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ORIGINAL ARTICLE

Inference for Type-I and Type-II Hybrid Censored Minimal Repair and Record Data

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Abstract
In this paper, hybrid censoring mechanisms are applied to minimal repair and record data. Based on the derivation of the joint distribution of such data under hybrid censoring, likelihood inference is discussed. For illustration, Type-I and Type-II hybrid censoring schemes are considered for exponential distributions. In particular, the exact (conditional) distribution of the maximum likelihood estimator is obtained for an exponential distribution. This result is used to construct exact (conditional) confidence intervals using the method of pivoting the cumulative distribution function. Finally, the results are illustrated using two data sets taken from the literature on minimal repair models. Although the discussion of the results is in terms of minimal repair models, the results can be applied directly to record value data. By utilizing a connection of minimal repair times to occurrence times of non-homogeneous Poisson processes, a nonparametric estimate for the intensity rate of the process and the underlying lifetime distribution under hybrid censoring is also proposed. The paper is supplemented by simulational results.

Keywords Minimal repair data · Record values · Hybrid censoring · Maximum likelihood estimation · Non-homogeneous Poisson process · Stochastic monotonicity · Exponential distribution

1 Introduction

The notion of minimal repair has been considered in the literature as a concept where a repairable item is repaired with minimal effort so that the item continues to function at the same level of wear and tear as before the repair. Repairs are usually assumed to be performed without time loss, since the repair time is usually supposed relatively short compared to the expected operating time of the item. A simple example of such

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a situation is the replacement of a broken V-belt in a car engine since the age of the engine can be considered the same after the repair. Minimal repair models have been discussed in various ways in the literature and many extensions have been provided. For instance, Barlow and Hunter [13] considered minimal repair in connection with maintenance strategies. In Ascher [6], the model has been discussed in terms of a bad-as-old concept. Ascher [4] commented on models for reliability of repaired equipment whereas Ascher and Feingold [5] focused on repairable systems reliability. Further results, extensions, and connections are presented in Block et al. [17] and Gupta and Kirmani [39]. Aven and Jensen [7] discussed a general minimal repair model in terms of point processes. A survey of results and models can be found in Tadj et al. [51].

Gupta and Kirmani [39] established a connection of the standard minimal repair model to record values and non-homogeneous Poisson processes (NHPP) (see also Pfeifer [50], Cramer [23]). This connection is quite useful in the statistical analysis of such data since it points out that record values from a common cumulative distribution function (cdf) F, minimal repair times with lifetime cdf F, as well as jump times of a NHPP with cumulative intensity rate $\Lambda = -\log \overline{F}$ can be treated identically in a probabilistic sense. In this respect, the results presented in the following can also be applied to the hybrid censored data that are generated by these models.

In this paper, we combine minimal repair models (or record data) with hybrid censoring strategies. A recent survey of hybrid censoring models and respective inferential results has been provided by Balakrishnan et al. [10] who mainly focus on models based on (progressively Type-II censored) order statistics (see also Balakrishnan and Kundu [12], Balakrishnan and Cramer [8]). These models have been entitled as *non-replacement* cases of truncated life tests. In addition to the above-mentioned reviews, the following works, among others, are of importance for the case of an exponential distribution mainly dealt with in this article: Chen and Bhattacharyya [21], Gupta and Kundu [40], Childs et al. [22], Chandrasekar et al. [20], Balakrishnan and Iliopoulos [11], Cramer and Balakrishnan[25].

However, early publications like Epstein [34] considered also (Type-I) hybrid censored in a *replacement* scenario. This means that as soon as the item fails, it will be replaced with an identical item of the same virtual age. Clearly, this can be interpreted as a minimal repair. This model has also been addressed in Ebrahimi [31] (two-parameter exponential distributed lifetimes), Fairbanks et al. [35] (confidence intervals), Draper and Guttman [29] (Bayesian inference), and Ebrahimi [32] (prediction), but does not appear to have received further attention in the literature.

In this paper, we will take this idea and combine it with the censoring mechanisms of hybrid censoring. In particular, we will use the structural approach proposed in Cramer [24] to present a general approach to the statistical analysis of such data under hybrid censoring. The approach is based on ordered data $X_{(1)} \leq \cdots \leq X_{(m)}$ which is subject to hybrid censoring. In the following, these data will be generated from the minimal repairs in a sample of exactly n items (for details see eq. 2). We will illustrate the power of the approach using the basic models of Type-I and Type-II hybrid censoring.

Thus, given minimal repair data $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq \cdots$, a Type-I hybrid censored sample with a desired number r of measurements and time threshold τ is obtained by terminating the life test at

$$W_{\mathsf{I}} = \min\{\tau, X_{(r)}\} = \tau \wedge X_{(r)}.$$

The corresponding sample is given the measurements $X_{(1)} \leq \cdots \leq X_{(M_{\mathsf{I}})}$ with random sample size $M_{\mathsf{I}} = \sum_{j=1}^{r} \mathbb{1}_{(-\infty,\tau]}(X_{(j)})$. Here, $\mathbb{1}_A : \mathbb{R} \to \{0, 1\}$ denotes the indicator function, that is, $\mathbb{1}_A(x) = 1 \iff x \in A$. Clearly, the resulting data is bounded by the threshold τ. The sample size M_{I} is at most r but may be zero.

For Type-II hybrid censoring, the test duration is given by

$$W_{\mathsf{II}} = \max\{\tau, X_{(r)}\} = \tau \vee X_{(r)}$$

which ensures a minimum sample size of r. In particular, we get the measurements $X_{(1)} \leq \cdots \leq X_{(M_{\mathsf{II}})}$ with random sample size $M_{\mathsf{II}} = r + \sum_{j=r+1}^{\infty} \mathbb{1}_{(-\infty,\tau]}(X_{(j)})$ However, both the observed sample size M_{II} and the test duration W_{II} are unbounded. Note that, in non-replacement models as discussed in detail in Balakrishnan et al. [10], the sample size is always bounded.

In Sect. 2, we introduce the data and the model in detail. We derive the joint distribution of the ordered repair time data and of the random number of measurements assuming a general life time distribution with cdf F. For subsequent use, we present also the respective results for exponentially distributed lifetimes. In particular, we establish the respective results for both Type-I and Type-II hybrid censoring. Sect. 3 addresses likelihood inference particularly in the exponential case for such data as well as nonparametric estimation of the intensity rate when the data is understood as occurrence times of a non-homogeneous Poisson process [see Sect. (2.2)]. After establishing the maximum likelihood estimators for the mean θ of an exponential distribution in terms of the total time on test, we derive their exact (conditional) cdf using the modularization approach in the form presented in Balakrishnan et al. [10] and Cramer [24]. Using the method of pivoting the cdf (see Balakrishnan et al. [9]) and illustrating its applicability, we show that it leads to exact (conditional) confidence intervals. In Sect. 4, the results are illustrated by two data sets taken from Whitaker and Samaniego [54] and Kumar and Klefsjö [44], respectively. For comparison, we provide also confidence intervals due to Fairbanks et al. [35] under Type-I hybrid censoring. The discussion is supplemented by some simulations illustrating the concepts and results. Further directions of research are sketched in Sect. 5 (conclusions and outlook).

Finally, we would like to emphasize that, although we consider the data to be repair times, it is possible to interpret them as record values, too (see also Gupta and Kirmani [39] and comments given above). This is not important for the derivation of the presented statistical and probabilistic results. It is all a matter of interpretation.

2 Models and Distributional Results

Throughout, we assume that the initial life times $X_1, \ldots, X_n$ are random variables (r.v.s) having an absolutely continuous cdf F with continuous probability density function (pdf) f. If exponential lifetimes are assumed, that is, $X_1 \sim \mathsf{Exp}(\theta)$, the corresponding pdf is given by $f(x) = \theta^{-1}e^{-x/\theta}$, $x \geq 0$. Thus, $\theta > 0$ denotes the mean lifetime of the items under study.

In the following, we study different minimal repair models with both a maximum or minimum number of r observations as well as a time threshold τ. The design of the hybrid censored life test is based on the time threshold τ which may be interpreted as either a maximum test duration or a desired test duration. In order to avoid trivialities, we assume that τ is included in the support of F, that is, $0 < F(\tau) < 1$. In this regard, the models can be considered as a (Type-I/Type-II) hybrid censoring of minimal repair times or as Type-I/Type-II hybrid censored data with replacement.

As mentioned above the considered models are designed by performing Type-I and Type-II censoring on a sequence of minimal repair times. Therefore, we introduce first the underlying basic minimal repair model (see Model 2.1).

Model 2.1 (*Minimal repair model in a sample of n items*) Consider a life test with n items whose lifetimes are monitored. If the first failure occurs at the failure time $X_{(n,1)}$, a minimal repair will be performed and the monitoring process will be continued. The failure times of different items are supposed independent.

This procedure will be repeated unless the experiment is terminated and, thus, results in an increasing sequence of failure/repair times

$$X_{(n,1)} \leq X_{(n,2)} \leq X_{(n,3)} \leq \cdots , \tag{2.1}$$

that is, $\left(X_{(n,k)}\right)_{k\in\mathbb{N}}$.

In order to analyse the above data, one has to discuss the distribution of the observed failure/repair times $\left(X_{(n,k)}\right)_{k\in\mathbb{N}}$. Clearly, $X_{(n,1)} = \min\{X_1, \dots, X_n\}$. But now, the failed component is repaired which affects the distribution of its lifetimes so that the resulting random variables are no longer identically distributed after the first repair. However, the independence is preserved. Therefore, we will discuss first the joint distribution of the first k repair times $X_{(n,1)}, \dots, X_{(n,k)}$ generated according to Model 2.1. To obtain a formal construction of the minimal repair model, we introduce the standard minimal repair model (SMR) and illustrate its relation to a NHPP. For brevity, we introduce the notation $\boldsymbol{t}_m = (t_1, \dots, t_m) \in \mathbb{R}^m$. The standard minimal repair model corresponds to the case $n = 1$ of Model 2.1, that is, we consider a life test with a single object whose lifetime is monitored. If a failure occurs then it will be instantly repaired in the sense of minimal repair so that always one object is under study. The corresponding sequence of failure/repair times is denoted by $X_{(1,1)} \leq X_{(1,2)} \leq X_{(1,3)} \leq \cdots$ or by $\left(X_{(1,k)}\right)_{k\in\mathbb{N}}$, respectively.

Block et al. [17] studied the standard minimal repair model under more general assumptions and provided a formal construction. Furthermore, Gupta and Kirmani [39] identified the failure/repair times of the standard minimal repair model with the record values of a sequence of $i.i.d.\ r.v.s$ with common cdf F. In order to establish this result, they considered the counting process generated by the corresponding failure/repair times, which turns out to be a NHPP. The corresponding counting process $(N(t))_{t\geq 0}$ is defined by

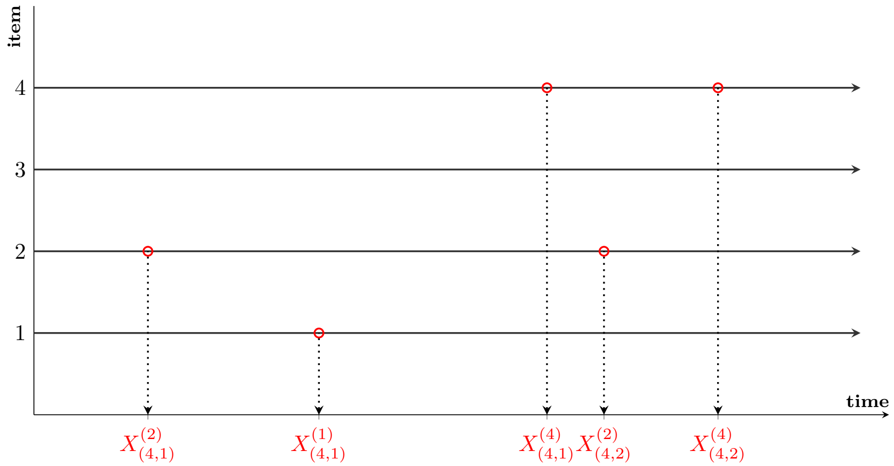

Fig. 1 Illustration of generating of a minimal repair sample starting with $n = 4$ items

$$N(t) = \sum_{j=1}^{\infty} \mathbb{1}_{[0,t]}(X_{(1,j)}), \quad t \geq 0. \tag{2.2}$$

Hence, the extensively studied topic of record values (see, for example, Arnold et al. [3], Nevzorov [49], David and Nagaraja [28]) can be applied to minimal repair data. In particular, the failure times $\{X_{(1,n)} \mid n \in \mathbb{N}\}$ from the SMR Model and the (upper) records in a sequence of i.i.d. r.v.s $(Y_n)_{n\in\mathbb{N}}$ are identically distributed. This follows immediately from Theorem 1 in Gupta and Kirmani s[39, Section 2]. Thus, we have the following proposition.

Proposition 2.1 *The repair times $(X_{(1,n)})_{n\in\mathbb{N}}$ in the SMR model and the (upper) record values $(Y_{L_n})_{n\in\mathbb{N}}$ in a sequence of i.i.d. r.v.s $(Y_n)_{n\in\mathbb{N}}$ with $Y_1 \sim F$ are identically distributed. $(L_n)_{n\in\mathbb{N}}$ denotes the sequence of record times.*

For fixed $k \in \mathbb{N}$, the joint pdf $f_{(1,\ldots,k)}$ of the r.v.s . $X_{(1,1)}, \ldots, X_{(1,k)}$ is given by

$$f_{(1,\ldots,k)}(\boldsymbol{t}_k) = \begin{cases} f(t_k) \prod_{i=1}^{k-1} \frac{f(t_i)}{\overline{F}(t_i)}, & t_1 < \cdots < t_k \\ 0, & \textit{otherwise} \end{cases}, \tag{2.3}$$

where $\overline{F}$ denotes the survival function of F.

We will see that a slight modification of this approach can be utilized to gain a similar identification for the Type-I/Type-II hybrid censored minimal repair models.

In a next step, we consider the minimal repair model 2.1. Note that if an object fails then a minimal repair is conducted, so that immediately after the failure again n items are under test. For illustration, the scenario including the repair times is depicted in Fig. 1 for $n = 4$ items on test.

The sequence of failures illustrated in Fig. 1 is given by

$$\begin{aligned} &\text{item 1:}\ \ X^{(1)}_{(4,1)} \\ &\text{item 2:}\ \ X^{(2)}_{(4,1)} \quad X^{(2)}_{(4,2)} \\ &\text{item 3:}\ \ --- \\ &\text{item 4:}\ \ X^{(4)}_{(4,1)} \quad X^{(4)}_{(4,2)} \end{aligned} \tag{2.4}$$

leading to the ordered sample $X^{(2)}_{(4,1)} \leq X^{(1)}_{(4,1)} \leq X^{(4)}_{(4,1)} \leq X^{(2)}_{(4,2)} \leq X^{(4)}_{(4,2)}$. Hence, the resulting data (ignoring the knowledge about the item that has failed) is given by $X_{(4,1)} \leq X_{(4,2)} \leq X_{(4,3)} \leq X_{(4,4)} \leq X_{(4,5)}$.

