Design of FOPID controller for higher order continuous interval system using improved approximation ensuring stability

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Abstract

This contribution deals with the design of a fractional-order proportional-integral-derivative (FOPID) controller through reduce-order modeling for continuous interval systems. First, a higher order interval plant (HOIP) is considered. The reduced-order interval plant (ROIP) for considered HOIP is derived by multipoint Padé approximation integrated with Routh table. Then, FOPID controller is designed for ROIP to satisfy the phase margin and gain cross over frequency. Thus obtained FOPID controller is implemented on HOIP also to validate the performance of designed FOPID on HOIP. A single-input-single-output (SISO) test system is taken up to elaborate the entire process of controller design. The outcomes affirm the validity of the designed FOPID controller. The designed FOPID controller produced stable results retaining the phase margin and gain cross-over frequency when implemented on HOIP. The results further proved that FOPID controller is working efficiently for ROIP and HOIP.

Keywords Fractional calculus · Fractional-order controller · Multipoint Padé method · Routh method · Optimization

1 Introduction

The burgeoning demands for accurate depiction and simulation of system dynamics require complex and higher order mathematical models [25]. In many industrial processes such as chemical plants, steam generators, nuclear reactors, air flight systems etc., the order of dynamic system may be quite large. As the order of plant increases, the controller design becomes complex and high order controllers need to be designed [4]. The realization of these controllers demands huge expenditure and poses the probable troubles of reliability, commissioning and maintenance. Therefore, control engineers prefer to work with low order controllers [9].

While modelling any engineering system, a nominal mathematical description (also known as nominal model) is generally obtained. The coefficients of numerator and

denominator parameters do have fixed values in case of nominal model. This is due to obtaining the nominal model at certain operating point. In case, the operating point of system shifts due to any of the reasons like, aging, parameter variations, sensor noises, environmental changes; it becomes difficult to describe the dynamics of system using earlier obtained nominal model. To tackle these problems, a model in which system parameters varies within certain range may be obtained. The model in which parameters varies in certain range is known as interval system. Recently, various engineering systems are modelled as interval systems [3]. Also, various techniques for model reduction of interval systems are presented in the literature [17].

The design of controllers [19] through interval arithmetic is extended to the interval systems. The design of controller using minimization of ISE via evolutionary

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approaches is appeared in literature [13]. In this technique, ISE of closed loop system is minimized using two interactive genetic algorithms for interval plants while the stability is ensured using Kharitonov polynomials. In [10], Hsu and Lin presented a technique based on hybrid particle swarm optimizer for designing of digital controller. Hsu et al. [11] presented a digital controller using resemblance of extremal gain/phase margin criteria for uncertain interval systems.

The controller design methodology based on two-degrees-of-freedom criterion for interval processes is also appeared in [19]. The pre-filter guarantees the robust performance and the stability is ensured using Hurwitz theorem [19] through making use of Kharitonov theorem [7, 24, 26]. Yeroglu and Tan [28] designed FOPID controller by combining the Bode envelopes with design criteria for the interval plants. The articles [19, 28] designed the controller for original systems. In case, the order of system is high, it is desirable to reduce the order of system for controller design. So, the controller design using reduced-order modelling should also be investigated in detail for interval systems.

In this contribution, a fractional-order proportional-integral-derivative (FOPID) controller [1, 22] is designed for reduced-order interval plant (ROIP) of higher-order continuous interval plant (HOIP) to satisfy the criteria of phase margin and gain cross over frequency. Then, this FOPID controller is implemented on HOIP. Multipoint Padé approximation integrated with Routh table derives the ROIP. The Routh table is utilized to derive the denominator of the interval model whereas multipoint Padé approximation is exploited for derivation of numerator [5]. Addition of Routh table assures for stability of ROIP. A SISO system is adopted to illustrate the entire process of controller designing. The results confirm the validity of the suggested method for designing of FOPID controller.

To present the proposed technique extensively, the paper comprises seven sections. Beginning with introductory part of the problem, the preliminaries are introduced in Sect. 2. The fractional-order calculus and FOPID controller are discussed in this section. The problem description is included in Sect. 3. The design procedure of FOPID controllers is given in Sect. 4. Section 5 covers order reduction procedure of high-order plant. Section 6 provides an illustrative test system, and lastly, Sect. 7 summarizes the work.

