrm{AS},\mathrm{S}\}$, $\mathrm{Shift} \in \{\mathrm{Location},\mathrm{Scale},\mathrm{Lehmann}\}$ and

$$\begin{aligned}
C_{\text{Score, Shift}}(f) &= \frac{d_{\text{Score, Shift}}(f)}{\sqrt{I_{\text{Score}}}},\\
I_{\text{Score}} &= \int_0^1 \phi^2_{\text{Score}}(u)du,\\
d_{\text{Score, Location}}(f) &= \int_0^1 \phi'_{\text{Score}}(u)f(F^{-1}(u))du,\\
d_{\text{Score, Scale}}(f) &= \int_0^1 \phi'_{\text{Score}}(u)f(F^{-1}(u))F^{-1}(u)du,\\
d_{\text{Score, Lehmann}}(f) &= -\int_0^1 \phi'_{\text{Score}}(u)u\log(u)du.
\end{aligned}$$

Similar to Kössler and Mukherjee (2020) and Mukherjee et al. (2021), to derive the asymptotic distribution and the asymptotic power of T_2 or T_4, we define the matrix $\boldsymbol{M}_{\mathrm{S}}(f)$ as

$$\boldsymbol{M}_{\mathrm{S}}(f) = \begin{pmatrix} C_{\mathrm{W,\,Location}}(f) & C_{\mathrm{W,\,Scale}}(f) & C_{\mathrm{W,\,Lehmann}}(f) \\ C_{\mathrm{AB,\,Location}}(f) & C_{\mathrm{AB,\,Scale}}(f) & C_{\mathrm{AB,\,Lehmann}}(f) \\ C_{\mathrm{S,\,Location}}(f) & C_{\mathrm{S,\,Scale}}(f) & C_{\mathrm{S,\,Lehmann}}(f) \end{pmatrix}.$$

Note that Kössler and Mukherjee (2020) defined the corresponding matrix for T_1 or T_3 to derive the asymptotic power as follows:

$$\boldsymbol{M}_{\mathrm{AS}}(f) = \begin{pmatrix} C_{\mathrm{W,\,Location}}(f) & C_{\mathrm{W,\,Scale}}(f) & C_{\mathrm{W,\,Lehmann}}(f) \\ C_{\mathrm{AB,\,Location}}(f) & C_{\mathrm{AB,\,Scale}}(f) & C_{\mathrm{AB,\,Lehmann}}(f) \\ C_{\mathrm{AS,\,Location}}(f) & C_{\mathrm{AS,\,Scale}}(f) & C_{\mathrm{AS,\,Lehmann}}(f) \end{pmatrix}.$$

For example, the matrices $\boldsymbol{M}_{\mathrm{AS}}(f)$ and $\boldsymbol{M}_{\mathrm{S}}(f)$ of the left- and right-skewed Gumbel distributions f_L and f_R, respectively, are as follows:

$$\boldsymbol{M}_{\mathrm{AS}}(f_L) = \begin{pmatrix} 0.8660 & -0.2341 & 0.8660 \\ 0.3345 & 0.9879 & -0.3345 \\ 0.6449 & -0.6649 & 1.0000 \end{pmatrix} \text{ and}$$

$$\boldsymbol{M}_{\mathrm{AS}}(f_R) = \begin{pmatrix} 0.8660 & 0.2341 & 0.8660 \\ -0.3345 & 0.9879 & -0.3345 \\ 1.0000 & -0.4228 & 1.0000 \end{pmatrix},$$

$$\boldsymbol{M}_{\mathrm{S}}(f_L) = \begin{pmatrix} 0.8660 & -0.2341 & 0.8660 \\ 0.3345 & 0.9879 & -0.3345 \\ 1.0000 & 0.4228 & 0.6449 \end{pmatrix} \text{ and}$$

$$\boldsymbol{M}_{\mathrm{S}}(f_R) = \begin{pmatrix} 0.8660 & 0.2341 & 0.8660 \\ -0.3345 & 0.9879 & -0.3345 \\ 0.6449 & 0.6649 & 0.6449 \end{pmatrix}.$$

In Sects. 3.1 and 3.2, we discuss the distributions of T_2 and T_4 under the alternative hypothesis H_1, respectively.

3.1 The distribution of T_2 under the alternative hypothesis

In addition, by a similar procedure to that of Kössler and Mukherjee (2020) and Mukherjee et al. (2021), we obtain the asymptotic distribution under the alternative hypothesis in Theorem 3.

Theorem 3 *Assume again that* $\boldsymbol{\Theta}'_N = \bar{\boldsymbol{\Theta}}'/\sqrt{N} = (\bar{\mu}, \bar{\sigma}, \bar{\delta})/\sqrt{N}$ *and* $n_1/N \to \lambda \in (0,1)$ *as* $\min(n_1, n_2) \to \infty$. *The statistic* T_2 *may be approximated by a linear function of a noncentral* χ^2 *distributed random variable* Z, *that is* $\mathrm{T}_2 \approx \beta_1(\boldsymbol{\Delta})Z + \beta_0(\boldsymbol{\Delta})$, $Z \sim \chi^2_{df(\boldsymbol{\Delta})}$ *for* $c_3^2(\boldsymbol{\Delta})/c_2^3(\boldsymbol{\Delta}) \le c_4(\boldsymbol{\Delta})/c_2^2(\boldsymbol{\Delta})$ *and*

$T_2 \approx \beta_1'(\mathbf{\Delta})Z' + \beta_0'(\mathbf{\Delta})$, $Z' \sim \chi^2_{df'(\mathbf{\Delta})}(nc)$ *for* $c_3^2(\mathbf{\Delta})/c_2^3(\mathbf{\Delta}) > c_4(\mathbf{\Delta})/c_2^2(\mathbf{\Delta})$. *The coefficients* β_0, β_1, β_0' *and* β_1' *are presented in Appendix 3.*

Proof See Appendix 3. □

Corollary 2 *By Theorem 3, the asymptotic power of* T_2 *is approximately*

$$\mathbb{P}(T_2 > t_{Q,1-\alpha}) \approx \mathbb{P}(\beta_1(\mathbf{\Delta})Z + \beta_0(\mathbf{\Delta}) > t_{Q,1-\alpha}) = \mathbb{P}\left(Z > \frac{t_{Q,1-\alpha} - \beta_0(\mathbf{\Delta})}{\beta_1(\mathbf{\Delta})}\right)$$
$$= 1 - F_Z\left(\frac{t_{Q,1-\alpha} - \beta_0(\mathbf{\Delta})}{\beta_1(\mathbf{\Delta})}\right),$$

where $t_{Q,1-\alpha}$ *be the* $1-\alpha$ *quantile of* T_2.

Here, we show the asymptotic power functions of T_1 and T_2 for the left-skewed Gumbel distribution and the right-skewed Gumbel distribution in Fig. 1.

Figure 1 indicates that we have to select AS or S before testing the hypothesis. Then, we discuss the selection of whether we use T_1 or T_2 in Sect. 4.

3.2 The distribution of T_4 under the alternative hypothesis

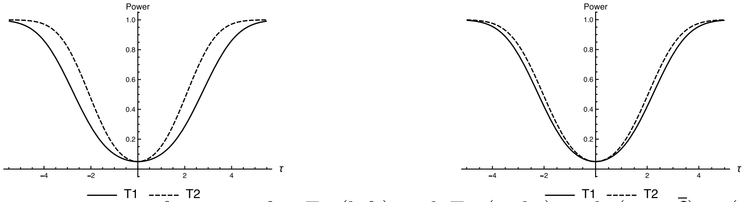
In this section, we derive the limiting distribution of T_4 under H_1. The limiting distribution of T_4 is a chi-square distribution with three degrees of freedom under the null hypothesis by Theorem 2. Hence, it is important to determine the noncentral parameter under H_1. Again, let the parameter vector $\mathbf{\Theta}' = \lim \mathbf{\Theta}_N' = (\mu, \sigma, \delta)$. As a similar procedure to that of Kössler and Mukherjee (2020), we define the noncentral parameter as $\mathrm{nc} = \lambda(1-\lambda)\mathbf{\Theta}'\boldsymbol{M}_{\mathrm{S}}(f)'\mathbf{\Sigma}_{\mathrm{S}}^{-1}\boldsymbol{M}_{\mathrm{S}}(f)\mathbf{\Theta}$. Then, we obtain the following theorem by a similar procedure to that of Kössler and Mukherjee (2020) and Mukherjee et al. (2021).

Theorem 4 *Assume that the parameter vector* $\mathbf{\Theta}_N' = (\bar{\mu}, \bar{\sigma}, \bar{\delta})/\sqrt{N}$ *and* $\lambda = n_1/N \in (0,1)$ *as* $\min(n_1, n_2) \to \infty$. *Then the limiting distribution of* T_4 *under the alternative hypothesis is noncentral chi-square distribution with three degrees of freedom and noncentral parameter* $nc = \lambda(1-\lambda)\mathbf{\Theta}'\boldsymbol{M}_{\mathrm{S}}(f)'\mathbf{\Sigma}_{\mathrm{S}}^{-1}\boldsymbol{M}_{\mathrm{S}}(f)\mathbf{\Theta}$, *where* $\mathbf{\Theta}' = \lim \mathbf{\Theta}_N' = (\mu, \sigma, \delta)$.

Proof See Appendix 4. □

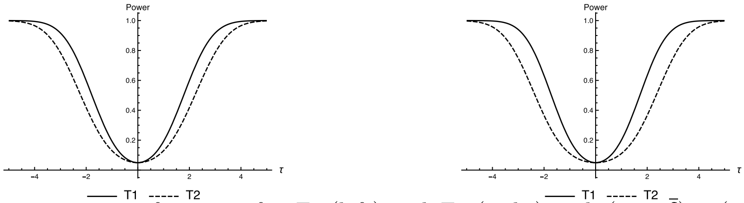
Here, we show the asymptotic power of T_3 and T_4 for the left-skewed Gumbel distribution and the right-skewed Gumbel distribution in Fig. 2.

Figure 2 again indicates that we have to select AS or S before testing the hypothesis. Then, we discuss the selection of T_3 or T_4 in Sect. 4.



Asymptotic power functions for F_L (left) and F_R (right) with $(\bar{\mu}, \bar{\sigma}, \bar{\delta}) = (\tau, \tau, \tau)$.

Asymptotic power functions for F_L (left) and F_R (right) with $(\bar{\mu}, \bar{\sigma}, \bar{\delta}) = (\tau, \tau, -\tau)$.



Asymptotic power functions for F_L (left) and F_R (right) with $(\bar{\mu}, \bar{\sigma}, \bar{\delta}) = (\tau, -\tau, \tau)$.

Asymptotic power functions for F_L (left) and F_R (right) with $(\bar{\mu}, \bar{\sigma}, \bar{\delta}) = (\tau, -\tau, -\tau)$.

Fig. 1 Asymptotic power functions of T_1 and T_2 for F_L (left) and F_R (right) with various $(\bar{\mu}, \bar{\sigma}, \bar{\delta})$

4 Maximum test and adaptive test

In practical analysis, we have to determine whether to use T_1 (T_3) or T_2 (T_4) before we treat the hypothesis test. Hence, we propose the maximum- and adaptive-type test based on T_1 (T_3) and T_2 (T_4).