Now, we are interested in the distribution of the failure/repair times given in (2.1). Therefore we consider the above construction for an arbitrary $n \in \mathbb{N}$. Clearly, the failure/repair times $X^{(i)}_{(n,1)} \leq X^{(i)}_{(n,2)} \leq X^{(i)}_{(n,3)} \leq \dots$ for object $i \in \{1, \dots, n\}$ can be interpreted as jump times of a NHPP $(N_i(t))_{t\geq 0}$. Furthermore, these NHPPs are independent by assumption. Therefore, the sequence of failure times $X_{(n,1)} \leq X_{(n,2)} \leq X_{(n,3)} \leq X_{(n,4)} \leq X_{(n,5)} \leq \dots$ can be seen as the jump times of the process resulting from the superposition of the n independent NHPPs $(N_i(t))_{t\geq 0}$, $1 \leq i \leq n$. It is well-known that the superposition of independent NHPPs with the same intensity rates $\lambda(\cdot)$ form a NHPP with intensity rate $n\lambda(\cdot)$ (see, e.g., Last and Penrose [45]). This means that the baseline distribution can be seen as the distribution of the minimum of the initial lifetimes, that is, $F_{1:n} = 1 - (1 - F)^n$ leading to the pdf $f_{1:n} = nf(1 - F)^{n-1}$. Thus, we directly get the following result from Proposition 2.1.

Corollary 2.2 *For fixed $k \in \mathbb{N}$, the r.v.s $X_{(n,1)} \leq X_{(n,2)} \leq X_{(n,3)} \leq \cdots \leq X_{(n,k)}$ generated from Model 2.1 have the joint pdf $f_{(1,\dots,k;n)}$ given by*

$$f_{(1,\dots,k;n)}(\boldsymbol{t}_k) = \begin{cases} n^k f(t_k)\overline{F}^{n-1}(t_k) \prod_{i=1}^{k-1} \frac{f(t_i)}{\overline{F}(t_i)}, & t_1 < \cdots < t_k \\ 0, & \textit{otherwise} \end{cases}. \tag{2.5}$$

Note that (2.5) equals the joint pdf of so-called n-record values (see, e.g., Dziubdziela and Kopociński [30], Kamps [43, Section 1.5], Arnold et al. [3]).

Remark 2.3 1. Corollary 2.2 shows that the situation can also be interpreted as follows. Instead of considering an (initial) sample of size n of i.i.d. r.v.s and performing minimal repairs to each item upon failure, one may consider a series system with n i.i.d. components and conduct a minimal repair to the series system. The resulting samples in both scenarios have the same distribution.

2. Notice that minimal repair times can be considered as particular sequential order statistics. Therefore, Type-I hybrid censored minimal repair times can be seen as a particular case of Type-I censored sequential order statistics which have been discussed in Burkschat et al. [18].
3. In the preceding construction, the information about the failed (and repaired) item has been ignored. If this information is available then we can handle the situation as n independent samples with the same baseline distribution. Note that the stopping rule for the observation process can now be set in two ways.

(a) The stopping is performed per sub-sample, that is, we require for each sub-sample a specific hybrid censoring scheme. For instance, we can consider Type-I hybrid censoring with a threshold τ_i and a maximum number m_i for the i-th sub-sample. Of course, one can choose $\tau_i = \tau$ or $m_i = m$ if appropriate. This construction leads to n independent samples of hybrid censored minimal repair times subject to possibly different hybrid censoring schemes. In particular, the sample sizes $M_{\text{I}i}$ for each sub-sample are independent. Note that, technically, these hybrid censoring schemes can be different for each sub-sample. In case of progressively censored data, such multi-sample situations have been considered in Górny and Cramer [38] (Type-I hybrid) and Jansen et al. [42] (Type-II hybrid) which have led to explicit but messy results. It would be interesting to see whether such results can also be established for hybrid censored minimal repair data.

(b) A hybrid censoring scheme is applied jointly to all sub-samples. In case of Type-I hybrid censoring, a threshold τ and a total number of observations m is fixed in advance leading to sample sizes $M_{\text{I}i}$ for each sub-sample. However, the $M_{\text{I}i}$'s are dependent now since they depend on all sub-samples! This situation can also be analysed in terms of a marked NHPP (resulting from the superposition as above) which is subject to hybrid censoring. The marks are given by the sub-sample number. For information on marked Poisson processes, see, e.g., Møller and Waagepetersen [48], Last and Penrose [45].

It will be interesting to analyse these models which will be subject of future research.

For exponential distributions, we find the following result (see also Ahsanullah [1], Arnold et al. [3]).

Corollary 2.4 (Exponential distribution) *Assuming exponentially distributed lifetimes, the joint pdf in Corollary 2.2 for fixed $k \in \mathbb{N}$ reads*

$$f_{(1,\ldots,k;n)}(\boldsymbol{t}_k) = \left(\frac{n}{\theta}\right)^k e^{-nt_k/\theta}, \quad t_1 < \cdots < t_k.$$

From Proposition 2.1, we find the marginal cdf of $X_{(n,i)}$ (see Arnold et al.[3, p.10]).

Theorem 2.5 *The cdf of $X_{(n,i)}$, for $i \in \mathbb{N}$, in Model 2.1 with common cdf F is given by*

$$F_{(n,i)}(t) = P(X_{(n,i)} \le t) = 1 - \overline{F}^n(t) \sum_{j=0}^{i-1} \frac{n^j}{j!}[-\ln \overline{F}(t)]^j, \quad t \in \mathbb{R}. \tag{2.6}$$

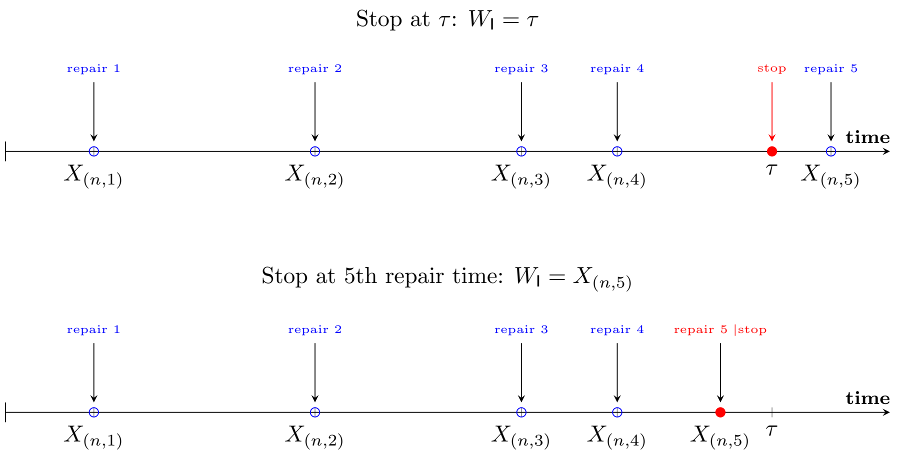

Fig. 2 Possible data scenarios for the minimal repair model with time censoring at $\tau > 0$ and maximum number $r = 5$ of repairs (with different choices of $X_{(n,5)}$)

2.1 Type-I Hybrid Censored Data

For a given sequence of increasingly ordered r.v.s $(Z_k)_{k\in\mathbb{N}}$ as well as parameters $r \in \mathbb{N}$ and $\tau > 0$, the corresponding Type-I hybrid censored sample $(Z_k^{(I)})_k$ is defined by

$$Z_1 \leq \cdots \leq Z_{M\wedge r} \quad \text{where } M = \sum_{i=1}^{\infty} \mathbb{1}_{(-\infty,\tau]}(Z_i). \tag{2.7}$$

This procedure is applied to minimal repair data generated from Model 2.1, that is, the life test is terminated when either the r-th repair occurs or the pre-determined maximum test duration $\tau > 0$ expires. This situation is depicted in Fig. 2 for $r > 4$ and some $\tau > 0$. The experimental time of such a life testing experiment is given by

$$W_{\mathsf{I}} = \min\{X_{(n,r)}, \tau\} = X_{(n,r)} \wedge \tau.$$

Note that this is obtained from Model 2.1 by bounding the test duration by τ. Denoting the number of repairs until τ by the random counter

$$M = \sum_{i=1}^{\infty} \mathbb{1}_{(-\infty,\tau]}(X_{(n,i)}), \tag{2.8}$$

it is clear that M maybe zero (generally with positive probability). To be more precise, this happens with probability

$$P(M = 0) = P(\tau < X_{(n,1)}) = P(\tau < X_{1:n}) = \overline{F}^n(\tau),$$

which is positive since $0 < F(\tau) < 1$ has been assumed. Therefore, the experimenter is faced with the problem that the life test may terminate without observing a repair. As a result, inference will be of conditional nature since we have to ensure to observe at least one failure or repair. Thus, we will assume subsequently that $M > 0$. In the Type-I hybrid censoring case, the sample size is given by

$$M_{\mathsf{I}} = M \wedge r,$$

that is, the maximum number of observed failures is bounded by r. Notice that the same problem is present for Type-I censoring and Type-I hybrid censoring without replacement, respectively. The probability mass function (pmf) of M results from the equivalence

$$M \geq m \iff X_{(n,m)} \leq \tau$$

and the marginal cdf of $X_{(n,m)}$ which is given in Theorem 2.5.

Lemma 2.6 *The pmf of the r.v.* $M_I = M \wedge r = \sum_{i=1}^{r} \mathbb{1}_{(-\infty,\tau]}(X_{(n,i)})$ *is given by*

$$P(M_I = m) = \begin{cases} \overline{F}^n(\tau), & m = 0 \\ \frac{n^m}{m!}\overline{F}^n(\tau)[-\log \overline{F}(\tau)]^m, & m \in \{1, \ldots, r-1\} \\ F_{(n,r)}(\tau), & m = r \end{cases}.$$

Note that $\{M > 0\} = \{M_I > 0\}$. *Further, the expectation of* M_I *is given by*

$$EM_I = rF_{(n,r)}(\tau) + [-\ln(\overline{F}^n(\tau)]\overline{F}_{(n,r-1)}(\tau) \xrightarrow{\tau\to\infty} r.$$

Proof The pmf follows directly from Lemma 2.5. The expectation of M_{I} is obtained by writing

$$\begin{aligned} EM_{\mathsf{I}} &= \sum_{i=1}^{r} P(X_{(n,i)} \leq \tau) = r - \overline{F}^n(\tau) \sum_{i=1}^{r}\sum_{j=0}^{i-1} \frac{[-\ln \overline{F}^n(\tau)]^j}{j!} \\ &= r - \overline{F}^n(\tau) \sum_{j=0}^{r-1}\sum_{i=j+1}^{r} \frac{[-\ln \overline{F}^n(\tau)]^j}{j!} \\ &= rF_{(n,r)}(\tau) + [-\ln(\overline{F}^n(\tau)]\overline{F}_{(n,r-1)}(\tau). \end{aligned}$$

□

In the next step, we present an expression for the joint cdf of the Type-I hybrid censored minimal repair times (see Fig. 3). It can be easily deduced from the general results presented in Cramer [24] (see also Burkschat et al. [18]) and Lemma 2.6 provided that

$$P(X_{(n,i)} \leq t_i, 1 \leq i \leq m, M = m),\ 1 \leq m \leq r, \quad P(X_{(n,i)} \leq t_i, 1 \leq i \leq r, M \geq r)$$

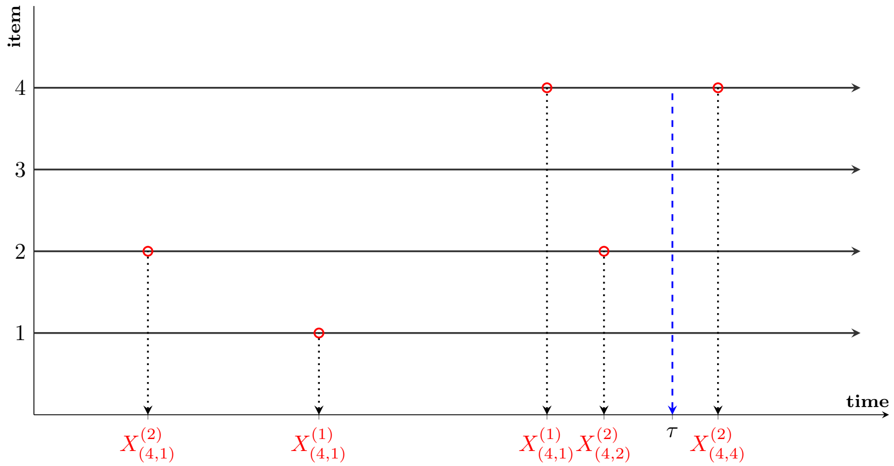

Fig. 3 Illustration of generating of a minimal repair sample starting with $n = 4$ items under Type-I hybrid censoring

are available. However, proceeding by analogy with the case of order statistics discussed in Cramer and Balakrishnan [25], we find similar expressions so that we omit the technical details. In particular, for $m \in \{1, \dots, r\}$, one has to replace the joint cdf of the first m order statistics by that one of the first m minimal repair times, that is, by

$$F_{(1,\dots,m;n)}(\boldsymbol{t}_m) = \int_{-\infty}^{t_1} \dots \int_{x_{m-1}}^{t_m} f_{(1,\dots,m;n)}(\boldsymbol{x}_m)\, dx_m \dots dx_1, \quad \boldsymbol{t}_m \in \mathbb{R}^m, \quad (2.9)$$

with $f_{(1,\dots,m;n)}$ given in (2.5). Furthermore, one has to take into account that the following identity is true for $1 \le m \le r-1$:

$$F_{(1,\dots,m+1;n)}(\boldsymbol{t}_m \wedge \tau, \tau) = F_{(1,\dots,m;n)}(\boldsymbol{t}_m \wedge \tau) - n^m \overline{F}^n(\tau) H_{(1,\dots,m)}(\boldsymbol{t}_m \wedge \tau) \quad (2.10)$$

where

$$H_{(1,\dots,m)}(\boldsymbol{v}_m) = \int_{-\infty}^{v_1} \dots \int_{x_{m-1}}^{v_m} \prod_{i=1}^{m} \frac{f(x_i)}{\overline{F}(x_i)}\, dx_m \dots dx_1, \quad \boldsymbol{v}_m \in \mathbb{R}^m. \quad (2.11)$$

This follows from (2.9) by evaluating the most inner integral. Thus, we find the following expressions for the cdf and pdf, respectively.

Theorem 2.7 *Let $(X_{(n,i)})_{i \in \mathbb{N}}$ be the minimal repair times generated from Model 2.1. Then, the joint cdf $F^{(I),M_I>0}_{(1,\dots,r;n)}$ of the Type-I hybrid censored repair times $X_{(n,1)}, \dots, X_{(n,M_I)}$ is given by*

$$F_{(1,\ldots,r;n)}^{(I),M_I>0}(\boldsymbol{t}_r) = P(X_{(n,i)} \le t_i, 1 \le i \le M_I)$$
$$= \overline{F}^n(\tau) \sum_{m=1}^{r-1} n^m H_{(1,\ldots,m)}(\boldsymbol{t}_m \wedge \tau) \prod_{j=m+1}^{r} \mathbb{1}_{[\tau,\infty)}(t_j) + F_{(1,\ldots,r;n)}(\boldsymbol{t}_r \wedge \tau).$$

where $F_{(1,\ldots,k;n)}$ denotes the joint cdf of $X_{(n,1)},\ldots,X_{(n,k)}$, $k \in \mathbb{N}$, and $H_{(1,\ldots,m)}$ is given in (2.11).

