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Mercator's Geometric Method in the Construction of His Projection from 1569

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Abstract
The geometric method Mercator used to construct his projection from 1569 was reconstructed. Modern mathematical means are required for the computational creation of Mercator's projection. It could be proven that Mercator worked out his projection exclusively with compass and linear. The methodological principles established by Ptolemy were used and further developed by Mercator. However, he did not describe his method and so an enigma of Mercator projection emerged.

Keywords History of map projections · Ptolemaios · Mercator · Rectification of the circle · Rhumb lines

Mercators Geometrische Methode bei der Konstruktion seiner Projektion von 1569

Zusammenfassung
Die geometrische Methode Mercators, die er bei der Konstruktion seiner Projektion von 1569 anwendete, wurde rekonstruiert. Es wurde nachgewiesen, dass Mercator seine Projektion, zu deren rechnerischen Erstellung moderne mathematische Mitteln erforderlich sind, ausschließlich mit Zirkel und Linear erarbeiten konnte. Dabei wurde von Mercator die von Ptolemaios erarbeiteten methodische Grundlagen zur Projektionslehre verwendet und weiter entwickelt. Seine Verfahrensweise beschrieb er jedoch nicht und so entstand ein Rätsel der Mercatorprojektion.

1 Introduction[1]

In 1569, Gerhard Mercator (1512–1594) published a world map with a new projection that is still used today, especially in maritime cartography. In modified form, it is the basis of official surveying in the Federal Republic of Germany (Mesenburg 2004, p. 186). With its rectilinear parallels and likewise rectilinear meridians, this projection resembles the projection of Marinos of Tyros (ca.70–130 AD). The critique of this projection inspired Klaudios Ptolemaios around 150 AD to construct better projections. However, Mercator's projection differs from Marinos' projection in that in it the distances of the parallels are uneven. The distances are successively increased as the latitude increases. Although this abandoned the principle of similarity intended by Ptolemy, it did optimise the functionality of the map for navigation, since the rhumb lines in Mercator's projection were represented as straight lines. Mercator was unable to create his projection mathematically, as the appropriate mathematical means were not yet available at the time. Consequently, it seemed inexplicable how Mercator could construct his projection using only a compass and ruler. He did not describe the method he used. Thus the enigma of Mercator's projection arose, for the solution of which several suggestions were made.

2 Hypotheses to the Construction of the Mercator Projection

There are numerous publications on the Mercator projection. A good overview of the Mercator projection in the history of projection theory of the sixteenth century was

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[1] Thanks are due to Frank Dickmann for the suggestion to elaborate this study in this detailed form.

given by John P. Snyder in 1993. Among others, reference was made to Erhardt Etzlaub (ca. 1460–1532) and Edward Wright (1561–1615) (Snyder 1993, pp. 47–49). Etzlaub used a similar projection to Mercator in a small map in 1511–13, increasing the distances between parallels. Wright first explained the mathematical basis of Mercator's projection in 1610. A very detailed presentation of the Mercator projection from a geodetic and cartographic point of view is given in the study by Peter Osborn (Osborne 2013).

Several hypotheses have emerged on the construction method of the Mercator projection. Raymond D'Hollander (2005) and Friedrich Wilhelm Krücken (2011) have published detailed analyses of these hypotheses. For this reason, only a chronological listing of the proposed solutions is given here: Nordenskiöld (1889), Breusing (1892), Günther (1908), Averdunk/ Müller-Reinhard (1914), Wagner (1915) Eckert (1925), Diercke (1929), Marguet (1931), Krücken (1994a, b, 2011), D'Hollander (2005) and Gaspar (2014). This list does not claim to be exhaustive. It is only meant to hint at the multitude of proposed solutions.

