Motion analysis method of multibody system with contact and plastic deformation using linear complementarity problem

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Abstract

This study proposes a method to analyse the motion of a multibody system with components subjected to large plastic deformations by contact and collision. The method is based on the mathematical similarity between the non-smooth behaviour of the contact and collision phenomena and the complementarity associated with the nonlinearity of plastic deformation. The material nonlinearity of plastic deformation is described by the complementarity condition for stress and strain. An analysis method of rigid non-smooth contact and collision phenomena was previously reported to solve a linear complementarity problem by using the complementarity condition for state transition. In this study, by extending the concept of state transition in the previously reported method, the complementarity of stress and strain related to plastic deformation is expressed as a dynamics problem and formulated as a linear complementarity problem incorporating contact. Hence, the problem can be analysed considering both contact and yield phenomena by solving one linear complementarity problem. The proposed method is evaluated using numerical analysis.

Article Highlights

- Mechanical system with contact and collision and plastic deformation phenomena is formulated in a unified manner.
- The proposed method is based on an efficient formulation and analysis of the conventional contact problem.
- The effectiveness of the proposed method is demonstrated by numerical analysis for different cases and different time steps.

Keywords Contact problem · Plasticity · Linear complementarity problem · Unilateral phenomenon

List of symbols

$\mathbf{A}$	$n \times n$ square matrix	g_{N2}	Gap between Body 1 and Body 2
A_i	Cross-sectional area of Body i	$\dot{g}_N$	A column vector of $\dot{g}_{Ns}$ ordered from the smallest s in the row direction
$A_{m\alpha}, A_{n\alpha}$	Areas of the contact surfaces of body m and n at contact point α	$\tilde{g}_N$	A column vector of $\dot{g}_{Nj}$ ordered from the smallest j in the row direction
$\mathbf{b}, \mathbf{x}, \mathbf{y}$	n -dimensional vectors	$\dot{\mathbf{g}}_N^R, \dot{\mathbf{g}}_N^L$	Variables associated with gap velocity introduced for decomposition of state transition 2 to Upr function
f_i	Stress-strain relationship of Body i		
g_N	Gap i.e., normal distance between bodies		
g_{N1}	Gap between the ground and Body 1		

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$\ddot{\mathbf{g}}_N^R, \ddot{\mathbf{g}}_N^L$	Variables associated with gap acceleration introduced for decomposition of state transition 1 to Upr function	ε_{pi}	Plastic strain of Body i
G	Acceleration of gravity	ε_{pij}	Plastic strain corresponding to ℓ_{ij}
$\mathbf{h}$	External force vector	$\varepsilon_{pm\alpha}, \varepsilon_{pn\alpha}$	Plastic strains of portion of body corresponding to $\ell_{m\alpha}$ and $\ell_{n\alpha}$ associated with contact point α
ℓ_{ij}	Length from the centre of gravity to contact point j of Body i	e_s	Coefficient of restitution at g_{Ns}
ℓ'_{ij}	Length from the centre of gravity to contact point j after plastic deformation of Body i	$\check{\mathbf{e}}$	A matrix of e_s ordered from the smallest s in the diagonal direction
L_i	Vertical length of Body i	λ_N	Normal contact force
m_i	Mass of Body i	$\lambda_{N\alpha}$	Contact force at point α
$\mathbf{M}$	Mass matrix	$\hat{\lambda}_N$	Contact force vector
S_D	A set consisting of gap numbers in the Deformation Phase	$\bar{\lambda}_N$	Column vector of λ_{ND}
S_j	A set consisting of gap numbers in State transition 2	$\hat{\lambda}_N$	A column vector of λ_{Nk} ordered from the smallest k in the row direction
S_k	A set of gap numbers in contact	$\hat{\lambda}_N^R, \hat{\lambda}_N^L$	Variables associated with normal contact force introduced for decomposition of state transition 1 to Upr function
S_s	A set consisting of gap numbers in Expansion Phase	$\sim$	
t_a	Start time of time step for numerical integral	λ_N	A column vector of λ_{Nj} ordered from the smallest j in the row direction
t_e	End time of time step for numerical integral	$\check{\lambda}_N$	A column vector of λ_{Ns} ordered from the smallest s in the row direction
t_m	Time at the midpoint of $[t_a, t_e]$	Λ_{MEs}	Impulse in the Expansion Phase
t_A	Compression Phase start time	$\check{\Lambda}_{AM}$	A column vector of Λ_{AMs} ordered from the smallest s in the row direction
t_M	Compression Phase end time	$\tilde{\Lambda}_{AM}^R, \tilde{\Lambda}_{AM}^L$	Variables associated with impulse introduced for decomposition of state transition 2 to Upr function
t_E	Expansion Phase or Deformation Phase end time	$\check{\Lambda}_{ME}$	A column vector of Λ_{MEs} ordered from the smallest s in the row direction
u_i	External force applied to Body i	$\check{\Lambda}_p$	A column vector of Λ_{ps} ordered from the smallest s in the row direction
$\mathbf{W}$	Jacobian	σ	Compressive stress
$\mathbf{W}$	Jacobian in the Deformation Phase	σ_y	Yield stress
$\tilde{\mathbf{W}}$	A matrix of column vectors w_k ordered from the smallest k in the column direction	σ_{yi}	Yield stress of Body i
$\check{\mathbf{W}}$	A matrix of column vectors w_j ordered from the smallest j in the column direction	σ_{yi0}	Maximum σ_{yi} for $\varepsilon_{i=0}$
$\check{\mathbf{W}}$	A matrix of column vectors w_s ordered from the smallest s in the column direction	$\sigma_{ym\alpha}, \sigma_{yn\alpha}$	Yield stresses of body m and n at contact point α
x_R, x_L	Variables for decomposition of Sgn function to Upr function	Φ	Yield function
y_i	Displacement of the centre of gravity of Body i from ground	$\Phi_{m\alpha}, \Phi_{n\alpha}$	Yield functions of body m and n associated with contact point α
Y_α	Yield force at the contact point α		
$\tilde{\mathbf{Y}}$	Column vector of Y_D		
$\check{\mathbf{Y}}$	A column vector of the yield force Y_i ordered from the smallest j in the row direction		
$\hat{\mathbf{Y}}$	A column vector of Y_k ordered from the smallest of k in the row direction		
ε	Compressive strain		
ε_i	Compressive strain of Body i		
ε_p	Plastic strain		

Subscript

a	Values at the time of t_a
m	Values at the time of t_m
e	Values at the time of t_e
A	Values at the time of t_A
M	Values at the time of t_M
E	Values at the time of t_E

1 Introduction

For several years the scientific community has studied the role of plastic deformation in the shock absorption by a cellular solid. The honeycomb-structure metal and metal foam are porous materials, and therefore they are considered cellular solids [1]. The pores in the cellular solid are responsible for its lightweight and high specific strength. The cellular solid deforms plastically at a constant stress under compression. Thus, the cellular solid has high energy absorption properties.

A crush honeycomb metal, which is also a cellular solid, was used in the landing gears of the lunar probe Apollo (NASA) [2] and Mars probe Viking (NASA) [3]. The metal was used to fill the landing leg and it crushed to absorb the landing impact to achieve soft landing. In addition, recent advances in casting technology and metal three-dimensional additive manufacturing made it possible to fabricate metal foam with diverse mechanical properties and shapes [4–7]. Therefore, metal foam is being considered for application in various mechanical systems, including the bumper and crash box of vehicles as well as landing gear [8]. To realize a shock absorption mechanism that utilizes the plastic deformation of cellular solids, it is important not only to ensure sufficient energy absorption but also to have the desired motion after shock absorption. For this purpose, it is necessary to consider both contact and collision phenomena and plastic deformation. For example, the design of a planetary lander must consider the plastic deformation of the shock absorber due to collision with the ground during landing. In addition, it is important to understand the effect of collision and plastic deformation on the attitude of the vehicle, because the lander must land in the required attitude to accomplish the mission. In shock absorption experiments using plastic deformation, it is impossible to reuse the shock absorber. The development of such a shock absorption system in repeated experiments is time consuming and expensive. Furthermore, it is difficult to discern the behaviour of spacecraft on the ground because of the large influence of natural factors such as gravity and atmosphere. Therefore, to shorten the development period and reduce the costs, the behaviour of the mechanical system was studied using numerical simulation.