For $m \in \{1,\ldots,r\}$, the pdf $f^{X_{(n,1)},\ldots,X_{(n,M_I)},M_I}$ of the Type-I hybrid censored repair times $X_{(n,1)},\ldots,X_{(n,M_I)}$ and the sample size M_I is given by

$$f^{X_{(n,1)},\ldots,X_{(n,M_I)},M_I}(\boldsymbol{t}_m, m) = \begin{cases} n^m \overline{F}^n(\tau) \prod_{i=1}^m \frac{f(t_i)}{\overline{F}(t_i)}, & t_1 < \cdots < t_m \le \tau, 0 < m < r \\ f_{(1,\ldots,r;n)}(\boldsymbol{t}_r), & t_1 < \cdots < t_r \le \tau, m = r \end{cases},$$

where $f_{(1,\ldots,r;n)}$ denotes the joint pdf of $X_{(n,1)},\ldots,X_{(n,r)}$, $r \in \mathbb{N}$.

Introducing the test duration $w = t_r \wedge \tau$, the pdf can be written as

$$f^{X_{(n,1)},\ldots,X_{(n,M_I)},M_I}(\boldsymbol{t}_m, m) = n^m \overline{F}^n(w) \prod_{i=1}^{m} \frac{f(t_i)}{\overline{F}(t_i)}, \quad t_1 \le \cdots \le t_m \le \tau, \quad (2.12)$$

so that the case-by-case definition is not needed. For exponential distributions, the situation simplifies considerably.

Corollary 2.8 (Exponential distribution) *Assuming exponentially distributed lifetimes, the joint pdf in Theorem 2.7 reads*

$$f^{X_{(n,1)},\ldots,X_{(n,M_I)},M_I}(\boldsymbol{t}_m, m) = \begin{cases} \left(\frac{n}{\theta}\right)^m e^{-n\tau/\theta}, & t_1 < \cdots < t_m \le \tau, 0 < m < r \\ \left(\frac{n}{\theta}\right)^r e^{-nt_r/\theta}, & t_1 < \cdots < t_r \le \tau, m = r \end{cases}.$$

It follows from (2.12) that the joint pdf depends only on the test duration $w = t_r \wedge \tau$ so that we can write

$$f^{X_{(n,1)},\ldots,X_{(n,M_I)},M_I}(\boldsymbol{t}_m, m) = \left(\frac{n}{\theta}\right)^m e^{-\mathsf{TTT}_r/\theta}. \quad (2.13)$$

Note that $\mathsf{TTT}_r = nw = n(t_r \wedge \tau)$ denotes the total time on test of the life test. This observation will be very useful considering likelihood inference.

2.2 Type-II Hybrid Censored Data

For a given sequence of increasingly ordered r.v.s $(Z_k)_{k\in\mathbb{N}}$ as well as parameters $r \in \mathbb{N}$ and $\tau > 0$, the Type-II hybrid censored sample is defined by

$$Z_1 \le \cdots \le Z_{M\vee r} \quad \text{where } M = \sum_{i=1}^{\infty} \mathbb{1}_{(-\infty,\tau]}(Z_i).$$

with M as in (2.7). Thus, $M_{\text{II}} = M \vee r$ denotes the random sample size of the Type-II hybrid censored sample. Clearly, we have $M_{\text{II}} \geq r$ so that at least r measurements are available. Moreover, since $(Z_k)_k$ is a sequence of r.v.s , it may be possible that M is infinite. In order to have a proper model, one may assume that

$$P(M < \infty) = 1.$$

Since, for any $\ell \in \mathbb{N}$,

$$P(M = \infty) = P(Z_i \leq \tau, i \in \mathbb{N}) \leq P(Z_\ell \leq \tau),$$

this holds, when $P(Z_\ell \leq \tau) \xrightarrow{\ell \to \infty} 0$.

As above, we consider minimal repairs in a sample of n items. The life test is terminated when both the r-th repair has occurred and the pre-determined maximum test duration $\tau > 0$ is expired.

Representing the repair times by $X_{(n,1)}, X_{(n,2)}, X_{(n,3)}, \ldots$ as in Fig. 3, the experimental time of the life test is given by

$$W_{\text{II}} = \max\{X_{(n,r)}, \tau\}.$$

Therefore, the test duration is bounded from below by τ which can be considered as a minimal test duration in this setting. The number of observed failures is given by $M_{\text{II}} = M \vee r$ with M as in (2.8). By similarity with Lemma 2.6, we get the pmf of M_{II}. Note that we can write

$$M_{\text{II}} = r + \sum_{i=r+1}^{\infty} \mathbb{1}_{(-\infty,\tau]}(X_{(n,i)}).$$

Lemma 2.9 *The pmf of the r.v.* $M_{II} = r + \sum_{i=r+1}^{\infty} \mathbb{1}_{(-\infty,\tau]}(X_{(n,i)})$ *is given by*

$$P(M_{II} = m) = \begin{cases} 0, & m < r \\ 1 - F_{(n,r+1)}(\tau), & m = r \\ \frac{n^m}{m!} \overline{F}^n(\tau)[-\log \overline{F}(\tau)]^m, & m > r \end{cases}.$$

Furthermore, we have $P(M_{II} < \infty) = 1$. *The expectation of* M_{II} *is given by*

$$EM_{II} = r\overline{F}_{(n,r+1)}(\tau) + [-\ln(\overline{F}^n(\tau)]F_{(n,r)}(\tau) \xrightarrow{\tau \to \infty} \infty.$$

Proof The representation of the pmf follows directly from the definition of M_{II} and the pmf of M. Since $0 < F(\tau) < 1$ has been assumed throughout this paper, we get using the marginal distribution of the ℓ-th minimal repair time (or record value) (see Theorem 2.5)

$$P(X_{(n,\ell)} \leq \tau) = 1 - \overline{F}^n(\tau) \sum_{i=0}^{\ell-1} \frac{[-n \log \overline{F}(\tau)]^i}{i!} \xrightarrow{\ell\to\infty} 1 - \overline{F}^n(\tau) \sum_{i=0}^{\infty} \frac{[\log \overline{F}^{-n}(\tau)]^i}{i!} = 0.$$

As mentioned above, this implies $P(M_{\text{II}} < \infty) = 1$.

The expectation of M_{II} is calculated by analogy with that of M_{I} in Lemma 2.6. It should be noted that it converges to ∞ when $\tau \to \infty$ since this implies $F(\tau) \to 1$. □

Since M_{II} is unbounded, we present in Theorem 2.10 the joint cdf of the data and the counter M (which is more convenient than M_{II} in that case). Notice that

$$\{M_{\text{II}} = k\} = \{M = k\}, k > r, \quad \{M_{\text{II}} \leq r\} = \{M < r\} \cup \{M = r\},$$

where the cases $\{M < r\}, \{M = r\}$ can be distinguished based on the observed data, that is, the sample $(X_{(n,i)})_{1 \leq i \leq M_{\text{II}}}, M_{\text{II}}$ can be constructed from the sample $(X_{(n,i)})_{1 \leq i \leq M \vee r}, M$ and vice versa.

Theorem 2.10 *Let $(X_{(n,i)})_{i \in \mathbb{N}}$ be the minimal repair times in a sample of size n. Then, for $m \geq r$, the cdf $F^{(II),M}_{(1,\ldots,r;n)}$ of the Type-II hybrid censored repair times $(X_{(n,i)})_{1 \leq i \leq M \vee r}$ and M is given by*

$$\begin{aligned} F^{(II),M}_{(1,\ldots,r;n)}(\boldsymbol{t}_m, m) &= P(X_{(n,i)} \leq t_i, 1 \leq i \leq m \vee r, M = m) \\ &= n^m \overline{F}^n(\tau) H_{(1,\ldots,m)}(\boldsymbol{t}_m \wedge \tau) \prod_{j=r+1}^{m} \mathbb{1}_{[\tau,\infty)}(t_j), \end{aligned}$$

where $F_{(1,\ldots,m;n)}$ denotes the joint cdf of $X_{(n,1)}, \ldots, X_{(n,m)}$, $m \in \mathbb{N}$, and $H_{(1,\ldots,m)}$ is defined in (2.11). Furthermore,

$$\begin{aligned} F^{(II),M<r}_{(1,\ldots,r;n)}(\boldsymbol{t}_r, r) &= P(X_{(n,i)} \leq t_i, 1 \leq i \leq r, M < r) \\ &= F_{(1,\ldots,r;n)}(\boldsymbol{t}_r) - F_{(1,\ldots,r;n)}(\boldsymbol{t}_r \wedge \tau). \end{aligned}$$

Proof Let $M = m \geq r$. Then, $P(X_{(n,i)} \leq t_i, 1 \leq i \leq m, M = m)$ results immediately from Theorem 2.7. In the case $M < r$, we obtain that

$$\begin{aligned} &P(X_{(n,i)} \leq t_i, 1 \leq i \leq r, M < r) = P(X_{(n,i)} \leq t_i, 1 \leq i \leq r, M \leq r - 1) \\ &= P(X_{(n,i)} \leq t_i, 1 \leq i \leq r, X_{(n,r)} > \tau) \\ &= F_{(1,\ldots,r;n)}(\boldsymbol{t}_r) - F_{(1,\ldots,r;n)}(\boldsymbol{t}_r \wedge \tau). \end{aligned}$$

Thus, we arrive at the given expression for the joint cdf. □

Since F is supposed absolutely continuous with pdf f, we get the following pdf from Theorem 2.10.

Corollary 2.11 *Let $(X_{(n,i)})_{i\in\mathbb{N}}$ be the minimal repair times in a sample of size n. Then, for $m \in \{r, r+1, \dots\}$, the pdf $f^{X_{(n,1)},\dots,X_{(n,M_{II})},M_{II}}$ of the Type-II hybrid censored repair times $X_{(n,1)},\dots,X_{(n,M_{II})}$ and the sample size M_{II} is given by*

$$f^{X_{(n,1)},\dots,X_{(n,M_{II})},M_{II}}(\boldsymbol{t}_m, m) = n^m \overline{F}^n(\tau) \prod_{i=1}^{m} \frac{f(t_i)}{\overline{F}(t_i)}, \quad t_1 < \cdots < t_m \leq \tau,$$

where $f_{(1,\dots,k;n)}$ denotes the joint pdf of $X_{(n,1)},\dots,X_{(n,k)}$, $k \in \mathbb{N}$. Furthermore,

$$f^{X_{(n,1)},\dots,X_{(n,r)},M_{II}}(\boldsymbol{t}_r, r) = f_{(1,\dots,r;n)}(\boldsymbol{t}_r), \quad t_1 < \cdots < t_{r-1} < t_r, \tau < t_r.$$

By analogy with (2.12), we can write the pdf in terms of the test duration $w = t_r \vee \tau$ as

$$f^{X_{(n,1)},\dots,X_{(n,M_{II})},M_{II}}(\boldsymbol{t}_m, m) = n^m \overline{F}^n(w) \prod_{i=1}^{m} \frac{f(t_i)}{\overline{F}(t_i)}, \quad t_1 \leq \cdots \leq t_m. \tag{2.14}$$

The expression in (2.14) simplifies considerably in case of exponentially distributed life times. In particular, the pdf depends only on the data via the total time on test $\mathsf{TTT}_r = nw = n(t_r \vee \tau)$.

Corollary 2.12 (Exponential distribution) *Assuming exponentially distributed lifetimes, the joint pdf in Corollary 2.11 reads*

$$f^{X_{(n,1)},\dots,X_{(n,M_{II})},M_{II}}(\boldsymbol{t}_m, m) = \left(\frac{n}{\theta}\right)^m e^{-\mathsf{TTT}_r/\theta}, \quad t_1 < \cdots < t_m,$$

Remark 2.13 Clearly, the preceding discussion can also be applied to other hybrid censoring schemes following the ideas presented in Cramer [24]. Further details will be provided in future research.

3 Likelihood Inference

In this section, we assume that the baseline cdf $F_{\boldsymbol{\theta}}$ depends on some unknown parameter $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^p$. First, we consider likelihood inference for Type-I and Type-II hybrid censored data. From (2.12) and (2.14), we find that the likelihood function is given by

$$L(\boldsymbol{\theta}; x_1, \dots, x_m, m) = n^m \overline{F}_{\boldsymbol{\theta}}^n(w) \prod_{i=1}^{m} \frac{f_{\boldsymbol{\theta}}(x_i)}{\overline{F}_{\boldsymbol{\theta}}(x_i)} \tag{3.1}$$

for realizations $x_1, \dots, x_m, m$ of $X_{(n,1)}, \dots, X_{(n,M_{\mathsf{HCS}})}, M_{\mathsf{HCS}}$, where $M_{\mathsf{HCS}} \in \{M_{\mathsf{I}}, M_{\mathsf{II}}\}$ and w denotes the observed test duration (that is, the realization of $W_{\mathsf{HCS}} \in \{W_{\mathsf{I}}, W_{\mathsf{II}}\}$ as mentioned above).

Remark 3.1 It follows from the results presented in Cramer [24] that the likelihood function under any hybrid censoring scheme has the form given in (3.1) with appropriately chosen values of m (observed sample size) and w (test duration) for the hybrid censoring scheme under consideration.

3.1 MLEs for Exponentially Distributed Lifetime Distributions

Clearly, the expression in (3.1) has the same mathematical form as the likelihood function for the first n record values so that the computation of the maximum likelihood estimates uses the same methods and leads to the same expressions (see general comments in Cramer [24]).

For illustration, we discuss the case of exponential distributions. In particular, the MLE for exponentially distributed lifetimes with mean θ is given by

$$\widehat{\theta} = \frac{1}{M_{\mathsf{HCS}}} \mathsf{TTT}_{M_{\mathsf{HCS}}} = \begin{cases} \frac{n}{M_{\mathsf{I}}}(X_{(M_{\mathsf{I}})} \wedge \tau), & \text{Type-I hybrid censoring} \\ \frac{n}{M_{\mathsf{II}}}(X_{(M_{\mathsf{II}})} \vee \tau), & \text{Type-II hybrid censoring} \end{cases} \tag{3.2}$$

provided it exists. Under Type-II hybrid censoring, this is always true. In case of Type-I hybrid censored data, one has to assume that $M_{\mathsf{I}} > 0$.

Remark 3.2 According to the comments provided in Cramer [24], results on Bayesian inference can also be easily established.

