Aerotecnica M&S 100 Years Ago: Enrico Pistolesi's Aerodynamics

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Dear Reader,

This issue #2 of vol. 100 is dedicated to the famous paper “The vortex theory” from Prof. Enrico Pistolesi, published 100 years ago in the Vol. 1, 1921 of *Aerotecnica*.

Enrico Pistolesi (1889–1968), enrolled in the Scuola Normale Superiore of Pisa for the Degree Course in Mathematics, graduated brilliantly in 1912, and in 1915, he obtained also the degree in Mechanical Engineering at the Polytechnic of Turin, where he taught Mathematical Analysis and where he met Prof. Modesto Panetti. In 1923, he was selected as full professor of Applied Mechanics at the School of Engineering of Pisa, where he remained until his death in 1968. In Pisa, he was also Dean of the Faculty of Engineering from 1944 to 1965. His scientific production is mainly devoted to Applied Mechanics and, above all, to Aeronautics, thanks to the collaboration with prof. Panetti, who in Turin had organized the first “aeronautical specialization course” and created the first wind tunnel in Italy (the fourth in the world). The mathematical perfection of 2D-hydrodynamics achieved in Italy thanks to the research of Levi-Civita and his school, had produced a gap compared to European countries, where there was more attention to applications. Pistolesi was the first to introduce the Prandtl's vortex theory of the finite span wing in Italy in 1921, applying it to the study of the propeller as reported in this issue of Volume 100. His research on propellers was then directly linked to those carried out on “fluid machines” (turbines, pumps and compressors) and also those related to the fluid dynamic field around an infinite array of thin profiles.

He was a member of the Accademia dei Lincei and a Pontifical Academic. In Pisa he also held the offices of Mayor of

the city and president of the Pisa province. He was also the founder and the director of the “Antonio Pacinotti” Higher School of Applied Sciences (Scuola S. Anna, today). Prof. Pistolesi was the Editor in Chief of *Aerotecnica* from 1924 to 1967.

1 Vortex Theory in Aerodynamics

Enrico Pistolesi

Presented on January 25th, 1921.

It is well known that physicists' ideas about the phenomenon of the actions that a fluid exerts on a body moving inside it has undergone a fundamental evolution. We also know that the main cause of this evolution has been precisely the experience tenaciously pursued to solve the age-old problem of human flight. Against the ideas of certain theorists attached to the old Newtonian views, according to which even swallows could not fly, the pioneers of mechanical flight have shown that flight is possible. Therefore, they are obliged to a profound revision of the old ideas in this field.

According to Newton, the action of a fluid on a moving body was due to fluid particles bouncing off the surface of the body, such as elastic spheres. It follows, for example, that the thrust on a sheet should be proportional to the square of the sine of the angle of attach.

Euler, in his research on fluid drag, came to the conclusion, completely disproved by the experience, that the action of a perfect fluid on a moving body should be zero.

It was not only in 1902 that Kutta imposed, in a particular case (the drag of an indefinite circular cylinder) the problem in its true form, which—generalized later by Joukowski—took the name of Kutta–Joukowski's theorem.

Let us say right away that the error of the ancient ways of considering the phenomenon was essentially that of supposing a conformation of the fluid stream around the immersed body very different from reality.

To explain clearly and fix the ideas, we certainly take into account the problem of the drag of an indefinite cylinder with the generators perpendicular to the asymptotic

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Fig. 1 (From the original paper)

direction of the stream, that is to say to the direction that the stream has at a very great distance from the cylinder, Fig. 1.

Now, let us see a closed circuit enclosing the cylinder inside and determining the circulation of the velocity along this contour. As it is well known, the circulation of a vector (V) along a contour is the integral, extended to the contour, of the product $V \cos(\alpha) ds$ being the contour path run in the positive sense.

This circulation Γ is, by the Stokes' theorem, the same for any contour including the cylinder.

Now, by the Kutta–Joukowski's theorem, the lift (thrust) per unit length exerted by the fluid on the hull is given by the product:

$$aV_0\Gamma,$$

where a is the density of the fluid, V_0 is the asymptotic speed, and Γ is the circulation.