Finite element method and multibody dynamics models are often used to analyse the nonlinearities of plastic deformation and contact, respectively. Static and dynamic mechanical properties, energy and impact absorption of a cellular solid have been investigated using numerical analysis and experiments using finite element models [9–12]. Zhijun et al. [9], for example, developed a mesoscopic model of a closed-cell metal foam as a group of

ellipsoids based on X-ray CT images. The analysis closely agreed with the static and dynamic experimental results, and the fracture behaviour of the cell wall was clarified. Using the finite element method, it is possible to consider material properties and the nonlinearity of contact, and to analyse the fracture behaviour in detail. However, the analysis of motion of a mechanical system consisting of many elements is computationally taxing.

Multibody dynamics enables the efficient motion analysis of mechanical systems composed of multiple mechanical elements. There have been several studies on multibody systems with contact. The non-smooth phenomenon of contact can be analysed with the method developed by Pfeiffer et al. [13]. This method describes the state transitions related to contact through a linear complementarity problem and enables the accurate and efficient analysis of a system with multiple contacts. In addition to studies on formulation of mechanical system with contact, there have been many studies on systems subjected to different physical phenomena such as smooth dynamics (e.g., flexible deformation) and non-smooth dynamics (e.g., contact problem), the handling of such phenomena requires attention, and various analytical algorithms, including numerical integration methods, have been proposed [14–16]. On the other hand, Nikravesh et al. [17] developed a mathematical model using plastic hinges for a connection between rigid bodies in a multibody system. This method enables the motion analysis of a mechanical system with plastic deformation. Since this plastic deformation model was provided for the joints of a rigid body, the plastic deformation of a large member is represented by connecting several smaller rigid bodies with plastic hinges [18, 19]. Since this model is based on rigid bodies, the contact phenomenon is treated as a contact between those rigid bodies. Therefore, a force larger than that observed in the actual phenomenon may be involved during plastic deformation due to contact. In the actual phenomenon, the contact and plastic deformation have an interactive effect on each other, including energy dissipation. Furthermore, in a system with multiple contacts, contact and yielding phenomena at one contact point may affect other contact points. However, it cannot be considered in that model.

This study focuses on the complementarity of contact and plastic deformation. Plastic deformation does not occur when the applied stress is less than the yield stress but starts when the stress is equal to the yield stress. From this relationship, the nonlinearity of the yield phenomenon can be described as the complementarity condition of stress and plastic strain. Using the similarity between the complementarity in plastic deformation and that in the contact phenomenon in the method by Pfeiffer et al. [13],

a mechanical system with plastic deformation and contact was formulated as a linear complementarity problem in this study. The contribution of this study is an analytical method that considers the interaction between contact and plastic deformation with multiple contacts by unifying the two phenomena.

The remainder of this paper is organized as follows. Section 2 outlines basic set-value problems. Section 3 describes contact phenomena and plastic deformation by complementarity conditions. Section 4 introduces an analysed model and formulates the model as linear complementarity problems by use of the conditions given in Sect. 3. Section 5 discusses the verification of the proposed method by the numerical analysis results. Finally, Sect. 6 concludes the paper and notes future tasks.

2 Basic set-value element

2.1 Linear complementarity problem

The linear complementarity problem involves determining a pair of vectors that satisfy the linear equality and complementarity conditions [20, 21]

$$\mathbf{y} = \mathbf{Ax} + \mathbf{b}, \quad (2.1)$$

$$\mathbf{y} \geq \mathbf{0}, \mathbf{x} \geq \mathbf{0}, \mathbf{y}^T \mathbf{x} = 0. \quad (2.2)$$

In other words, the solution to be determined consists of $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^n$ for known $\mathbf{A}$, $\mathbf{b}$, where $\mathbf{A}$ is an $n \times n$ square matrix and $\mathbf{b}$ is an n -dimensional vector. Further, $\mathbf{x} \geq \mathbf{0}$ implies that all elements of vector $\mathbf{x}$ are greater than or equal to 0 [13]. For convenience, the following expression is introduced for Eq. (2.2)

$$\mathbf{0} \leq \mathbf{y} \perp \mathbf{x} \geq \mathbf{0} \quad (2.3)$$

where $\mathbf{y} \perp \mathbf{x}$ represents $\mathbf{y}^T \mathbf{x} = 0$. The linear complementarity problem can be efficiently solved by an optimisation method, such as the Lemke method [20].

2.2 Multi-valued function for complementarity

The Upr function, which is a multi-valued function for the complementarity, is shown in the following equation, and the relation is illustrated in Fig. 1a.

$$\text{Upr}(x) := \begin{cases} \{0\} & x > 0 \\ (-\infty, 0] & x = 0 \\ \emptyset & x < 0 \end{cases} \quad (2.4)$$

The complementarity condition in the linear complementarity problem can be expressed using the Upr function as follows [22, 23]

$$-y \in \text{Upr}(x) \Leftrightarrow y \geq 0, x \geq 0, xy = 0. \quad (2.5)$$

Sgn function is introduced as

$$\text{Sgn}(x) := \begin{cases} \{+1\} & x > 0 \\ [-1, +1] & x = 0 \\ \{-1\} & x < 0 \end{cases} \quad (2.6)$$

The Sgn function is illustrated in Fig. 1b. In the general signum function $\text{sgn}(x)$, $\text{sgn}(0) = 0$ is defined; however, $\text{Sgn}(0)$ takes a multi-value. In addition, $-y \in \text{Sgn}(x)$ can be represented as follows using two Upr functions by introducing x_R and x_L .

$$-y \in \text{Sgn}(x) \Leftrightarrow \begin{cases} -y \in +\text{Upr}(x_R) + 1 \\ -y \in -\text{Upr}(x_L) - 1 \\ x = x_R - x_L \end{cases} \quad (2.7)$$

Therefore, from Eq. (2.5), Eq. (2.7) can be represented as the complementarity condition.

$$-y \in \text{Sgn}(x) \Leftrightarrow \begin{cases} 1 + y \geq 0, & x_R \geq 0, & (1 + y)x_R = 0 \\ 1 - y \geq 0, & x_L \geq 0, & (1 - y)x_L = 0 \\ x = x_R - x_L \end{cases} \quad (2.8)$$

Equation (2.8) is illustrated in Fig. 1c. Thus, the problem represented by the Sgn function can be described by the

Fig. 1 Multifunction for complementarity: **a** Upr; **b** Sgn; **c** Decomposition of Sgn

complementarity condition. In this study, the state transition of contact and plastic deformation can be represented in the form of a Sgn function, which can be treated in a unified manner using the linear complementarity problem.

3 Complementarity conditions for contact and plastic deformation

3.1 Complementarity for contact phenomena

The contact state between bodies is represented by the Signorini's condition [24]. The relative normal distance between bodies is the gap g_N , and the normal contact force is λ_N , as shown in Fig. 2a. Based on the non-penetration condition, $g_N \geq 0$ is satisfied. When $g_N = 0$, the objects are in contact. Because there is no attraction between objects, $\lambda_N \geq 0$ is satisfied. When the objects are in contact ($g_N = 0$), a normal contact force greater than 0 is generated ($\lambda_N > 0$). On the other hand, when the objects are not in contact ($g_N > 0$), the normal contact force is zero ($\lambda_N = 0$). Therefore, the contact state is represented as

$$\begin{cases} g_N = 0 \wedge \lambda_N \geq 0 \\ g_N > 0 \wedge \lambda_N = 0 \end{cases} \quad (3.1)$$

Equation (3.1) is the complementarity, and the product of the gap and the contact force is 0. Thus, Signorini's condition is represented by the following complementarity condition, and the relation is illustrated as shown in Fig. 2b.

$$g_N \geq 0, \lambda_N \geq 0, g_N \lambda_N = 0 \quad (3.2)$$

However, in this condition, the impact and reaction forces during plastic deformation are not considered.

3.2 Complementarity for plastic deformation

This study targets materials such as metal foam, that exhibit shock absorption and other properties through a

Fig. 2 **a** Normal relative gap and normal contact force; **b** Signorini's normal contact law

Fig. 3 **a** Stress–strain curve under compressive load; **b** Simplified stress–strain curve

large compressive plastic deformation. Figure 3a shows the typical stress–strain curve for a generic metallic material under compressive stress loading and unloading. First, a load is applied that monotonically increases from 0 to σ_y ($O_0P_0Z_0$). Then, after being unloaded (Z_0O_1), it is loaded again, reaching a higher stress than σ_y ($O_1P_1Z_1$). That is, it follows the path $O_0P_0Z_0O_1P_1Z_1$ in the figure. O_0P_0 is almost a straight line in the elastic region; however, when it is beyond P_0 , the slope of the stress–strain curve changes significantly. When unloading at Z_0 , the strain recovers elastically to the unloaded state O_1 . However, O_1 is different from the initial state O_0 , and permanent strain is observed. This permanent strain is represented by the plastic strain ϵ_p . Then, when the load is reapplied, the material deforms linearly elastically in agreement with O_0P_0 . If unloaded before reaching P_1 , it will recover to its O_1 state, and no new plastic strain is generated. However, when load is applied beyond the elastic limit, i.e., beyond P_0 and P_1 , new plastic strain is generated.