The critical examination of the hypotheses can only be limited to a few basic remarks at this point. It can be ruled out that Mercator would have worked out his projection with the help of rhumb tables. To convert curved rhumb lines into straight lines, one needs the differential calculus, which, however, was not invented until the end of the seventeenth century (Wußing 2008, p. 353, pp. 427–468). Pedro Nuñez (1502–1578) developed the first approach to theory of rhumb lines (Nuñez 1566). However, his work does not contain formulae, but sentences with mathematical relations (D'Hollander 2005, p. 51). Curiously, Nuñez published only a blank table[2] (Nuñez 1566, p. 172). The view that Mercator pragmatically derived the rectilinear representation of the rhumb lines from his terrestrial globe (1541) cannot be understood cartographically either. The prerequisite of constructing the projection only with compass and ruler is best fulfilled by the proposal of Friedrich Wilhelm Krücken (Krücken 2009, p. 160), for which he published the basic principles as early as 1994 (Krücken 1994a). However, this method does not fulfil the requirements of a geometric solution for two reasons. Firstly, the irrational and transcendental circle number π is used here. Secondly, the increasing distances between the parallels deviate from the Mercator projection in such a way that the deviation in the direction of the pole always increases.

The general problem with the reconstruction attempts mentioned here is that the historical context of Mercator's cartographic way of thinking has not been sufficiently taken into account. The primary starting point was always the result, i.e. the projection structure, and less the possible methodological considerations with the help of which this result was achieved.

3 Sources for the Construction of the Mercator Projection

One source was written by Mercator himself. It is an explanation on the map of 1569 in Latin.[3] Here Mercator states his intention to construct a projection in which the meridians and the parallel appear in a new proportion to each other. However, he did not explain how he achieved this new relation. He has only hinted at his method, with the following sentence being of particular importance: "*Quibus consideratis gradus latitudinum versus utrumque polum paulatim auximus pro incremento parallelorum supra rationem quam habent ad aequinoctialem*" (Mercator 1569a). In Wikipedia, the translation is as follows: "*It is for these reasons that we have progressively increased the degrees of latitude towards each pole in proportion to the lengthening of the parallels with reference to the equator*". An earlier translation from 1926 by Emerson D. Fite and Archibald Freeman includes Snyder's study: "*Taking all this into consideration, we have somewhat increased the degrees of latitude toward each pole, in proportion to the increase of the parallels beyond the ratio they really have to the equator*" (Snyder 1993, p 47). The word "*paulatim*" is particularly noteworthy in this text, its translation, besides "*gradually*", "*progressively*" and "*successively*", can also be translated as "*piece by piece*" or "*one after the other*". Consequently, this text could be interpreted more conclusively as follows: "*Having considered the [length of the] latitudes in the direction of the both poles, we have one after the other increased the [length of the] latitudes according to their relation to the equator*".

A second source is Mercator's "Vita", written by Walter Ghim (1530–1611) shortly after Mercator's death. This biography was printed in the atlas with Mercator's maps published in 1595 (Ghim 1595; Geske 1962). Ghim, who was elected mayor of Duisburg several times, was a neighbour and also a friend of Mercator. He associated Mercator's method with squaring the circle: „*A very short time passed*[4] *when he published a new work, namely a very accurate map of the entire world, presented in very large format to the eyes and sight of the contemplating scholars, travellers and sailors, and presented it with a new invention, which is very suitable for expanding the surface of the sphere in the plane,*

[2] D'Hollander published this table with the values he calculated (D'Hollander 2005, S. 50).

[3] The Latin original and its English translation are available in Wikipedia: https://en.wikipedia.org/wiki/Mercator_1569_world_map.