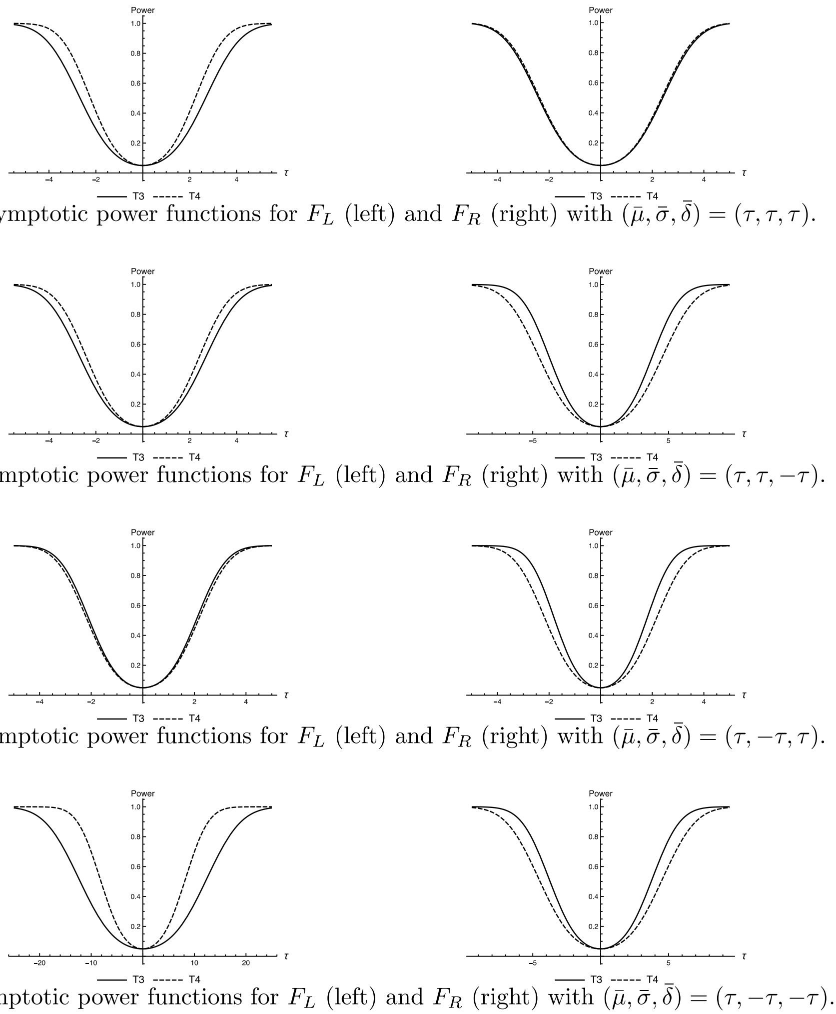


Fig. 2 Asymptotic power functions of T_3 and T_4 for F_L (left) and F_R (right) with various $(\bar{\mu}, \bar{\sigma}, \bar{\delta})$

4.1 Maximum test

A simple way to solve this problem is to use the larger of the two statistics as the test statistic. Then, we propose two maximum test statistics as follows:

$$\begin{aligned} T^{(1)}_{\max} &= \max\{T_1, T_2\}, \\ T^{(2)}_{\max} &= \max\{T_3, T_4\}. \end{aligned}$$

Herein, we list the critical values of $T^{(1)}_{\max}$ and $T^{(2)}_{\max}$ for selected sample sizes in Table 1. We can derive the exact distribution of test statistics for small sample sizes using all possible permutations. However, deriving the exact distribution is

Table 1 Critical values of $T^{(1)}_{max}$ and $T^{(2)}_{max}$

n_1	n_2	$\mathbb{P}(T^{(1)}_{max} \geq t_1)$	t_1	$\mathbb{P}(T^{(2)}_{max} \geq t_2)$	t_2
30	15	0.100	7.7336	0.100	6.6962
		0.050	9.9333	0.050	8.0155
		0.025	12.1283	0.025	9.2720
		0.010	14.9487	0.010	10.8786
30	30	0.100	7.7801	0.100	6.5899
		0.050	10.0266	0.050	7.9297
		0.025	12.2492	0.025	9.2397
		0.010	15.1372	0.010	10.9479
50	25	0.100	7.8009	0.100	6.8288
		0.050	10.0871	0.050	8.2394
		0.025	12.3458	0.025	9.5891
		0.010	15.3791	0.010	11.2941
50	50	0.100	7.8177	0.100	6.7640
		0.050	10.1579	0.050	8.1362
		0.025	12.4568	0.025	9.4950
		0.010	15.4916	0.010	11.2655
100	50	0.100	7.8353	0.100	6.9125
		0.050	10.1807	0.050	8.4159
		0.025	12.5367	0.025	9.8669
		0.010	15.6894	0.010	11.7068
100	100	0.100	7.8528	0.100	6.9401
		0.050	10.2121	0.050	8.3908
		0.025	12.6089	0.025	9.7817
		0.010	15.7519	0.010	11.5916
200	100	0.100	7.8464	0.100	6.9930
		0.050	10.2259	0.050	8.5363
		0.025	12.6259	0.025	10.0108
		0.010	15.7965	0.010	11.9570
200	200	0.100	7.8470	0.100	7.0283
		0.050	10.2510	0.050	8.5361
		0.025	12.6589	0.025	9.9906
		0.010	15.8052	0.010	11.8875

difficult when the sample sizes are moderate to large. Therefore, we estimate the critical value of $T^{(1)}_{max}$ and $T^{(2)}_{max}$ for the moderate to large sample sizes by 1,000,000 times permutations. An approximate permutation test with a random sample of, e.g., 1,000,000 permutations, can evaluate the p-value fairly for practical applications.

4.2 Adaptive test

In this section, we introduce a selection rule. Hogg et al. (2018, pp. 622) gave the selector statistic as

$$\hat{Q}_1 = \frac{\hat{U}_{0.05} - \hat{M}_{0.5}}{\hat{M}_{0.5} - \hat{L}_{0.05}}.$$

The statistic $\hat{Q}_1$ measures skewness, and $\hat{L}_\gamma$, $\hat{M}_\gamma$ and $\hat{U}_\gamma$ denote the average of the smallest, middle and largest γN order statistics, respectively. If $\hat{Q}_1 > 2$, then the right tail of the distribution seems longer than the left tail; that is, there is an indication that the distribution is skewed to the right. On the other hand, if $\hat{Q}_1 < 1/2$, the sample indicates that the distribution may be skewed to the left. Let be

- $\mathrm{N}(\mu, \sigma^2)$: normal distribution with mean μ and variance σ^2.
- $\mathrm{G_L}(\mu, \sigma)$: left-skewed Gumbel distribution with location μ and scale σ.
- $\mathrm{G_R}(\mu, \sigma)$: right-skewed Gumbel distribution with location μ and scale σ.
- $\mathrm{U}(a, b)$: uniform distribution with interval (a, b).
- $\mathrm{Exp}(\nu)$: exponential distribution with rate ν.
- t_ν: t distribution with ν degrees of freedom.
- $\mathrm{C}(\mu, \sigma)$: Cauchy distribution with median μ and scale σ.
- χ^2_ν: chi square distribution with ν degrees of freedom.
- $\mathrm{LG}(\mu, \sigma)$: logistic distribution with location μ and scale σ.
- $\mathrm{LA}(\mu, \sigma)$: Laplace distribution with location μ and scale σ.
- $\mathrm{Ga}(\eta, \sigma)$: gamma with shape η and rate σ.
- $\mathrm{BE}(a, b)$: beta distribution with shapes a and b.
- $\mathrm{CN1}(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2)$: mixture normal distribution with $0.9\mathrm{N}(\mu_1, \sigma_1^2) + 0.1\mathrm{N}(\mu_2, \sigma_2^2)$.
- $\mathrm{CN2}(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2)$: mixture normal distribution with $0.5\mathrm{N}(\mu_1, \sigma_1^2) + 0.5\mathrm{N}(\mu_2, \sigma_2^2)$.

We list the theoretical value of Q_1 for the specific distributions in Table 2. The values of Q_1 for some distributions may be found in Neuhäuser et al. (2004).

The Savage and Anti-Savage tests are useful location tests for the left-skewed and right-skewed distributions, respectively. When the theoretical value of Q_1 is greater (less) than 1, the selector statistic indicates that the distribution is right-skewed (left-skewed) distribution. Large (small) values of Q_2 in Hogg et al. (2018, pp. 623) indicate

Table 2 Theoretical values of Q_1 of various distributions

Distribution	Q_1	Distribution	Q_1
$\mathrm{N}(0, 1)$	1.0000	$\mathrm{LG}(0, 1)$	1.0000
$\mathrm{G_L}(0, 1)$	0.4893	$\mathrm{LA}(0, 1)$	1.0000
$\mathrm{G_R}(0, 1)$	2.0439	$\mathrm{Ga}(5, 1)$	1.8787
$\mathrm{U}(0, 1)$	1.0000	$\mathrm{BE}(2, 2)$	1.0000
$\mathrm{Exp}(2)$	4.5689	$\mathrm{BE}(0.5, 2)$	4.6395
t_2	1.0000	$\mathrm{BE}(2, 0.5)$	0.2155
$\mathrm{C}(0, 1)$	1.0000	$\mathrm{CN1}(0, 1, 0, 9)$	1.0000
χ^2_3	3.3515	$\mathrm{CN2}(1, 4, -1, 1)$	1.6748

that the distribution is heavy-tailed (light-tailed) distribution. In this paper, we replace the Anti-Savage test with the Savage test. The tail-weight does not play any role here. Therefore, we use only one statistic for simplicity instead of Q_1 and Q_2. Hogg et al. (2018, pp. 623) indicated that the distribution's right tail seems longer than the left tail when $\hat{Q}_1$ is large (2 or more). On the other hand, the distribution may be skewed to the left when $\hat{Q}_1 < 1/2$. We then used these values as a cutoff point. In this paper, we denote the combined sample $V = (X_1, X_2)$ and propose the selector statistic based on $\hat{Q}_1$ and Table 2 as follows:

$$\mathcal{I}(\mathrm{S}^*, \mathrm{AS}^*) = \begin{cases} \mathrm{AS}^*, & \text{if } \hat{Q}_1(X_1) > 2 \text{ and } \hat{Q}_1(X_2) > 2 \\ & \text{or } \hat{Q}_1(X_1) > 2 \text{ and } \hat{Q}_1(X_2) < \frac{1}{2} \text{ and } \hat{Q}_1(V) > 2 \\ & \text{or } \hat{Q}_1(X_1) < \frac{1}{2} \text{ and } \hat{Q}_1(X_2) > 2 \text{ and } \hat{Q}_1(V) > 2, \\ \mathrm{S}^*, & \text{if } \hat{Q}_1(X_1) < \frac{1}{2} \text{ and } \hat{Q}_1(X_2) < \frac{1}{2} \\ & \text{or } \hat{Q}_1(X_1) > 2 \text{ and } \hat{Q}_1(X_2) < \frac{1}{2} \text{ and } \hat{Q}_1(V) < \frac{1}{2} \\ & \text{or } \hat{Q}_1(X_1) < \frac{1}{2} \text{ and } \hat{Q}_1(X_2) > 2 \text{ and } \hat{Q}_1(V) < \frac{1}{2}, \\ \text{randomly select } \mathrm{AS}^* \text{ or } \mathrm{S}^*, \text{ each with probability } \frac{1}{2}, & \text{otherwise} \end{cases}$$

where

$$\mathrm{S}^* = \left(\frac{\mathrm{S} - \mathrm{E[S]}}{\sqrt{\mathrm{V[S]}}} \right)^2, \quad \mathrm{AS}^* = \left(\frac{\mathrm{AS} - \mathrm{E[AS]}}{\sqrt{\mathrm{V[AS]}}} \right)^2.$$

Then, we suggest two statistics AD_1 and AD_2 for the adaptive tests based on a new selector by

$$\begin{aligned} \mathrm{AD}_1 &= \left(\frac{\mathrm{W} - \mathrm{E[W]}}{\sqrt{\mathrm{V[W]}}} \right)^2 + \left(\frac{\mathrm{AB} - \mathrm{E[AB]}}{\sqrt{\mathrm{V[AB]}}} \right)^2 + \mathcal{I}(\mathrm{S}^*, \mathrm{AS}^*), \\ \mathrm{AD}_2 &= \boldsymbol{T}^*_{\mathrm{M}_2} \boldsymbol{\Sigma}^{*-1}_{\mathrm{M}_2} \boldsymbol{T}^{*\prime}_{\mathrm{M}_2}, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{T}^*_{\mathrm{M}_2} &= \left(\frac{\mathrm{W} - \mathrm{E[W]}}{\sqrt{\mathrm{V[W]}}}, \frac{\mathrm{AB} - \mathrm{E[AB]}}{\sqrt{\mathrm{V[AB]}}}, \mathcal{I}(\sqrt{\mathrm{S}^*}, \sqrt{\mathrm{AS}^*}) \right), \\ \boldsymbol{\Sigma}^*_{\mathrm{M}_2} &= \begin{pmatrix} 1 & 0 & \rho_{\mathrm{W},\mathcal{I}(\mathrm{S,AS})} \\ 0 & 1 & \rho_{\mathrm{AB},\mathcal{I}(\mathrm{S,AS})} \\ \rho_{\mathrm{W},\mathcal{I}(\mathrm{S,AS})} & \rho_{\mathrm{AB},\mathcal{I}(\mathrm{S,AS})} & 1 \end{pmatrix}. \end{aligned}$$

Note that the principle of Hogg et al. (2018, pp. 622–623) is based on the independence of rank and order statistics of the full sample. However, in our selection rule, in $\hat{Q}_1(X_1)$ and $\hat{Q}_1(X_2)$ we use order statistics of both single samples separately. Therefore the independence property is not satisfied and we have to check whether the use of the (exact or asymptotic) critical values for T_1 and T_3 are applicable to that for AD_1 and AD_2.