By using these features, we formulated a mathematical model of a plastically deformable body. A few differences are observed between the Z_0O_1 and O_1P_1 paths in Fig. 3a; however, the differences are insignificant. Therefore, this difference was ignored, and it is assumed that Z_0 and P_1 are identical. In addition, the object of analysis is a material that exhibits properties such as shock absorption under large compressive plastic deformation, like metal foam. Thus, only the compressive strain is considered, not the tensile strain. Furthermore, since this study focuses on the case where the elastic strain is sufficiently smaller than the plastic strain, the object is assumed to be a rigid plastic body. That is, its plastic strain is equal to the total strain, and it behaves as a rigid body when it is in the elastic region. A simplified compressive stress–strain curve based on the stated assumptions is shown in Fig. 3b.

The compressive stress applied to the object is σ , the yield stress is σ_y , and the plastic strain is ϵ_p . When $\sigma < \sigma_y$, plastic deformation has not occurred yet. That is, the plastic strain rate $\dot{\epsilon}_p = 0$. On the other hand, when $\sigma = \sigma_y$, either the elastic unloading state ($\dot{\epsilon}_p = 0$) or the plastic

loading state ($\dot{\epsilon}_p > 0$) take place. These are formulated as the yield criterion in the following equation [25].

$$\begin{cases} \sigma < \sigma_y \wedge \dot{\epsilon}_p = 0 \\ \sigma = \sigma_y \wedge \begin{cases} \dot{\epsilon}_p = 0 & \text{(Elastic unloading)} \\ \dot{\epsilon}_p > 0 & \text{(Plastic loading)} \end{cases} \end{cases} \quad (3.3)$$

Here, the yield function Φ is introduced as.

$$\Phi = \sigma - \sigma_y \quad (3.4)$$

By using the yield function, Eq. (3.3) is represented in the complementarity condition of.

$$\begin{cases} -\Phi = 0 \wedge \dot{\epsilon}_p \geq 0 \\ -\Phi > 0 \wedge \dot{\epsilon}_p = 0 \end{cases} \Leftrightarrow -\Phi \geq 0, \dot{\epsilon}_p \geq 0, -\Phi \dot{\epsilon}_p = 0. \quad (3.5)$$

In the complementarity conditions of contact and plastic deformation in Eqs. (3.2) and (3.5), respectively, the collision phenomenon and the associated plastic deformation were not considered. Therefore, in the proposed method, which will be introduced later, these phenomena are expressed by expanding to the complementarity condition using acceleration and velocity of the gap and formulating the state transition of contact and deformation.

4 Formulation using state transition of contact and plastic deformation

Pfeiffer et al. [13] formulated the analysis method for the rigid body contact problem with focus on two characteristic state transitions associated with contact: transition from contact to separation, and transition related to

impact. In this study, plastic deformation was represented by introducing a state transition related to yield for each of these two state transitions. In this section, we formulate these two state transitions using a model composed of basic elements.

4.1 Definition of the model composed of basic elements

A unique feature of the method of Pfeiffer et al. is the possibility of efficient, accurate analysis when contact occurs at multiple points simultaneously. In this study, we assumed yield at one contact point affected the contact and yield at other contact points as well as the overall system. Therefore, this study introduces a basic model comprising two bodies and the ground, as shown in Fig. 4a. The two bodies are defined as Body 1 and Body 2, respectively; the gap between the ground and Body 1 is g_{N1} and the gap between Body 1 and Body 2 is g_{N2} . The mass of Body i ($i = 1, 2$) is m_i , the vertical length is L_i , the cross-sectional area is A_i , and the distance from the centre of gravity to the contact point j ($j = 1, 2$) is ℓ_{ij} . The external force of u_i is applied to Body i vertically upward, and the acceleration due to gravity G points vertically downward. Each body is a rigid plastic body with a different stress–strain model, and the ground is rigid. The stress–strain model for each is shown in Fig. 4b. Moreover, the displacement of the centre of gravity of each Body relative to the ground is defined as the generalised coordinate y_i .

4.1.1 Equation of motion

The relationship between the gap vector $\mathbf{g}_N = [g_{N1} \ g_{N2}]^T$ and the generalised coordinate $\mathbf{y} = [y_1 \ y_2]^T$ and its time derivatives is as follows:

Fig. 4 **a** Model composed of basic elements; **b** Model of stress–strain

$$\mathbf{g}_N = \mathbf{W}^T \mathbf{y} + \mathbf{c}, \quad (4.1)$$

$$\dot{\mathbf{g}}_N = \mathbf{W}^T \dot{\mathbf{y}}, \quad (4.2)$$

$$\ddot{\mathbf{g}}_N = \mathbf{W}^T \ddot{\mathbf{y}}. \quad (4.3)$$

Here, $\dot{*}$ and $\ddot{*}$ represent the first and second derivatives of $*$ with respect to time, respectively, $\mathbf{W}$ is the Jacobian, and $\mathbf{c}$ is a term associated with the size of the body:

$$\mathbf{W} = [w_1 \ w_2], \quad w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad (4.4)$$

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\ell_{11} \\ -\ell_{12} - \ell_{22} \end{bmatrix}. \quad (4.5)$$

The equation of motion for the system in Fig. 4 with the external and contact forces is.

$$\mathbf{M}\ddot{\mathbf{y}} - \mathbf{h} - \mathbf{W}\lambda_N = \mathbf{0}, \quad (4.6)$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{h}$ is the external force vector, and λ_N is the contact force.

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad (4.7)$$

$$\mathbf{h} = \mathbf{F}_G + \mathbf{F}_E = \begin{bmatrix} m_1 G \\ m_2 G \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad (4.8)$$

$$\lambda_N = \begin{bmatrix} \lambda_{N1} \\ \lambda_{N2} \end{bmatrix}. \quad (4.9)$$

The equation of motion in Eq. (4.6) represents both contact and non-contact states since the contact force vector is zero when no contact force is generated.

4.1.2 Definition of basic elements for plastic deformation

The stress-strain model for each body is derived using the following equation and is shown in Fig. 4b.

$$\sigma_i = \begin{cases} [0, \sigma_{yi0}] & \varepsilon_i = 0 \\ f_i(\varepsilon_i) & 0 \leq \varepsilon_i \end{cases} \quad (4.10)$$

Since the model is a rigid plastic body, the total strain ε_i is equal to the plastic strain ε_{pi} . Consequently, the yield stress σ_{yi} is represented as

$$\sigma_{yi} = \begin{cases} \sigma_{yi0} & \varepsilon_{pi} = 0 \\ f_i(\varepsilon_{pi}) & 0 \leq \varepsilon_{pi} \end{cases}. \quad (4.11)$$

In the contact problem, point contacts are considered, and the contact force is treated as a concentrated load. However, to consider the yielding phenomenon, it is necessary to treat the contact force as stress. We supposed that bodies n and m shown in Fig. 5a contact each other at point α . In the following equations, we consider the possibility of plastic deformation due to the contact force $\lambda_{N\alpha}$ at point α . It is assumed that bodies n and m are in contact through contact surfaces $A_{n\alpha}$ and $A_{m\alpha}$, respectively. In other words, point contact at the contact point α is regarded as surface contact between $A_{n\alpha}$ and $A_{m\alpha}$. It is also assumed that the contact surface itself does not deform. The previous considerations are introduced in Eq. (3.4) to express the yield function for the contact point α of each body:

$$\Phi_{n\alpha} = \frac{\lambda_{N\alpha}}{A_{n\alpha}} - \sigma_{yn\alpha}(\varepsilon_{pn\alpha}), \quad (4.12)$$

$$\Phi_{m\alpha} = \frac{\lambda_{N\alpha}}{A_{m\alpha}} - \sigma_{ym\alpha}(\varepsilon_{pm\alpha}), \quad (4.13)$$

where, $\varepsilon_{pn\alpha}$ is the plastic strain of $\ell_{n\alpha}$. As shown in Eq. (3.5), the yield function $-\Phi$ is greater than or equal to zero. By rearranging Eqs. (4.12) and (4.13), the following conditions can be obtained:

$$\lambda_{N\alpha} \leq \sigma_{yn\alpha} A_{n\alpha}, \quad (4.14)$$

$$\lambda_{N\alpha} \leq \sigma_{ym\alpha} A_{m\alpha}. \quad (4.15)$$

When $\sigma_{yn\alpha} A_{n\alpha} < \sigma_{ym\alpha} A_{m\alpha}$, the condition in Eq. (4.14) gives $\lambda_{N\alpha} \leq \sigma_{yn\alpha} A_{n\alpha}$. Hence, $-\Phi_{m\alpha} > 0$. Thus, the contact force $\lambda_{N\alpha}$ is not given. Similarly, on the other hand, when $\sigma_{yn\alpha} A_{n\alpha} > \sigma_{ym\alpha} A_{m\alpha}$, the contact force required to yield is not given for Body n . On the other hand, when $\sigma_{yn\alpha} A_{n\alpha} = \sigma_{ym\alpha} A_{m\alpha}$, $\Phi_{n\alpha} = \Phi_{m\alpha} = 0$ may be satisfied.