[4] The short time here refers to the period that passed from the publication of the "Chronolgie" in 1568 to 1569.

and which so corresponds to the squaring of the circle[5] *that nothing seems to be missing except the lack of proof, as I have heard from his mouth several times.* "[6] (English translation of the German translation by Geske, pp. 50–51.) Squaring the circle with compass and ruler was considered an unsolvable problem since antiquity. Mercator's contemporaries also pointed this out (Nuñez, 1541, p. 46–50). In the reconstruction attempts, therefore, this source remained completely unnoticed. Only Krücken commented that the "squaring of the circle" mentioned here could have a relationship to Mercator's method (Krücken 1994b, p. 210). The reference to squaring the circle was considered a metaphor pointing to the difficulty of the problem. At that time, there was no clear distinction between the squaring of the circle, i.e. the transformation of the circle into an equal-area square, and the rectification of the circle into a square with the same circumference. Ghim's reference to the squaring of the circle could not be interpreted correctly for another reason. The knowledge that besides the true rectification of the circle, a modified rectification of the circle is also possible was no longer available (Pápay 2022).

In a true rectification, both the circle and the square must be present. Ptolemy devised a modified rectification of the circle in which the circle remains "invisible". This involved rectifying the small circles of the sphere in such a constellation that certain large circles were already rectified by flattening the half of the sphere (Pápay 2022). To "visualise" the small circles, one would have needed the π. For this reason it is also evident that here we are dealing with a modified but very remarkable rectification. Neither Ptolemy nor Mercator needed a "real" rectification to work out their projections. The modified rectification was completely sufficient for this. When Mercator communicated about his method with friends and acquaintances, he may have meant such a modified rectification of the spherical quadrilaterals. These quadrilaterals are enclosed by four circles, namely by two meridians and two parallels. Mercator's intention was thus to transform the spherical quadrilaterals into such rectilinear and circumferential quadrilaterals in which the proportions between the meridian distances and the parallel distances are maintained even after scaling. Mercator's statements on his projection were thus correctly notified by Ghim. By "squaring the circle", Mercator probably meant a modified type of rectification, for which no theoretical justification could be found in geometry. This discrepancy apparently caused him not only pride, but at the same time an ambivalent pensiveness.

4 Reconstruction of Mercator's Geometric Method

4.1 Mercator's Motivation

The study "Chronologia…" published in 1569 testifies to Mercator's profound knowledge of the science of antiquity (Mercator 1569b). Here Ptolemy is mentioned several times alongside Plato and Hipparchus. In the preface to this work, Mercator noted that he intended to write a work on cosmography. This plan was not realised, but it can be assumed that he did preliminary work on it. It can therefore be assumed that Mercator's method in creating his projection was conceptually influenced by ancient scholarship and especially by Ptolemy. When attempting to reconstruct Mercator's methodological approach, the question arose whether the search for a rectilinear representation of the rhumb lines was Mercator's only intention. The cartographic prerequisites for solving this problem were limited merely to the fact that this required straight lines of meridians arranged in even intervals and likewise straight lines of latitude, but arranged unevenly. It can be assumed that Mercator was also preoccupied with another view of the problem, the solution of which, using only compass and ruler, promised considerably more prospects of success. Mercator was also practically involved with Ptolemaic cartography. For several regional maps, Mercator used the same projection as Ptolemy, in which the meridians appeared as vertical lines and the parallels as horizontal lines. The relation between the meridians and the parallels varied according to latitude. Nevertheless, a good relation could only be achieved with regard to the middle parallel. The length of the middle parallel therefore had to be determined anew from map to map.[7] This meant that in a regional map with the distance of the parallel from the mid-latitude circle, this relation was distorted both in the direction of the north and in the direction of the south. Consequently, their projection was only useful for mapping relatively small areas. Regional maps with this projection could also not be merged. Thus, a different view of the problem may have emerged for Mercator, which, at least initially, may not have had any direct relation to the straightforward representation of the rhumb lines.

[5] What is meant here, of course, is not the compass but the circle.

[6] This passage in the source: „*Brevissimo temporis curriculo intercedente novum opus, scilicet universi orbis exactissimam descriptionem, in amplissima forma intuentium doctorum hominum ac peregrinantium et navigantum oculis conspectuique exhibuit atque proposuit – inventione nova et convenientissima sphaeram in plano extendendo, quae sic quadraturae circuli respondet, ut nihil deesse videatur, practerqauam quod demonstratione careat, ut ex illius ore aliquoties audivi.*"

[7] For a reconstruction of Ptolemy's geometric method, see Fig. 16 in (Papay 2022).