3.2 Nonparametric Estimates for the Intensity Rate and the Cumulative Distribution Function

Since the model can also be considered in terms of superposed NHPPs, one can consider estimation of the intensity rate $\lambda(\cdot)$. Using the parametric approach for exponentially distributed lifetimes, we get

$$\widehat{\lambda} = 1/\widehat{\theta} \quad \text{and} \quad \widehat{F}(t) = 1 - e^{-\widehat{\lambda} t}, t \geq 0, \tag{3.3}$$

as parametric estimates for the intensity and the underlying distribution function, respectively.

First note that inferential results for a NHPP under hybrid censoring are not available so far. However, utilizing the previous results we can construct a nonparametric estimator restricting the observation window of the test to the interval $[0, W_{\mathsf{HCS}})$. We consider the superposed NHPP.

For illustration, we restrict the discussion to Type-I hybrid censoring. Other hybrid censoring schemes, can be handled similar by suitable adaptions. A more detailed analysis will be subject of future research. Let $w_{\mathsf{I}} = x_m \wedge \tau$ and m_{I} be the realizations of W_{I} and M_{I}, respectively. Then, a piecewise-linear Nelson-Aalen type estimator of the cumulative intensity function $\Lambda = -\ln(1 - F)$ under Type-I hybrid censoring is constructed by

1. Type-I case, that is, $w_{\mathsf{I}} = \tau \leq x_m$ and $m_{\mathsf{I}} = \sum_{i=1}^{\infty} \mathbb{1}_{(-\infty,\tau]}(x_i)(< m)$:

$$\widehat{\Lambda}_n(t) = \frac{m_{\mathsf{I}}}{m_{\mathsf{I}}+1} \begin{cases} \frac{i}{n} + \frac{t-x_i}{n(x_{i+1}-x_i)}, & x_i < t \leq x_{i+1}, i = 0, \ldots, m_{\mathsf{I}}-1 \\ \frac{m_{\mathsf{I}}}{n} + \frac{t-x_{m_{\mathsf{I}}}}{n(\tau-x_{m_{\mathsf{I}}})}, & x_{m_{\mathsf{I}}} < t \leq \tau \end{cases}, \quad (3.4)$$

   where $x_0 = 0$, and, as above, and $x_1, \ldots, x_m$ are realizations of the ordered occurrence times $X_{(n,1)}, \ldots, X_{(n,M_{\mathsf{I}})}$. As mentioned in Leemis [46], the factor $m_{\mathsf{I}}/(m_{\mathsf{I}}+1)$ takes into account that stopping at the threshold τ leads to $m_{\mathsf{I}}+1$ gaps.
2. Type-II case, that is, $w_{\mathsf{I}} = x_m < \tau$ and $m_{\mathsf{I}} = m$:

$$\widehat{\Lambda}_n(t) = \frac{i}{n} + \frac{t-x_i}{n(x_{i+1}-x_i)}, \quad x_i < t \leq x_{i+1}, i = 0, \ldots, m_{\mathsf{I}}-1. \quad (3.5)$$

The corresponding estimate of the cumulative distribution function is given by $\widehat{F}_n^*(t) = 1 - e^{-\widehat{\Lambda}_n(t)}$, $t \geq 0$. Note that a Type-I censored version of a NHPP has also been discussed in Henderson [41] (see also Lewis and Shedler [47], Leemis [46]). Furthermore, the intensity rate $\lambda(\cdot)$ of the NHPP is estimated by

(i) Type-I case:

$$\widehat{\lambda}_n(t) = \begin{cases} \frac{m_{\mathsf{I}}}{(m_{\mathsf{I}}+1)n(x_{i+1}-x_i)}, & x_i < t \leq x_{i+1}, i = 0, \ldots, m_{\mathsf{I}}-1 \\ \frac{m_{\mathsf{I}}}{(m_{\mathsf{I}}+1)n(\tau-x_{m_{\mathsf{I}}})}, & x_{m_{\mathsf{I}}} < t \leq \tau \end{cases}. \quad (3.6)$$

(ii) Type-II case:

$$\widehat{\lambda}_n(t) = \frac{1}{n(x_{i+1}-x_i)}, \quad x_i < t \leq x_{i+1}, i = 0, \ldots, m_{\mathsf{I}}-1. \quad (3.7)$$

Applications of these estimates to Type-I hybrid censored data are presented in Sect. 4.

3.3 Distribution of the MLE for Exponentially Distributed Lifetimes

As pointed out in (3.2), the MLE of θ can be written in terms of the total time on test statistic. Thus, we get a quite similar expression as for hybrid censored order statistics' data. Moreover, the distribution of the MLE can be found using the modularization approach (see, e.g., Górny and Cramer [37], Balakrishnan et al. [10], Cramer [24]). In case of Type-I hybrid censoring, the conditional cdf can be taken directly from Burkschat et al. [18]. However, since the derivation is quite forward, we present a direct proof which also yields a simpler representation of the conditional cdf. For convenience, we denote by $g_{m,\theta}$ and $G_{m,\theta}$ the pdf and cdf of a gamma distribution with scale parameter $\theta > 0$ and shape parameter $m \in \mathbb{N}$, that is, $g_{m,\theta}$ is defined by

$$g_{m,\theta}(t) = \frac{1}{(m-1)!\theta^m} t^{m-1} e^{-t/\theta}, \quad t \geq 0.$$

Theorem 3.3 *For $x \geq 0$, the conditional cdf $G^{I}_{r,\theta}(\cdot \mid M \geq 1)$ of the MLE under Type-I hybrid censoring is given by,*

$$P_\theta(\widehat{\theta} \leq x \mid M \geq 1) = G^{I}_{r,\theta}(x \mid M \geq 1)$$
$$= \frac{1}{1 - e^{-n\tau/\theta}} \begin{cases} G_{r,\theta}(rx), & rx < n\tau \\ G_{k+1,\theta}(n\tau), & kx < n\tau \leq (k+1)x, 0 \leq k < r \end{cases}. \tag{3.8}$$

Proof We get by the law of total probability (w.r.t. the random counter M) that

$$\begin{aligned} P_\theta(\widehat{\theta} \leq x, M \geq 1) &= \sum_{m=1}^{r-1} P_\theta(\widehat{\theta} \leq x, M = m) + P_\theta(\widehat{\theta} \leq x, M \geq r) \\ &= \sum_{m=1}^{r-1} P_\theta(nX_{(n,r)} \wedge \tau \leq mx, X_{(n,m)} \leq \tau < X_{(n,m+1)}) \\ &\qquad + P_\theta(X_{(n,m)} \wedge \tau \leq rx/n, X_{(n,r)} \leq \tau) \\ &= \sum_{m=1}^{r-1} \mathbb{1}_{(0,mx]}(n\tau) P_\theta(X_{(n,m)} \leq \tau < X_{(n,m+1)}) \\ &\qquad + P_\theta(X_{(n,r)} \leq rx/n, X_{(n,r)} \leq \tau) \\ &= \sum_{m=1}^{r-1} \mathbb{1}_{(0,mx]}(n\tau)[F_{(n,m)}(\tau) - F_{(n,m+1)}(\tau)] + F_{(n,r)}(rx/n \wedge \tau) \end{aligned}$$

From the pdf

$$f_{(n,m)}(t) = f(t)\overline{F}^{n-1}(t)\frac{n^m}{(r-1)!}\big[-\ln \overline{F}(t)\big]^{m-1} = \frac{n^m}{(m-1)!}\frac{t^{m-1}}{\theta^m}e^{-nt/\theta}, \quad t > 0,$$

we conclude that, for $x \geq 0$,

$$F_{(n,m)}\left(\frac{x}{n}\right) = \int_0^{\frac{x}{n}} \frac{n^m}{(m-1)!}\frac{t^{m-1}}{\theta^m}e^{-nt/\theta}dt = \int_0^x \frac{u^{m-1}}{(m-1)!\theta^m}e^{-u/\theta}du = G_{m,\theta}(x).$$

Hence, we get

$$P_\theta(\widehat{\theta} \leq x, M \geq 1) = \sum_{m=1}^{r-1} \mathbb{1}_{(0,mx]}(n\tau)[G_{m,\theta}(n\tau) - G_{m+1,\theta}(n\tau)] + G_{r,\theta}(rx \wedge n\tau).$$

Furthermore, by Theorem 2.5, we get

$$G_{m,\theta}(n\tau) - G_{m+1,\theta}(n\tau) = \theta g_{m+1,\theta}(n\tau) = \frac{(n\tau/\theta)^m}{m!}e^{-n\tau/\theta}. \tag{3.9}$$

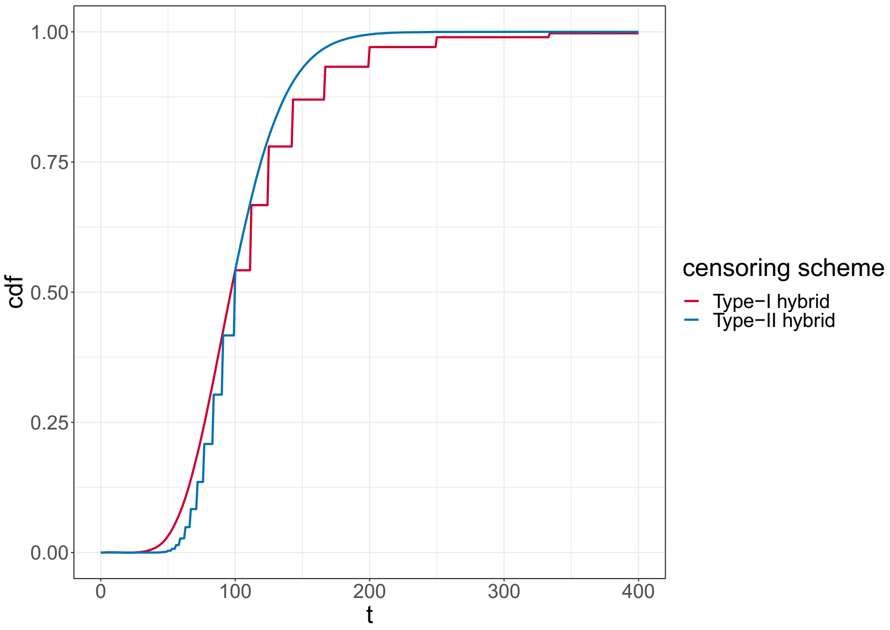

Fig. 4 Plots of (conditional) cdf of the maximum likelihood estimator $\widehat{\theta}$ under Type-I and Type-II hybrid censoring with $n = 2, \tau = 500, r = 10, \theta = 100$

Depending on the value of $n\tau$, we can split this expression into the following cases

$$
\begin{aligned}
&P_\theta(\widehat{\theta} \le x, M \ge 1)\\
&\quad = \begin{cases} G_{r,\theta}(rx), & rx < n\tau \\ \sum_{m=k+1}^{r-1}[G_{m,\theta}(n\tau) - G_{m+1,\theta}(n\tau)] + G_{r,\theta}(n\tau), & kx < n\tau \le (k+1)x, 0 \le k < r \end{cases}\\
&= \begin{cases} G_{r,\theta}(rx), & rx < n\tau \\ \sum_{m=k+1}^{r-1} \frac{(n\tau/\theta)^m}{m!} e^{-n\tau/\theta} + G_{r,\theta}(n\tau), & kx < n\tau \le (k+1)x, 0 \le k < r \end{cases}\\
&= \begin{cases} G_{r,\theta}(rx), & rx < n\tau \\ G_{k+1,\theta}(n\tau), & kx < n\tau \le (k+1)x, 0 \le k < r \end{cases}.
\end{aligned}
$$

Consequently one gets the expression in (3.8) by dividing this formula by the probability $P_\theta(M \ge 1) = 1 - P_\theta(M = 0)$. □

Note that the corresponding distribution of the MLE $\widehat{\theta}$ can be split into a continuous part with support on $[0, n\tau/r]$ and a discrete part with point masses at the points $n\tau, n\tau/2, \ldots, n\tau/(r-1)$.

A plot of a density function (red curve) is depicted in Fig. 4 with $n = 2, \tau = 500, r = 10, \theta = 100$. The jumps are located at 1000, 500, 1000/3, 250, 200, 1000/6, 1000/7, 125, 1000/9.

In order to construct exact conditional confidence intervals, we make use of the method of pivoting the cdf. Therefore, we have to ensure that the (conditional) cdf of the MLE is monotonic in $\theta > 0$ and that the associated limits satisfy certain conditions (see Balakrishnan et al. [9]). Fortunately, these conditions can be concluded again from the results of Burkschat et al. [18] or by straightforward derivations using the representations of the conditional cdf in Theorem 3.3. The stochastic monotonicity of the MLE can alternatively be deduced from van Bentum and Cramer [14].

Theorem 3.4 *For the conditional MLE* $\widehat{\theta}$ *under Type-I hybrid censoring and arbitrary* $n\tau > x \geq 0$, $P_\theta(\widehat{\theta} > x \mid M \geq 1)$ *is a monotone increasing function in* $\theta > 0$ *with*

$$\lim_{\theta \to 0_+} P_\theta(\widehat{\theta} > x \mid M \geq 1) = 0$$

and

$$\lim_{\theta \to \infty} P_\theta(\widehat{\theta} > x \mid M \geq 1) = \begin{cases} 1 - \frac{x}{n\tau}, & r = 1 \\ 1, & r > 1 \end{cases}.$$

For $x \geq n\tau$, $P_\theta(\widehat{\theta} > x \mid M \geq 1) = 0$ *for all* $\theta > 0$.

Therefore, we get the following result.

Theorem 3.5 *Let* $k = \lceil n\tau/\widehat{\theta} \rceil - 1$ *with* $\lceil z \rceil = \min\{\ell \in \mathbb{Z} \mid z \leq \ell\}$. *For fixed* $\alpha_1, \alpha_2 > 0$, *with* $\alpha_1 + \alpha_2 = \alpha \in (0, 1)$, *a conditional* $100(1-\alpha)\%$ *confidence interval* (θ_L, θ_U) *for* θ *is obtained by solving the following two non-linear equations. In particular, if* $r\widehat{\theta} < n\tau$ *then one has to solve*

$$\alpha_1(1 - e^{-n\tau/\theta_U}) = G_{r,\theta_U}(r\widehat{\theta}) \quad \text{and} \quad (1-\alpha_2)(1 - e^{-n\tau/\theta_L}) = G_{r,\theta_L}(r\widehat{\theta})$$

else

$$\alpha_1(1 - e^{-n\tau/\theta_U}) = G_{k+1,\theta_U}(n\tau) \quad \text{and} \quad (1-\alpha_2)(1 - e^{-n\tau/\theta_L}) = G_{k+2,\theta_L}(n\tau).$$

Theorem 3.5 follows directly from the method of pivoting the cdf (see, for example, Casella and Berger [19]) and Theorem 3.3.

Remark 3.6

(i) To utilize Theorem 3.5, one has to ensure the solvability of the above equations. For $r > 1$, this is guaranteed by Theorem 3.4, while the given equations do not have to have a solution in the case $r = 1$. Therefore, one has to modify the bounds of the confidence interval in this case slightly (see Balakrishnan et al. [9]). However, if possible, the problematic case $r = 1$ should be excluded by the design of the experiment.