5 Numerical results

5.1 Robustness

We investigate the robustness for various distributions to propose the adaptive test based on the selector statistic. In this paper, we compare the performances of T_1, T_2, T_3, T_4, $T^{(1)}_{max}$, $T^{(2)}_{max}$, AD_1 and AD_2. We use the exact critical values listed in Table 3 when the sample sizes are $(n_1, n_2) = (10, 10), (10, 5)$. On the other hand, when the sample sizes are $(n_1, n_2) = (100, 50), (100, 100), (200, 100), (200, 200)$, we use the 95% point of the limiting null distribution given in Theorems 1 and 2. In addition, from the results of Kössler and Mukherjee (2020) and Theorems 1 and 2 in this paper, the limiting null distributions of T_1 and T_3 are same as those of T_2 and T_4, respectively. Therefore, we use the asymptotic critical values for T_1, T_3, AD_1 and AD_2. For $T^{(1)}_{max}$ and $T^{(2)}_{max}$, we use the estimated critical values listed in Table 1.

We show the type-I error rates (5%) for various test statistics in Table 4. Although the competing statistics T_1, T_2, T_3, T_4, $T^{(1)}_{max}$ and $T^{(2)}_{max}$ are distribution-free under the null hypothesis, the independence between the selector and AD_1 or AD_2 is not guaranteed. Therefore, we investigated the type-I error rates of AD_1 and AD_2 to ensure they are safely used. The simulated results are based on 1,000,000 replications of Monte-Carlo simulations.

Table 4 indicates that the type-I errors of statistics are around the significance level as expected because exact critical values are used for small sample sizes. For larger sample sizes, since the type-I error begins to converge, the estimated critical values for some selected sample sizes and the asymptotic critical values are useful. In practice, an approximate permutation test with a random sample is possible for evaluating p value for other sample sizes. From the numerical results, we can also see that the null distribution of T_1 and T_2 and the null distribution of T_3 and T_4 are the same. Then, we can select T_1 or T_2 and T_3 or T_4 by our selecting rule.

We also estimated rejection probabilities when populations deviate from the assumed location-scale-shape family. However, we only comment on the results to save the space. In this paper, we focus on the following pairs of distributions, $N(0,1)$ with $LG(0, \frac{\sqrt{3}}{\pi})$, $N(0, \frac{3}{2})$ with t_6, $U(0, \sqrt{\frac{3}{5}}) + \frac{5-\sqrt{15}}{10}$ with $Be(2,2)$ and $LG(0, \frac{3}{\sqrt{2\pi}})$ with t_6. In these cases, both populations' mean, variance and skewness are same but the distributions are different. Therefore, our proposed test may not be suitable if the alternative is not in the location-scale-shape family. But we should also note that the Kolmogorov–Smirnov, the Cramér–von Mises or the Anderson–Darling tests

Table 3 The exact and asymptotic critical values ($\alpha = 0.05$)

(n_1, n_2)	T_1	T_2	T_3	T_4	$T^{(1)}_{max}$	$T^{(2)}_{max}$	AD_1	AD_2
(10, 10)	8.7025	8.7025	6.8516	6.8516	9.5411	7.4987	8.7025	6.8516
(10, 5)	8.1898	8.1898	6.6579	6.6579	8.8160	7.2161	8.1898	6.6579
(∞, ∞)	9.1715	9.1715	7.8147	7.8147	–	–	9.1715	7.8147

–, the asymptotic critical value is not known

Table 4 Type-I error of various test statistics for various densities

(n_1, n_2)		N(0, 1)	$G_L(0, 1)$	$G_R(0, 1)$	U(0, 1)	Exp(2)	t_2	C(0, 1)	χ^2_3
(10, 10)	T_1	0.0501	0.0502	0.0501	0.0500	0.0501	0.0502	0.0503	0.0498
	T_2	0.0501	0.0502	0.0500	0.0503	0.0502	0.0499	0.0503	0.0499
	T_3	0.0502	0.0501	0.0498	0.0499	0.0500	0.0501	0.0503	0.0500
	T_4	0.0502	0.0501	0.0503	0.0500	0.0499	0.0502	0.0503	0.0496
	$T^{(1)}_{max}$	0.0503	0.0500	0.0501	0.0501	0.0502	0.0499	0.0503	0.0498
	$T^{(2)}_{max}$	0.0501	0.0500	0.0499	0.0498	0.0500	0.0504	0.0503	0.0498
	AD_1	0.0510	0.0525	0.0525	0.0505	0.0536	0.0512	0.0512	0.0534
	AD_2	0.0462	0.0466	0.0465	0.0469	0.0486	0.0469	0.0484	0.0477
(10, 5)	T_1	0.0505	0.0507	0.0506	0.0504	0.0505	0.0506	0.0503	0.0508
	T_2	0.0504	0.0509	0.0505	0.0502	0.0505	0.0505	0.0501	0.0509
	T_3	0.0511	0.0509	0.0504	0.0505	0.0507	0.0507	0.0505	0.0506
	T_4	0.0508	0.0509	0.0507	0.0503	0.0505	0.0505	0.0506	0.0508
	$T^{(1)}_{max}$	0.0501	0.0505	0.0501	0.0501	0.0501	0.0500	0.0497	0.0506
	$T^{(2)}_{max}$	0.0504	0.0505	0.0504	0.0502	0.0501	0.0503	0.0502	0.0505
	AD_1	0.0513	0.0524	0.0522	0.0508	0.0532	0.0517	0.0510	0.0533
	AD_2	0.0471	0.0473	0.0471	0.0470	0.0487	0.0476	0.0488	0.0482
(100, 50)	T_1	0.0482	0.0482	0.0479	0.0479	0.0476	0.0481	0.0484	0.0478
	T_2	0.0480	0.0480	0.0478	0.0476	0.0476	0.0480	0.0486	0.0479
	T_3	0.0453	0.0453	0.0452	0.0452	0.0449	0.0454	0.0451	0.0449
	T_4	0.0455	0.0453	0.0449	0.0453	0.0449	0.0456	0.0450	0.0450
	$T^{(1)}_{max}$	0.0500	0.0501	0.0500	0.0505	0.0500	0.0500	0.0502	0.0505
	$T^{(2)}_{max}$	0.0499	0.0497	0.0503	0.0500	0.0498	0.0499	0.0498	0.0500
	AD_1	0.0483	0.0497	0.0493	0.0478	0.0476	0.0483	0.0490	0.0482
	AD_2	0.0456	0.0450	0.0447	0.0455	0.0449	0.0454	0.0458	0.0463
(100, 100)	T_1	0.0482	0.0483	0.0481	0.0478	0.0483	0.0484	0.0483	0.0486
	T_2	0.0485	0.0483	0.0483	0.0481	0.0484	0.0485	0.0484	0.0482
	T_3	0.0455	0.0454	0.0452	0.0454	0.0453	0.0452	0.0456	0.0451
	T_4	0.0454	0.0455	0.0454	0.0452	0.0452	0.0451	0.0455	0.0449
	$T^{(1)}_{max}$	0.0501	0.0499	0.0497	0.0504	0.0496	0.0497	0.0499	0.0501
	$T^{(2)}_{max}$	0.0500	0.0497	0.0496	0.0498	0.0495	0.0494	0.0495	0.0499
	AD_1	0.0484	0.0497	0.0495	0.0480	0.0483	0.0486	0.0485	0.0486
	AD_2	0.0454	0.0453	0.0449	0.0453	0.0453	0.0468	0.0474	0.0451
(200, 100)	T_1	0.0486	0.0485	0.0481	0.0484	0.0486	0.0485	0.0484	0.0482
	T_2	0.0486	0.0485	0.0481	0.0483	0.0485	0.0482	0.0483	0.0480
	T_3	0.0476	0.0474	0.0469	0.0467	0.0472	0.0471	0.0472	0.0472
	T_4	0.0474	0.0471	0.0468	0.0470	0.0471	0.0479	0.0471	0.0474
	$T^{(1)}_{max}$	0.0500	0.0500	0.0498	0.0498	0.0504	0.0501	0.0501	0.0503
	$T^{(2)}_{max}$	0.0498	0.0500	0.0499	0.0497	0.0499	0.0500	0.0496	0.0499
	AD_1	0.0485	0.0497	0.0492	0.0483	0.0486	0.0483	0.0486	0.0481
	AD_2	0.0473	0.0466	0.0463	0.0467	0.0472	0.0471	0.0482	0.0472

Table 4 (continued)