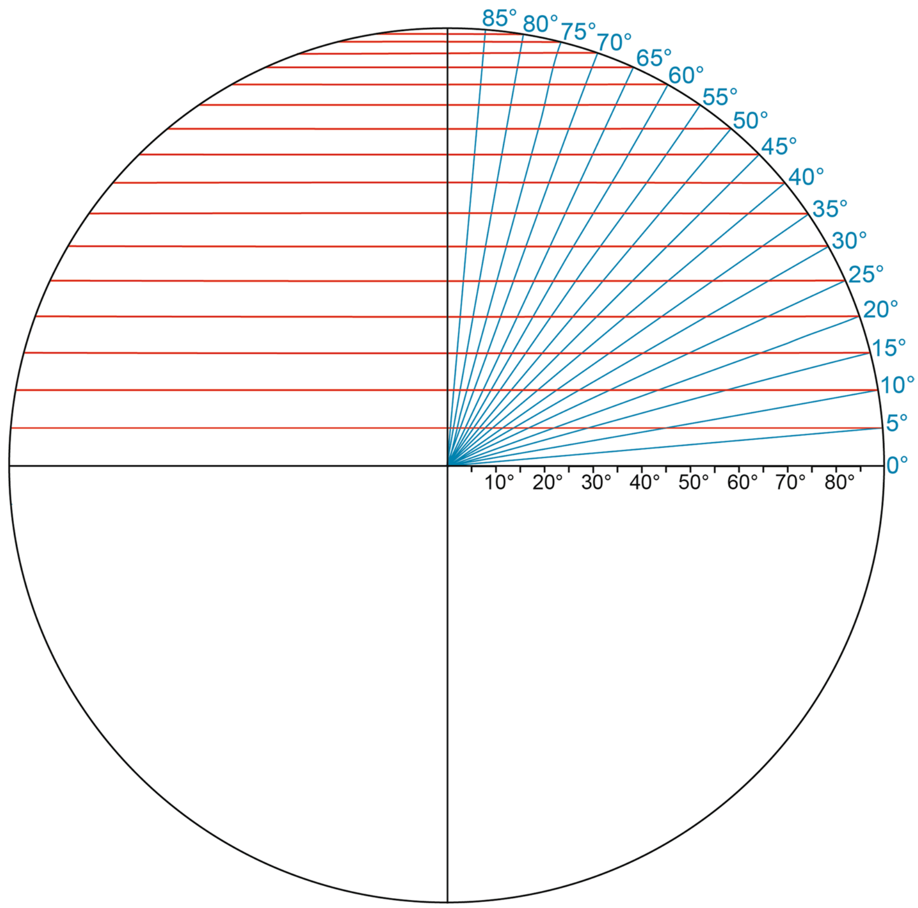

Fig. 1 The starting point is the transfer the image of the globe onto the flat surface, in which the diameter of 180 units corresponding to 180°. Here, only the central meridian and the equator are equidistant. The method of transformed great circle was already used by Ptolemy to create his projections and served him to rectify the parallels, i.e. to transform the parallels into straight lines whose length corresponds to the circumference of the parallels (cf. Pápay 2022, Fig. 8)