(ii) In order to apply the pivoting method, it is important to take into account that the (conditional) distribution of the maximum likelihood estimator has point masses under hybrid censoring. As pointed out in Casella and Berger [19, p. 434], these point masses have to be considered in the sense that we do not consider the

survival function but the probability $P_\theta(\widehat{\theta} \geq x \mid M \geq 1)$ to determine the lower bound θ_L when the value $\widehat{\theta}_{\text{obs}}$ of the maximum likelihood estimator satisfies $\widehat{\theta}_{\text{obs}} > n\tau/r$. This ensures the desired level which, due to the point masses, may be somewhat higher than required.
Rewriting the gamma cdf in Theorem 3.5 as a cdf of a χ^2-distribution, we get $G_{\ell,\theta}(x) = F_{\chi^2(2\ell)}(2x/\theta)$ illustrating that the conditions are connected to $\chi^2(2\ell)$-distributions. This shows the similarity to Fairbank's construction of confidence intervals (see Fairbanks et al. [35] and Remark 3.7).

(iii) In general, one has to calculate the interval bounds θ_L and θ_U numerically. Examples are presented in Sect. 4.

Remark 3.7 Fairbanks et al. [35] proposed $(1-\alpha)$-confidence intervals for Type-I hybrid censored data given the observed sample size M_{I}. In particular, they gave the following confidence intervals (written in our notation)

$$\begin{aligned} &\left[\frac{2n\tau}{\chi^2_{1-\alpha/2}(2)}, \infty\right), && \text{if } M_{\text{I}} = 0, \\ &\left[\frac{2m\widehat{\theta}}{\chi^2_{1-\alpha/2}(2m+2)}, \frac{2m\widehat{\theta}}{\chi^2_{\alpha/2}(2m)}\right], && \text{if } M_{\text{I}} = m \in \{1, \ldots, r-1\}, \\ &\left[\frac{2r\widehat{\theta}}{\chi^2_{1-\alpha/2}(2r)}, \frac{2r\widehat{\theta}}{\chi^2_{\alpha/2}(2r)}\right], && \text{if } M_{\text{I}} = r. \end{aligned}$$

A comparison to the confidence intervals constructed by the method of pivoting the cdf is given in the data examples in Sect. 4.

In case of Type-II hybrid censoring, we get similar (but unconditional) results. First, we present an explicit expression of the cdf of the MLE.

Theorem 3.8 *For $x \geq 0$, the cdf $G^{II}_{r,\theta}$ of the MLE $\widehat{\theta}$ under Type-II hybrid censoring is given by*

$$P_\theta(\widehat{\theta} \leq x) = G^{II}_{r,\theta}(x) = \begin{cases} G_{r,\theta}(rx), & n\tau \leq rx \\ G_{k+1,\theta}(n\tau), & kx < n\tau \leq (k+1)x, k \geq r \end{cases}. \tag{3.10}$$

Proof We get directly by using the law of total probability (w.r.t. the random counter M), that

$$\begin{aligned} P_\theta(\widehat{\theta} \leq x) &= P_\theta(\widehat{\theta} \leq x, M < r) + \sum_{m=r}^{\infty} P_\theta(\widehat{\theta} \leq x, M = m) \\ &= P_\theta(X_{(n,r)} \vee \tau \leq rx/n, X_{(n,r)} > \tau) \\ &\quad + \sum_{m=r}^{\infty} P_\theta(nX_{(n,r)} \vee \tau \leq mx, X_{(n,m)} \leq \tau < X_{(n,m+1)}) \\ &= P_\theta(X_{(n,r)} \leq rx/n, X_{(n,r)} > \tau) + \sum_{m=r}^{\infty} P_\theta(\tau \leq mx/n, X_{(n,m)} \leq \tau < X_{(n,m+1)}) \end{aligned}$$

$$
\begin{aligned}
&= [F_{(n,r)}(rx/n) - F_{(n,r)}(\tau)]\mathbb{1}_{(0,rx]}(n\tau) \\
&\quad + \sum_{m=r}^{\infty} \mathbb{1}_{(0,mx]}(n\tau)[F_{(n,m)}(\tau) - F_{(n,m+1)}(\tau)]
\end{aligned}
$$

Arguing as in the proof of Theorem 3.3, we obtain with the cdf of the gamma distribution the following expression for the cdf of the MLE:

$$
\begin{aligned}
P_\theta(\widehat{\theta} \le x) = [G_{r,\theta}(rx) - G_{r,\theta}(n\tau)]\mathbb{1}_{(0,rx]}(n\tau) \\
+ \sum_{m=r}^{\infty} \mathbb{1}_{(0,mx]}(n\tau)[G_{m,\theta}(n\tau) - G_{m+1,\theta}(n\tau)].
\end{aligned}
$$

Furthermore, using (3.9), we arrive at the desired representation of the cdf $G^{\text{II}}_{r,\theta}$. Depending on the value of $n\tau$, we can split the above expression again into the following cases

$$
\begin{aligned}
P_\theta(\widehat{\theta} \le x) &= \begin{cases} G_{r,\theta}(rx), & n\tau \le rx \\ \sum_{m=k+1}^{\infty}[G_{m,\theta}(n\tau) - G_{m+1,\theta}(n\tau)], & kx < n\tau \le (k+1)x, k \ge r \end{cases} \\
&= \begin{cases} G_{r,\theta}(rx), & n\tau \le rx \\ \sum_{m=k+1}^{\infty} \frac{(n\tau/\theta)^m}{m!} e^{-n\tau/\theta}, & kx < n\tau \le (k+1)x, k \ge r \end{cases} \\
&= \begin{cases} G_{r,\theta}(rx), & n\tau \le rx \\ G_{k+1,\theta}(n\tau), & kx < n\tau \le (k+1)x, k \ge r \end{cases}.
\end{aligned}
$$

Consequently one gets the expression in (3.10). □

Remark 3.9 The representation of the cdf in (3.10) shows that the MLE has both a continuous part with support $(n\tau/r, \infty)$ and a discrete part with point masses at $n\tau/k$, $k = r, r+1, \ldots$. Notice that the cdf has infinitely many jumps in the interval $[0, n\tau/r]$. A plot of a density function (blue curve) is depicted in Fig. 4 with $n = 2$, $\tau = 500$, $r = 10$, $\theta = 100$. The jumps are located at 100, 1000/11, 1000/12, 1000/13, ...

It is worth mentioning that the situation is in some sense reflected to the Type-I hybrid censoring scheme where one has a continuous part first followed by a finite number of jumps.

The stochastic monotonicity of the MLE follows directly from Bentum and Cramer [14, Example 1] so that it remains to show that the limits of the cdf w.r.t. θ are 0 and 1, respectively. This follows directly from (3.10) by noticing

$$
G_{m,\theta}(z) = G_{m,1}(z/\theta),
$$

that is, θ is a scale parameter. Hence, $\lim_{\theta\to 0} G_{m,1}(z/\theta) = 1$, $\lim_{\theta\to\infty} G_{m,1}(z/\theta) = 0$. Then, considering the expression in (3.10) for a given x, the result follows immediately. Furthermore, it should be noted that the monotonicity in θ can also be directly

Table 1 Repair times of the air conditioner data from a Boeing aircraft taken from Whitaker and Samaniego [54, Table 2]

	Repair times								
System 1	197	385	464	552	598	603	608	644	667
System 2	139	349	446	476	499	512	526		

Table 2 Repair times of hydraulic subsystems of load-haul dump machines used in mining taken from Kumar and Klefsjö [44] (for present selection of LHD machine repair times, see comments in Beutner and Cramer [16])

	Repair times										
System 1 (LHD9)	249	461	665	847	963	993	1017	1049	1087	1097	
System 2 (LHD11)	211	293	468	547	664	690	694	699	759	798	833

established from the representation of the cdf in Theorem 3.8 using properties of the gamma distribution. Summarizing, we get the following theorem.

Theorem 3.10 *For the MLE $\widehat{\theta}$ under Type-II hybrid censoring and arbitrary $x \geq 0$, $P_\theta(\widehat{\theta} > x)$ is a monotone increasing function in $\theta > 0$ with*

$$\lim_{\theta \to 0_+} P_\theta(\widehat{\theta} > x) = 0, \quad \lim_{\theta \to \infty} P_\theta(\widehat{\theta} > x) = 1$$

Exact confidence intervals can be established by the method of pivoting the cdf by replacing $G^{\mathsf{I}}_{r,\theta}(\cdot \mid M \geq 1)$ by $G^{\mathsf{II}}_{r,\theta}$ and a modification of the case distinction in Theorem 3.5. Notice that the cdf of a gamma distribution is continuous in the scale parameter. The results are illustrated in Sect. 4 by two data examples.

4 Illustration

4.1 Data Analysis

We illustrate the above procedures by two data sets given in Tables 1 and 2. These data have been analysed by various authors.

According to our model, the two samples are considered as realizations of minimal repair times from independent and exponentially distributed lifetimes. Furthermore, the data is pooled so that we have a single increasing sequence of values. Pooling of such data in a nonparametric prediction context has been discussed in Beutner and Cramer [15] in case of two independent samples (for more than two samples, we refer to Amini and Balakrishnan [2]).

In order to apply our methods to the data, we introduce several thresholds τ as well as desired sample sizes r and compare the resulting estimates assuming exponential life times. Note that the number of samples is given by $n = 2$ in both data sets. Thus, the MLEs are given by (see (3.2))

$$\widehat{\theta} = \begin{cases} \frac{2}{M_{\text{I}}}(X_{(M_{\text{I}})} \wedge \tau), & \text{Type-I hybrid censoring} \\ \frac{2}{M_{\text{II}}}(X_{(M_{\text{II}})} \vee \tau), & \text{Type-II hybrid censoring} \end{cases}$$

Example 4.1 (Air conditioner data) The ordered pooled sample of repair times taken from Table 1 is given by

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
139	197	349	385	446	464	476	499	512	526	552	598	603	608	644	667

The resulting total time on tests, number of observed measurements (in brackets), and maximum likelihood estimates for θ are summarized in Table 3. Notice that the MLE for θ given the two (independent) samples is given by

$$\widehat{\theta}_c = \frac{1}{16}(667 + 526) = 74.5625$$

(see Cramer and Kamps [26], Beutner and Cramer [16]). For exponential distributions, an exact $(1 - \alpha)$-confidence interval for θ is given by

$$\left[\frac{32\widehat{\theta}_c}{\chi^2_{1-\alpha/2}(32)}, \frac{32\widehat{\theta}_c}{\chi^2_{\alpha/2}(32)}\right] \overset{\alpha=.05}{=} \left[\frac{32 \times 74.5625}{49.4804}, \frac{32 \times 74.5625}{18.2908}\right]$$
$$= [48.2211, 130.4483].$$

Despite the rather small sample sizes present in the censored data, the results for both estimates and conditional confidence intervals are quite reasonable compared to the full available information without hybrid censoring, that is, the data taken from Tables 1 and 2. The aspect is also discussed using simulations in Sect. 4.2. In particular, the results improve as the desired sample size r and/or the time thresholds increase. This is quite natural since this is associated with a larger observation window, so that more information is available in the estimation process. Note that the results are worse for small thresholds under Type-I hybrid censoring, i.e. for a short maximum test duration, which usually results in few or no measurements. Furthermore, the results are better under Type-II hybrid censoring which is also due to more information provided by the sampling process. Finally, it turns out that the (conditional) confidence intervals for Type-I hybrid censored data due to Fairbanks et al. [35] are close to our results. For a larger threshold τ and larger values of r, they are almost identical.

Estimators of the intensity rate and the cumulative distribution function based on Type-I hybrid censored data are depicted in Figs. 5 and 6. First, it should be noted that the nonparametric piecewise-linear estimate of the distribution function and the parametric estimate based on an exponential distribution assumption are quite close, despite the small sample sizes. The parametric estimate of the distribution function majorizes (in these examples) the nonparametric estimate so that early failures are somewhat more likely assuming an exponential distribution. The nonparametric intensity rate estimate is close to the constant estimate obtained from the exponential assumption. However, at the end of the observation window it indicates an increase in the rate

Table 3 Test durations, number of observed measurements (in brackets), and maximum likelihood estimates for θ based on the data in Table 1 for some values of the threshold τ and the sample size r under Type-I and Type-II hybrid censoring

	Sample sizes r								
	5 Thresholds τ			10 Thresholds τ			15 Thresholds τ		
	100	500	750	100	500	750	100	500	750
Type-I									
W_{I}	100 (0)	446 (5)	446 (5)	100 (0)	500 (8)	526 (10)	100(0)	500 (8)	644 (15)
$\widehat{\theta}$	na	178.4	178.4	na	125	105.2	na	125	85.87
.95-CI	na	[87.1, 579.0]	[87.1, 560.0]	na	[63.4, 291.4]	[61.6, 219.4]	na	[63.4, 291.4]	[54.8, 153.4]
.95-CI (FB)	[54.2, ∞)	[87.1, 549.4]	[87.1, 549.4]	[54.2, ∞)	[63.4, 289.5]	[61.6, 219.4]	[54.2, ∞)	[63.4, 289.5]	[54.8, 153.4]
Type-II									
W_{II}	446 (5)	500 (8)	750 (16)	526 (10)	526 (10)	750 (16)	644 (15)	644 (15)	750 (16)
$\widehat{\theta}$	178.4	125	93.75	105.2	105.2	93.75	85.87	85.87	93.75
.95-CI	[87.1, 549.4]	[63.4, 289.5]	[57.7, 164.0]	[61.6, 219.4]	[61.6, 219.4]	[57.7, 164.0]	[54.8, 153.4]	[54.8, 153.4]	[57.7, 164.0]

For Type-I hybrid censoring, the confidence intervals due to Fairbanks et al. [35] are provided for comparison. *na* means that the MLE does not exist (of course, confidence intervals constructed by the method of pivoting the cdf do not exist in these cases, too). The confidence intervals are constructed symmetrically, that is, $\alpha_1 = \alpha_2 = \alpha/2$

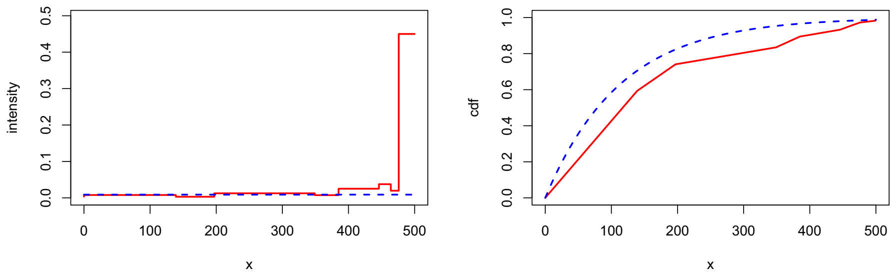

Fig. 5 Plots of intensity rate estimators (left) and corresponding estimated cumulative distribution function (right) for air conditioner data under Type-I hybrid censoring with $\tau = 500$ (leading to the Type-I censoring case) and $m = 10$. The solid red line corresponds to the nonparametric estimation (see (3.4) and (3.6)) whereas the dashed blue line is obtained from the estimate given in (3.3). The corresponding distribution function is estimated under an exponential assumption

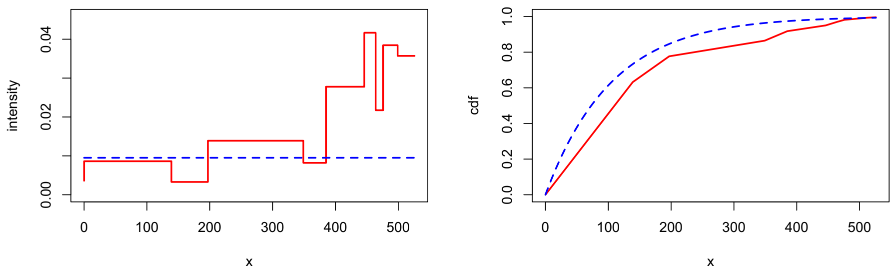

Fig. 6 Plots of intensity rate estimators (left) and corresponding estimated cumulative distribution function (right) for air conditioner data under Type-I hybrid censoring with $\tau = 750$ (leading to the Type-II censoring case) and $m = 10$. The solid red line corresponds to the nonparametric estimation (see (3.4) and (3.6)) whereas the dashed blue line is obtained from the estimate given in (3.3). The corresponding distribution function is estimated under an exponential assumption

which, of course, can not be detected by the exponential model. This might indicate an increasing failure rate in contrast to the constant rate assumed for exponentially distributed lifetimes.