The direction of the thrust is perpendicular to V_0 , i.e., perpendicular to the asymptotic speed vector or, if the body is in motion within the originally motionless air, to the direction of the motion. In other words, there is thrust, but also not drag. If then $\Gamma = 0$, the thrust is zero as well.

Euler's error consisted precisely into assuming Γ always equal to zero, i.e., always zero circulation.

Now, let us suppose to remove the cylinder and to replace it with the fluid in such physical and mechanical conditions that the stream conformation is not altered around it. In this hypothesis, the Stokes' theorem warns us that in the fluid so arranged there must be vortexes. Stokes' theorem states that circulation along any closed contour that can be reduced to a point without leaving the fluid is equal to twice the sum of the intensities of vortex tubes enclosed by the contour.

As far as the conformation of the stream is concerned, the infinitely long cylinder can be replaced by a system of infinitely long vortex tubes contained within the space occupied by the solid, with such intensity that their sum equals half of the circulation. These vortexes will be called, with Prandtl, "adherent vortexes". Indeed, when we consider the fluid stream at such a large distance from

Fig. 2 (From the original paper)

Fig. 3 (From the original paper)

the solid that the transverse dimensions of it are negligible compared to the distance, this one can be replaced by a single vortex tube, whose intensity equals to the half the circulation Γ .

The replacement of a system of vortexes to the body immersed in a fluid in motion is more than a simple mathematical artifice. If, as experience seems to confirm, in immediate contact with a surface the relative velocity is null, while at a very small distance from the surface itself, there is a finite relative speed, there is a surface of discontinuity always containing vortexes. In this hypothesis, a very thin vorticity layer will form around the body, which we can conceive as a belt of vortex filaments (Fig. 2) each bringing the contribution $V ds$ to the circulation, where ds is an elementary portion of the body contour.

As already observed, if the Kutta–Joukowski theorem gives an explanation of the phenomenon of the thrust normal to the motion, it leaves in the dark with regard to drag to advancement, i.e., drag parallel to the direction of motion, which, according to the theorem itself, should be zero, apart from the contribution due to viscosity.

In this case too, the clear discrepancy with the evidence of experience (that is not removed, in general, taking friction drag into account) derives from an imperfect representation of the phenomenon. The error, this time, is to assume that the stream is a direct and uniform one, i.e., invariable over time. The reality is different, and Karman has the merit to have put it in its true light. He, in a 1911 paper, revised in 1912, assumed that the drag to advancement of an indefinitely long cylindrical body in a stream is due to the vortexes which originate in the body stream and take on the appearance of a double array of zigzag vortex tubes (Fig. 3).

The vortexes obtained in the water, in the tank, in experiences performed at the Aerodynamic Laboratory of the

Imperial School in Moscow, show that real things are proceeding in this way. Similar results were obtained by Prandtl in Göttingen.

Karman showed that only the zigzag arrangement is stable and determined the relationship between the distance h between the two rows and the distance l between two successive nuclei of each row. This ratio was found in the first memory (1911) equal to 0.367; in the second one (1912) he modified the procedure used before and found $h/l = 0.283$.

This is a point that deserves further study, since we do not have the experience to give reliable indications in this regard. Now, the vortex double row does not stand still with respect to the stream, but is provided with a uniform translational motion, opposite to the stream motion and the velocity u of the relative motion is $\frac{I}{l\sqrt{8}} = 0.354\frac{I}{l}$, where I is the intensity of each vortex of the two rows.

Then, applying the theorem of momentum and taking also the pressure differences into account, called L the chord of the wing (or in general a characteristic dimension of the immersed body), you will have

$$\chi_x = K_x/a = \left[0.799 \frac{u}{V_0} - 0.323 \left(\frac{u}{V_0} \right)^2 \right] \frac{l}{L}.$$

This formula agrees with the experimental data very well, but unfortunately, it depends on the $\frac{u}{V_0}$ and $\frac{l}{L}$ ratios, which cannot be determined by pure calculation until now. However, it is time to move on to an interesting application of vortex theory to the case of finite length (i.e. span) wings.