2 Preliminaries

The basic description of fractional calculus and fractional order systems is elaborated in this section. This section also describes FOPID controller.

2.1 Fractional order calculus

In fractional calculus, the continuous integro-differential operator [18] of order r is defined as

$${}_u D_v^r = \begin{cases} \frac{d^r}{dt^r} & \Re(r) > 0 \\ 1 & \Re(r) = 0 \\ \int_u^v (d\tau)^{-r} & \Re(r) < 0 \end{cases} \quad (1)$$

with the limits of the operation bounded by u and v . The integration and differentiation of any order can be performed which can assume any value; rational, irrational or complex. The frequently used integro-differential definitions are given as follows.

The Riemann–Liouville (RL) definition [1] is given as

$${}_u D_v^r f(t) = \frac{1}{\Gamma(n-r)} \frac{d^n}{dt^n} \int_u^v \frac{f(\tau)}{(t-\tau)^{r-n+1}} d\tau \quad (2)$$

where r takes any values between $n-1$ and n , and $\Gamma(\cdot)$ is a Gamma function.

The Grunwald–Letnikov definition [9] of differentiation is given as

$${}_u D_v^r f(t) = \lim_{h \rightarrow 0} h^{-k} \sum_{q=0}^{\frac{(v-u)}{h}} (-1)^q \binom{r}{q} f(t - qh) \quad (3)$$

where $(-1)^q \binom{r}{q}$ are the binomial coefficients $C_q^{(r)}$, ($q = 0, 1, \dots$) and $[\cdot]$ denotes the integer part. The expressions given in [9] are used to obtain the binomial coefficients.

The Caputo expression [2, 9] is defined as

$${}_u D_v^r = \frac{1}{\Gamma(n-r)} \int_u^v \frac{f^n(\tau)}{(t-\tau)^{r-n+1}} d\tau \quad (4)$$

where $n-1 < r < n$.

2.2 The Laplace transform of fractional order systems

The Laplace transformations of the integro-differential expressions [9] can be given as

$$L\left\{\frac{d^r f(t)}{dt^r}\right\} = s^r L\{f(t)\} - \sum_{p=0}^{n-1} s^p \left[\frac{d^{r-1-p} f(t)}{dt^{r-1-p}} \right]_{t=0} \quad (5)$$

where r is an integer number and $(n-1) < r < n$. If all the derivatives of $f(t)$ are zero, the expression given by (5) can be written as

$$L\left\{\frac{d^r f(t)}{dt^r}\right\} = s^r L\{f(t)\} \quad (6)$$

Consider a SISO system given as

$$\begin{aligned} \hat{a}_n \frac{d^{\hat{a}_n} y(t)}{dt^{\hat{a}_n}} + \hat{a}_{n-1} \frac{d^{\hat{a}_{n-1}} y(t)}{dt^{\hat{a}_{n-1}}} + \dots + \hat{a}_0 \frac{d^{\hat{a}_0} y(t)}{dt^{\hat{a}_0}} \\ = \hat{b}_m \frac{d^{\hat{\beta}_m} u(t)}{dt^{\hat{\beta}_m}} + \dots + \hat{b}_0 \frac{d^{\hat{\beta}_0} u(t)}{dt^{\hat{\beta}_0}} \end{aligned} \quad (7)$$

where output and input of the system are designated as $y(t)$ and $u(t)$, respectively. By taking Laplace transform [9] of (7), following transfer function is obtained

$$F(s) = \frac{Y(s)}{U(s)} = \frac{\hat{b}_m s^{\hat{\beta}_m} + \hat{b}_{m-1} s^{\hat{\beta}_{m-1}} + \dots + \hat{b}_0 s^{\hat{\beta}_0}}{\hat{a}_n s^{\hat{a}_n} + \hat{a}_{n-1} s^{\hat{a}_{n-1}} + \dots + \hat{a}_0 s^{\hat{a}_0}} \quad (8)$$

where $\hat{a}_n > \hat{a}_{n-1} > \dots > \hat{a}_0 \geq 0$, $\hat{\beta}_m > \hat{\beta}_{m-1} > \dots > \hat{\beta}_0 \geq 0$ and $\hat{a}_p$ for $p = 0, 1, \dots, n$ and $\hat{\beta}_p$ for $p = 0, 1, \dots, m$ are constants.