(n_1, n_2)		N(0, 1)	$G_L(0,1)$	$G_R(0,1)$	U(0, 1)	Exp(2)	t_2	C(0, 1)	χ^2_3
(200, 200)	T_1	0.0483	0.0486	0.0485	0.0485	0.0485	0.0486	0.0485	0.0484
	T_2	0.0483	0.0487	0.0483	0.0483	0.0483	0.0488	0.0483	0.0483
	T_3	0.0478	0.0468	0.0476	0.0477	0.0475	0.0476	0.0473	0.0477
	T_4	0.0476	0.0472	0.0477	0.0474	0.0476	0.0476	0.0475	0.0478
	$T^{(1)}_{max}$	0.0500	0.0502	0.0500	0.0499	0.0499	0.0498	0.0499	0.0500
	$T^{(2)}_{max}$	0.0500	0.0499	0.0501	0.0501	0.0500	0.0497	0.0501	0.0497
	AD_1	0.0484	0.0497	0.0495	0.0484	0.0485	0.0487	0.0485	0.0484
	AD_2	0.0478	0.0467	0.0471	0.0475	0.0475	0.0486	0.0496	0.0477
		LG(0, 1)	LA(0, 1)	Ga(5, 1)	BE(2, 2)	BE(0.5, 2)	BE(2, 0.5)	CN1(0, 1, 0, 9)	CN2(1, 4, −1, 1)
(10, 10)	T_1	0.0495	0.0505	0.0502	0.0501	0.0500	0.0499	0.0501	0.0499
	T_2	0.0498	0.0502	0.0505	0.0501	0.0502	0.0499	0.0500	0.0499
	T_3	0.0501	0.0501	0.0498	0.0496	0.0501	0.0500	0.0504	0.0503
	T_4	0.0504	0.0500	0.0495	0.0500	0.0500	0.0497	0.0501	0.0509
	$T^{(1)}_{max}$	0.0497	0.0503	0.0504	0.0501	0.0501	0.0501	0.0501	0.0499
	$T^{(2)}_{max}$	0.0505	0.0500	0.0497	0.0496	0.0498	0.0497	0.0502	0.0506
	AD_1	0.0506	0.0516	0.0524	0.0506	0.0531	0.0530	0.0510	0.0524
	AD_2	0.0463	0.0457	0.0462	0.0466	0.0491	0.0487	0.0467	0.0469
(10, 5)	T_1	0.0509	0.0504	0.0507	0.0507	0.0511	0.0508	0.0507	0.0509
	T_2	0.0508	0.0502	0.0506	0.0508	0.0511	0.0505	0.0506	0.0511
	T_3	0.0504	0.0507	0.0504	0.0501	0.0503	0.0506	0.0506	0.0509
	T_4	0.0505	0.0506	0.0504	0.0507	0.0503	0.0507	0.0507	0.0507
	$T^{(1)}_{max}$	0.0505	0.0499	0.0504	0.0504	0.0508	0.0503	0.0502	0.0506
	$T^{(2)}_{max}$	0.0504	0.0504	0.0502	0.0504	0.0499	0.0502	0.0502	0.0506
	AD_1	0.0520	0.0516	0.0521	0.0513	0.0540	0.0536	0.0516	0.0527
	AD_2	0.0466	0.0465	0.0467	0.0468	0.0488	0.0492	0.0471	0.0476
(100, 50)	T_1	0.0479	0.0481	0.0480	0.0486	0.0480	0.0475	0.0479	0.0478
	T_2	0.0479	0.0479	0.0479	0.0485	0.0480	0.0475	0.0481	0.0479
	T_3	0.0452	0.0449	0.0449	0.0451	0.0459	0.0448	0.0450	0.0449
	T_4	0.0452	0.0450	0.0451	0.0453	0.0453	0.0451	0.0449	0.0450
	$T^{(1)}_{max}$	0.0501	0.0501	0.0499	0.0504	0.0498	0.0499	0.0501	0.0501
	$T^{(2)}_{max}$	0.0497	0.0497	0.0498	0.0498	0.0498	0.0498	0.0499	0.0494
	AD_1	0.0479	0.0482	0.0488	0.0485	0.0480	0.0475	0.0481	0.0487
	AD_2	0.0450	0.0446	0.0442	0.0453	0.0459	0.0451	0.0444	0.0463

Table 4 (continued)

		LG(0, 1)	LA(0, 1)	Ga(5, 1)	BE(2, 2)	BE(0.5, 2)	BE(2, 0.5)	CN1(0, 1, 0, 9)	CN2(1, 4, −1, 1)
(100, 100)	T_1	0.0480	0.0484	0.0482	0.0476	0.0482	0.0485	0.0485	0.0484
	T_2	0.0481	0.0483	0.0482	0.0478	0.0480	0.0484	0.0485	0.0484
	T_3	0.0455	0.0449	0.0451	0.0451	0.0455	0.0452	0.0456	0.0453
	T_4	0.0455	0.0454	0.0454	0.0448	0.0452	0.0453	0.0453	0.0456
	$T^{(1)}_{max}$	0.0498	0.0498	0.0500	0.0496	0.0499	0.0497	0.0498	0.0500
	$T^{(2)}_{max}$	0.0496	0.0500	0.0499	0.0491	0.0494	0.0492	0.0495	0.0497
	AD_1	0.0479	0.0484	0.0490	0.0476	0.0482	0.0484	0.0485	0.0490
	AD_2	0.0457	0.0455	0.0448	0.0450	0.0455	0.0453	0.0464	0.0477
(200, 100)	T_1	0.0479	0.0480	0.0488	0.0485	0.0479	0.0488	0.0484	0.0480
	T_2	0.0478	0.0482	0.0483	0.0489	0.0481	0.0486	0.0484	0.0481
	T_3	0.0470	0.0469	0.0471	0.0472	0.0474	0.0470	0.0476	0.0472
	T_4	0.0470	0.0472	0.0472	0.0470	0.0474	0.0470	0.0474	0.0470
	$T^{(1)}_{max}$	0.0498	0.0497	0.0501	0.0498	0.0502	0.0500	0.0501	0.0496
	$T^{(2)}_{max}$	0.0498	0.0494	0.0497	0.0496	0.0500	0.0497	0.0504	0.0494
	AD_1	0.0478	0.0481	0.0493	0.0487	0.0479	0.0486	0.0484	0.0483
	AD_2	0.0470	0.0472	0.0459	0.0471	0.0474	0.0470	0.0470	0.0490
(200, 200)	T_1	0.0484	0.0485	0.0485	0.0483	0.0482	0.0486	0.0482	0.0481
	T_2	0.0483	0.0483	0.0483	0.0483	0.0480	0.0482	0.0483	0.0484
	T_3	0.0477	0.0474	0.0473	0.0473	0.0473	0.0475	0.0478	0.0476
	T_4	0.0478	0.0474	0.0474	0.0475	0.0477	0.0477	0.0477	0.0475
	$T^{(1)}_{max}$	0.0502	0.0494	0.0502	0.0501	0.0500	0.0499	0.0501	0.0499
	$T^{(2)}_{max}$	0.0499	0.0494	0.0502	0.0500	0.0505	0.0497	0.0501	0.0497
	AD_1	0.0483	0.0484	0.0489	0.0484	0.0482	0.0482	0.0481	0.0482
	AD_2	0.0479	0.0475	0.0463	0.0472	0.0473	0.0477	0.0479	0.0504

designed for general alternatives are also not very practical in these cases and future researches on this are highly warranted.

5.2 Power comparison

Kössler and Mukherjee (2020) and Mukherjee et al. (2021) compared the powers of T_1 and T_3 with various existing statistics and they showed the validity of T_1 or T_3 for various distributions. Therefore, we focus on comparing the power of T_1, T_2, T_3, T_4, $T^{(1)}_{max}$, $T^{(2)}_{max}$, AD_1, AD_2, the statistic of Boos (1986), abbreviated by BOOS, the Kolmogorov–Smirnov statistic (Gibbons & Chakraborti, 2021) (KS), the Cramér–von Mises statistic (Anderson, 1962) (CvM), and the Anderson–Darling statistic (Pettitt, 1976) (A–D), for $(n_1, n_2) = (10, 10)$ and (10, 5) in this section. Note that the test statistic of Boos (1986) is a combination of several multisample linear rank statistics. It capitalizes the notion of the Legendre polynomials to construct a multisample statistic. Based on the first three Legendre polynomials, the test statistic of Boos (1986) is a test statistic for location, scale and skew parameters as follows:

$$\begin{aligned}\text{BOOS} =& \frac{12}{n_1 n_2 (N+1)} \left\{ \sum_{i=1}^{N} \left(i - \frac{N+1}{2} \right) V_i \right\}^2 \\ &+ \frac{180}{n_1 n_2 (N+1)(N^2-4)} \left\{ \sum_{i=1}^{N} \left[\left(i - \frac{N+1}{2} \right)^2 - \frac{N^2-1}{12} \right] V_i \right\}^2 \\ &+ \frac{7}{n_1 n_2 (N+1)(N^2-4)(N^2-9)} \left\{ \sum_{i=1}^{N} \left[20 \left(i - \frac{N+1}{2} \right)^3 \right. \right. \\ &\left. \left. -(3N^2-7)\left(i - \frac{N+1}{2} \right) \right] V_i \right\}^2 .\end{aligned}$$

There are broadly seven possible types of shifts, three of which are an isolated shift in one of the three parameters, location, scale and shape; another three are mixed shifts involving any two out of three parameters, and a situation with a shift in all the three parameters. We use the normal and the logistic distributions as examples for symmetric distributions. In addition, we use left-skewed and right-skewed Gumbel distributions as examples of asymmetric distributions. The power patterns of five statistics under various alternatives are similar to these four distributions. Therefore, to save space, we only display the results of normal, logistic, left-skewed Gumbel and right-skewed Gumbel distribution. We show the results for the symmetric distributions and the asymmetric distributions in Tables 5, 6, 7, 8, respectively.

The statistics T_1 and T_2 (or T_3 and T_4) differ only by the components AS and S designed for deviations in shape parameters. Therefore, there is no difference in the powers of T_1, T_2, AD_1 and $T^{(1)}_{max}$ (or T_3, T_4, AD_2 and $T^{(2)}_{max}$) for the pure location or the pure scale parameter in symmetric distributions with equal sample sizes. From Tables 5, 6, 7 and 8, altogether T_1 is better than T_2 which is in accordance with the theory since test AS is optimal for Lehmann alternatives. Further, in most cases the power of AD_1 (AD_2) is between that of T_1 and T_2 (T_3 and T_4) as expected. Tests based on the Euclidean distance seem to be better than that based on the Mahalanobis distance. The winner of our study is test $T^{(1)}_{max}$ densely followed by T_1. The adaptive test AD_1 is on the third place. The other tests are worse. The max-test includes the better of the two tests AS and S for the current situation. The Adaptive test might be considered as a competitor which might be improved by a possibly better selecting rule. Nevertheless, we suggest to apply the Max test $T^{(1)}_{max}$ as it is simpler and better than the adaptive test in many situations. Furthermore, compared to the goodness-of-fit tests, in many cases, the power of $T^{(1)}_{max}$ is greater than or similar to the maximal power of the three tests CvM, KS, and A–D. In fact, we computed the ranks of the statistics according to the simulated power. For each alternative configuration, the worst power is assigned the rank one, that with the largest power is assigned the rank 12. Thus, the test statistics with the largest rank sums are the best. In Table 9, we list the rank sums of statistics over all 17 considered alternatives for $(n_1, n_2) = (10, 10)$ and $(10, 5)$.

Table 9 shows that $T^{(1)}_{max}$ has the largest rank sum for normal, logistic, left-skewed Gumbel and right-skewed Gumbel distributions. Further, the rank sum of AD_1 (AD_2) is between that of T_1 and T_2 (T_3 and T_4).

Table 5 Power of various tests for normal and logistic distributions $(n_1, n_2) = (10, 10)$ with $\alpha = 0.05$