Consider a flat circuit c embracing the wing; on it, there will be a circulation Γ corresponding, approximately, to the value of the thrust coefficient K_y relevant to the elementary portion of the wing corresponding to the circuit c ; that is

$$\Gamma = K_y / a l V_0,$$

where l is the wingspan. Now, let us move the circuit c out of the wing tip, in c' . Since the thrust is null in this position, we could conclude that the circulation in c is zero.

However, the Stokes' theorem tells us that a variation of circulation along a closed contour moving in a fluid can occur only in the case that the contour crosses some vortex pipes, whose intensity is the half of the variation of circulation. It follows that circuit c must have cut by vortex pipes of an overall intensity equal to $\Gamma/2$. If circulation Γ remains constant along the wing, i.e., if the thrust is evenly distributed over the entire span L of the wing, the vortices should detach from the wing tips (Fig. 4), forming two vortex tip plumes.

More generally, Γ (and, therefore, K_y) will decrease from the centerline to the tips of the wing, and thus, they will have to start vortex filaments from the wing for all, or almost, its span (Fig. 5). Thus, if moving from z to $z + dz$,

Fig. 4 (From the original paper)

Fig. 5 (From the original paper)

the circulation varies from $d\Gamma$ to $(\Gamma + d\Gamma)$, a vortex filament of intensity $1/2 d\Gamma$ should detach at point z .

It can be proved that the axis of these vortex filaments has the direction of the stream, i.e., the vortices follow the streamlines (fluid filaments). Therefore, they are straight

Fig. 6 (From the original paper)

Fig. 7 (From the original paper)

and parallel to the stream direction. We will call them “free vortexes” as opposed to the “adherent vortexes” defined before.

They produce a special velocity field, and we need the Biot–Savart formula to calculate it. The velocity induced at point P by a vortex element ds of intensity I (Fig. 6) is given by

$$\frac{I \, ds}{2\pi} \times \frac{1}{r^2} \sin \varphi,$$

where r is the distance of the point from the vortex element and φ is the angle that the vector, which goes from the element to the point, and makes with the axis of the vortex.

The direction of this induced velocity is perpendicular to the plane containing the point and the vortex axis. It is easy to deduce that the velocity induced by a segment AB of a straight vortex (Fig. 7) is given by the following expression:

$$\frac{I}{2\pi a} (\cos \varphi_1 + \cos \varphi_2).$$

In particular, if it is a half-line (Fig. 8), the speed induced at a point P located in the normal plane through A will be expressed by

Fig. 8 (From the original paper)

Fig. 9 (From the original paper)

$$\frac{I}{2\pi a},$$

and the one induced at a P point very far from that plan will be

$$2 \frac{I}{2\pi a} = \frac{I}{\pi a},$$

the double of before.

Applying these concepts to the free vortexes flowing from the wing tips, we can conclude that a particular velocity, due to the action of these vortexes, is added to the speed that would be obtained in the case of a wing with an infinite span. This speed increase is not generally constant along the whole span; constancy is obtained when it is supposed that the thrust is distributed elliptically along the span, that is, the thrust diagram is represented by a semi-ellipse, as shown in Fig. 9.

In this hypothesis, the effect of free vortexes is to tilt down the stream of an angle:

$$\varphi = \frac{2}{\pi} \chi_y \frac{S}{L \frac{I}{2\pi a}},$$

where S is the area and L the span of the wing. If we define as aspect ratio the following

$$\frac{L^2}{S},$$

and we indicate it with λ , it results

Fig. 10 (From the original paper)

$$\phi = \frac{2}{\pi \lambda} \chi_y,$$

This angle, in short, is equivalent to a rotation of the entire stream from V_0 in V'_0 (Fig. 10) and has two effects: a decrease of the incidence, represented precisely by ϕ and an angular variation of the resultant by an angle equal to ϕ with respect to the direction of the asymptotic stream.

