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ORIGINAL PAPER

On the functional basis of isotropic vector and tensor functions by Shariff (2023)

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Abstract

In the paper Shariff (*Q. J. Mech. Appl. Math.* 76:143–161, 2023) a functional basis of a system of vectors and symmetric tensors is proposed. The functional basis is expressed in terms of eigenvalues and eigenvectors of the first tensor and includes a smaller number of terms in comparison to the classical irreducible representation (see, e.g., Boehler, *J. Appl. Math. Mech.* 57:323–327, 1977; Pennisi and Trovato, *Int. J. Eng. Sci.* 25:1059–1065, 1987). In the present contribution, we show that elements of the functional basis by Shariff (*Q. J. Mech. Appl. Math.* 76:143–161, 2023) do not represent isotropic invariants of the vector and tensor arguments and cannot thus be referred to as the functional basis. To this end, a counterexample with two symmetric tensors is considered. Under an arbitrary orthogonal transformation the functional basis (Shariff, *Q. J. Mech. Appl. Math.* 76:143–161, 2023) of these two tensors should remain constant but it does change in contrast to the classical representation.

Keywords Functional basis · Symmetric tensors · Isotropic invariants · Eigenvectors · Eigenvalues

A functional basis of a vector and tensor system is a set of their isotropic invariants such that every invariant can uniquely be expressed in terms the basis. Thus, by the very definition elements of the functional basis should represent isotropic invariants of the given system of vectors and tensors, which is not the case for the functional basis by Shariff [4].

To prove this statement we consider a simple special case of two symmetric tensors $\mathbf{A}_1$ and $\mathbf{A}_2$. Any scalar-valued isotropic function (invariant) of these tensors should satisfy the condition

$$f\left(\mathbf{Q}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_2\mathbf{Q}^{\mathrm{T}}\right) = f\left(\mathbf{A}_1, \mathbf{A}_2\right), \quad \forall \mathbf{Q} \in Orth^3, \tag{1}$$

where $Orth^3$ denotes a group of all orthogonal transformations within the three dimensional Euclidean space. According to the classical invariant theory a functional basis of these tenors can be given by ten following invariants (see, e.g., [1, 2])

$$\mathrm{tr}\mathbf{A}_1,\ \mathrm{tr}\mathbf{A}_1^2,\ \mathrm{tr}\mathbf{A}_1^3,\ \mathrm{tr}\mathbf{A}_2,\ \mathrm{tr}\mathbf{A}_2^2,\ \mathrm{tr}\mathbf{A}_2^3,\ \mathrm{tr}\left(\mathbf{A}_1\mathbf{A}_2\right),\ \mathrm{tr}\left(\mathbf{A}_1^2\mathbf{A}_2\right),\ \mathrm{tr}\left(\mathbf{A}_1\mathbf{A}_2^2\right),\ \mathrm{tr}\left(\mathbf{A}_1^2\mathbf{A}_2^2\right), \tag{2}$$

where tr• denotes the trace of a tensor. In contrast, according to Shariff [4] the functional basis of this tensor system can be represented by a smaller number (nine) of terms as follows

$$\lambda_i, A_{ij}^{(2)}, \quad i = 1, 2, 3,\ j = 1, \dots i, \tag{3}$$

where λ_i denote eigenvalues of $\mathbf{A}_1$ with the spectral decomposition $\mathbf{A}_1 = \sum_{i=1}^{3} \lambda_i \boldsymbol{v}_i \otimes \boldsymbol{v}_i$, while $A_{ij}^{(2)}$, $(i, j = 1, 2, 3)$ represent components of $\mathbf{A}_2$ with respect to the basis formed by the unit eigenvectors $\boldsymbol{v}_i$ of $\mathbf{A}_1$. Accordingly, $\mathbf{A}_2 = \sum_{i,j=1}^{3} A_{ij}^{(2)} \boldsymbol{v}_i \otimes \boldsymbol{v}_j$.

In the following, we show that the terms $A_{ij}^{(2)}$, $i = 1, 2, 3$, $j = 1, \dots i$ in (3) do not represent isotropic invariants according to (1). To this end, we consider a counterexample with

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$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \tag{4}$$

where $\boldsymbol{e}_i$ form an orthonormal basis such that $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$, $(i, j = 1, 2, 3)$, where δ_{ij} denotes the Kronecker symbol. Accordingly, the vectors $\boldsymbol{v}_i$ can be set as

$$\boldsymbol{v}_i = \boldsymbol{e}_i, \quad i = 1, 2, 3 \tag{5}$$

so that the basis (3) becomes

$$\lambda_1 = \lambda_2 = \lambda_3 = 1, \quad A_{11}^{(2)} = 1, A_{21}^{(2)} = 2, A_{22}^{(2)} = 2, A_{31}^{(2)} = 3, A_{32}^{(2)} = 4, A_{33}^{(2)} = 3. \tag{6}$$

Consider now an orthogonal transformation by an (orthogonal) tensor

$$\mathbf{Q} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \tag{7}$$

in which $\mathbf{A}_1' = \mathbf{Q}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}} = \mathbf{A}_1$ but

$$\begin{aligned} \mathbf{A}_2' = \mathbf{Q}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ &= \begin{bmatrix} 2 & -2 & 4 \\ -2 & 1 & -3 \\ 4 & -3 & 3 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \end{aligned} \tag{8}$$

Due to the fact that $\mathbf{A}_1' = \mathbf{A}_1$ the eigenvectors (5) remain unchanged so that

$$\begin{aligned} A_{11}^{\prime(2)} = 2 \neq A_{11}^{(2)}, \quad A_{21}^{\prime(2)} = -2 \neq A_{21}^{(2)}, \quad & A_{22}^{\prime(2)} = 1 \neq A_{22}^{(2)}, \\ & A_{31}^{\prime(2)} = 4 \neq A_{31}^{(2)}, \quad A_{32}^{\prime(2)} = -3 \neq A_{32}^{(2)} \end{aligned} \tag{9}$$

in view of (6). These five terms do change under the orthogonal transformation by (7) and thus do not represent isotropic invariants of two symmetric tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ (4) according to (1). On the contrary, one can easily check that the invariants (2) do remain constant under this transformation and satisfy (1). Thus, the terms proposed in [4] do not represent a functional basis and cannot also be used in a functional basis of vector and tensor valued functions.

The above statement can also be argued by means of the following thought experiment. Consider two students which independent of each other are asked to calculate invariants according (3). They get the following sets of tensors

1st student: $\mathbf{A}_1$ and $\mathbf{A}_2$ (4),

2nd student: $\mathbf{A}_1$ and $\mathbf{A}_2'$ (8)

and do not know about each other. Most probably, they both will use the given orthonormal basis $\boldsymbol{e}_i$, $(i = 1, 2, 3)$, with respect to which the components of $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_2'$ are defined. Thus, they will set $\boldsymbol{v}_i = \boldsymbol{e}_i$, $i = 1, 2, 3$ and get different sets of the elements in the factional basis (3). However, in this case the invariants should be the same as predicted by (2) according to the classical invariant theory. Alternatively, the students can use arbitrary orthonormal bases since every of them represents a set of eigenvectors of the identity tensor $\mathbf{A}_1$. However, in this case the probability that the bases chosen by student 1 and 2 are accidentally related to each other by the orthogonal transformation (7) and their invariants (3) will thus coincide is zero.

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Data Availability There are no external data supporting the findings of this study.

Declarations

Conflict of interest The author declares no competing interests.

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