2.3 Fractional-order proportional-integral-derivative (FOPID) controller

The FOPID controller is given as

$$\tilde{C}(s) = K_1 + \frac{K_2}{s^w} + K_3 s^z, \quad (w, z > 0) \quad (9)$$

which is commonly known as $PI^w D^z$ controller, where the real numbers w and z , respectively, are designated as orders of integration and differentiation. Clearly, FOPID controllers have two additional variables when compared to classical PID controller [6, 8, 15, 21, 23]. Due to this, FOPID controllers can be designed such that these perform better control and are less sensitive to parameter variations [20].

3 Problem description

Consider an n th-order continuous interval system $\bar{G}(s)$ for closed-loop system of Fig. 1 given as

Fig. 1 The feedback control having controller

$$G(s) = \frac{[\underline{a}_0, \underline{a}_0] + [\underline{a}_1, \underline{a}_1]s + [\underline{a}_2, \underline{a}_2]s^2 + \dots + [\underline{a}_{n-1}, \underline{a}_{n-1}]s^{n-1}}{[\underline{b}_0, \underline{b}_0] + [\underline{b}_1, \underline{b}_1]s + [\underline{b}_2, \underline{b}_2]s^2 + \dots + [\underline{b}_n, \underline{b}_n]s^n} = \frac{\bar{A}(s)}{\bar{B}(s)} \quad (10)$$

The interval polynomial $\bar{A}(s)$ of the numerator has interval coefficients $[\underline{a}_p, \underline{a}_p]$ for $(p = 0, 1, \dots, n-1)$, with $\underline{a}_p$ and $\underline{a}_p$ as the lower and upper limits, respectively. Similarly, the interval polynomial $\bar{B}(s)$ of denominator has interval coefficients $[\underline{b}_p, \underline{b}_p]$ for $(p = 0, 1, \dots, n)$, with combined with two design criteria $\underline{b}_p$ and $\underline{b}_p$ as the lower and upper limits, respectively.

Figure 2 depicts the feedback system without controller. HOIP is reduced to ROIP and is represented in Fig. 3. In Fig. 3, the k th-order model $\hat{G}(s)$ of $G(s)$ is given as

$$\begin{aligned} \hat{G}(s) &= \frac{[\underline{c}_0, \underline{c}_0] + [\underline{c}_1, \underline{c}_1]s + [\underline{c}_2, \underline{c}_2]s^2 + \dots + [\underline{c}_{k-1}, \underline{c}_{k-1}]s^{k-1}}{[\underline{d}_0, \underline{d}_0] + [\underline{d}_1, \underline{d}_1]s + [\underline{d}_2, \underline{d}_2]s^2 + \dots + [\underline{d}_k, \underline{d}_k]s^k} \\ &= \frac{\hat{C}(s)}{\hat{D}(s)} \end{aligned} \quad (11)$$

The interval polynomial $\hat{C}(s)$ of the numerator has interval coefficients $[\underline{c}_p, \underline{c}_p]$ for $(p = 0, 1, \dots, k-1)$ with $\underline{c}_p$ and $\underline{c}_p$ as the lower and upper limits, respectively. Similarly, the interval polynomial $\hat{D}(s)$ of the denominator has interval coefficients $[\underline{d}_p, \underline{d}_p]$ for $(p = 0, 1, \dots, k)$ with $\underline{d}_p$ and $\underline{d}_p$ as the lower and upper limits, respectively.

The system in Fig. 2 is approximated by that of Fig. 3, where $H(s)$ is assumed to be same in both the cases. Fig. 4 represents ROIP with controller $\tilde{C}(s)$ that is designed for ROIP. Thus designed controller is implemented on HOIP as shown in Fig 5.

Fig. 2 The feedback control having HOIP

Fig. 3 The feedback control having ROIP

Fig. 4 The feedback control having ROIP and controller

Fig. 5 The feedback control having HOIP and controller

4 Design of fractional order controller

In this section, design of $PI^w D^z$ controller for ROIP is presented. Five unknown parameters of $PI^w D^z$ controller are calculated to satisfy gain margin and phase margin of interval system. All the design parameters of $PI^w D^z$ controller are computed by solving two non-linear equations. Bode envelopes of (12) are combined with two design criteria to obtain the $PI^w D^z$ controller parameters [28].