Fig. 5 **a** Definition of contact point and surface; **b** Definition of plastic deformation

Therefore, the relationship between the yield function Φ_α and the body capable of plastic deformation can be expressed as

$$\begin{cases} \Phi_\alpha = \Phi_{n\alpha} & \sigma_{yn\alpha} A_{n\alpha} < \sigma_{ym\alpha} A_{m\alpha} & (\text{Body } n \text{ can yield}) \\ \Phi_\alpha = \Phi_{n\alpha} = \Phi_{m\alpha} & \sigma_{yn\alpha} A_{n\alpha} = \sigma_{ym\alpha} A_{m\alpha} & (\text{Body } n, m \text{ can yield}) \\ \Phi_\alpha = \Phi_{m\alpha} & \sigma_{ym\alpha} A_{m\alpha} > \sigma_{yn\alpha} A_{n\alpha} & (\text{Body } m \text{ can yield}) \end{cases} \quad (4.16)$$

Then, the yield force Y_α at the contact point α is defined as

$$Y_\alpha = \min(\sigma_{yn\alpha} A_{n\alpha}, \sigma_{ym\alpha} A_{m\alpha}) \quad (4.17)$$

By the application of Y_α , state transition to the yield phenomenon will be formulated in the next section.

In the method by Pfeiffer et al. [13], no body-to-body penetration is considered. However, in this study, by allowing penetration, the amount of plastic deformation is calculated from the amount of penetration. The details of the penetration are explained in the next section. Figure 5b displays a schematic diagram of the case when $\Phi_\alpha = \Phi_{n\alpha} = 0$, i.e., when body n yields at the contact point α . Thus, the amount of plastic deformation is calculated as the amount of penetration of a gap with a yield function of zero. The plastic strain $\varepsilon_{pn\alpha}$ at $\ell_{n\alpha}$ is given by

$$\varepsilon_{pn\alpha} = \int_{\ell_{n\alpha}}^{\ell'_{n\alpha}} \frac{d\ell}{\ell} = \ln\left(\frac{\ell'_{n\alpha}}{\ell_{n\alpha}}\right) \quad (4.18)$$

This model will be used in the next section to formulate the theory of the proposed method.

4.2 State transition from contact to detachment or plastic deformation (state transition 1)

The first state transition is the transition from the contact state to the detachment state, or the yield state. An overview of this state transition is illustrated in Fig. 6. In the contact state, the gap $g_N = 0$, and contact force $\lambda_N \geq 0$. In addition, the gap acceleration $\ddot{g}_N = 0$. When the gap acceleration $\ddot{g}_N$ becomes greater than zero at the contact state, the contact force λ_N becomes 0 because of the transition from the contact state to the non-contact state. Since the contact force acts in the direction of compression with respect to the body, the body begins to undergo compressive plastic

Fig. 6 State transitions 1 for detachment and plastic deformation when the stress due to the contact force equals the yield stress and the gap acceleration $\ddot{g}_N \leq 0$. In the general rigid-body contact problem, the gap acceleration in the contact state is non-negative owing to non-penetration conditions. However, the transition to compressive plastic deformation is the same as the transition to the penetration state. Therefore, by allowing the gap acceleration $\ddot{g}_N$ to take a negative value, we express transition to the yield state. This transition is called 'State transition 1'.

To formulate 'State transition 1', we focus on the gap in the contact state. Let S_k be a set of gap numbers in contact and let the following equation be true for g_{Nk} ($k \in S_k$).

$$g_{Nk} = 0, \quad \dot{g}_{Nk} = 0 \quad (4.19)$$

In this case, the equation of motion in Eq. (4.6) is rewritten as

$$\mathbf{M}\ddot{\mathbf{y}} - \mathbf{h} - \hat{\mathbf{W}}\hat{\lambda}_N = \mathbf{0}, \quad (4.20)$$

where $\hat{\mathbf{W}}$ is a matrix of column vectors w_k ordered from the smallest k in the column direction, and $\hat{\lambda}_N$ is a column vector of λ_{Nk} ordered from the smallest k in the row direction. The gap acceleration and contact force in 'State transition 1' is summarised as

$$\begin{cases} \ddot{g}_{Nk} \geq 0, & \lambda_{Nk} = 0 & (\text{detachment}) \\ \ddot{g}_{Nk} = 0, & 0 \leq \lambda_{Nk} \leq Y_k & (\text{contact}) \\ \ddot{g}_{Nk} \leq 0, & \lambda_{Nk} = Y_k & (\text{plastic deformation}) \end{cases} \quad (4.21)$$

Here, Y_k is the yield force at gap k (contact point) defined in Eq. (4.17). By introducing $\ddot{\mathbf{g}}_N^R$ and $\ddot{\mathbf{g}}_N^L$, such as Eq. (2.8), we can represent Eq. (4.21) in terms of the Sgn function, and rewrite it as

$$-\lambda_{Nk} \in \frac{Y_k}{2} \{ \text{Sgn}(\ddot{g}_{Nk}) + 1 \} \Leftrightarrow \begin{cases} \hat{\lambda}_N \geq \mathbf{0}, & \ddot{\mathbf{g}}_N^R \geq \mathbf{0}, \lambda_N^T \ddot{\mathbf{g}}_N^R = 0 \\ \hat{\mathbf{Y}} - \hat{\lambda}_N \geq \mathbf{0}, & \ddot{\mathbf{g}}_N^L \geq \mathbf{0}, (\hat{\mathbf{Y}} - \hat{\lambda}_N)^T \ddot{\mathbf{g}}_N^L = 0. \\ \ddot{\mathbf{g}}_N = \ddot{\mathbf{g}}_N^R - \ddot{\mathbf{g}}_N^L \end{cases} \quad (4.22)$$

In addition, $\hat{\lambda}_N^R$ and $\hat{\lambda}_N^L$ are defined as

$$\hat{\lambda}_N^R := \hat{\lambda}_N, \quad (4.23)$$

$$\hat{\lambda}_N^L := \hat{Y} - \hat{\lambda}_N, \quad (4.24)$$

where, $\hat{Y}$ is a column vector of Y_k ordered from the smallest of k in the row direction. Equation (4.22) is then described by the complementarity condition of

$$\mathbf{0} \leq \begin{pmatrix} \ddot{\mathbf{g}}_N^R \\ \hat{\lambda}_N^L \end{pmatrix} \perp \begin{pmatrix} \hat{\lambda}_N^R \\ \ddot{\mathbf{g}}_N^L \end{pmatrix} \geq \mathbf{0} \quad (4.25)$$

Eliminating $\hat{\lambda}_N$ from the equation of motion in Eq. (4.20) using Eq. (4.23) gives

$$\mathbf{M}\ddot{\mathbf{y}} - \mathbf{h} - \hat{\mathbf{W}}\hat{\lambda}_N^R = \mathbf{0} \quad (4.26)$$

From the definition of gap acceleration Eq. (4.3), the gap acceleration for a gap k in contact is

$$\ddot{\mathbf{g}}_N = \hat{\mathbf{W}}^T \ddot{\mathbf{y}}. \quad (4.27)$$

Substituting $\ddot{\mathbf{g}}_N$ of Eq. (4.22) into Eq. (4.27) gives

$$\ddot{\mathbf{g}}_N^R = \hat{\mathbf{W}}^T \ddot{\mathbf{y}} + \ddot{\mathbf{g}}_N^L \quad (4.28)$$