Projecting the new overall resultant along V_0 and the normal to V'_0 , since ϕ is small, one can disregard the influence of the inclination on the F'_x component and we have, by approximation:

$$F_y = F'_y,$$

$$F_x = F'_x + F_{xi},$$

where F_{xi} (induced drag) is the projection of F'_y on V_0 . It follows that the effect of free vortices is to add an induced drag coefficient χ_{xi} to the one, χ_x , of the wing:

$$\chi_{xi} = \frac{2}{\pi \lambda} \chi_y^2,$$

that is

$$\chi_x = \chi_{x\infty} + \frac{2}{\pi \lambda} \chi_y^2.$$

As regards the incidence, to which it corresponds χ_y in the previous formula, it will be: effective, α_∞ , and apparent $\alpha_\infty + \phi$, that is

$$\alpha = \alpha_\infty + \frac{2}{\pi l} \chi_y^2.$$

This is the very interesting result of Prandtl; this result was then showed by Betz and experimentally verified by M. Munk at the Göttingen Laboratory.

For practical use, note that the formula:

Fig. 11 (From the original paper)

$$\chi_{xi} = \frac{2}{\pi \lambda} \chi_y^2,$$

is the equation of a parabola (Fig. 11) with a parameter proportional to the aspect ratio l . Thus, the polars of the same profile for two different aspect ratios are congruent each other by subtracting from each the ordinates of the relevant parabolas. Of course, from the foregoing we deduce the formulae for moving from one elongation to another one:

$$\chi'_x = \chi_x + \frac{2}{\pi} \left(\frac{1}{\lambda} - \frac{1}{\lambda'} \right) \chi_y^2,$$

$$\alpha' = \alpha + \frac{2}{\pi} \left(\frac{1}{\lambda} - \frac{1}{\lambda'} \right) \chi_y.$$

These results apply strictly only in the case of the elliptical distribution of the thrust, in which case it is shown that the minimum value of the induced drag is achieved; but they apply by approximation in all cases of practice.

Fig. 12 (From the original paper)

The combination of end and wake vortices (zig, zag) has a characteristic aspect of ladder,¹ Fig. 12.

This was directly observed at Mc Cook Field, USA where interesting experiments were carried out using a small wind tunnel, where it was possible to reach the speed of 200 m/s. By condensation of water vapor and appropriate lighting it was possible to see and photograph the vortices.

However, the applications don't finish here. Vortex theory extends to the determination of the mutual influence of two parallel wings, as we have in a biplane. Without introducing formulae, which would not be beneficial to the clarity of the exposition, let us show the principle on which this determination is based.

¹ The figure is taken from Report No. 28 of the National Advisory Committee for Aeronautics (Washington 1920) George De Bothezat: *An introduction to the study of the laws of air resistance of aerofoils*.

Fig. 13 (From the original paper)

We replace each wing with the set of an adherent vortex, parallel to the same wing, and two extremity-free vortices (Fig. 13). The arrows indicate the angular velocity vector for vortices, both free and adherents.

Now, if we want to determine the perturbation that the wing, e.g., lower, exerts on the upper one, we have to consider the speed range produced at the upper (indeed each point of the upper wing) by the free and adherent vortices of the lower wing. This influence can be divided as follows:

1. Variation of the horizontal speed;
2. Variation of the vertical speed (and therefore of the incidence);
3. Curving of the streamline.

The first one will increase the thrust and drag coefficients in the same ratio; the second one will change the incidence. As for the third one, assumed that a curvature produced in the current keeping the wing unchanged is equivalent to a curvature of the wing (opposite to the first one) keeping the current unchanged. It is necessary to know experimentally how the coefficients of thrust and resistance vary by changing the average curvature of the wing. This is passably known for the thrust, but unknown for the resistance.

The adherent vortex produces an increment of the horizontal velocity on the upper wing and, if this is not exactly over the lower one, also a vertical component; free vortices produce both an increase in the vertical velocity and a speed parallel to the generatrix of the wing, which is not taken into account. Both free vortices and adherent vortices produce a change in the curvature of the stream.

The previous research is due to Betz. Later on also Fuchs, Munk, Sanden, Wieselberger and Prandtl himself have worked extensively on the same topic. This led to the production of formulae suitable for practical use and to extend the previous considerations to the case of the elliptical distribution of thrust over the wings.