(μ, σ, δ)	T_1	T_2	T_3	T_4	AD_1	AD_2	$T_{max}^{(1)}$	$T_{max}^{(2)}$	BOOS	CvM	KS	A–D
	The null distribution is N(0, 1)											
(1, 0, 0)	0.4900	0.4899	0.3418	0.3417	0.4911	0.3415	0.4848	0.3104	0.3220	0.5035	0.4087	0.5266
(0, 2, 0)	0.8014	0.8014	0.8804	0.8803	0.8022	0.8804	0.8299	0.8719	0.9179	0.2297	0.2903	0.4431
(0, 0, −1.5)	0.7843	0.6795	0.5705	0.6337	0.7316	0.6025	0.7453	0.5652	0.5732	0.7324	0.6841	0.7557
(1, 0, −2)	0.6535	0.4754	0.4407	0.5363	0.5639	0.4893	0.6004	0.4645	0.4567	0.5400	0.5345	0.5669
(1, 0, 0.5)	0.8590	0.8286	0.6944	0.7151	0.8442	0.7046	0.8413	0.6602	0.6894	0.8518	0.7751	0.8681
(1, −1, 0)	0.9131	0.7604	0.7830	0.8570	0.8360	0.8204	0.8868	0.8099	0.8024	0.8171	0.8407	0.8374
(1, 1, 0)	0.3117	0.5010	0.5067	0.4151	0.4068	0.4613	0.4529	0.4451	0.4790	0.2519	0.1732	0.2862
(0, 1, −1)	0.8756	0.6741	0.7642	0.8467	0.7726	0.8066	0.8426	0.7996	0.8001	0.6781	0.7495	0.7174
(0, 1, 1)	0.7018	0.8447	0.7376	0.6574	0.7730	0.6979	0.8076	0.6740	0.6706	0.7693	0.6359	0.7900
(1, 1, −1.5)	0.9296	0.7682	0.8485	0.9098	0.8467	0.8803	0.9071	0.8760	0.8750	0.7594	0.8327	0.7985
(1, 1, 1.5)	0.9996	0.9999	0.9988	0.9984	0.9997	0.9986	0.9998	0.9977	0.9984	0.9998	0.9983	0.9999
(1, −1, −1.5)	0.2196	0.1358	0.1456	0.1981	0.1789	0.1716	0.1896	0.1649	0.1729	0.1531	0.1527	0.1646
(1, −1, 1.5)	0.9970	0.9752	0.9825	0.9911	0.9859	0.9868	0.9954	0.9856	0.9820	0.9825	0.9915	0.9864
(−1, 1, −1.5)	0.9939	0.9590	0.9708	0.9848	0.9758	0.9781	0.9910	0.9759	0.9718	0.9709	0.9836	0.9768
(−1, 1, 1.5)	0.7147	0.8321	0.7038	0.6344	0.7734	0.6695	0.7961	0.6384	0.6420	0.7750	0.6403	0.7956
(−1, −1, −1.5)	0.9687	0.9886	0.9553	0.9401	0.9787	0.9477	0.9834	0.9334	0.9389	0.9798	0.9381	0.9839
(−1, −1, 1.5)	0.6000	0.8214	0.8187	0.7223	0.7097	0.7712	0.7814	0.7684	0.7874	0.5424	0.4256	0.6004
	The null distribution is LG(0, 1)											
(1, 0, 0)	0.2042	0.2038	0.1487	0.1489	0.2057	0.1482	0.2044	0.1415	0.1394	0.2148	0.1753	0.2260
(0, 2, 0)	0.7712	0.7708	0.8569	0.8569	0.7707	0.8571	0.8014	0.8460	0.8966	0.2200	0.2718	0.4062
(0, 0, −1.5)	0.7843	0.6795	0.5704	0.6338	0.7275	0.6056	0.7454	0.5651	0.5730	0.7322	0.6839	0.7557
(1, 0, −2)	0.8389	0.7006	0.6452	0.7192	0.7599	0.6886	0.8008	0.6530	0.6413	0.7539	0.7470	0.7773
(1, 0, 0.5)	0.5833	0.5238	0.3957	0.4325	0.5556	0.4137	0.5561	0.3832	0.3949	0.5716	0.5005	0.5923

Table 5 (continued)

(μ, σ, δ)	T_1	T_2	T_3	T_4	AD_1	AD_2	$T^{(1)}_{max}$	$T^{(2)}_{max}$	BOOS	CvM	KS	A–D
(1, −1, 0)	0.6584	0.4303	0.5062	0.6201	0.5451	0.5636	0.6042	0.5535	0.5638	0.4604	0.5061	0.4910
(1, 1, 0)	0.2494	0.3686	0.4028	0.3426	0.3099	0.3731	0.3380	0.3552	0.3979	0.1538	0.1119	0.1802
(0, 1, −1)	0.8549	0.6463	0.7306	0.8189	0.7426	0.7788	0.8179	0.7664	0.7667	0.6636	0.7272	0.6988
(0, 1, 1)	0.6808	0.8143	0.6969	0.6209	0.7458	0.6602	0.7765	0.6327	0.6338	0.7501	0.6206	0.7692
(1, 1, −1.5)	0.9437	0.8020	0.8629	0.9177	0.8639	0.8944	0.9243	0.8855	0.8804	0.8157	0.8708	0.8433
(1, 1, 1.5)	0.9961	0.9983	0.9900	0.9876	0.9972	0.9889	0.9976	0.9834	0.9879	0.9980	0.9901	0.9984
(1, −1, −1.5)	0.0689	0.0850	0.0799	0.0657	0.0787	0.0723	0.0760	0.0724	0.0801	0.0634	0.0451	0.0723
(1, −1, 1.5)	0.9671	0.8647	0.9075	0.9442	0.9155	0.9262	0.9551	0.9202	0.9099	0.8741	0.9228	0.8946
(−1, 1, −1.5)	0.9865	0.9276	0.9459	0.9706	0.9532	0.9601	0.9802	0.9553	0.9503	0.9483	0.9657	0.9575
(−1, 1, 1.5)	0.8493	0.9131	0.8093	0.7690	0.8800	0.7900	0.8919	0.7574	0.7712	0.8887	0.7903	0.9011
(−1, −1, −1.5)	0.8463	0.9090	0.7938	0.7518	0.8798	0.7720	0.8853	0.7361	0.7433	0.8737	0.7550	0.8908
(−1, −1, 1.5)	0.3814	0.4831	0.5642	0.4979	0.4324	0.5316	0.4667	0.5167	0.5842	0.1717	0.1427	0.2229

Table 6 Power of various tests for normal and logistic distributions $(n_1, n_2) = (10, 5)$ with $\alpha = 0.05$

(μ, σ, δ)	T_1	T_2	T_3	T_4	AD_1	AD_2	$T_{max}^{(1)}$	$T_{max}^{(2)}$	BOOS	CvM	KS	A–D
	The null distribution is N(0, 1)											
(1, 0, 0)	0.3044	0.3755	0.2035	0.2333	0.3413	0.2176	0.3403	0.1931	0.2068	0.3489	0.2838	0.3690
(0, 2, 0)	0.6619	0.6618	0.7244	0.7248	0.6625	0.7246	0.6985	0.7188	0.7370	0.2566	0.2364	0.2808
(0, 0, −1.5)	0.6475	0.5285	0.4392	0.4620	0.5879	0.4505	0.6075	0.3984	0.4185	0.5435	0.4883	0.5785
(1, 0, −2)	0.5375	0.4036	0.3649	0.4037	0.4704	0.3844	0.4978	0.3510	0.3628	0.3979	0.3662	0.4310
(1, 0, 0.5)	0.6075	0.6596	0.4212	0.4687	0.6341	0.4439	0.6275	0.3924	0.4394	0.6690	0.5836	0.6874
(1, −1, 0)	0.4739	0.4114	0.2699	0.4951	0.4437	0.3790	0.4189	0.4475	0.5059	0.6239	0.6735	0.5957
(1, 1, 0)	0.3216	0.4232	0.3996	0.3605	0.3733	0.3793	0.4057	0.3631	0.3806	0.2229	0.1713	0.2493
(0, 1, −1)	0.7332	0.5977	0.6280	0.6795	0.6648	0.6542	0.7046	0.6387	0.6430	0.5173	0.5039	0.5483
(0, 1, 1)	0.5825	0.7101	0.5642	0.5264	0.6456	0.5454	0.6741	0.5010	0.5141	0.5820	0.4761	0.6160
(1, 1, −1.5)	0.7992	0.6754	0.7077	0.7542	0.7362	0.7313	0.7740	0.7204	0.7217	0.5887	0.5768	0.6170
(1, 1, 1.5)	0.9869	0.9948	0.9719	0.9687	0.9908	0.9702	0.9927	0.9500	0.9611	0.9880	0.9680	0.9914
(1, −1, −1.5)	0.0699	0.0724	0.0594	0.0990	0.0720	0.0787	0.0623	0.0842	0.0790	0.0939	0.1019	0.0900
(1, −1, 1.5)	0.8060	0.7370	0.5511	0.8395	0.7715	0.6906	0.7599	0.8099	0.8609	0.9222	0.9594	0.9067
(−1, 1, −1.5)	0.9405	0.8775	0.8776	0.9007	0.9082	0.8896	0.9274	0.8780	0.8772	0.8587	0.8465	0.8747
(−1, 1, 1.5)	0.5746	0.6960	0.5258	0.4970	0.6351	0.5114	0.6583	0.4605	0.4785	0.5838	0.4742	0.6184
(−1, −1, −1.5)	0.8405	0.8480	0.7343	0.6498	0.8446	0.6902	0.8323	0.6569	0.7312	0.9083	0.8239	0.9080
(−1, −1, 1.5)	0.1729	0.2608	0.3969	0.2181	0.2178	0.3056	0.1742	0.3800	0.4252	0.3211	0.2483	0.2846
	The null distribution is LG(0, 1)											
(1, 0, 0)	0.1294	0.1697	0.0942	0.1201	0.1508	0.1062	0.1494	0.1027	0.1045	0.1555	0.1301	0.1641
(0, 2, 0)	0.6339	0.6344	0.6995	0.7000	0.6349	0.6997	0.6717	0.6916	0.7124	0.2480	0.2276	0.2721
(0, 0, −1.5)	0.6467	0.5279	0.4388	0.4608	0.5840	0.4504	0.6070	0.3977	0.4175	0.5428	0.4873	0.5778
(1, 0, −2)	0.6969	0.5675	0.5110	0.5481	0.6270	0.5311	0.6594	0.4867	0.4993	0.5680	0.5270	0.6022
(1, 0, 0.5)	0.3291	0.3755	0.2007	0.2614	0.3537	0.2291	0.3459	0.2146	0.2320	0.3948	0.3438	0.4058

Table 6 (continued)

(μ, σ, δ)	T_1	T_2	T_3	T_4	AD_1	AD_2	$T^{(1)}_{max}$	$T^{(2)}_{max}$	BOOS	CvM	KS	A–D
(1, −1, 0)	0.2007	0.1596	0.1434	0.2714	0.1814	0.2054	0.1545	0.2467	0.2701	0.2747	0.3353	0.2510
(1, 1, 0)	0.2649	0.3282	0.3276	0.3026	0.2982	0.3149	0.3241	0.3006	0.3220	0.1591	0.1237	0.1801
(0, 1, −1)	0.7136	0.5752	0.5999	0.6525	0.6410	0.6276	0.6836	0.6076	0.6139	0.5042	0.4939	0.5344
(0, 1, 1)	0.5574	0.6804	0.5247	0.4939	0.6181	0.5094	0.6433	0.4651	0.4803	0.5628	0.4572	0.5949
(1, 1, −1.5)	0.8223	0.7020	0.7237	0.7687	0.7573	0.7482	0.7973	0.7320	0.7336	0.6400	0.6292	0.6666
(1, 1, 1.5)	0.9578	0.9781	0.9155	0.9116	0.9676	0.9134	0.9716	0.8732	0.8977	0.9626	0.9187	0.9703
(1, −1, −1.5)	0.0593	0.0432	0.0589	0.0492	0.0518	0.0537	0.0478	0.0548	0.0507	0.0511	0.0424	0.0545
(1, −1, 1.5)	0.5323	0.4390	0.3157	0.6161	0.4868	0.4585	0.4566	0.5871	0.6586	0.6989	0.8029	0.6650
(−1, 1, −1.5)	0.9148	0.8345	0.8340	0.8642	0.8713	0.8505	0.8981	0.8330	0.8332	0.8099	0.7977	0.8295
(−1, 1, 1.5)	0.6976	0.7945	0.6218	0.6047	0.7448	0.6130	0.7634	0.5551	0.5796	0.7119	0.6029	0.7415
(−1, −1, −1.5)	0.6358	0.6142	0.4745	0.3938	0.6255	0.4308	0.6069	0.3951	0.4585	0.7029	0.5755	0.7082
(−1, −1, 1.5)	0.0988	0.1248	0.2374	0.1997	0.1128	0.2186	0.0700	0.2216	0.2022	0.0734	0.0649	0.0601

Table 7 Power of various tests for G_L and G_R $(n_1, n_2) = (10, 10)$ with $\alpha = 0.05$