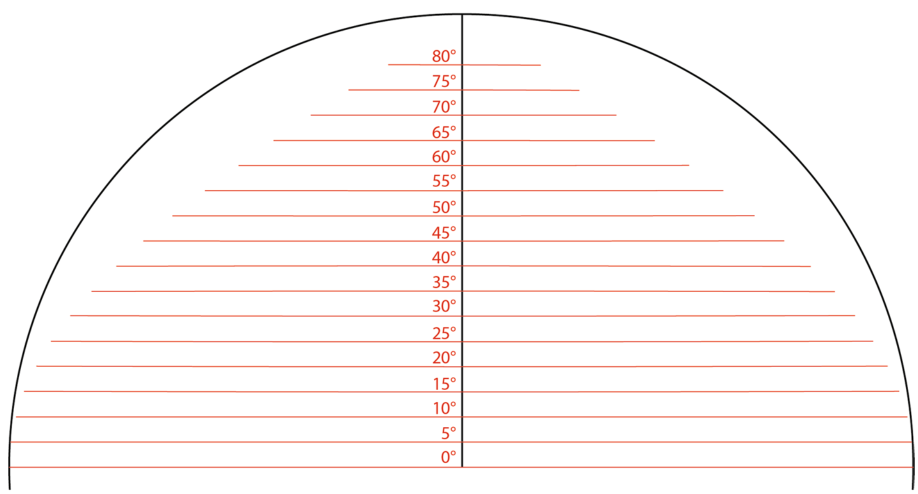

Fig. 2 The red lines represent the rectified parallels. Their length corresponds to half the circumference of the respective parallels. The modified rectification was possible because the transformed great circle contains two great circles, the equator and the central meridian, which are already rectified

4.2 Determining the Length of the Parallel

The problem was to create a map of the world with constant meridian distances, preserving the relations between the meridians and the parallels. This demanded the determination of the length of the parallels. Mercator had two ways of solving this problem, by mathematical calculation or by applying a geometric method. The formulas of spherical geometry required for this and for a simplifying calculation with trigonometric functions were already known to Hipparchus and Ptolemy. Accordingly, Mercator would be able to choose the mathematical calculation. However, he probably chose the more elegant path in Plato's sense by solving geometric problems with only a compass and ruler. The method for this was already used by Ptolemy in the construction of his projections, but not explicitly described. Recently, this

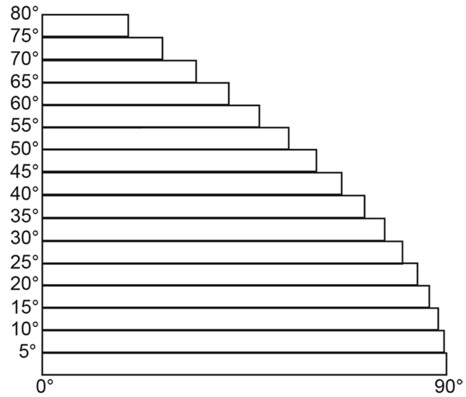

Fig. 3 Transforming the spherical quadrilaterals at 5° intervals into rectangular quadrilaterals with the same perimeter

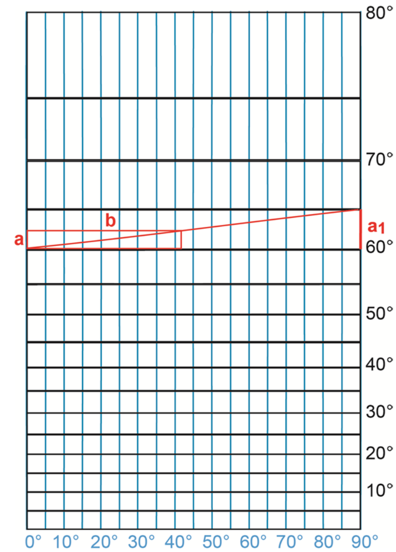

Fig. 4 The blue lines represent the meridians in Mercator projection. The red quadrilateral shows the rectified spherical quadrilateral between 60° and 65° before scaling. The side length of the converted quadrilateral denoted by "a" is 5 degrees, i.e. 5 units. By increasing the side denoted by "b" to 90 units, the distance between the parallel of 60° and 65° denoted by "a_1" is determined

method has been reconstructed (Pápay 2022). Since Mercator was very intensively involved with Ptolemy's theory, it seems very likely that he knew this method, which was based on the transfer the image of the globe onto the flat surface. This enabled him to rectify the circular arc of the parallels. Figures 1 and 2 present this method.