Eliminating $\ddot{\mathbf{y}}$ from Eqs. (4.26) and (4.28) and rearranging them yields

$$\ddot{\mathbf{g}}_N^R = \hat{\mathbf{W}}^T \mathbf{M}^{-1} \mathbf{h} + \hat{\mathbf{W}}^T \mathbf{M}^{-1} \hat{\mathbf{W}} \hat{\lambda}_N^R + \ddot{\mathbf{g}}_N^L \quad (4.29)$$

From Eqs. (4.23) and (4.24), $\hat{\lambda}_N^L$ is represented by the following equation using $\hat{\lambda}_N^R$.

$$\hat{\lambda}_N^L = \hat{Y} - \hat{\lambda}_N^R. \quad (4.30)$$

From Eqs. (4.29) and (4.30), the linear relationship is obtained as

$$\begin{pmatrix} \ddot{\mathbf{g}}_N^R \\ \hat{\lambda}_N^L \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{W}}^T \mathbf{M}^{-1} \hat{\mathbf{W}} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \hat{\lambda}_N^R \\ \ddot{\mathbf{g}}_N^L \end{pmatrix} + \begin{pmatrix} \hat{\mathbf{W}}^T \mathbf{M}^{-1} \mathbf{h} \\ \hat{Y} \end{pmatrix}. \quad (4.31)$$

The linear complementarity problem for 'State transition 1' is obtained by combining the complementarity condition of Eq. (4.25) and the linear relation Eq. (4.31). The contact force $\hat{\lambda}_N$ of the gap in contact is calculated by solving this linear complementarity problem and using Eqs. (4.23) and (4.24). The linear complementarity problem can be solved using a numerical solution method such as Lemke's method [20].

4.3 State transition of collision and plastic deformation with relative velocity (state transition 2)

The second state transition is the case where a relative velocity exists between the bodies. This state transition consists of two cases. The first is a state transition in which an impact force is generated to transition from the non-contact state to the contact state and back to the non-contact state. The second is the state transition to the yield state due to collision. An overview of this state transition is illustrated in Fig. 7. Figure 7 focuses on the gap g_{N1} between Body 1 and the ground only for ease of understanding. Initially, the gap velocity in the non-contact state ($\{A\}$) is $\dot{g}_N < 0$. Then, $g_{N1} = 0$ and impact force begins to occur in ($\{B\}$), and Body 1 shifts the phase of compression. This is defined as the 'Compression Phase' ($\{B\} \rightarrow \{C\}, \{C'\}$). If Body 1 is a rigid body, an impulse producing $\dot{g}_N = 0$ is given in the Compression Phase ($\{C\}$). Subsequently, Body 1 shifts to the 'Expansion Phase' in which the compressed body, Body 1, expands to its original size ($\{C\} \rightarrow \{D\}$). According to Poisson's impact law, in the Expansion Phase, a constant multiple of the impulse given in the compression phase is given to Body 1 and

Fig. 7 State transitions 2 for collision and plastic deformation

the gap velocity becomes $\dot{g}_N > 0$. The case of state transition $\{A\} \rightarrow \{B\} \rightarrow \{C\} \rightarrow \{D\}$ is consistent with the collision problem in the method of Pfeiffer et al. [13]. If Body 1 is a rigid plastic body, it begins to undergo plastic deformation when the stress due the contact force acting in the Compression Phase matches the yield stress ($\{C\}$). This is defined as the 'Deformation Phase'. In the Deformation Phase, no rebound phenomenon as in the Expansion Phase occurs, and the force due to the yield stress continues to generate plastic deformation. The aforementioned state transition between collision and plastic deformation is defined as 'State transition 2'.

To formulate 'State transition 2', we focus on the gap in which this transition occurs. Let S_j be a set consisting of gap numbers in State transition 2 and the following equation hold for g_{Nj} ($j \in S_j$)

$$g_{Nj} = 0. \quad (4.32)$$

In this case, the equation of motion in Eq. (4.6) is rewritten as

$$\mathbf{M}\ddot{\mathbf{y}} - \mathbf{h} - \tilde{\mathbf{W}}\tilde{\lambda}_N = \mathbf{0}, \quad (4.33)$$

where $\tilde{\mathbf{W}}$ is a matrix of column vectors w_j ordered from the smallest j in the column direction, and $\tilde{\lambda}_N$ is a column vector of λ_{Nj} ordered from the smallest j in the row direction. At first the exchange of impulses in the Compression Phase was considered. The Compression Phase start time is t_A , the end time is t_M , and the equation of motion (Eq. (4.33)) is integrated from t_A to t_M as

$$\mathbf{M}(\dot{\mathbf{y}}_M - \dot{\mathbf{y}}_A) - \mathbf{H}_{AM} - \tilde{\mathbf{W}}\tilde{\Lambda}_{AM} = \mathbf{0}, \quad (4.34)$$

where $\dot{\mathbf{y}}_M = \dot{\mathbf{y}}(t_M)$ and $\dot{\mathbf{y}}_A = \dot{\mathbf{y}}(t_A)$, and the other integral terms and the integral of the yield force are defined as

$$\int_{t_A}^{t_M} \mathbf{h} dt = \mathbf{H}_{AM}, \quad \int_{t_A}^{t_M} \tilde{\lambda}_N dt = \tilde{\Lambda}_{AM}, \quad \int_{t_A}^{t_M} \tilde{\mathbf{Y}} dt = \tilde{\Sigma}_{AM}. \quad (4.35)$$

Here, $\tilde{\mathbf{Y}}$ is a column vector of the yield force Y_j ordered from the smallest j in the row direction. Further $\tilde{\Sigma}_{AM}$ is necessary for the consideration of complementarity condition, which will be discussed later. When $\Lambda_{AMj} = \Sigma_{AMj}$, the gap velocity becomes $\dot{g}_{Nj}(t_M) < 0$ as plastic deformation begins. However, as State transition 1 occurs, the transition to the yield state is expressed by allowing a negative gap velocity. When an impulse of $0 < \Lambda_{AMj} < \Sigma_{AMj}$ is given, the yield phenomenon does not occur and $\dot{g}_{Nj}(t_M) = 0$. If $\Lambda_{AMj} = 0$, the impulse was not given in the Compression Phase; therefore, $\dot{g}_{Nj}(t_M) \geq 0$. The relationship between $\dot{g}_{Nj}(t_M)$ and Λ_{AMj} can then be given by

$$\begin{cases} \dot{g}_{Nj}(t_M) \geq 0, \Lambda_{AMj} = 0 & (\text{no collision}) \\ \dot{g}_{Nj}(t_M) = 0, 0 \leq \Lambda_{AMj} \leq \Sigma_{AMj} & (\text{collision}) \\ \dot{g}_{Nj}(t_M) \leq 0, \Lambda_{AMj} = \Sigma_{AMj} & (\text{plastic deformation}) \end{cases}. \quad (4.36)$$

Equation (4.36) can be represented in terms of the Sgn function, and can be rewritten as the equation below by defining $\dot{\mathbf{g}}_N^R$ and $\dot{\mathbf{g}}_N^L$, such as in Eq. (2.8), where $\dot{\mathbf{g}}_N$ is a column vector of $\dot{g}_{Nj}$ ordered from the smallest j in the row direction.

$$-\Lambda_{AMj} \in \frac{\Sigma_{AMj}}{2} \{ \text{Sgn}(\dot{g}_{Nj}(t_M)) + 1 \} \\ \Leftrightarrow \begin{cases} \tilde{\Lambda}_{AM} \geq \mathbf{0}, & \dot{\mathbf{g}}_N^R(t_M) \geq \mathbf{0}, & \tilde{\Lambda}_{AM}^T \dot{\mathbf{g}}_N^R(t_M) = 0 \\ \tilde{\Sigma}_{AM} - \tilde{\Lambda}_{AM} \geq \mathbf{0}, & \dot{\mathbf{g}}_N^L(t_M) \geq \mathbf{0}, & (\tilde{\Sigma}_{AM} - \tilde{\Lambda}_{AM})^T \dot{\mathbf{g}}_N^L(t_M) = 0, \\ \dot{\mathbf{g}}_N = \dot{\mathbf{g}}_N^R - \dot{\mathbf{g}}_N^L \end{cases} \quad (4.37)$$

where $\tilde{\Lambda}_{AM}^R$ and $\tilde{\Lambda}_{AM}^L$ are defined as

$$\tilde{\Lambda}_{AM}^R := \tilde{\Lambda}_{AM}, \quad (4.38)$$

$$\tilde{\Lambda}_{AM}^L := \tilde{\Sigma}_{AM} - \tilde{\Lambda}_{AM}. \quad (4.39)$$