ROIP of first order can be given as

$$\hat{G}(s) = \frac{[\underline{c}_0, c_0]}{[\underline{d}_0, d_0] + [\underline{d}_1, d_1]s} \quad (12)$$

where $\underline{c}_0$ and c_0 , respectively, are the lower and upper limits of interval coefficient $[c_0, c_0]$ of the numerator of $\hat{G}(s)$. Similarly, $\underline{d}_0$ and $\underline{d}_1$ are the lower limits, and d_0 and d_1 are the upper limits of interval coefficients $[d_0, d_0]$ and $[d_1, d_1]$.

The significant frequency domain characteristics for design of FOPID controller are namely; phase margin (ϕ_{pm}) and gain cross-over frequency (ω_{gc}). ϕ_{pm} and ω_{gc} constitute key measures of robustness and stability. Therefore, the FOPID controller is designed to obtain the desired ϕ_{pm} and ω_{gc} . The expressions [16, 27] given in (13) and (14) should be satisfied for a specified ω_{gc} and ϕ_{pm} .

$$20 \times \log_{10} |\tilde{C}(j\omega_{gc})\hat{G}(j\omega_{gc})| = 0 \quad (13)$$

$$\angle(\tilde{C}(j\omega_{gc})\hat{G}(j\omega_{gc})) = -\pi + \phi_{pm} \quad (14)$$

4.1 Design of fractional order controller by use of Bode envelope

The transfer functions given in (15) and (16), respectively, offer the minimum and maximum plots of the gain.

$$\hat{G}_1(s) = \frac{\underline{c}_0}{\underline{d}_1 s + \underline{d}_0} \quad (15)$$

$$\hat{G}_2(s) = \frac{c_0}{\underline{d}_1 s + \underline{d}_0} \quad (16)$$

Similarly, the transfer functions given by (17) and (18), respectively, provide the minimum and maximum plots of the phase.

$$\hat{G}_3(s) = \frac{c_0}{\underline{d}_1 s + \underline{d}_0} \quad (17)$$

$$\hat{G}_4(s) = \frac{\underline{c}_0}{\underline{d}_1 s + \underline{d}_0} \quad (18)$$

For designing of $PI^w D^z$ controller for (12), $\hat{G}_2(s)$ and $\hat{G}_4(s)$ should satisfy (13) and (14), respectively. Thus, (13) and (14) become

$$20 \times \log_{10} |\tilde{C}(j\omega_{gc})\hat{G}_2(j\omega_{gc})| = 0 \quad (19)$$

$$\angle(\tilde{C}(j\omega_{gc})\hat{G}_4(j\omega_{gc})) = -\pi + \phi_{pm} \quad (20)$$

Putting the value of $\tilde{C}(s)$ from (9); (19) and (20) become (21) and (22), respectively.

$$20 \times \log_{10} \left| \left(\frac{c_0}{\sqrt{(\underline{d}_1 \omega_{gc})^2 + \underline{d}_0^2}} \right) \times (\sqrt{r^2 + s^2}) \right| = 0 \quad (21)$$

$$\tan^{-1} \left[\frac{s}{r} \right] - \tan^{-1} \left(\frac{\underline{d}_1 \omega_{gc}}{\underline{d}_0} \right) = -\pi + \phi_{pm} \quad (22)$$

where

$$r = K_1 + K_2 \omega_{gc}^{-w} \cos\left(\frac{\pi}{2} w\right) + K_3 \omega_{gc}^z \cos\left(\frac{\pi}{2} z\right)$$

$s = -K_2 \omega_{gc}^{-w} \sin\left(\frac{\pi}{2} w\right) + K_3 \omega_{gc}^z \sin\left(\frac{\pi}{2} z\right)$ The 'fmincon' optimization function of MATLAB is used for solving (21) and (22) to derive $\tilde{C}(s)$.

5 Derivation of approximant for high order interval system

The high-order interval system given in (10) is approximated to (11) by multipoint Padé approximation integrated with Routh table [5, 14]. The inclusion of Routh table for reduction of (10) results in stable approximation of (11).