Fig. 14 (From the original paper)

It is interesting to remark a theorem of reciprocity that links the mutually induced drags of the two wings. If the two wings are located in the same plane normal to the direction of the stream, the drag induced by the lower wing on the upper wing equals the drag induced by the upper wing on the lower one; if the aforementioned circumstance does not occur, that is, if the wings are scaled, the two induced resistances are no longer equal to each other, but their sum is yet constant.

Also interesting is the application of the previous research to the determination of the influence that an infinite plane parallel to the wing exerts on a wing itself, e.g., the ground (Fig. 14) and it can be easily proved that this influence is equal to a wing positioned symmetrically to the first, as in a mirror.

These considerations may be extended to the case where the wall is not flat, and everyone can understand the great importance they present in relation to tests on models in the wind tunnels.

It remains to be said how previous research can be usefully applied to the theory of propellers.

In Germany, this application has been initiated by Prandtl and his collaborators, but this research is not known among us yet.

We believe, however, that they will not be very different from what we are going to set out briefly.

Let us first consider, to fix the ideas, a two-bladed propeller and suppose these very thin ones (Fig. 15), so that we can neglect their transversal dimensions.

Now, if we conduct a closed circuit around one of the blades, we will have on it a circuit Γ which (assimilating an elementary section of the blade to a section of a wing), multiplied by the speed, the density and by dr will give the elementary action on the section dr of the blade.

This circulation will vary, generally speaking, along the blade; if it remains constant, for considerations similar to those for the wings, a swirling plume of intensity $\Gamma/2$ will escape from the tip A of the blade, and similarly from B; in addition, a swirling plume of intensity $2\Gamma/2 = \Gamma$ should start from O.

Fig. 15 (From the original paper)

The latter one, for obvious reasons of symmetry, will extend straight; as for the other two, it is likely that an infinite distance they have a helical trend, but it is not easy to determine exactly what trend they will have in the vicinity of the blades.

Of course, the pitch of these helical vortices will not be strictly the same of the propeller advancement, because they have a proper motion in the opposite direction to the rotation of the propeller.

If we consider the velocity field induced by these vortices on the propeller disk and deduce its average axial and tangential velocity, we can easily see that:

1. The increment of the average axial velocity due to adherent vortices, both on the entire disk and for each radius, is zero;
2. The average increment of the tangential speed produced by the external free vortices is zero;
3. Therefore, the average increment of the tangential speed is that produced by the central vortex;
4. The average axial increment at the disk is only that produced by the two external free vortices and depends on the fact that they are tilted on the direction of advancement.

However, if we want to consider a simpler situation, let us look at the classic case of propeller with infinite blades with infinitesimal width, able to produce a wake absolutely congruent to itself with respect to a rotation around the axis.

This wake looks like an air mass animated by a speed of translation and a speed of rotation, one and the other generally variable with the radius. This mass of air is not cylindrical, but after a contracted section $S'-S'$ at a certain distance from the propeller (Fig. 16) it increases in section

Fig. 16 (From the original paper)

Fig. 17 (From the original paper)

due to viscosity. To simplify, we consider a strictly cylindrical wake and, also, we assume that the axial velocity is constant at all the points in the wake and equal to $V + v$; we also assume that the velocity, tangential and also constant, i.e. the angular velocity (ω) is inversely proportional to the radius r .

On the surface which limits externally the wake we will thus have a speed jump, equal axially to v , tangentially equal to $R\omega$, which will rise to a swirling layer, that can be ideally decomposed into a system of ring vortices (Fig. 17), due to the difference in axial velocity and in a system of vortices arranged according to the generatrices, due to the difference of tangential speed (Fig. 18).

Fig. 18 (From the original paper)

By studying the velocity field produced by these vortices, it can be shown that the first system, in front of the propeller—i.e., at the plane of it—produces an increment of the axial velocity equal to the half that appears at an infinite distance from the propeller, that is, the total increase, whereas the second system does not produce any increase of speed.

Moreover, in addition to the previous ones, there is still an axial vortex (Fig. 18), the total intensity of which equals the sum of the intensities of the second system. It produces an increment of the velocity tangential to the disk equal to half the total increase (i.e., at an infinite distance from the disk); this vortex together with the second system produces no speed increase outside the wake, in total.