Example 4.2 (Load-haul dump machine data) Proceeding as in Example 4.1, we start with the ordered pooled data generated from the two samples in Table 2.

1	2	3	4	5	6	7	8	9	10	11
211	249	293	461	468	547	664	665	690	694	699
12	13	14	15	16	17	18	19	20	21	
759	798	833	847	963	993	1017	1049	1087	1097	

The MLE for θ given the two (independent) samples is given by

$$\widehat{\theta}_c = \frac{1}{21}(1097 + 833) = 91.9048$$

(see Cramer and Kamps [26], Beutner and Cramer [16]). For exponential distributions, an exact $(1-\alpha)$-confidence interval is given by

$$\left[\frac{42\widehat{\theta}_c}{\chi^2_{1-\alpha/2}(42)}, \frac{42\widehat{\theta}_c}{\chi^2_{\alpha/2}(42)}\right] \overset{\alpha=.05}{=} \left[\frac{42\times 91.9048}{61.7768}, \frac{42\times 91.9048}{25.9987}\right]$$
$$= [62.4830, 148.4692].$$

The conclusions drawn from Table 4 are very similar to those obtained for the data analysed in Example 4.1 so that we do not provide further comments here.

Estimators of the intensity rate and the cumulative distribution function based on Type-I hybrid censored data are depicted in Figs. 7 and 8. For both intensity rate and cumulative distribution function, the behaviour of the estimates is similar to that already described for the air conditioner data in Example 4.1. In particular, the impact on the observation window caused by the thresholds $\tau = 500$ and $\tau = 750$ is more evident in this example than in Example 4.1.

4.2 Simulations

In order to assess properties of the point estimator, we have conducted a simulation study. For brevity, we restrict ourselves to the Type-I hybrid censoring scheme but, of course, similar simulations can be performed for other hybrid censoring schemes. We consider a situation similar to the load-haul dump machine data, that is, we assume

(i) maximum sample sizes $m \in \{1, 2, \ldots, 25\}$,
(ii) threshold $\tau \in \{50, 60, \ldots, 1200\}$,
(iii) number of simulation runs $N = 5000$ for each combination of m and τ,
(iv) a true parameter $\theta = 91.9048$, and $n = 2$ (superposed) minimal repair samples.

Similar results have been observed for other input parameters. The evaluations are based on the same seeds for any m and τ. This enables us to study the impact of both the sample size and the threshold on the estimates. A histogram and kernel density estimate of the estimates $\widehat{\theta}_\mathsf{I}$ is presented in Fig. 9 for $m = 20$ and selected values of τ. The plots clearly illustrate the point masses of the distribution. For comparison, the density function of a scaled χ^2-distribution is additionally provided. The degrees of freedom df are taken as rounded values of $2m_\mathsf{I}$ where m_I is computed as mean sample size from the simulation (see also Table 5). Comparing the continuous kernel density estimate and the plotted χ^2-density function, it turns out that the curves are somewhat close (particularly on the right tail), that is, the χ^2-density function looks like a smoothed version of the kernel density estimate. The difference gets smaller for increasing τ which corresponds to less cases corresponding to Type-I censoring (see Table 5, column 6). In fact, this might illustrate why Fairbanks' construction of confidence intervals works although it ignores the existence of point masses in the distribution of the maximum likelihood estimator.

For a better visualization, the results are presented in terms of heatmaps generated by the R package 'gplots' (see Warnes et al. [53]). In particular, we provide the proportion

Table 4 Test durations, number of observed measurements (in brackets), and maximum likelihood estimates for θ based on the data in Table 2 for some values of the threshold τ and the sample size r under Type-I and Type-II hybrid censoring

	Sample sizes r								
	5 Thresholds τ			10 Thresholds τ			20 Thresholds τ		
	200	500	1000	200	500	1000	200	500	1000
Type-I									
W_{I}	200 (0)	468 (5)	468 (5)	200 (0)	500 (5)	694 (10)	200(0)	500 (5)	1000 (17)
$\widehat{\theta}$	na	187.2	187.2	na	200	138.8	na	200	117.65
.95-CI	na	[91.4, 610.9]	[91.4, 581.6]	na	[85.7, 657.9]	[81.2, 289.5]	na	[85.7, 657.9]	[73.5, 202.0]
.95-CI (FB)	$[108.4, \infty)$	[91.4, 576.5]	[91.4, 576.5]	$[108.4, \infty)$	[85.7, 616.0]	[81.2, 289.4]	$[108.4, \infty)$	[85.7, 616.0]	[73.5, 202.0]
Type-II									
W_{II}	468 (5)	500 (5)	1000 (17)	694 (10)	694 (10)	1000 (17)	1087 (20)	1087 (20)	1087 (20)
$\widehat{\theta}$	187.2	200	117.65	138.8	138.8	117.65	108.7	108.7	108.7
.95-CI	[91.4, 576.5]	[97.6, 616.0]	[73.5.0, 202.0]	[81.2, 289.4]	[81.2, 289.4]	[73.5, 202.0]	[73.3, 178.0]	[73.3, 178.0]	[73.3, 178.0]

For Type-I hybrid censoring, the confidence intervals due to Fairbanks et al. [35] are provided for comparison. *na* means that the MLE does not exist (of course, confidence intervals constructed by the method of pivoting the cdf do not exist in these cases, too). The confidence intervals are constructed symmetrically, that is, $\alpha_1 = \alpha_2 = \alpha/2$

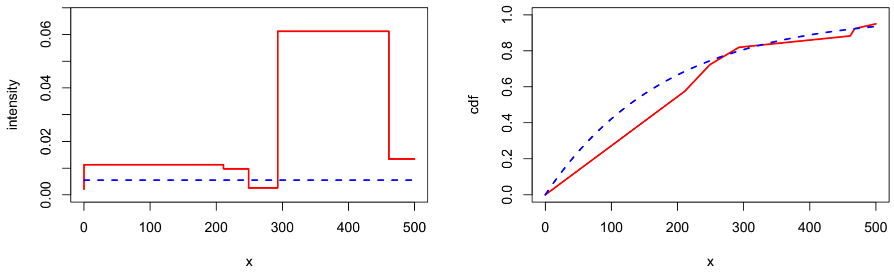

Fig. 7 Plots of intensity rate estimators (left) and corresponding estimated cumulative distribution function (right) for load haul dump machine data under Type-I hybrid censoring with $\tau = 500$ (leading to the Type-I censoring case) and $m = 10$. The solid red line corresponds to the nonparametric estimation (see (3.4) and (3.6)) whereas the dashed blue line is obtained from the estimate given in (3.3). The corresponding distribution function is estimated under an exponential assumption

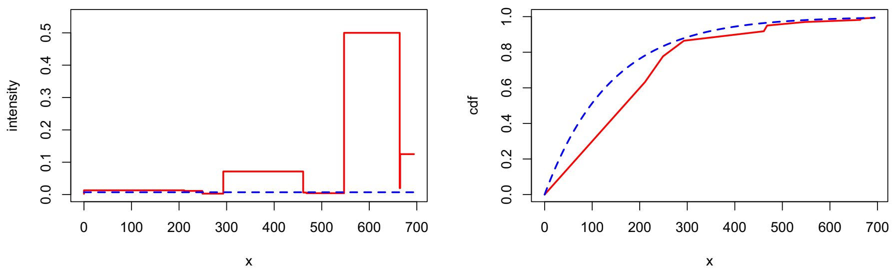

Fig. 8 Plots of intensity rate estimators (left) and corresponding estimated cumulative distribution function (right) for load haul dump machine data under Type-I hybrid censoring with $\tau = 750$ (leading to the Type-II censoring case) and $m = 10$. The solid red line corresponds to the nonparametric estimation (see (3.4) and (3.6)) whereas the dashed blue line is obtained from the estimate given in (3.3). The corresponding distribution function is estimated under an exponential assumption

of non-existing MLEs (Fig. 10) as well as bias (Fig. 11), standard deviation (Fig. 12), and mean squared error (Fig. 13) of the maximum likelihood estimates, respectively.

It turns out that, as expected, the observed proportions of non-existing MLEs rapidly decrease with an increasing threshold τ. In particular, the proportion is less than 5% and 1% for $\tau \geq 140$ and $\tau \geq 210$, respectively. Its zero for $\tau \geq 560$. If $\tau = 100$ which is close to the true value of θ then the MLE does not exist in 11.42% of the simulation runs. For early Type-I censoring ($\tau = 50$), the MLE does not exist in 33.46% of the runs. The heatmap in Fig. 11 clearly illustrates that the threshold has a huge impact on the bias of the estimator. If τ is small (e.g., $\tau \leq 140$) then the MLE tends to underestimate the true parameter. Interestingly, this seems to be rather independent of the choice of m (at least if $m \geq 3$). Then, the bias is increasing-decreasing (as a function of τ). Furthermore, it should be mentioned that, for larger τ, the maximum likelihood estimates seem to have a lower bias for smaller m. The histogram provided in the upper left corner of Fig. 11 supports these observations. Although the bias has a wide range, most of the scenarios show a smaller bias.

For the deviation, the situation is slightly different (see Figs. 12 and 13). Except for the case $m = 1$, where the deviation increases with τ, the deviation decreases

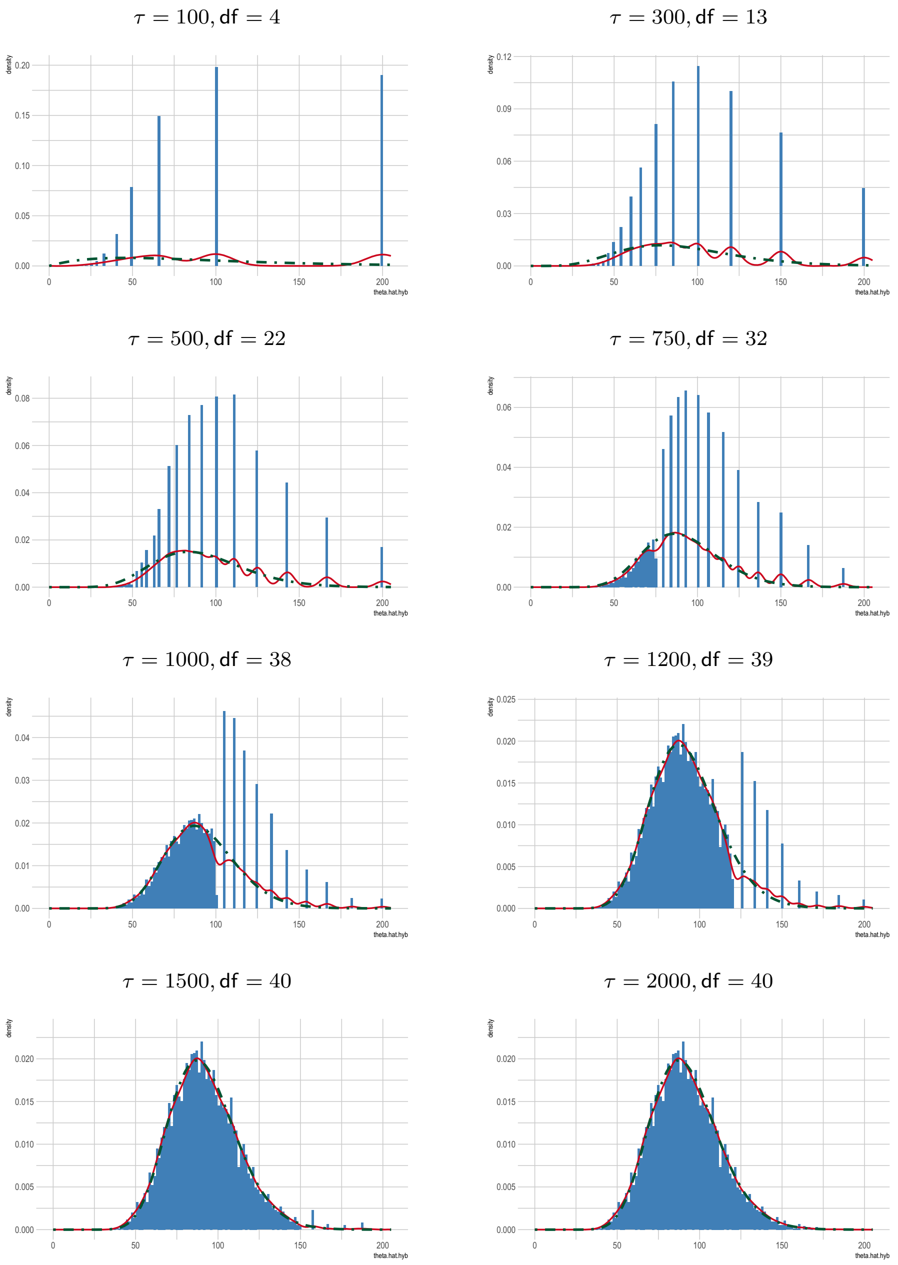

Fig. 9 Histogram and kernel density estimate (solid red line) of the maximum likelihood estimates $\widehat{\theta}_1$ for $m = 20$ and thresholds $\tau \in \{100, 300, 500, 750, 1000, 1200, 1500, 2000\}$. The dot-dashed green curve represents the density function of a $\chi^2(\mathrm{df})$-distribution. For better comparison, the domain is restricted to the interval [0, 200] although estimates might be outside this range

Table 5 Results of a simulation study and comparison of confidence intervals under Type-I hybrid censoring