5.1 Derivation of the denominator

$\hat{D}(s)$, the denominator of the interval model, is derived from the Routh array given in Table 1. The Routh array is formed from the denominator of (10). The denominator of (10) is given as

$$\bar{B}(s) = [\underline{b}_0, b_0] + [\underline{b}_1, b_1]s + [\underline{b}_2, b_2]s^2 + \dots + [\underline{b}_n, b_n]s^n \quad (23)$$

The entries in top two rows of Table 1 are taken from (23), while the entries in the successive rows are completed using (24).

$$\delta_{p,q} = \left(\delta_{p-2,q+1} \times \delta_{p-1,1} - \delta_{p-2,q} \times \delta_{p-1,q+1} \right) / \delta_{p-1,1} \quad (24)$$

where

$$p \geq 3, 1 \leq q \leq \left(\frac{n-p+3}{2} \right) \quad (25)$$

The entries of row numbers $(n+1-k)$ and $(n+2-k)$ of Table 1 are utilized to obtain the k th-order denominator, $\hat{D}(s)$, of the interval model as given below

$$\hat{D}(s) = \delta_{n+1-k,1}s^k + \delta_{n+2-k,1}s^{k-1} + \delta_{n+1-k,2}s^{k-2} + \dots \quad (26)$$

5.2 Derivation of the numerator

Multipoint Padé approximation [5] is utilized to obtain the numerator of (11). In multipoint Padé approximation [5], multiple expansion points are chosen to make equal the

responses of (10) and (11). The expansions for (10) and (11) are equated as

$$G(\rho_q) = \hat{G}(\rho_q) \quad (27)$$

around $2k$ expansion points ρ_q , where $q = 1, 2, \dots, 2k$.

Using (27), the interval coefficients of the numerator of model (11) are derived. The expression (27) [5] changes to (28)

$$\alpha(s) = \beta(s), \quad s = \rho_q \quad (28)$$

where $q = 1, 2, \dots, 2k$ with

$$\alpha(s) = \hat{C}(s)\bar{B}(s) \quad (29)$$

$$\beta(s) = \bar{A}(s)\hat{D}(s) \quad (30)$$

The polynomials given in (31) and (32), are obtained from (29) and (30), respectively.

$$\alpha(s) = [\underline{\alpha}_0, \alpha_0] + [\underline{\alpha}_1, \alpha_1]s + [\underline{\alpha}_2, \alpha_2]s^2 + \dots + [\underline{\alpha}_{n+k-1}, \alpha_{n+k-1}]s^{n+k-1} \quad (31)$$

$$\beta(s) = [\underline{\beta}_0, \beta_0] + [\underline{\beta}_1, \beta_1]s + [\underline{\beta}_2, \beta_2]s^2 + \dots + [\underline{\beta}_{n+k-1}, \beta_{n+k-1}]s^{n+k-1} \quad (32)$$

where

$$\begin{aligned} [\underline{\alpha}_{n+k-1}, \alpha_{n+k-1}] &= [\underline{b}_{n-1}, b_{n-1}] [\underline{c}_k, c_k] \\ [\underline{\alpha}_{n+k-2}, \alpha_{n+k-2}] &= [\underline{b}_{n-2}, b_{n-2}] [\underline{c}_{k-1}, c_{k-1}] \\ &\quad + [\underline{b}_{n-1}, b_{n-1}] [\underline{c}_{k-1}, c_{k-1}] \\ &\quad \vdots \\ [\underline{\alpha}_0, \alpha_0] &= [\underline{b}_0, b_0] [\underline{c}_k, c_k] \\ &\quad + \dots + [\underline{b}_{n-1}, b_{n-1}] [\underline{c}_0, c_0] \end{aligned} \quad (33)$$

and

$$\begin{aligned} [\underline{\beta}_{n+k-1}, \beta_{n+k-1}] &= [\underline{a}_{n-1}, a_{n-1}] [\underline{d}_k, d_k] \\ [\underline{\beta}_{n+k-2}, \beta_{n+k-2}] &= [\underline{a}_{n-2}, a_{n-2}] [\underline{d}_{k-1}, d_{k-1}] \\ &\quad + [\underline{a}_{n-1}, a_{n-1}] [\underline{d}_{k-1}, d_{k-1}] \\ &\quad \vdots \\ [\underline{\beta}_0, \beta_0] &= [\underline{a}_0, a_0] [\underline{d}_k, d_k] \\ &\quad + \dots + [\underline{a}_{n-1}, a_{n-1}] [\underline{d}_0, d_0] \end{aligned} \quad (34)$$