(μ, σ, δ)	T_1	T_2	T_3	T_4	AD_1	AD_2	$T^{(1)}_{max}$	$T^{(2)}_{max}$	BOOS	CvM	KS	A–D
	The null distribution is $G_L(0, 1)$											
(1, 0, 0)	0.3673	0.4564	0.3233	0.2702	0.4218	0.2933	0.4163	0.2724	0.2736	0.4112	0.3015	0.4351
(0, 2, 0)	0.8285	0.7089	0.8272	0.8828	0.7621	0.8594	0.8114	0.8524	0.8967	0.3170	0.4227	0.4806
(0, 0, −1.5)	0.7834	0.6786	0.5696	0.6326	0.7218	0.6076	0.7446	0.5642	0.5721	0.7316	0.6834	0.7549
(1, 0, −2)	0.8330	0.6486	0.6574	0.7509	0.7167	0.7192	0.7932	0.6885	0.6807	0.7175	0.7340	0.7414
(1, 0, 0.5)	0.8072	0.8357	0.6803	0.6607	0.8248	0.6694	0.8171	0.6229	0.6535	0.8274	0.7134	0.8463
(1, −1, 0)	0.9069	0.8292	0.7624	0.8157	0.8651	0.7916	0.8851	0.7660	0.7787	0.8848	0.8511	0.8958
(1, 1, 0)	0.2969	0.3902	0.4370	0.4043	0.3547	0.4189	0.3727	0.3989	0.4517	0.1340	0.1040	0.1686
(0, 1, −1)	0.8950	0.7091	0.7678	0.8610	0.7802	0.8278	0.8646	0.8148	0.8142	0.7661	0.8012	0.7931
(0, 1, 1)	0.4865	0.6691	0.5485	0.4536	0.5854	0.4987	0.6155	0.4751	0.4578	0.5391	0.3927	0.5694
(1, 1, −1.5)	0.9571	0.8279	0.8854	0.9417	0.8725	0.9239	0.9414	0.9158	0.9148	0.8545	0.8952	0.8804
(1, 1, 1.5)	0.9959	0.9990	0.9934	0.9906	0.9975	0.9919	0.9984	0.9888	0.9893	0.9976	0.9885	0.9982
(1, −1, −1.5)	0.0620	0.0727	0.0743	0.0732	0.0721	0.0728	0.0695	0.0743	0.0770	0.0598	0.0496	0.0639
(1, −1, 1.5)	0.9998	0.9977	0.9977	0.9988	0.9986	0.9983	0.9997	0.9978	0.9973	0.9990	0.9994	0.9993
(−1, 1, −1.5)	0.9938	0.9625	0.9650	0.9818	0.9729	0.9766	0.9904	0.9709	0.9693	0.9805	0.9837	0.9841
(−1, 1, 1.5)	0.4361	0.5622	0.4121	0.3400	0.5042	0.3745	0.5112	0.3452	0.3394	0.4794	0.3387	0.5096
(−1, −1, −1.5)	0.8527	0.9213	0.8145	0.7679	0.8957	0.7873	0.8985	0.7582	0.7554	0.8760	0.7498	0.8958
(−1, −1, 1.5)	0.7286	0.8480	0.8850	0.8466	0.7917	0.8650	0.8310	0.8581	0.8964	0.3586	0.3066	0.5106
	The null distribution is $G_R(0, 1)$											
(1, 0, 0)	0.4562	0.3664	0.2696	0.3234	0.4213	0.2932	0.4161	0.2721	0.2736	0.4109	0.3624	0.4347
(0, 2, 0)	0.7092	0.8283	0.8826	0.8282	0.7624	0.8599	0.8116	0.8529	0.8971	0.3162	0.2762	0.4797
(0, 0, −1.5)	0.7840	0.6795	0.5709	0.6333	0.7406	0.5988	0.7452	0.5647	0.5729	0.7324	0.6839	0.7557
(1, 0, −2)	0.4555	0.3667	0.2703	0.3238	0.4213	0.2936	0.4162	0.2725	0.2737	0.4111	0.3628	0.4346
(1, 0, 0.5)	0.7844	0.6798	0.5713	0.6339	0.7408	0.5995	0.7454	0.5657	0.5734	0.7325	0.6841	0.7558

Table 7 (continued)

(μ, σ, δ)	T_1	T_2	T_3	T_4	AD_1	AD_2	$T_{max}^{(1)}$	$T_{max}^{(2)}$	BOOS	CvM	KS	A–D
(1, −1, 0)	0.8012	0.5803	0.6870	0.7690	0.7038	0.7245	0.7596	0.7126	0.7119	0.5415	0.6381	0.5868
(1, 1, 0)	0.3030	0.4964	0.4657	0.3492	0.3936	0.4133	0.4431	0.4020	0.4170	0.3497	0.2529	0.3724
(0, 1, −1)	0.8009	0.5800	0.6867	0.7688	0.7034	0.7249	0.7590	0.7123	0.7116	0.5407	0.6377	0.5861
(0, 1, 1)	0.8292	0.9069	0.8157	0.7622	0.8651	0.7914	0.8852	0.7658	0.7787	0.8850	0.7916	0.8959
(1, 1, −1.5)	0.8589	0.6574	0.7502	0.8235	0.7689	0.7842	0.8245	0.7738	0.7674	0.6356	0.7282	0.6768
(1, 1, 1.5)	0.9998	0.9998	0.9984	0.9983	0.9998	0.9984	0.9998	0.9971	0.9983	0.9999	0.9987	0.9999
(1, −1, −1.5)	0.4431	0.3132	0.4283	0.4763	0.3908	0.4494	0.4134	0.4301	0.4776	0.1652	0.2156	0.2030
(1, −1, 1.5)	0.9619	0.8529	0.8993	0.9374	0.9118	0.9172	0.9489	0.9112	0.8996	0.8567	0.9137	0.8797
(−1, 1, −1.5)	0.9881	0.9348	0.9567	0.9752	0.9635	0.9655	0.9832	0.9622	0.9539	0.9398	0.9702	0.9512
(−1, 1, 1.5)	0.9062	0.9507	0.8800	0.8472	0.9266	0.8651	0.9377	0.8411	0.8566	0.9405	0.8731	0.9476
(−1, −1, −1.5)	0.9943	0.9968	0.9841	0.9814	0.9955	0.9828	0.9959	0.9752	0.9821	0.9968	0.9864	0.9975
(−1, −1, 1.5)	0.3535	0.5513	0.5007	0.3808	0.4453	0.4471	0.4973	0.4351	0.4434	0.4210	0.3119	0.4435

Table 8 Power of various tests for G_L and G_R $(n_1, n_2) = (10, 5)$ with $\alpha = 0.05$

(μ, σ, δ)	T_1	T_2	T_3	T_4	AD_1	AD_2	$T^{(1)}_{max}$	$T^{(2)}_{max}$	BOOS	CvM	KS	A–D
	The null distribution is $G_L(0, 1)$											
(1, 0, 0)	0.2789	0.3775	0.2323	0.2244	0.3370	0.2270	0.3403	0.2008	0.2121	0.2963	0.2240	0.3225
(0, 2, 0)	0.6750	0.6173	0.6843	0.7244	0.6461	0.7062	0.6855	0.6975	0.7137	0.2960	0.2915	0.3235
(0, 0, −1.5)	0.6472	0.5278	0.4391	0.4609	0.5801	0.4509	0.6073	0.3980	0.4177	0.5432	0.4876	0.5781
(1, 0, −2)	0.6960	0.5584	0.5339	0.5811	0.6170	0.5614	0.6609	0.5244	0.5324	0.5394	0.5120	0.5729
(1, 0, 0.5)	0.6012	0.6911	0.4657	0.4704	0.6519	0.4671	0.6530	0.4032	0.4392	0.6351	0.5255	0.6661
(1, −1, 0)	0.6217	0.6186	0.4068	0.5377	0.6227	0.4758	0.6076	0.4659	0.5129	0.7197	0.6713	0.7185
(1, 1, 0)	0.2928	0.3487	0.3583	0.3434	0.3288	0.3492	0.3514	0.3381	0.3624	0.1555	0.1278	0.1774
(0, 1, −1)	0.7667	0.6366	0.6419	0.6951	0.6933	0.6725	0.7386	0.6444	0.6490	0.5870	0.5717	0.6199
(0, 1, 1)	0.4077	0.5452	0.4142	0.3731	0.4837	0.3917	0.5046	0.3622	0.3744	0.3997	0.3090	0.4331
(1, 1, −1.5)	0.8513	0.7418	0.7595	0.8052	0.7874	0.7869	0.8301	0.7672	0.7675	0.6790	0.6679	0.7077
(1, 1, 1.5)	0.9537	0.9786	0.9318	0.9239	0.9667	0.9276	0.9720	0.9005	0.9109	0.9561	0.9159	0.9653
(1, −1, −1.5)	0.0745	0.0840	0.0726	0.0809	0.0834	0.0763	0.0829	0.0753	0.0815	0.0644	0.0543	0.0705
(1, −1, 1.5)	0.9589	0.9327	0.8196	0.9450	0.9448	0.8855	0.9433	0.9228	0.9521	0.9897	0.9927	0.9869
(−1, 1, −1.5)	0.9454	0.8880	0.8704	0.8925	0.9115	0.8837	0.9335	0.8578	0.8630	0.8808	0.8636	0.8976
(−1, 1, 1.5)	0.3391	0.4596	0.3044	0.2804	0.4054	0.2915	0.4181	0.2576	0.2713	0.3480	0.2605	0.3810
(−1, −1, −1.5)	0.6316	0.6144	0.4695	0.3846	0.6231	0.4188	0.6021	0.3862	0.4558	0.7091	0.5672	0.7125
(−1, −1, 1.5)	0.2100	0.3051	0.4840	0.3984	0.2648	0.4364	0.1728	0.4750	0.4769	0.1607	0.1358	0.1356
	The null distribution is $G_R(0, 1)$											
(1, 0, 0)	0.2132	0.2471	0.1203	0.1776	0.2304	0.1436	0.2217	0.1426	0.1498	0.2701	0.2438	0.2768
(0, 2, 0)	0.6162	0.6743	0.7237	0.6835	0.6450	0.7058	0.6843	0.6967	0.7128	0.2953	0.2512	0.3226
(0, 0, −1.5)	0.6471	0.5287	0.4390	0.4612	0.5962	0.4480	0.6075	0.3979	0.4179	0.5441	0.4880	0.5788
(1, 0, −2)	0.3774	0.2787	0.2244	0.2321	0.3368	0.2269	0.3400	0.2007	0.2123	0.2966	0.2569	0.3229
(1, 0, 0.5)	0.4423	0.4659	0.2586	0.3492	0.4549	0.2961	0.4389	0.2865	0.3237	0.5348	0.4962	0.5387

Table 8 (continued)

(μ, σ, δ)	T_1	T_2	T_3	T_4	AD_1	AD_2	$T_{max}^{(1)}$	$T_{max}^{(2)}$	BOOS	CvM	KS	A–D
(1, −1, 0)	0.2496	0.1824	0.1953	0.3415	0.2216	0.2564	0.1779	0.3270	0.3663	0.3244	0.4404	0.2936
(1, 1, 0)	0.3118	0.4250	0.3573	0.3163	0.3662	0.3375	0.3958	0.3163	0.3291	0.2719	0.2058	0.2994
(0, 1, −1)	0.6525	0.5117	0.5537	0.6016	0.5913	0.5753	0.6217	0.5665	0.5756	0.4199	0.4054	0.4484
(0, 1, 1)	0.6933	0.7910	0.6257	0.6060	0.7382	0.6156	0.7606	0.5596	0.5856	0.7068	0.6014	0.7359
(1, 1, −1.5)	0.7089	0.5682	0.6044	0.6525	0.6471	0.6261	0.6773	0.6178	0.6242	0.4875	0.4725	0.5155
(1, 1, 1.5)	0.9894	0.9939	0.9617	0.9630	0.9914	0.9621	0.9920	0.9360	0.9577	0.9918	0.9746	0.9940
(1, −1, −1.5)	0.1006	0.0836	0.1604	0.1857	0.0946	0.1717	0.0589	0.1759	0.1569	0.0722	0.1139	0.0611
(1, −1, 1.5)	0.5085	0.4160	0.3030	0.5931	0.4676	0.4229	0.4310	0.5661	0.6387	0.6723	0.7868	0.6388
(−1, 1, −1.5)	0.9140	0.8328	0.8435	0.8715	0.8767	0.8564	0.8960	0.8518	0.8503	0.8000	0.7883	0.8170
(−1, 1, 1.5)	0.7761	0.8539	0.6960	0.6835	0.8119	0.6895	0.8289	0.6289	0.6603	0.7908	0.6917	0.8159
(−1, −1, −1.5)	0.9525	0.9462	0.8731	0.8463	0.9503	0.8595	0.9476	0.8160	0.8638	0.9637	0.9238	0.9685
(−1, −1, 1.5)	0.1616	0.1829	0.2299	0.1150	0.1733	0.1772	0.1551	0.1994	0.2130	0.2583	0.1948	0.2422