					Proportion	Confidence interval pivoting method				Confidence interval Fairbanks			
r	τ	W_I	m_I	$\widehat{\theta}$	Type-I	Lower	Upper	Length	Coverage	Lower	Upper	Length	Coverage
3	100	87.121	1.898	104.943	0.628	17.172	1353.816	1336.644	0.976	29.115	2585.889	2559.870	0.976
3	200	124.867	2.733	104.855	0.184	31.800	1318.860	1287.060	0.976	36.225	1344.833	1309.677	0.962
3	300	134.828	2.942	97.160	0.043	36.471	833.756	797.286	0.976	37.764	738.525	701.011	0.963
3	500	137.655	2.996	92.422	0.003	38.022	525.402	487.380	0.950	38.158	472.024	433.912	0.949
3	750	137.937	3.000	91.958	0.000	38.186	468.672	430.486	0.949	38.185	445.914	407.729	0.949
3	1000	137.937	3.000	91.958	0.000	38.185	457.369	419.184	0.949	38.185	445.914	407.729	0.949
3	1500	137.937	3.000	91.958	0.000	38.185	449.542	411.357	0.949	38.185	445.914	407.729	0.949
3	2000	137.937	3.000	91.958	0.000	38.185	447.293	409.108	0.949	38.185	445.914	407.729	0.949
3	2500	137.937	3.000	91.958	0.000	38.185	446.504	408.319	0.949	38.185	445.914	407.729	0.949
5	100	98.484	2.139	109.914	0.932	14.100	1378.412	1364.312	0.976	30.007	2598.871	2571.850	0.976
5	200	174.679	3.812	114.856	0.561	29.032	1293.263	1264.231	0.975	39.923	1324.418	1285.510	0.961
5	300	211.666	4.619	103.760	0.216	38.668	706.659	667.992	0.976	43.445	646.968	603.762	0.962
5	500	228.591	4.971	93.578	0.019	44.362	333.813	289.451	0.951	44.863	318.879	274.061	0.946
5	750	229.814	4.999	92.036	0.001	44.834	286.214	241.380	0.946	44.888	283.893	239.005	0.946
5	1000	229.884	5.000	91.954	0.000	44.892	283.849	238.957	0.946	44.892	283.198	238.306	0.946
5	1500	229.884	5.000	91.954	0.000	44.892	283.308	238.416	0.946	44.892	283.198	238.306	0.946
5	2000	229.884	5.000	91.954	0.000	44.892	283.221	238.329	0.946	44.892	283.198	238.306	0.946
5	2500	229.884	5.000	91.954	0.000	44.892	283.204	238.312	0.946	44.892	283.198	238.306	0.946
10	100	100.000	2.177	110.343	1.000	14.179	1379.467	1365.288	0.990	30.078	2599.793	2572.693	0.974
10	200	199.626	4.356	119.134	0.986	30.185	1298.678	1268.493	0.986	41.060	1329.200	1289.139	0.971
10	300	293.678	6.394	112.199	0.873	41.702	694.803	653.101	0.974	47.010	641.568	594.790	0.960
10	500	418.472	9.093	99.364	0.357	52.031	273.530	221.500	0.960	52.530	260.889	208.403	0.955

Table 5 continued

					Proportion	Confidence interval pivoting method				Confidence interval Fairbanks			
r	τ	W_I	m_I	$\widehat{\theta}$	Type-I	Lower	Upper	Length	Coverage	Lower	Upper	Length	Coverage
10	750	457.535	9.920	93.280	0.041	53.854	198.251	144.397	0.954	53.909	197.392	143.484	0.954
10	1000	460.995	9.997	92.275	0.002	53.973	192.525	138.552	0.946	53.973	192.506	138.533	0.946
10	1500	461.141	10.000	92.228	0.000	53.983	192.328	138.345	0.946	53.983	192.327	138.344	0.946
10	2000	461.141	10.000	92.228	0.000	53.983	192.327	138.344	0.946	53.983	192.327	138.344	0.946
10	2500	461.141	10.000	92.228	0.000	53.983	192.327	138.344	0.946	53.983	192.327	138.344	0.946
15	100	100.000	2.177	110.343	1.000	14.179	1379.467	1365.288	0.990	30.078	2599.793	2572.693	0.974
15	200	200.000	4.365	119.179	1.000	30.194	1298.750	1268.556	0.986	41.068	1329.272	1289.202	0.971
15	300	299.915	6.531	112.796	0.996	41.854	695.549	653.695	0.982	47.167	642.338	595.402	0.968
15	500	490.546	10.683	102.758	0.861	53.616	273.240	219.624	0.962	54.115	260.593	206.522	0.957
15	750	642.195	13.948	96.073	0.341	57.851	182.914	125.064	0.957	57.905	182.065	124.160	0.956
15	1000	683.818	14.878	92.592	0.053	58.607	166.581	107.974	0.958	58.607	166.570	107.963	0.958
15	1500	689.133	15.000	91.887	0.000	58.676	164.176	105.501	0.951	58.676	164.176	105.501	0.951
15	2000	689.151	15.000	91.887	0.000	58.677	164.174	105.497	0.951	58.677	164.174	105.497	0.951
15	2500	689.151	15.000	91.887	0.000	58.677	164.174	105.497	0.951	58.677	164.174	105.497	0.951
20	100	100.000	2.177	110.343	1.000	14.179	1379.467	1365.288	0.990	30.078	2599.793	2572.693	0.974
20	200	200.000	4.365	119.179	1.000	30.194	1298.750	1268.556	0.986	41.068	1329.272	1289.202	0.971
20	300	300.000	6.533	112.801	1.000	41.854	695.554	653.701	0.982	47.167	642.343	595.408	0.968
20	500	499.617	10.896	103.191	0.990	53.735	273.627	219.891	0.966	54.234	260.980	206.789	0.961
20	750	729.134	15.856	98.321	0.781	59.131	182.487	123.356	0.951	59.185	181.638	122.452	0.951
20	1000	867.540	18.907	94.199	0.320	61.186	158.959	97.773	0.952	61.186	158.948	97.762	0.952
20	1500	916.615	19.985	91.794	0.007	61.824	150.350	88.527	0.945	61.824	150.350	88.527	0.945
20	2000	917.247	20.000	91.725	0.000	61.828	150.165	88.337	0.945	61.828	150.165	88.337	0.945
20	2500	917.247	20.000	91.725	0.000	61.828	150.165	88.337	0.945	61.828	150.165	88.337	0.945

Table 5 continued

r	τ	$W_\|$	$m_\|$	$\widehat{\theta}$	Proportion	Confidence interval pivoting method				Confidence interval Fairbanks			
					Type-I	Lower	Upper	Length	Coverage	Lower	Upper	Length	Coverage
25	100	100.000	2.177	110.343	1.000	14.179	1379.467	1365.288	0.990	30.078	2599.793	2572.693	0.974
25	200	200.000	4.365	119.179	1.000	30.194	1298.750	1268.556	0.986	41.068	1329.272	1289.202	0.971
25	300	300.000	6.533	112.801	1.000	41.854	695.554	653.701	0.982	47.167	642.343	595.408	0.968
25	500	500.000	10.905	103.210	1.000	53.738	273.649	219.911	0.966	54.237	261.003	206.809	0.961
25	750	748.136	16.277	98.909	0.969	59.334	182.932	123.598	0.952	59.389	182.083	122.694	0.952
25	1000	966.784	21.078	95.812	0.724	62.237	158.433	96.195	0.953	62.237	158.422	96.185	0.953
25	1500	1138.291	24.787	92.301	0.077	64.199	143.187	78.988	0.950	64.199	143.187	78.988	0.950
25	2000	1147.756	24.999	91.826	0.000	64.284	141.896	77.613	0.945	64.284	141.896	77.613	0.945
25	2500	1147.795	25.000	91.824	0.000	64.284	141.890	77.606	0.945	64.284	141.890	77.606	0.945
30	100	100.000	2.177	110.343	1.000	14.179	1379.467	1365.288	0.990	30.078	2599.793	2572.693	0.974
30	200	200.000	4.365	119.179	1.000	30.194	1298.750	1268.556	0.986	41.068	1329.272	1289.202	0.971
30	300	300.000	6.533	112.801	1.000	41.854	695.554	653.701	0.982	47.167	642.343	595.408	0.968
30	500	500.000	10.905	103.210	1.000	53.738	273.649	219.911	0.966	54.237	261.003	206.809	0.961
30	750	749.902	16.316	98.967	0.998	59.347	182.995	123.648	0.963	59.402	182.145	122.744	0.963
30	1000	995.180	21.698	96.421	0.943	62.476	158.825	96.349	0.954	62.476	158.814	96.339	0.954
30	1500	1324.756	28.813	93.310	0.300	65.799	140.314	74.515	0.951	65.799	140.313	74.515	0.951
30	2000	1377.715	29.964	92.027	0.014	66.222	136.462	70.240	0.946	66.222	136.462	70.240	0.946
30	2500	1379.241	30.000	91.949	0.000	66.232	136.283	70.051	0.946	66.232	136.283	70.051	0.946

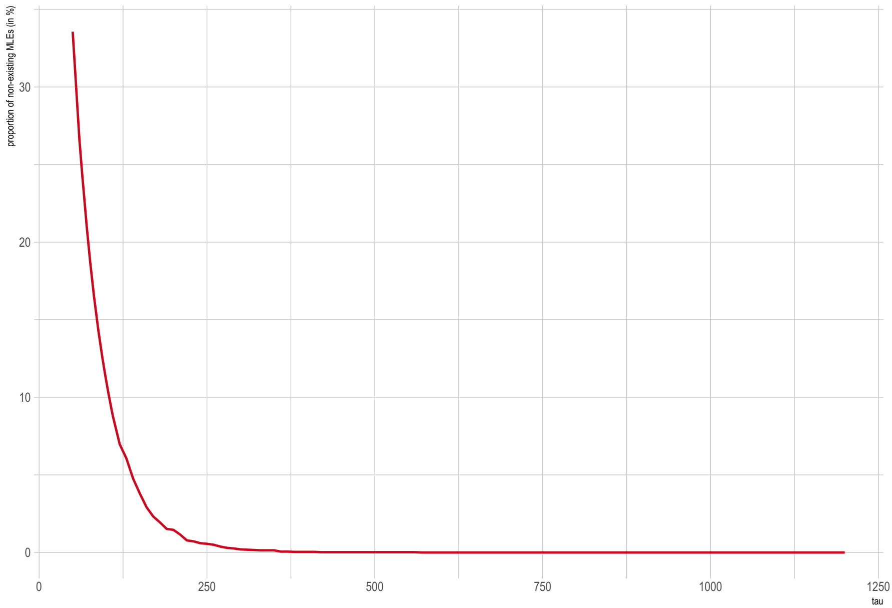

Fig. 10 Proportion of non-existing maximum likelihood estimates as a function of the threshold τ

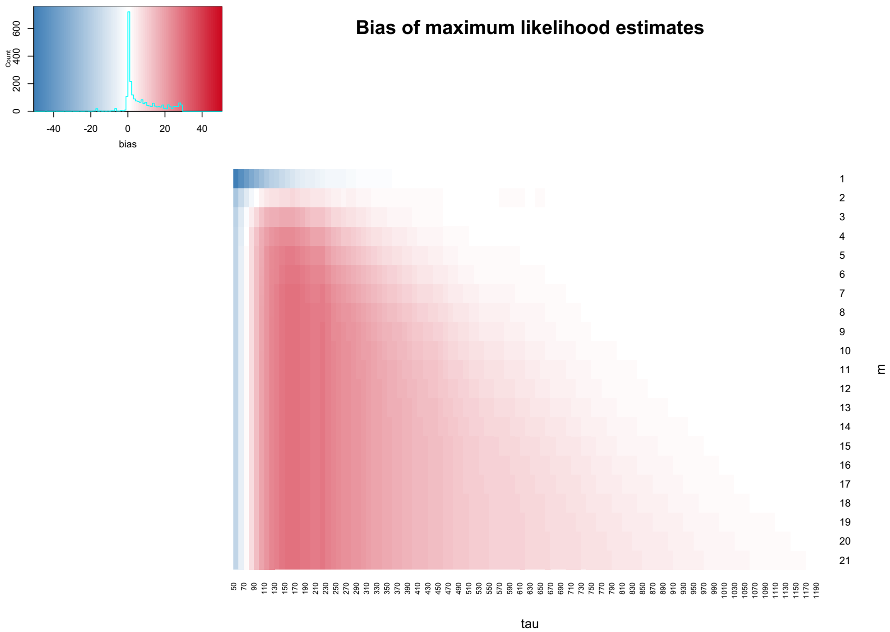

Fig. 11 Heatmap of bias of maximum likelihood estimates (true value $\theta = 91.9048$; see beginning of Sect. 4.2). The threshold is fixed in each column, whereas the desired sample size m is given in the rows. Note that only cases are included where the MLE exists (see Fig. 10)

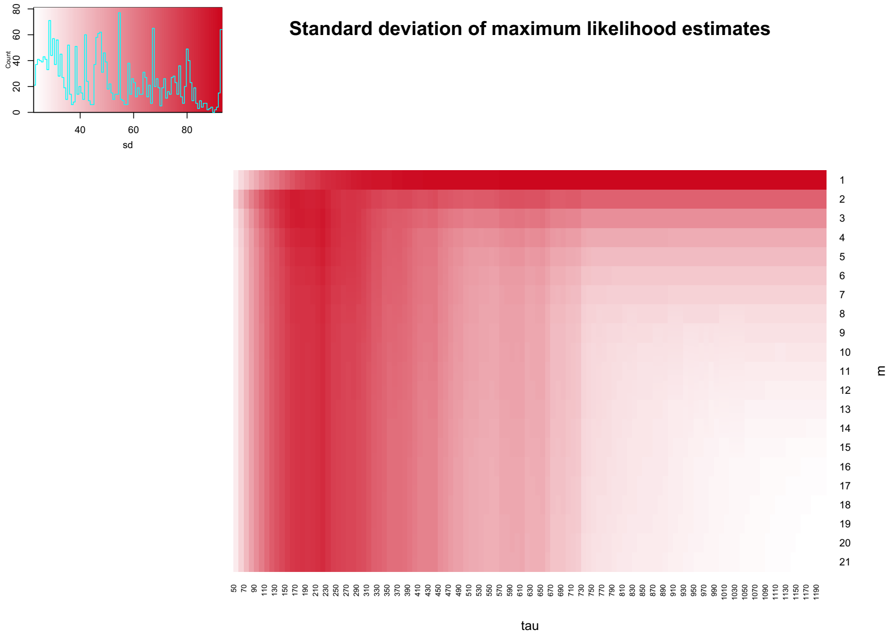

Fig. 12 Heatmap of standard deviation of maximum likelihood estimates (true value $\theta = 91.9048$; see beginning of Sect. 4.2). The threshold is fixed in each column, whereas the desired sample size m is given in the rows. Note that only cases are included where the MLE exists (see Fig. 10)

with increasing τ and m for the other cases. The heatmaps show also some ’column structure’ which indicates that the deviation seems to be quite robust against the desired sample size m whereas the threshold seems to have a larger impact. Moreover, the heatmap shows that the deviations do not generally improve for an increasing τ. These patterns may be caused by the fact that the distribution of the MLE has point masses at certain values. Hence, the relative position of the point masses and the thresholds have a larger impact on the deviation.