4.3 Determining the Increasing Distances of the Parallels

Mercator's real innovation consisted in working out the geometric method for determining the increasing distances of the parallels. To preserve the relation between the meridian distances and the parallel distances, it was initially necessary to convert the spherical quadrilaterals into rectilinear quadrilaterals. A spherical quadrilateral is a quadrilateral on the globe formed by the circular arcs of two meridians and two parallels. Two circular arcs are meridian segments whose length is constant, in the Fig. 3 it is 5 units corresponding to 5 degrees. There were several possibilities for the transformation of the spherical quadrilaterals into rectilinear quadrilaterals, e.g. they could have been transformed into trapezoids with a perimeter equal to the perimeter of the spherical quadrilaterals. Such a transformation seemed very tempting at first in the attempts to reconstruct Mercator's method, but then did not lead to the same increasing distances of the parallel as is the case in Mercator's projection. Further experiments showed that in order to achieve the angular accuracy of his projection, Mercator converted the spherical quadrilaterals into rectilinear quadrilaterals whose perimeter corresponded to the spherical quadrilaterals (Fig. 3). There are several possibilities, but only the following procedure leads to success. The length of the lower parallel and that of the upper parallel are connected and divided into two equal parts. These parts form both the lower and the upper boundary lines of the right-angled quadrilateral. The lateral boundary lines of the quadrilateral are always constant. In Fig. 3, the length is 5°, i.e. 5 units. In this figure, the length of the transformed spherical quadrilateral is 90°, i.e. 90 units. The transformations can be carried out geometrically, i.e. using only a compass and ruler. Mathematical calculations can also be used to perform the transformations. However, Mercator chose the geometric solution with a fairly high degree of certainty, as it corresponded to Plato's demand for geometry. The written sources mentioned above also suggest a geometrical approach rather than a mathematical calculation.

In the next step, the rectilinear quadrilaterals were enlarged so that their length was unified to the equatorial length, which in Fig. 4 is 90 units corresponding to 90°. The transformation of the quadrilaterals begins at the equator and continues successively in the direction of the pole, as Mercator described: "*Having considered the [length of the] latitudes in the direction of the both poles, we have one after the other increased the [length of the] latitudes according to their relation to the equator*" (Mercator 1569a). The resulting change in the height of the quadrilaterals determines the increasing distances of the parallels (Fig. 4). The scaled rectification of the spherical quadrilaterals made the rectilinear representation of the rhumb lines possible.

4.4 Determining the Parallel Distances Per One Degree

There were several possibilities for Mercator for determining the parallel distances per one degree. It is quite

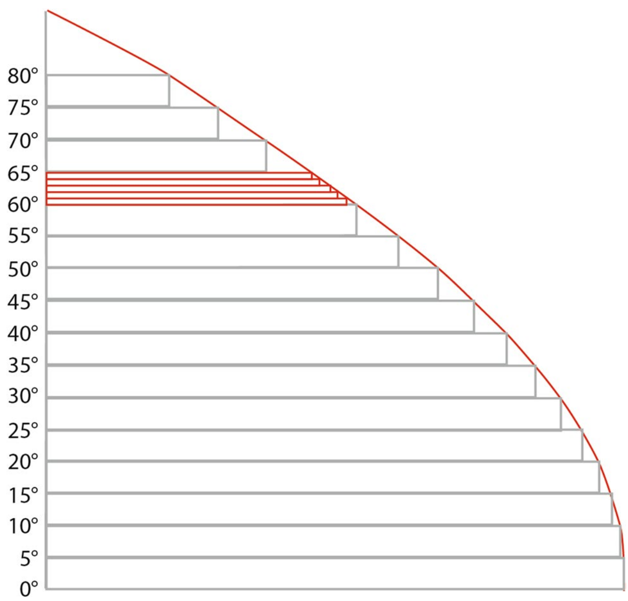

Fig. 5 Simplified method for determining the distances of the parallels per 1°. (Shown between the parallels of 60° and 65° as an example.)