Finally, it should be considered that the two bands of the propeller disc have different tangential velocities, which gives rise to a system of adherent vortices with radial axes (Fig. 19) which are exactly equivalent of circulation Γ in the case of the two-bladed propeller.

These vortices produce: (i) no axial velocity increment and (ii) an increment in tangential velocity at the disk opposed to that produced by the central vortex.

In summary we have, on the disc:

1. No increment in tangential speed, because that due to radial, vortices equal the speed induced by the axial one.
2. An increment of the axial velocity equals to the half the total increment at infinite.

These results even though have been obtained by simplifying hypotheses (however, they could be extended to

Fig. 19 (From the original paper)

Fig. 20 (From the original paper)

cases, where the distribution of velocity is different from that in question) sheds a vivid light on the most debated question on the theory of propellers. Let the audience allows us to put it out in a nutshell.

It is known that the only way to calculate the thrust and power of a propeller according to its design is to assimilate the element of blade to an element of wing (Fig. 20).

Once the characteristic coefficients of the profile and magnitude direction of the relative velocity are known, you can calculate the elementary action which, decomposed in the axial and tangential components, allow to calculate the thrust and the torque.

Fig. 21 (From the original paper)

This method, due to Drzewiecki, was first applied by simply taking for the relative speed the speed W composed of the axial speed, V , and the rotational speed $r\Omega$.

Later, however, various researchers in Aerodynamics, notably Fage, Riach and later De Bothezat and others, questioned the correctness of this assumption and argued that the increase of the speed, both axial and tangential, had to be considered in order to obtain the true value of W and the true incidence (see Fig. 20). This theory by English and American authors called Inflow Theory ("teoria del risucchio" in Italian language).

The conflict of different opinion in this field is far from being over and this learned dispute is still going on between Drzewiecki, De Bothezat and others and reported on several the treaties and journals. There is also a third intermediate opinion, between Drzewiecki's theory and the Inflow Theory. This is supported by Wood and others and to this opinion agrees, in his latest book, Drzewiecki: the resulting method of calculation is called cascade method and consists in assimilating the propeller to a multiplane of infinite planes (Fig. 21), each of them influencing the other, like the wings of a multiplane.

I believe that the points made earlier make it possible to say a word, perhaps decisive, on this subject.

These considerations tell us that the relative speed must indeed be calculated by taking the increases into account and that the assumption, which was also made by this author at the time, that these increases are simply due to the modification that the presence of the wing produces in the velocity field is unfounded: in other words, that they are due to adherent vortices. In fact, we have seen that the two semi-increments $V/2$ and $\omega/2$ are due instead to free vortices which re-inclose the wake; they are therefore comparable in all to the vertical velocity produced by the end vortices in the case of the finite span wing. It should be noted that, as far as the tangential increase is concerned, this effect must be considered, despite the fact that it is apparently cancelled out by the action of the adherent vortices.

The proponents of the Inflow Theory are therefore right: but only in part, because, to take proper account of the increments, it is necessary to refer to the χ_x and χ_y that compete with the wing of infinite aspect ratio and not already, as

those authors have always practiced, to the experimental results performed over wings of finite span.²

Regarding the cascade method, it also has a foundation of truth in this respect, since the reciprocal disturbances of the wings of a multiplane are also exactly the effects of the speed increases produced by the vortexes, both free and adherent.

Based on these concepts, when the increments are small, the value of the relative velocity may also be left unchanged and equal to W , provided to define an appropriate elongation to the section considered. From this point of view, the inflow theory is not different from the Drzewiecki's theory.

The elongation thus will become, instead of a global feature of the propeller, a function of the radius.

Of course, when the propeller has, as in practice, a finite number of blades, things get very complicated, the

increments are not constant on the entire disk, but vary from point to point and the influence of the adherent vortexes is no longer negligible.

However, the principle set out may also apply by approximation to this case, possibly with the use of correction coefficients.

We believe that these deductions are well founded; in any case, they will have to be carried out further and the results submitted to the inexorable examination of the experimental test.

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² Generally, this elongation was assumed to be equal to the average elongation of the blade.