Equation (4.37) is then described by the complementarity condition of

$$\mathbf{0} \leq \begin{pmatrix} \dot{\mathbf{g}}_N^R(t_M) \\ \tilde{\Lambda}_{AM}^L \end{pmatrix} \perp \begin{pmatrix} \tilde{\Lambda}_{AM}^R \\ \dot{\mathbf{g}}_N^L(t_M) \end{pmatrix} \geq \mathbf{0}. \quad (4.40)$$

Eliminating $\tilde{\Lambda}_{AM}$ from the equation of motion Eq. (4.34) using Eq. (4.38) gives

$$\mathbf{M}(\dot{\mathbf{y}}_M - \dot{\mathbf{y}}_A) - \mathbf{H}_{AM} - \tilde{\mathbf{W}}\tilde{\Lambda}_{AM}^R = \mathbf{0}. \quad (4.41)$$

From the definition of gap velocity Eq. (4.2), the gap velocity at t_M is

$$\dot{\mathbf{g}}_N(t_M) = \tilde{\mathbf{W}}^T \dot{\mathbf{y}}_M. \quad (4.42)$$

Substituting $\dot{\mathbf{g}}_N(t_M)$ of Eq. (4.37) into Eq. (4.42) gives

$$\dot{\mathbf{g}}_N^R(t_M) = \tilde{\mathbf{W}}^T \dot{\mathbf{y}}_M + \dot{\mathbf{g}}_N^L(t_M). \quad (4.43)$$

Eliminating $\dot{\mathbf{y}}_M$ from Eqs. (4.41) and (4.43) and rearranging yields

$$\dot{\mathbf{g}}_N^R(t_M) = \tilde{\mathbf{W}}^T \mathbf{M}^{-1} \mathbf{H}_{AM} + \tilde{\mathbf{W}}^T \mathbf{M}^{-1} \tilde{\mathbf{W}} \tilde{\Lambda}_{AM}^R + \tilde{\mathbf{W}}^T \dot{\mathbf{y}}_A + \dot{\mathbf{g}}_N^L(t_M) \quad (4.44)$$

From Eqs. (4.38) and (4.39), $\tilde{\Lambda}_{AM}^L$ is represented by the following equation using $\tilde{\Lambda}_{AM}^R$

$$\tilde{\Lambda}_{AM}^L = \tilde{\Sigma}_{AM} - \tilde{\Lambda}_{AM}^R \quad (4.45)$$

From Eqs. (4.44) and (4.45), the linear relationship is obtained as

$$\begin{pmatrix} \dot{\tilde{\mathbf{g}}}_N^R(t_M) \\ \dot{\tilde{\Lambda}}_{AM}^L \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{W}}^T \mathbf{M}^{-1} \tilde{\mathbf{W}} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \tilde{\Lambda}_{AM}^R \\ \dot{\tilde{\mathbf{g}}}_N^L(t_M) \end{pmatrix} + \begin{pmatrix} \tilde{\mathbf{W}}^T \mathbf{M}^{-1} \mathbf{H}_{AM} + \tilde{\mathbf{W}}^T \dot{\mathbf{y}}_A \\ \tilde{\Sigma}_{AM} \end{pmatrix} \quad (4.46)$$

By combining the complementarity condition Eq. (4.40) and the linear relation Eq. (4.46), we obtained the linear complementarity problem for the Compression Phase. Subsequently the contact impulse $\tilde{\Lambda}_{AM}$ of the gap and gap velocity $\dot{\tilde{\mathbf{g}}}_N(t_M)$ at t_M were calculated solving this linear complementarity problem and using Eqs. (4.37) and (4.38). As aforementioned, the phase after the Compression Phase is divided into two, depending on the value of $\dot{\tilde{\mathbf{g}}}_N(t_M)$ the Expansion Phase, which is the transition to the non-contact state, and the Deformation Phase, which is the plastic deformation.

On the one hand, if $\dot{\tilde{\mathbf{g}}}_N(t_M) < 0$ is satisfied, Deformation Phase begins. Let $\mathbf{S}_D$ be a set consisting of gap numbers in the Deformation Phase and the following equation hold true for $\dot{g}_{ND}$ ($D \in \mathbf{S}_D$).

$$\dot{g}_{ND} < 0 \quad (4.47)$$

Since the force due to yield stress continues to be generated during plastic deformation, the contact force in the Deformation Phase is.

$$\tilde{\lambda}_N = \bar{\mathbf{Y}}, \quad (4.48)$$

where $\tilde{\lambda}_N$ and $\bar{\mathbf{Y}}$ are the column vectors of λ_{ND} and Y_D , respectively, ordered from the smallest D in the row direction. Therefore, the impulse generated between the Deformation Phase start time t_M and the end time t_E is given by.

$$\bar{\Lambda}_N = \int_{t_M}^{t_E} \tilde{\lambda}_N dt \quad (4.49)$$

On the other hand, if $\dot{\tilde{\mathbf{g}}}_N(t_M) \geq 0$ is satisfied, the Expansion Phase begins. Let $\mathbf{S}_s$ be a set consisting of gap numbers in Expansion Phase and the following equation hold true for $\dot{g}_{Ns}$ ($s \in \mathbf{S}_s$).

$$\dot{g}_{Ns} \geq 0 \quad (4.50)$$

In this case, the equation of motion Eq. (4.6) is rewritten as

$$\mathbf{M}\ddot{\mathbf{y}} - \mathbf{h} - \tilde{\mathbf{W}}\tilde{\lambda}_N = 0, \quad (4.51)$$

where $\tilde{\mathbf{W}}$ is a matrix of column vectors w_s ordered from the smallest s in the column direction, and $\tilde{\lambda}_N$ is a column vector of λ_{Ns} ordered from the smallest s in the row direction. Integrating the equation of motion Eq. (4.51) from

the Expansion Phase start time t_M to the end time t_E , the following is obtained:

$$\mathbf{M}(\dot{\mathbf{y}}_E - \dot{\mathbf{y}}_M) - \mathbf{H}_{ME} - \tilde{\mathbf{W}}\tilde{\Lambda}_{ME} = 0, \quad (4.52)$$

where $\dot{\mathbf{y}}_E = \dot{\mathbf{y}}(t_E)$ and $\dot{\mathbf{y}}_M = \dot{\mathbf{y}}(t_M)$, and the other integral terms are defined as

$$\int_{t_M}^{t_E} \mathbf{h} dt = \mathbf{H}_{ME}, \quad \int_{t_M}^{t_E} \tilde{\lambda}_n dt = \tilde{\Lambda}_{ME}. \quad (4.53)$$

From the definition of gap velocity in Eq. (4.2), the gap velocities at t_M and t_E are given by

$$\dot{\tilde{\mathbf{g}}}(t_M) = \tilde{\mathbf{W}}^T \dot{\mathbf{y}}_M, \quad \dot{\tilde{\mathbf{g}}}(t_E) = \tilde{\mathbf{W}}^T \dot{\mathbf{y}}_E, \quad (4.54)$$

where, $\dot{\tilde{\mathbf{g}}}_N$ is a column vector of $\dot{g}_{Ns}$ ordered from the smallest s in the row direction. Subtracting each side of Eq. (4.54) yields

$$\dot{\tilde{\mathbf{g}}}(t_E) = \tilde{\mathbf{W}}^T (\dot{\mathbf{y}}_E - \dot{\mathbf{y}}_M) + \dot{\tilde{\mathbf{g}}}_N(t_M) \quad (4.55)$$

According to Poisson's law, impulse in the Expansion Phase is given as a constant multiple of the impulse in the Compression Phase as

$$\Lambda_{MEs} = \epsilon_s \Lambda_{AMs}, \quad (4.56)$$

where ϵ_s is the coefficient of restitution at g_{Ns} . Λ_{AMs} and Λ_{MEs} are elements related to the g_{Ns} of $\tilde{\Lambda}_{AM}$ and $\tilde{\Lambda}_{ME}$, respectively. When multiple-point contacts occur, penetration may take place owing to the influence of contact at other points. It is necessary to modify Poisson's law to ensure the impenetrability condition at the end of the impact [13]. If $\dot{g}_{Ns}(t_E) < 0$, penetration has occurred; therefore, $\dot{g}_{Ns}(t_E) \geq 0$ must be satisfied. The magnitude of $\dot{g}_{Ns}(t_E)$ depends on the impulse in the Expansion Phase. In the case of multiple contacts, the Poisson impulse may not be sufficiently large to prevent penetration because of contact at other points. Therefore, it is necessary to allow a greater impulse than the original Poisson impulse.