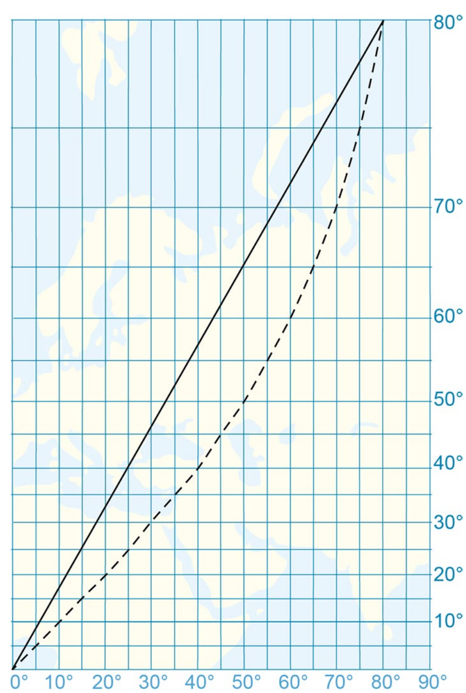

Fig. 6 Map in Mercator projection. The black solid line shows a loxodrome (rhumb line), the black dashed line an orthodrome

possible that Mercator determined the distances for each degree using the method described above. However, he could also have used a less complicated procedure, the principle of which is shown in Fig. 5. However, one cannot exclude the possibility that Mercator used the rhumb lines for the detailed division of degrees.

4.5 Map in the Mercator projection

Figure 6 shows the map in the Mercator projection with the degree grid of Fig. 4. Here it is shown that in this projection, the orthodromes appear as curved lines. The actually spiral-shaped loxodromes, the rhumb lines are shown as straight lines.

5 Comparison of the Mercator Map with the Mathematically Calculated Map Grid

Today, the mathematical calculation of the Mercator projection is based on the following equations: $y = \text{arc}\ \lambda * R/m$ $x = \ln \tan (45 + \varphi/2) * \text{R/m}$ (Mesenburg, 2004, S. 189). The theoretical values calculated in this way were used by Peter Mesenburg to examine the accuracy of the Mercator map. It was found that the distances of the latitude circles deviated slightly from the mathematically calculated nominal values, with increasing distance from the equator (Mesenburg 2004, p. 190). However, these differences are so small that the accuracy of the latitudinal distances also meets today's requirements (Mesenburg 2004, p. 190). From a theoretical point of view, however, such a systematic deviation is significant because it also confirms the construction method outlined above. In this method, the determination of the distances of the latitude circles begins at the equator and is continued successively in the direction of the pole. Very small deviations can thus add up to ever larger differences. In addition, the length of the quadrilaterals used to determine the distances of the latitude circles is always shorter in the direction of the pole. This means that slightly larger differences could potentially occur when determining the distances than in the area near the equator.

6 Conclusions

The geometric solution path for constructing a projection with rectilinear rhumb lines consists of two stages. In the first stage, the length of the parallel is determined. In the second stage, the maintenance of the correct relations between the length of the meridians and the parallel, with the extension of the latter to equatorial length, is determined. In Mercator's time, it was already possible to calculate both stages mathematically, but only separately, i.e. the calculation was only possible in two stages. A calculation with common formulae for both stages only became available after the invention of differential calculus. In solving the problem in the first stage, Mercator was able to use a special geometric method, the rectification of the latitude circles, which he could take over from Ptolemy. The solution to the problem situation in the second stage could undoubtedly only be found through geometric thinking. The geometric method used by Ptolemy and its use and innovative further development by Mercator remained unknown until now in both cartography and geometry. The reconstruction of these methods undertaken here illuminates the Mercator projection in a new light and questions the classification of this projection as a cylindrical projection. Mercator did not need a cylinder as an intermediate projection surface for the construction of his projection. For this reason, the question arises whether Mercator's projection can be called a cylinder projection only because of its similarity to it.

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