Table 9 Rank sums of the twelve tests

	Null distributions			
	N(0, 1)	LG(0, 1)	$G_L(0, 1)$	$G_R(0, 1)$
T_1	285	287	285	285
T_2	235	236	254	233
T_3	173	189	185	178
T_4	227	219	219	211
AD_1	257	256	263	260
AD_2	205	215	211	208
$T^{(1)}_{max}$	306	298	317	295
$T^{(2)}_{max}$	146	157	143	154
BOOS	184	192	183	189
CvM	217	206	203	209
KS	156	150	133	177
A–D	261	247	256	252

6 Illustration

In this section, we illustrate the proposed test with real data sets. We use the Aspartate Aminotransferase (AST) and the γ-glutamyl transpeptidase (GGT) blood concentration data. The data are derived from Ralf et al. (2020, https://doi.org/10.24432/C5D612). Values of blood tests are essential in the study of non-invasive techniques. AST is involved in protein metabolism and serves as an indicator of liver function and heart muscle damage when measured. GGT plays a critical role in antioxidant stress and is an indicator of atherosclerosis, renal disorders, Parkinson's disease, and multidrug resistance in cancer cells. Here, we examine whether AST and GGT in blood levels are comparable between patients with hepatitis and cirrhosis. Figures 3 and 4 give an impression of their distributions. The figures show that our model is adequate here, differences in location or shape may occur.

We compute the p value based on 1,000,000 permutations. As before, we use the standardized form of the Wilcoxon, Ansari–Bradley, Anti-Savage, Savage and Lepage (L) tests to compute the p value of the isolated tests. We present the computational details of AST in Table 10.

From Table 10, we see that, for this example, W, T_1, AD_1, CvM, KS and A–D reject the null hypothesis at the level of 0.05, but the others do not. In more detail, from Fig. 3, the range of AST for hepatitis is similar to that of cirrhosis. In addition, the skewness of AST for hepatitis and cirrhosis are not much different. In fact, the skewness of AST for hepatitis and cirrhosis are 2.146 and 1.371, respectively. Therefore, in this case, it looks similar to the case (1, 0, 0) for $G_R(0, 1)$ in Table 8. From our simulation results, $AD_1, T^{(1)}_{max}$, CvM, KS and A–D are useful statistics for this case.

Additionally, we present the computational details of GGT in Table 11.

Table 11 reveals that W, AS, T_1, AD_1, CvM and KS reject the null hypothesis at the level of 0.1 in this example; the others do not. From Fig. 4, we see that the location

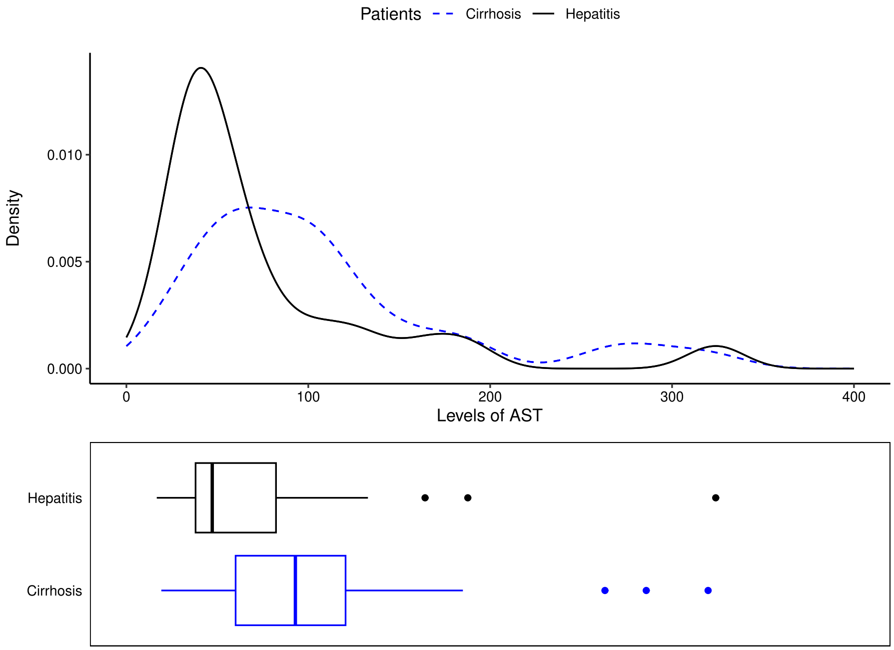


Fig. 3 Density estimation and box plot for AST

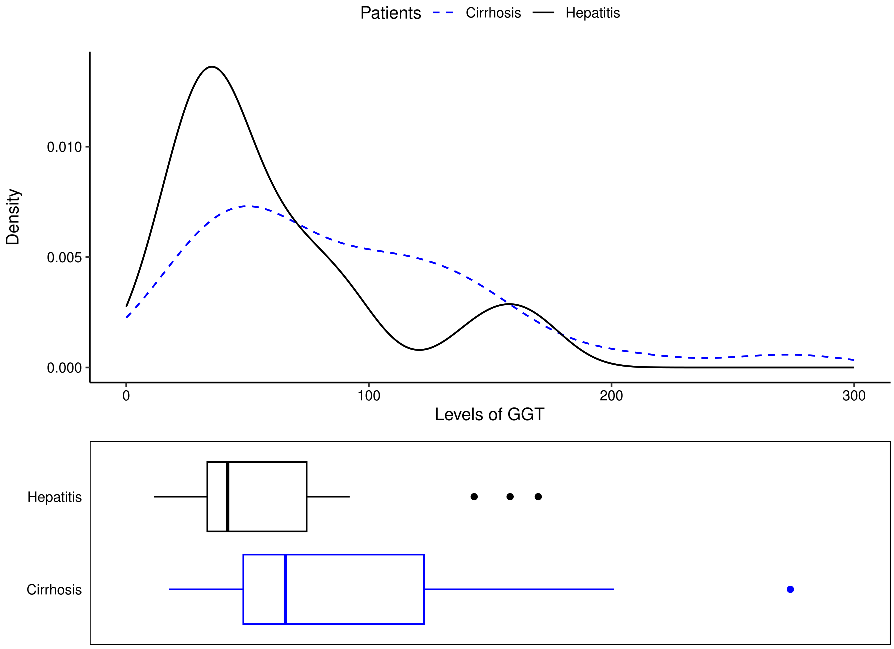


Fig. 4 Density estimation and box plot for GGT

Table 10 AST data

Hepatitis											
33.1	67.0	164.2	187.7	37.8	39.0	45.0	96.2	60.9	31.6	48.4	32.0
53.5	77.6	31.1	39.0	38.1	132.8	324.0	63.2	16.7	38.3	46.0	114.4
Cirrhosis											
60.0	35.6	60.2	263.1	101.9	113.0	19.2	102.0	185.0	66.6	319.8	123.0
80.3	181.8	110.1	65.2	95.4	143.2	54.0	90.4	55.7	36.3	30.4	150.0
285.8	110.3	44.4	99.0	62.0	80.0						
Test Statistic	W	AB	AS	S	L	T_1	T_2	T_3	T_4		
p values	0.0277	0.4120	0.0524	0.1789	0.0622	0.0460	0.0850	0.0770	0.0750		
Test Statistic	AD_1	AD_2	$T^{(1)}_{max}$	$T^{(2)}_{max}$	BOOS	CvM	KS	A–D			
p values	0.0465	0.0770	0.0644	0.1052	0.0546	0.0197	0.0141	0.0279			

Table 11 GGT data

Hepatitis											
18.9	65.0	90.4	40.2	35.9	37.0	43.0	48.1	33.1	34.4	68.2	40.6
57.9	143.4	27.6	158.2	92.1	76.4	392.2	491.0	11.5	24.7	22.3	169.8
Cirrhosis											
99.0	133.4	151.0	61.0	65.6	138.0	105.6	201.0	399.5	28.5	93.7	35.9
17.6	273.7	56.9	28.5	53.6	400.3	107.0	46.8	146.3	112.0	142.5	49.7
101.1	650.9	35.9	64.2	50.0	34.0						
Test Statistic	W	AB	AS	S	L	T_1	T_2	T_3	T_4		
p values	0.0703	0.5864	0.0597	0.1798	0.1675	0.0928	0.1615	0.3114	0.3115		
Test Statistic	AD_1	AD_2	$T_{max}^{(1)}$	$T_{max}^{(2)}$	BOOS	CvM	KS	A–D			
p values	0.0932	0.3114	0.1236	0.4116	0.2803	0.0769	0.0972	0.1101			

and scale of GGT for hepatitis differ from those of cirrhosis. However, the skewness of GGT for hepatitis is similar to that of cirrhosis. In fact, the skewness of GGT for hepatitis and cirrhosis are 2.304 and 2.223, respectively. This case looks similar to the case of $(1, -1, 0)$ for $\mathrm{G_R}(0, 1)$ in Table 8.

Note that the presentation of p values is only for illustration of the considered tests. A clear decision based on all nine tests cannot be drawn. Since we suggested to apply test $\mathrm{T}^{(1)}_{\max}$ the null hypothesis is rejected for the AST data at the 0.1 level, but for the GGT data not.

7 Concluding remarks

For the past 50 years, there have been studies on simultaneously testing the two-sample location and the scale parameters. However, only a few pieces of literature consider simultaneous location, scale, and shape parameter testing. This paper introduced the distribution-free and robust adaptive tests AD_1 and AD_2 to test for the location, scale, or shape simultaneously. The test proposed by Kössler and Mukherjee (2020) is a combination of the standardized Wilcoxon, the standardized Ansari–Bradley and the standardized Anti-Savage tests.

In this paper, we additionally consider test statistics where the Anti-Savage test is replaced with the Savage test. We derive the asymptotic distribution of the proposed tests. In addition, we suggest a selection rule for an adaptive test. Moreover, we considered max-type tests. We investigate the behavior of the power of the tests for small sample sizes via Monte Carlo simulation. In average, the max-type test based on the Euclidean distances is shown to be the best. We also discussed specific data from biomedical experiments.

For future research, we may think of combining the Wilcoxon rank-sum test, the Ansari–Bradley test, the Anti-Savage test and the Savage test as the quad-aspect test statistics, as suggested by a referee. Another research topic may be, for example, a simple new study where the Ansari–Bradley test is replaced with another scale test. More generally, other rank scores for location and scales are also worth considering.