$$\Lambda_{MEs} \geq \epsilon_s \Lambda_{AMs} \quad (4.57)$$

where $\Lambda_{Ps} = \Lambda_{MEs} - \epsilon_s \Lambda_{AMs}$. If the Poisson impulse is sufficiently large, the gap velocity at the end of the Expansion Phase can have a positive value, that is, when $\Lambda_{MEs} - \epsilon_s \Lambda_{AMs} = 0$, $\dot{g}_{Ns}(t_E) > 0$. On the other hand, if the Poisson impulse is small, an impulse that achieves $\dot{g}_{Ns}(t_E) = 0$ must occur to prevent the penetration. This

implies that, $\Lambda_{ps} = \Lambda_{MEs} - \epsilon_s \Lambda_{AMs} > 0$. Thus, $\dot{g}_{Ns}(t_E)$ and Λ_{ps} share a complementarity relationship. $\check{\Lambda}_p$ is defined as

$$\check{\Lambda}_p = \check{\Lambda}_{ME} - \check{\epsilon} \check{\Lambda}_{AM}, \quad (4.58)$$

where $\check{\Lambda}_p$ is a column vector of Λ_{ps} ordered from the smallest s in the row direction, $\check{\Lambda}_{AM}$ is a column vector of Λ_{AMs} ordered from the smallest s in the row direction, and $\check{\epsilon}$ is a matrix of ϵ_s ordered from the smallest s in the diagonal direction. Thus, the complementarity condition in the Expansion Phase is given by

$$\mathbf{0} \leq \check{g}_N(t_E) \perp \check{\Lambda}_p \geq \mathbf{0}. \quad (4.59)$$

Eliminating $\check{\Lambda}_{ME}$ from Eqs. (4.52) and (4.58), and then eliminating $\dot{y}_E - \dot{y}_M$ by using Eq. (4.55), gives the following equation.

$$\check{g}_N(t_E) = \check{W}^T M^{-1} \check{W} \check{\Lambda}_p + \check{W}^T M^{-1} H_{ME} + \check{W}^T M^{-1} \check{W} \check{\epsilon} \check{\Lambda}_{AM} + \check{g}_N(t_M) \quad (4.60)$$

By combining the complementarity condition Eq. (4.59) and the linear relation Eq. (4.60), we obtain the linear complementarity problem for the Expansion Phase. Then, the contact impulse $\check{\Lambda}_{ME}$ is calculated by solving this linear complementarity problem. The impulse $\check{\Lambda}_N$ of the Deformation Phase given by Eq. (4.49) and the impulse $\check{\Lambda}_{ME}$ of the Expansion Phase are combined to obtain the impulse at times t_M to t_E . Furthermore, the impulses $\check{\Lambda}_{AM}$, $\check{\Lambda}_{ME}$ and $\check{\Lambda}_{ME}$ of the Compression, Deformation and Expansion Phases, respectively, are combined to derive the impulse $\check{\Lambda}_N$ in State transition 2.

$$\check{W} \check{\Lambda}_N = \check{W} \check{\Lambda}_{AM} + \check{W} \check{\Lambda}_{ME} + \check{W} \check{\Lambda}_{ME}, \quad (4.61)$$

where $\check{W}$ is the Jacobian in the Deformation Phase.

5 Numerical analysis

5.1 Moreau's time-stepping method

The time-stepping method is a discrete numerical analysis solution approach suitable for simulating non-smooth systems [26]. In this section, we carry out discretisation based on Moreau's time-stepping method [27]. The equation of motion Eq. (4.6) is integrated over a finite time Δt , where the start of Δt is t_a and the end is t_e . The integral of each term in the equation of motion is

$$\int_{\Delta t} M \ddot{y} dt \approx M_m \Delta \dot{y} = M_m (\dot{y}_e - \dot{y}_a), \quad (5.1)$$

$$\int_{\Delta t} h dt = \Delta h \approx h_m \Delta t, \quad (5.2)$$

$$\int_{\Delta t} W \lambda_N dt = \int_{\Delta t} \hat{W} \hat{\lambda}_N dt + \check{W} \check{\Lambda}_N \approx \hat{W}_a \hat{\lambda}_{Na} \Delta t + \check{W}_a \check{\Lambda}_N, \quad (5.3)$$

where t_m is the time at the midpoint of $[t_a, t_e]$, and subscripts a and m are the values at the time of t_a and t_m , respectively.

$$t_m = t_a + \frac{1}{2} \Delta t. \quad (5.4)$$

In addition, the times t_A , t_M , and t_E of State transition 2 correspond to t_a , t_m , and t_e . With this discretisation, the equation of motion Eq. (4.6) is expressed as

$$M_m (\dot{y}_e - \dot{y}_a) - h_m \Delta t - \hat{W}_a \hat{\lambda}_{Na} \Delta t - \check{W}_a \check{\Lambda}_{Na} = \mathbf{0}. \quad (5.5)$$

Consequently, the velocity $\dot{y}_e$ at the end of the time step $t_e = t_a + \Delta t$ is given by

$$\dot{y}_e = \dot{y}_a + M_m^{-1} h_m \Delta t + M_m^{-1} \hat{W}_a \hat{\lambda}_{Na} \Delta t + M_m^{-1} \check{W}_a \check{\Lambda}_{Na}. \quad (5.6)$$

As a result, the generalised coordinate y_e at the end of time step t_e is

$$y_e = y_a + \frac{\Delta t}{2} (\dot{y}_a + \dot{y}_e). \quad (5.7)$$

At the start of time step t_A , the gap and gap speed are calculated from the position and velocity, and the contact state is determined. The contact force term $\hat{W}_a \hat{\lambda}_{Na} \Delta t$ is derived by solving the linear complementarity problem of State transition 1 for gaps that satisfied Eq. (4.19). Then, transition to the detachment, contact, and plastic deformation states can be detected. In addition, the contact force term $\check{W}_a \check{\Lambda}_{Na}$ is derived by solving the linear complementarity problem of State transition 2 for the gap that satisfies Eq. (4.23). Thus, state transitions to collision and plastic deformation can be detected. The contact impulse is then derived by detecting the state transition and solving the linear complementarity problem at each time step, and the state quantity is calculated using Eqs. (5.6) and (5.7). Thus, the graph of the system against time can be calculated.

5.2 Numerical analysis for the model composed of basic elements

Based on the proposed method, two cases of numerical analysis were performed on the model with the basic elements defined in the previous section. Table 1 and Fig. 8 show the parameters used in the numerical

Table 1 Parameters for numerical analysis

		Case 1	Case 2
Time step	Δt [s]	1.0×10^{-4}	
Inertial properties	m_1 [kg]	7.85	0.270
	m_2 [kg]	7.85	
Initial conditions	y_{10} [m]	0.05	
	y_{20} [m]	0.20	
	$\dot{y}_{10}$ [m/s]	0.0	
	$\dot{y}_{20}$ [m/s]	-18.0	
	L_{10}, L_{20} [m]	0.100	
	A_1, A_2 [m ²]	0.010	
Force elements	G [m/s ²]	9.81	
	u_1, u_2 [N]	0.00	
	ϵ_1, ϵ_2	0.50	
Contact parameters			
Mechanical property	σ_{10} [MPa]	490	2.53
	σ_{20} [MPa]	490	490
	ϵ_d	-	0.50
	E_{pl} [MPa]	-	1.00
	E_d [MPa]	-	103

analysis, their schematic diagrams, and stress–strain model. To validate the fundamental properties of the method, for each body a linear model was adopted as the stress–strain model. In Case 1, Body 1 and 2 were assumed to be dense metal, and stress–strain model of rigid perfect body was used. The mechanical properties were determined by referring to carbon steel for machine construction S45C. In Case 2, Body 1 was assumed to be metal foam and Body 2 was assumed to be dense metal. There are three characteristic regions in the stress–strain relation of metal foam under compressive deformation—elastic region, plateau region and densification region [1]. When compressive stress reaches the yield point, the cell walls buckle continuously, and plastic deformation proceeds under constant stress. This region is called the plateau region, and the

strain lasts up to approximately 0.5. As the fracture of cell walls progresses, the metal foam becomes closer to a dense material, and the stress at deformation increases. This region is called the densification region. In this analysis, a linear approximation of the plateau and densification regions were used. The mechanical properties were determined by referring to those of an open-cell foam aluminum alloy A6101 with a porosity of approximately 90%. Numerical analysis was performed using MATLAB.