Table 1 Routh array

$\delta_{1,1} = [\underline{b}_n, b_n]$	$\delta_{1,2} = [\underline{b}_{n-2}, b_{n-2}]$	$\delta_{1,3} = [\underline{b}_{n-4}, b_{n-4}]$	$\vdots$
$\delta_{2,1} = [\underline{b}_{n-1}, b_{n-1}]$	$\delta_{2,2} = [\underline{b}_{n-3}, b_{n-3}]$	$\delta_{2,3} = [\underline{b}_{n-5}, b_{n-5}]$	$\vdots$
$\delta_{3,1}$	$\delta_{3,2}$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\delta_{n,1}$	$\delta_{n,2}$	$\vdots$	$\vdots$
$\delta_{n+1,1}$			

Table 2 Routh like table

$\begin{bmatrix} \alpha_{n+k-1}, \alpha_{n+k-1} \end{bmatrix}$	$\begin{bmatrix} \alpha_{n+k-2}, \alpha_{n+k-2} \end{bmatrix}$	$\begin{bmatrix} \alpha_{n+k-3}, \alpha_{n+k-3} \end{bmatrix}$	$\cdots$	$\begin{bmatrix} \alpha_0, \alpha_0 \end{bmatrix}$
1	Δ_{2k-1}	Δ_{2k-2}	$\cdots$	Δ_0
$\begin{bmatrix} l_{n+k-2}, l_{n+k-2} \end{bmatrix}$	$\begin{bmatrix} l_{n+k-3}, l_{n+k-3} \end{bmatrix}$	$\begin{bmatrix} l_{n+k-4}, l_{n+k-4} \end{bmatrix}$	$\cdots$	$\begin{bmatrix} l_0, l_0 \end{bmatrix}$
1	Δ_{2k-1}	Δ_{2k-2}	$\cdots$	Δ_0
$\begin{bmatrix} m_{n+k-3}, m_{n+k-3} \end{bmatrix}$	$\begin{bmatrix} m_{n+k-4}, m_{n+k-4} \end{bmatrix}$	$\begin{bmatrix} m_{n+k-5}, m_{n+k-5} \end{bmatrix}$	$\cdots$	$\begin{bmatrix} m_0, m_0 \end{bmatrix}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\begin{bmatrix} g_{n+k-4}, g_{n+k-4} \end{bmatrix}$	$\begin{bmatrix} g_{n+k-5}, g_{n+k-5} \end{bmatrix}$	$\begin{bmatrix} g_{n+k-6}, g_{n+k-6} \end{bmatrix}$	$\cdots$	$\begin{bmatrix} g_0, g_0 \end{bmatrix}$
1	Δ_{2k-1}	Δ_{2k-2}	$\cdots$	Δ_0
$\begin{bmatrix} h_{n+k-5}, h_{n+k-5} \end{bmatrix}$	$\begin{bmatrix} h_{n+k-6}, h_{n+k-6} \end{bmatrix}$	$\begin{bmatrix} h_{n+k-7}, h_{n+k-7} \end{bmatrix}$	$\cdots$	$\begin{bmatrix} h_0, h_0 \end{bmatrix}$

The polynomial corresponding to $2k$ expansion points is given as

$$E(s) = \prod_{p=1}^{2k} (s - \rho_p) = s^{2k} + \Omega_{2k-1}s^{2k-1} + \cdots + \Omega_1s + \Omega_0 \quad (35)$$

In the multipoint Padé approximation, Routh like table (Table 2) is obtained from (31) and (35).

Coefficients of (31) and (35), occupy top two rows of Table 2, while successive rows get entries using (36). Similarly, from (32) and (35), another Routh like table is derived. The numerator of (11) is acquired by matching the like entries of the two tables formed [5].

$$\begin{aligned} \begin{bmatrix} l_{n+k-p-2}, l_{n+k-p-2} \end{bmatrix} &= \begin{bmatrix} \alpha_{n+k-p-2}, \alpha_{n+k-p-2} \end{bmatrix} \\ &\quad - \Omega_{2k-p-1} \begin{bmatrix} \alpha_{n+k-1}, \alpha_{n+k-1} \end{bmatrix} \\ p &= 0, 1, 2, \dots, (n+k-2) \\ \begin{bmatrix} m_{n+k-p-3}, m_{n+k-p-3} \end{bmatrix} &= \begin{bmatrix} l_{n+k-p-3}, l_{n+k-p-3} \end{bmatrix} \\ &\quad - \Omega_{2k-p-1} \begin{bmatrix} l_{n+k-2}, l_{n+k-2} \end{bmatrix} \\ p &= 0, 1, 2, \dots, (n+k-3) \\ &\vdots \\ \begin{bmatrix} h_{n+k-p-5}, h_{n+k-p-5} \end{bmatrix} &= \begin{bmatrix} g_{n+k-p-5}, g_{n+k-p-5} \end{bmatrix} \\ &\quad - \Omega_{2k-p-1} \begin{bmatrix} g_{n+k-4}, g_{n+k-4} \end{bmatrix} \\ p &= 0, 1, 2, \dots, (2k-1) \end{aligned} \quad (36)$$