In case of the confidence intervals, we provide the simulation results given in Table 5. The results illustrate that the coverage probabilities are quite close to the desired level and usually are larger so that the confidence intervals can be considered as conservative (as expected). Furthermore, the differences for the confidence intervals based on the pivoting method and Fairbanks’ proposal are—at least for larger thresholds—quite small. In particular, they yield (on average) the same bounds in many cases. For smaller thresholds, the pivoting methods yields smaller average length of the confidence interval with a comparable coverage probability. On the other hand, Fairbanks’ approach yields one-sided unbounded confidence intervals when the experiment is terminated without observing a minimal repair. In this case, the pivoting method does not provide a confidence interval. Therefore, in this case, one should use Fairbanks’ confidence interval. Finally, it should be mentioned that the lengths of the confidence intervals decrease with both increasing desired sample size r and increasing threshold τ.

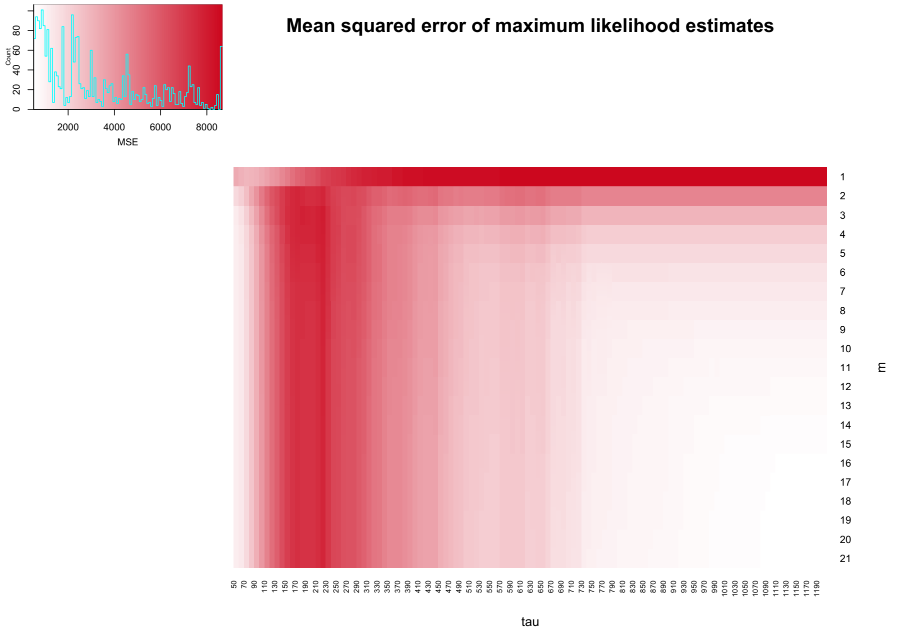

Fig. 13 Heatmap of mean squared error of maximum likelihood estimates (true value $\theta = 91.9048$; see beginning of Sect. 4.2). The threshold is fixed in each column, whereas the desired sample size m is given in the rows. Note that only cases are included where the MLE exists (see Fig. 10)

5 Conclusions and Outlook

We have obtained results for hybrid censored minimal repair data (or, equivalently, for hybrid censored record values or jump times of a NHPP). In particular, we have shown how results established for Type-I and Type-II censored minimal repair time data can be utilized under hybrid censoring. Specifically, we have illustrated the power of the present approach assuming exponentially distributed life times. In this case, we showed the stochastic monotonicity of the MLE. Using this property, we constructed exact (conditional) confidence intervals for the mean of the life time with the method of pivoting the cumulative distribution.

An obvious extension of the discussed model with exponentially distributed lifetimes is the consideration of Weibull distributions, which also lead to explicit representations for the maximum likelihood estimators of the distribution parameters using results of Wang and Ye [52] (or, in terms of the power law process, see, e.g., Crow [27], Finkelstein [36], Engelhardt and Bain [33]). To be more precise, for Weibull distributed data with parameters $\theta > 0$ and $\beta > 0$, the likelihood function is given by

$$L(\theta, \beta; x_1, \ldots, x_m, m) = \left(\frac{n\beta}{\theta^\beta}\right)^m e^{-nw^\beta/\theta^\beta} \left(\prod_{i=1}^{m} x_i\right)^{\beta-1}. \tag{5.1}$$

From the presented results as well as the likelihood function in Wang and Ye [52, eq. (2)], we find immediately the respective maximum likelihood estimators, that is,

$$\widehat{\beta} = \frac{M_{\mathsf{HCS}}}{\sum_{i=1}^{M_{\mathsf{HCS}}} \ln(W_{\mathsf{HCS}}/X_{(n,i)})}, \quad \widehat{\theta} = W_{\mathsf{HCS}}\Big(\frac{n}{M_{\mathsf{HCS}}}\Big)^{1/\widehat{\beta}}$$

(provided $M_{\mathsf{HCS}} \geq 1$). Related results for minimal repair/record data under hybrid censoring for Weibull distributions will be studied in a future paper.

Another interesting direction of future research is the multi-sample case sketched in Remark 2.3. Following the approaches presented in Górny and Cramer [38] and Jansen et al.[42], it seems to be possible to establish corresponding results for minimal repair/record data subject to hybrid censoring. Moreover, the case of joint hybrid censoring seems to be of interest but, at a first glance, more challenging than the multiple sample case with sample-wise hybrid censoring.

Furthermore, it would be interesting to study the nonparametric approach in more detail. In particular, the derivation of asymptotic results for hybrid censoring schemes will be worth to address. This will supplement studies already available for Type-I and Type-II censored jump times of a NHPP (see, e.g., Henderson [41] and references cited therein).

Finally, it would be of interest to study large sample results. This will be particularly useful in the case where a larger number of measurements is available since the construction of the exact confidence intervals is computationally expensive.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Ahsanullah M (1981) Record values of exponentially distributed random variables. Statist. Papers 22:121–127
2. Amini M, Balakrishnan N (2013) Nonparametric meta-analysis of independent samples of records. Comp. Statist. Data Anal. 66:70–81. https://doi.org/10.1016/j.csda.2013.03.019
3. Arnold BC, Balakrishnan N, Nagaraja HN (1998) Records. John Wiley & Sons, Inc. https://doi.org/10.1002/9781118150412

4. Ascher H (1979) Comments on 'models for reliability of repaired equipment'. IEEE Transactions on Reliability R-28, 119–119. https://doi.org/10.1109/tr.1979.5220515
5. Ascher H, Feingold H (1984) Repairable Systems Reliability: Modeling, Inference. Misconceptions and their Causes, Marcel Dekker, New York
6. Ascher HE (1968) Evaluation of repairable system reliability using the 'bad-as-old' concept. IEEE Transactions on Reliability R-17, 103–110. https://doi.org/10.1109/tr.1968.5217523
7. Aven T, Jensen U (2000) A general minimal repair model. Journal of Applied Probability 37:187–197. https://doi.org/10.1239/jap/1014842276
8. Balakrishnan N, Cramer E (2014) The Art of Progressive Censoring. Applications to Reliability and Quality, Birkhäuser, New York
9. Balakrishnan N, Cramer E, Iliopoulos G (2014) On the method of pivoting the CDF for exact confidence intervals with illustration for exponential mean under life-test with time constraints. Statistics & Probability Letters 89:124–130. https://doi.org/10.1016/j.spl.2014.02.022
10. Balakrishnan N, Cramer E, Kundu D (2023) Hybrid Censoring Know-How - Designs and Implementations. Academic Press, Cambridge
11. Balakrishnan N, Iliopoulos G (2009) Stochastic monotonicity of the MLE of exponential mean under different censoring schemes. Ann. Inst. Statist. Math. 61:753–772
12. Balakrishnan N, Kundu D (2013) Hybrid censoring: Models, inferential results and applications (with discussions). Comput. Statist. Data Anal. 57:166–209
13. Barlow R, Hunter L (1960) Optimum preventive maintenance policies. Operations Research 8:90–100. https://doi.org/10.1287/opre.8.1.90
14. van Bentum T, Cramer E (2019) Stochastic monotonicity of MLEs of the mean for exponentially distributed lifetimes under sequential hybrid censoring. Statist. Probab. Letters 148:1–8
15. Beutner E, Cramer E (2010) Nonparametric meta-analysis for minimal-repair systems. Aust. N. Z. J. Stat. 52:383–401. https://doi.org/10.1111/j.1467-842X.2010.00591.x
16. Beutner E, Cramer E (2011) Confidence intervals for quantiles in a minimal repair set-up. Int. J. Appl. Math. Statist. 24:86–97
17. Block HW, Borges WS, Savits TH (1985) Age-dependent minimal repair. Journal of Applied Probability 22:370–385. https://doi.org/10.2307/3213780
18. Burkschat M, Cramer E, Górny J (2016) Type-I censored sequential k-out-of-n systems. Applied Mathematical Modelling 40:8156–8174. https://doi.org/10.1016/j.apm.2016.03.052
19. Casella G, Berger R (2002) Statistical inference. Thomson Learning, Australia Pacific Grove, CA
20. Chandrasekar B, Childs A, Balakrishnan N (2004) Exact likelihood inference for the exponential distribution under generalized Type-I and Type-II hybrid censoring. Naval Res. Logist. 51:994–1004. https://doi.org/10.1002/nav.20038
21. Chen SM, Bhattacharyya GK (1988) Exact confidence bounds for an exponential parameter under hybrid censoring. Comm. Statist. Theory Methods 16:2429–2442. https://doi.org/10.1080/03610928708829516
22. Childs A, Chandrasekar B, Balakrishnan N, Kundu D (2003) Exact likelihood inference based on Type-I and Type-II hybrid censored samples from the exponential distribution. Ann. Inst. Statist. Math. 55:319–330
23. Cramer E (2023) Ordered and censored lifetime data in reliability: An illustrative review. WIREs Comput. Stat. 15:e1571. https://doi.org/10.1002/wics.1571
24. Cramer E (2023b) Structure of hybrid censoring schemes and its implications. Submitted for publication
25. Cramer E, Balakrishnan N (2013) On some exact distributional results based on Type-I progressively hybrid censored data from exponential distributions. Statist. Methodology 10:128–150
26. Cramer E, Kamps U (2001) Estimation with sequential order statistics from exponential distributions. Ann. Inst. Statist. Math. 53:307–324
27. Crow L (1974) Reliability analysis for complex repairable systems. In: Proschan F, Serfling R (eds) booktitleReliability and Biometry. SIAM, Philadelphia, pp 379–410
28. David HA, Nagaraja HN (2003) Order Statistics, edition3rd edn. John Wiley & Sons, Hoboken, New Jersey
29. Draper N, Guttman I (1987) Bayesian analysis of hybrid life tests with exponential failure times. Annals of the Institute of Statistical Mathematics 39:219–225. https://doi.org/10.1007/BF02491461
30. Dziubdziela W, Kopociński B (1976) Limiting properties of the k-th record values. Appl. Math. 15:187–190

31. Ebrahimi N (1986) Estimating the parameters of an exponential distribution from a hybrid life test. Journal of Statistical Planning and Inference 14:255–261. https://doi.org/10.1016/0378-3758(86)90163-1
32. Ebrahimi N (1992) Prediction intervals for future failures in the exponential distribution under hybrid censoring. IEEE Transactions on Reliability 41:127–132
33. Engelhardt M, Bain LJ (1978) Prediction intervals for the Weibull process. Technometrics 20:167–169. https://doi.org/10.1080/00401706.1978.10489642
34. Epstein B (1954) Truncated life tests in the exponential case. Ann. Math. Stat. 25:555–564
35. Fairbanks K, Madsen R, Dykstra R (1982) A confidence interval for an exponential parameter from a hybrid life test. Journal of the American Statistical Association 77:137–140. https://doi.org/10.1080/01621459.1982.10477776
36. Finkelstein JM (1976) Confidence bounds on the parameters of the Weibull process. Technometrics 18:115. https://doi.org/10.2307/1267924
37. Górny J, Cramer E (2018) Modularization of hybrid censoring schemes and its application to unified progressive hybrid censoring. Metrika 81:173–210
38. Górny J, Cramer E (2020) Type-I hybrid censoring of multiple samples. J. Comp. Appl. Math. 366:112404
39. Gupta RC, Kirmani S (1988) Closure and monotonicity properties of nonhomogeneous Poisson processes and record values. Probability in the Engineering and Informational Sciences 2:475–484. https://doi.org/10.1017/s0269964800000188
40. Gupta RD, Kundu D (1998) Hybrid censoring schemes with exponential failure distribution. Comm. Statist. Theory Methods 27:3065–3083. https://doi.org/10.1080/03610929808832273
41. Henderson SG (2003) Estimation for nonhomogeneous Poisson processes from aggregated data. Operations Research Letters 31:375–382. https://doi.org/10.1016/S0167-6377(03)00027-0
42. Jansen M, Górny J, Cramer E (2022) Exact likelihood inference for an exponential parameter under a multi-sample Type-II progressive hybrid censoring model. American Journal of Mathematics and Management Science 41:101–127
43. Kamps U (1995) A Concept of Generalized Order Statistics. Teubner, Stuttgart
44. Kumar U, Klefsjö B (1992) Reliability analysis of hydraulic systems of LHD machines using the power law process model. Reliab. Engrg. Syst. Safety 35:217–224. https://doi.org/10.1016/0951-8320(92)90080-5
45. Last G, Penrose M (2017) Lectures on the Poisson Process. Cambridge University Press. https://doi.org/10.1017/9781316104477
46. Leemis LM (1991) Nonparametric estimation of the cumulative intensity function for a nonhomogeneous Poisson process. Management Science 37:886–900. https://doi.org/10.1287/mnsc.37.7.886
47. Lewis PAW, Shedler GS (1976) Statistical analysis of non-stationary series of events in a data base system. IBM Journal of Research and Development 20:465–482. https://doi.org/10.1147/rd.205.0465
48. Møller J, Waagepetersen RP (2004) Statistical inference and simulation for spatial point processes. volume 100 of *series Monogr. Stat. Appl. Probab*. Boca Raton, FL: Chapman and Hall/CRC
49. Nevzorov VB, (2001) Records: Mathematical Theory. Translations of Mathematical Monographs Vol. 194, American Mathematical Society, Providence, Rhode Island
50. Pfeifer D (1982) The structure of elementary pure birth processes. J. Appl. Probab. 19:664–667
51. Tadj L, Ouali MS, Yacout S, Ait-Kadi D (eds) (2011) Replacement Models with Minimal Repair. Springer, London
52. Wang BX, Ye ZS (2015) Inference on the Weibull distribution based on record values. Comp. Statist. Data Anal. 83:26–36. https://doi.org/10.1016/j.csda.2014.09.005
53. Warnes GR, Bolker B, Bonebakker L, Gentleman R, Huber W, Liaw A, Lumley T, Maechler M, Magnusson A, Moeller S, Schwartz M, Venables B (2022). gplots: Various R Programming Tools for Plotting Data. https://CRAN.R-project.org/package=gplots. r package version 3.1.3
54. Whitaker LR, Samaniego FJ (1989) Estimating the reliability of systems subject to imperfect repair. Journal of the American Statistical Association 84:301–309. https://doi.org/10.1080/01621459.1989.10478770