Case 1 is an analysis in which Body 2 (S45C) collides with Body 1 (S45C) on the ground in free fall. Figure 9 illustrates the analysis result of Case 1. Figure 9a shows the position–time graph of the centre of gravity y_i and the contact points for each body and Fig. 9b shows the contact force λ_{Nj} and the yield force Y_j . Figure 9c is the plastic strain graph against time, and ϵ_{p11} , ϵ_{p12} , and ϵ_{p22} are the plastic strains corresponding to ℓ_{11} , ℓ_{12} and ℓ_{22} , respectively. According to these results, Body 2 fell freely and bounced back from Body 1. Since the contact force at the time of collision is less than the yield force, no state transition to plastic deformation occurred in either gap, and consequently, neither Body 1 nor Body 2 deformed plastically.

Case 2 is an analysis in which Body 2 (S45C) collides with Body 1 (metal foam) on the ground at the speed of 18 m/s. Figure 10 illustrates the analysis result of Case 2. Figure 10a shows the graph of the coordinate y_i of the centre of gravity and the contact points for each body. Figure 10b shows the gap velocity $\dot{g}_{Nj}$, and Fig. 10c shows the contact force λ_{Nj} and the yield force Y_j . Figure 10d is the graph of plastic strain. Figure 10e is the graph of mechanical energy of the whole system and the energy spent to plastically deform Body 1. According to these results, both ℓ_{11} and ℓ_{12} of Body 1 start changing owing to the collision of Body 2. As it can be noticed, ℓ_{12} is more plastically deformed than ℓ_{11} . This is because the gap velocity $\dot{g}_{N2}$ between Bodies 1 and 2 is large immediately after the collision, and the plastic strain ϵ_{p12} owing to contact with Body 2 is large. Figure 10c shows that the contact force during plastic deformation matches the yield force Y_j . Body 2 is

Fig. 8 a Initial condition; b Stress–strain model

Fig. 9 Case 1: **a** Coordinates; **b** Contact force; **c** Yield strain

decelerated by the contact force and Body 1 is accelerated. As a result, the rate of change of ϵ_{p11} increases and the rate of change of ϵ_{p12} decreases. Subsequently, the strain of ϵ_{p11} exceeds $\epsilon_d (= 0.5)$, at which the plateau region ends, and enters the densification region at approximately 0.0055 s. Then, the yield force Y_1 increases and the deformation of ϵ_{p11} stops, and only ϵ_{p12} plastically deforms, as Fig. 10d shows. Then, the deformation of ϵ_{p12} progresses and enters the densification region, and when $Y_2 \geq Y_r$, the plastic deformation of ϵ_{p11} starts again at approximately 0.006 s. As indicated by this behaviour, the contact forces at the two contact points influence each other's plastic deformation behaviour. When the gap velocity becomes sufficiently small while repeating the previously described behavior, a contact force smaller than Y_i is generated, which indicates the completion of plastic deformation. The final plastic strain is approximately 0.55 for both ϵ_{p11} and ϵ_{p12} . As revealed by the energy balance graph in Fig. 10e, the whole kinetic energy of Body 2 is converted into the plastic strain energy of Body 1. Note that the contact forces take discrete values in several time regions as shown in Fig. 10c, and the generation mechanism of the discrete values can be explained as follows. When the gap velocity $\dot{g}_N(t_M)$ is negative in state transition 2, Deformation Phase begins and the transition to the phase is repeated until $\dot{g}_N(t_M)$ becomes non-negative. In general, the gap velocity rarely becomes exactly zero on discrete integration time step, and the gap velocity has a very small positive value

Fig. 10 Case 2: **a** Coordinates; **b** Gap velocity; **c** Contact force; **d** Yield strain; **e** Energy

when Deformation Phase ends. As a result, bodies associated with gaps of positive gap velocities are separated by a small distance. After that, gravity causes the gap to become smaller, the collision phenomenon occurs again, and Compression Phase of state transition 2 begins. Since the collision occurs at a very small gap velocity, the system does not enter Deformation Phase, but rather Expansion Phase in which a stress smaller than the yield stress is instantaneously generated. Although the generation of such discrete values has little influence on the overall system as long as the time step is small enough, solving this phenomenon is a challenging task and will contribute

Fig. 11 Calculation times for 0.01[s] analysis with respect to different time steps

to improvement of computational cost of the proposed method.

The time step in the above results is 1.0×10^{-4} [s], and the computation times required for the 0.01 [s] analysis are 2.739[s] and 2.573[s] for Case 1 and Case 2, respectively. In addition, Fig. 11 shows a comparison of the computation times for different time steps but the conditions other than the time step are the same as in Table 1. Note that the CPU used for numerical analysis is Intel Core i7-8565U (1.99 GHz), and the computation times are averages of 10 analyses. As shown in the figure, the computation time increases exponentially as the time step becomes smaller. It is also confirmed that the behaviors are quantitatively and qualitatively similar when the time step is less than 1.0×10^{-4} [s], but the behavior changes significantly when the time step becomes coarser, and the accuracy of the analysis deteriorates; this is the reason why the step time is set to 1.0×10^{-4} [s] in the aforementioned analyses. Since the appropriate time step depends on the analysis conditions, it is necessary to determine the time step by preliminary analysis in each case. However, the contact and deformation phenomena are formulated in a unified manner, and compared to co-simulation, which combines different methods of contact analysis and deformation analysis, the proposed method is easier to evaluate and improves computational efficiency.

Case 1, although Body 1 and Body 2 contact instantaneously, can be regarded as a single point contact between Body1 and Ground for most of the analysis time. On the other hand, Case 2 is subjected to simultaneous contact at two points during most of the analysis time, i.e., one contact is between Body1 and Ground and the other contact

is between Body 1 and Body 2. As can be seen from Fig. 11, there is no significant difference in the analysis time for both Case 1 and Case 2 for different step times. This indicates that the influence of multi-point simultaneous contact on the computation cost is not significant. This is one of the advantages of the proposed method, compared to multi-point simultaneous contact analysed by the classical method. Note that, since this is a basic study focusing on the concept of state transition, the basic integration method, Moreau's time-stepping method, is employed. Various other integration methods have been developed, and it is considered that the introduction of such integration methods can further improve the calculation time and analysis accuracy.

Note that the metal foam introduced in Case2 has a porous structure, and interpenetration may occur when such materials collide with each other and deform. The occurrence of interpenetration depends on the microscopic characteristics such as the porous structure. On the other hand, the proposed method is based on the macroscopic material properties and does not consider the effect of interpenetration; however, by considering such effects, the proposed method can be made more practical.

6 Conclusions and future tasks

In this study, the motion of a multibody system was analysed using a method in which the components are subjected to large compressive deformations with contact. The method is a formulation of the contact phenomena and plastic deformation, resulting in a linear complementarity problem. First, the complementarity conditions for contact and plastic deformation were derived. The yield phenomenon was analysed by expanding each of the contact-detachment and collision state transitions in the rigid-body contact problem developed by Pfeiffer et al. [13]. As a result, the contact and yield phenomena could be described as a unified complementarity condition and formulated as a linear complementarity problem by deriving a linear relationship from the equation of motion. The contact force was calculated by solving this linear complementarity problem. In addition, the numerical analysis was performed with the same algorithm as a general contact problem by performing the discretisation based on Moreau's time-stepping method. Through numerical analysis, the method was evaluated in terms of the variation of plastic strain and energy balance against time during the collision phenomenon. In addition, the effectiveness of the proposed method was demonstrated by comparing the computation time for different time steps for the cases where single point contact is dominant and multiple point

contact is dominant, showing that the influence of increasing the number of contact points on the computation time is not critical. Since the proposed method can be used to analyse the non-smooth behaviour of the system under two different physical phenomena, (contact and collision, and yielding phenomena) in a unified dynamics framework, it is easier to adjust the parameters that govern the computational efficiency of numerical analysis, such as step time, compared to co-simulation method.

On the other hand, the method proposed in this study is a surface contact approach in linear motion, which is a preliminary basic method for the analysis of practical contact problems on structures. In order to develop a practical method it is necessary to consider the volume contact approach and the effect of frictional force in the tangential direction on the contact points.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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