Appendix 1: Proof of Theorem 1

The statistic T_2 can be written as a quadratic form of three independently and identically distributed standard normal variables, say W_i, $i = 1, 2, 3$. Thus $\mathrm{T}_2 = \sum_{i=1}^{3} \lambda_i W_i^2 := Q$ (say). Let $t_{Q,1-\alpha}$ and $t_{Z,1-\alpha}$ be the $1-\alpha$ quantiles; μ_Q and μ_Z be the means; and σ_Q and σ_Z be the standard deviations of the distribution of Q and Z, respectively. We apply the χ^2 approximation proposed by Liu et al. (2009) cf. also Yamaguchi and Murakami (2023). The eigenvalues of $\mathbf{\Sigma}_\mathrm{S}$ are

$$\lambda_1 = 1.9284, \quad \lambda_2 = 1, \quad \lambda_3 = 0.0716.$$

The quadratic form Q is approximated by a χ^2 distribution suitably shifted and scaled. The degrees of freedom and the location and scale parameters are to be determined. Denote

$$c_k = \sum_{i=1}^{3} \lambda_i^k, \quad k = 1, 2, 3, 4, \quad s_1^2 = \frac{c_3^2}{c_2^3}, \quad s_2 = \frac{c_4}{c_2^2}.$$

In our case, we have

$$s_1^2 = 0.6335, \quad \text{and} \quad s_2 = 0.6645, \quad \text{that is,} \quad s_1^2 < s_2.$$

Following Liu et al. (2009), the *df* of Q are given by

$$df = \frac{1}{s_1^2} = 1.5786,$$

and noncentrally parameter by zero. To determine location and scale, we note that

$$\mu_Q = \mathrm{E}[Q] = c_1 = 3, \quad \sigma_Q = \sqrt{\mathrm{V}[Q]} = \sqrt{2c_2} = 3.0737.$$

Let

$$\mu_Z = \mathrm{E}[Z] = df = 1.5786, \quad \sigma_Z = \sqrt{\mathrm{V}[Z]} = \sqrt{2df} = 1.7769$$

be expectation and variance of Z, where $Z \sim \chi^2_{df}$ and $df = 1.5786$. For approximation of the quadratic form Q, we shall have

$$\begin{aligned}
\frac{Q - \mu_Q}{\sigma_Q} &\approx \frac{Z - \mu_Z}{\sigma_Z} \\
Q &\approx \frac{\sigma_Q}{\sigma_Z} Z - \frac{\sigma_Q}{\sigma_Z} \mu_Z + \mu_Q \\
&= \frac{\sqrt{2c_2}}{\sqrt{2df}} Z - \frac{\sqrt{2c_2}}{\sqrt{2df}} \cdot df + c_1 \\
&= \sqrt{c_2} s_1 Z - \frac{\sqrt{c_2}}{s_1} + c_1 \\
&= \frac{c_3}{c_2} Z - \frac{c_2^2}{c_3} + c_1 \\
&\approx 1.7299Z + 0.2692.
\end{aligned}$$

Then, the proof is completed. □

Appendix 2: Proof of Theorem 2

This assertion follows immediately from Lemma 1 and from the decomposition

$$\mathrm{T}_4 = \boldsymbol{T}_{\mathrm{M}_2}\boldsymbol{\Sigma}_{\mathrm{M}_2}^{-1}\boldsymbol{T}'_{\mathrm{M}_2} = \boldsymbol{T}_{\mathrm{M}_2}\boldsymbol{\Sigma}_{\mathrm{M}_2}^{-1/2}\boldsymbol{\Sigma}_{\mathrm{M}_2}^{-1/2}\boldsymbol{T}'_{\mathrm{M}_2} = \boldsymbol{Y}'\boldsymbol{Y},$$

where $\boldsymbol{Y} \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{I})$. □

Appendix 3: Proof of Theorem 3

Recall that $\mathrm{T}_2 = \boldsymbol{T}_{\mathrm{M}_2}\boldsymbol{T}'_{\mathrm{M}_2}$. Then, we obtain the following lemma by replacing μ_j, $j = 1, 2, 3$ in Lemma 4.2 of Kössler and Mukherjee (2020) with $\boldsymbol{\gamma}$ given in Lemma 2.

Lemma 2 *Under the sequence* $\boldsymbol{\Theta}'_N = \bar{\boldsymbol{\Theta}}'/\sqrt{N} = (\bar{\mu}, \bar{\sigma}, \bar{\delta})/\sqrt{N}$ *and with* $n_1/N \to \lambda \in (0, 1)$ *as* $\min(n_1, n_2) \to \infty$ *the limiting distribution of* $\boldsymbol{T}'_{\mathrm{M}_2}$ *is* $\mathrm{N}(\boldsymbol{\gamma}', \boldsymbol{\Sigma}_{\mathrm{S}})$, *with the asymptotic expectation* $\boldsymbol{\gamma} = (\gamma_{\mathrm{W}}, \gamma_{\mathrm{AB}}, \gamma_{\mathrm{S}}) = (\gamma_{\mathrm{W}}(f), \gamma_{\mathrm{AB}}(f), \gamma_{\mathrm{S}}(f))$ *with*

$$\gamma_{\mathrm{W}} = \gamma_{\mathrm{W}}(f) = -\sqrt{\lambda(1-\lambda)} \cdot \frac{\bar{\mu} d_{\mathrm{W,Location}} + \bar{\sigma} d_{\mathrm{W,\ Scale}} + \bar{\delta} d_{\mathrm{W,\ Lehmann}}}{\sqrt{I_W}},$$

$$\gamma_{\mathrm{AB}} = \gamma_{\mathrm{AB}}(f) = -\sqrt{\lambda(1-\lambda)} \cdot \frac{\bar{\mu} d_{\mathrm{AB,\ Location}} + \bar{\sigma} d_{\mathrm{AB,\ Scale}} + \bar{\delta} d_{\mathrm{AB,\ Lehmann}}}{\sqrt{I_{AB}}},$$

$$\gamma_{\mathrm{S}} = \gamma_{\mathrm{S}}(f) = -\sqrt{\lambda(1-\lambda)} \cdot \frac{\bar{\mu} d_{\mathrm{S,\ Location}} + \bar{\sigma} d_{\mathrm{S,\ Scale}} + \bar{\delta} d_{\mathrm{S,\ Lehmann}}}{\sqrt{I_S}}.$$

Obviously, the asymptotic expectation can be written as

$$\boldsymbol{\gamma} = -\sqrt{\lambda(1-\lambda)}\boldsymbol{M}_{\mathrm{S}}(f)\bar{\boldsymbol{\Theta}}.$$

Recall λ_i are the eigenvalues of $\boldsymbol{\Sigma}_{\mathrm{S}}$ in (5). Let $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, and $\boldsymbol{U}$ be the matrix of eigenvectors of $\boldsymbol{\Sigma}_{\mathrm{S}}$. With transformation

$$\boldsymbol{W} = \boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{U}'\boldsymbol{T}'_{\mathrm{M}_2},$$

we convert the vector $\boldsymbol{T}'_{\mathrm{M}_2}$ to $\boldsymbol{W} \sim \mathrm{N}(\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{U}'\boldsymbol{\gamma}', \boldsymbol{I})$. Therefore, we have for the expectation

$$\boldsymbol{\Delta} := \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} = \mathrm{E}[\boldsymbol{W}] = \boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{U}'\boldsymbol{\gamma}' = -\sqrt{\lambda(1-\lambda)}\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{U}'\boldsymbol{M}_{\mathrm{S}}(f)\bar{\boldsymbol{\Theta}}$$
$$= -\sqrt{\lambda(1-\lambda)} \begin{pmatrix} 0.4750 & -0.1835 & 0.5092 \\ 0.3603 & 0.9328 & 0 \\ -2.4650 & 0.9522 & 2.6425 \end{pmatrix} \boldsymbol{M}_{\mathrm{S}}(f) \begin{pmatrix} \bar{\mu} \\ \bar{\sigma} \\ \bar{\delta} \end{pmatrix}.$$

Let us introduce

$$c_k(\boldsymbol{\Delta}) = \sum_{i=1}^{3} \lambda_i^k + k \sum_{i=1}^{3} \lambda_i^k \Delta_i^2, \quad s_1^2(\boldsymbol{\Delta}) = \frac{c_3^2(\boldsymbol{\Delta})}{c_2^3(\boldsymbol{\Delta})}, \quad s_2(\boldsymbol{\Delta}) = \frac{c_4(\boldsymbol{\Delta})}{c_2^2(\boldsymbol{\Delta})}.$$

In addition, define that

if $\quad s_1^2(\boldsymbol{\Delta}) \le s_2(\boldsymbol{\Delta}),$

$$df(\boldsymbol{\Delta}) = \frac{1}{s_1^2(\boldsymbol{\Delta})},$$
$$\beta_0(\boldsymbol{\Delta}) = c_1(\boldsymbol{\Delta}) - \frac{c_2^2(\boldsymbol{\Delta})}{c_3(\boldsymbol{\Delta})},$$
$$\beta_1(\boldsymbol{\Delta}) = \frac{c_3(\boldsymbol{\Delta})}{c_2(\boldsymbol{\Delta})},$$

if $\quad s_1^2(\boldsymbol{\Delta}) > s_2(\boldsymbol{\Delta}),$

$$df'(\boldsymbol{\Delta}) = a^2(\boldsymbol{\Delta}) - 2nc(\boldsymbol{\Delta}),$$
$$a(\boldsymbol{\Delta}) = \frac{1}{s_1(\boldsymbol{\Delta}) - \sqrt{s_1^2(\boldsymbol{\Delta}) - s_2(\boldsymbol{\Delta})}},$$
$$nc(\boldsymbol{\Delta}) = s_1^2(\boldsymbol{\Delta}) a^3(\boldsymbol{\Delta}) - a^2(\boldsymbol{\Delta}),$$
$$\beta_0'(\boldsymbol{\Delta}) = c_1(\boldsymbol{\Delta}) - \{df'(\boldsymbol{\Delta}) + nc(\boldsymbol{\Delta})\} \frac{\sqrt{c_2(\boldsymbol{\Delta})}}{a(\boldsymbol{\Delta})},$$
$$\beta_1'(\boldsymbol{\Delta}) = \frac{\sqrt{c_2(\boldsymbol{\Delta})}}{a(\boldsymbol{\Delta})}.$$

□

Appendix 4: Proof of Theorem 4

Recall that $\boldsymbol{T}_{\mathrm{M}_2} = \left(\frac{\mathrm{W}-\mathrm{E}[\mathrm{W}]}{\sqrt{\mathrm{V}[\mathrm{W}]}}, \frac{\mathrm{AB}-\mathrm{E}[\mathrm{AB}]}{\sqrt{\mathrm{V}[\mathrm{AB}]}}, \frac{\mathrm{S}-\mathrm{E}[\mathrm{S}]}{\sqrt{\mathrm{V}[\mathrm{S}]}} \right)$ is the vector of the three components of T_4, $\boldsymbol{\eta} = \mathrm{E}[\boldsymbol{T}_{\mathrm{M}_2}]$ is its expectation vector, and $\boldsymbol{\Sigma}_{\mathrm{M}_2}$ the correlation matrix of this vector. Recall that

$$\mathrm{T}_4 = \boldsymbol{T}_{\mathrm{M}_2} \boldsymbol{\Sigma}_{\mathrm{M}_2}^{-1} \boldsymbol{T}_{\mathrm{M}_2}'.$$

This statistic is, under the alternative $\boldsymbol{\Theta}$, asymptotically χ^2 distributed with three degrees of freedom and noncentrality parameter

$$nc(f) := \boldsymbol{\eta} \boldsymbol{\Sigma}_{\mathrm{S}}^{-1} \boldsymbol{\eta}' = \lambda(1-\lambda) \boldsymbol{\Theta}' \boldsymbol{M}_{\mathrm{S}}(f)' \boldsymbol{\Sigma}_{\mathrm{S}}^{-1}(f) \boldsymbol{M}_{\mathrm{S}}(f) \boldsymbol{\Theta},$$

which can be seen from the decomposition

$$\mathrm{T}_4 = \boldsymbol{T}_{\mathrm{M}_2}\boldsymbol{\Sigma}_{\mathrm{M}_2}^{-1}\boldsymbol{T}'_{\mathrm{M}_2} = \boldsymbol{T}_{\mathrm{M}_2}\boldsymbol{\Sigma}_{\mathrm{M}_2}^{-1/2}\boldsymbol{\Sigma}_{\mathrm{M}_2}^{-1/2}\boldsymbol{T}'_{\mathrm{M}_2} = \boldsymbol{Y}'\boldsymbol{Y},$$

where $\boldsymbol{Y} = \boldsymbol{\Sigma}_{\mathrm{M}_2}^{-1/2}\boldsymbol{T}'_{\mathrm{M}_2}$ and $\boldsymbol{Y} \sim \mathrm{N}(\boldsymbol{\Sigma}_{\mathrm{M}_2}^{-1/2}\boldsymbol{\eta}', \boldsymbol{I})$. □

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