6 SISO test system

The proposed methodology is illustrated with a third-order test system. A third-order SISO test system [12] is considered as

$$G(s) = \frac{[2, 4] + [1, 3]s + [1, 2]s^2}{[1, 1] + [4, 4.5]s + [21, 22]s^2 + [4, 28]s^3} = \frac{\bar{A}(s)}{\bar{B}(s)} \quad (37)$$

The desired model for (37) is written as

$$\hat{G}(s) = \frac{[c_0, c_0]}{[d_1, d_1]s + [d_0, d_0]} \quad (38)$$

6.1 Determination of denominator

Using (24), Table 3 for denominator of (37) becomes

Direct truncation of Table 3 gives the denominator of (38)

$$\hat{D}(s) = [2.545, 4.523]s + [0.562, 1.777] \quad (39)$$

From (38) and (39), it is clear that

$$[d_0, d_0] = [0.562, 1.777], [d_1, d_1] = [2.545, 4.523] \quad (40)$$

For determination of the numerator of (38), a polynomial (41) is obtained by choosing $s = 0$ and $s = 3$ as two expansion points. Hence, (35) becomes

$$E(s) = s^2 - 3s \quad (41)$$

Using (31) and (32), the polynomials (42) and (43) are obtained for (37) and (38).

$$\begin{aligned} \alpha(s) &= [c_0, c_0][4, 28]s^3 + [c_0, c_0][21, 22]s^2 + [c_0, c_0][4, 4.5]s \\ &\quad + [c_0, c_0][1, 1] \end{aligned} \quad (42)$$

$$\begin{aligned} \beta(s) &= [2.545, 4.523][1, 2]s^3 + \{[2.545, 4.523][1, 3] \\ &\quad + [0.562, 1.777][1, 2]\}s^2 + \{[2, 4][2.545, 4.523] \\ &\quad + [0.562, 1.777][1, 3]\}s + [0.562, 1.777] \times [2, 4] \end{aligned} \quad (43)$$

Routh like tables, Table 4 for (41) and (42), and Table 5 for (41) and (43) are obtained by utilizing Table 2.

Table 3 Routh array

[4, 28]	[4, 4.5]
[21, 22]	[1, 1]
[2.545, 4.523]	
[0.562, 1.777]	

Table 4 Routh like table for $\alpha(s)$ and $E(s)$

$[c_0 c_0][4, 28]$	$[c_0 c_0][21, 22]$	$[c_0 c_0][4, 4.5]$	$[c_0 c_0][1, 1]$
1	-3	0	0
$[c_0 c_0][21, 22] + 3[c_0 c_0][4, 28]$	$[c_0 c_0][4, 4.5]$	$[c_0 c_0][1, 1]$	

The similar entries of the third rows of Table 4 and Table 5 are equated to obtain (44).

$$\begin{aligned} [c_0, c_0][21, 22] + 3[c_0, c_0][4, 28] &= \{[2.545, 4.523][1, 3] \\ &+ [.562, 1.777][1, 2]\} + 3[2.545, 4.523][1, 2] \\ [c_0, c_0][1, 1] &= [.562, 1.777][2, 4] \end{aligned} \quad (44)$$

From (44), the value of $[c_0, c_0]$ is obtained as

$$[c_0, c_0] = [1.124, 7.108] \quad (45)$$

On substitution of the interval coefficients from (39) and (45), the first-order model (38) becomes (48).

$$\hat{G}(s) = \frac{[1.124, 7.108]}{[2.545, 4.523]s + [0.562, 1.777]} \quad (46)$$

6.2 Design of FOPID controller

The FOPID controller is designed for (43). For this problem, (16) and (18) are identified as

$$\hat{G}_2(s) = \frac{7.108}{2.545s + 0.562} \quad (47)$$

$$\hat{G}_4(s) = \frac{1.124}{4.523s + 0.562} \quad (48)$$

Gain cross-over frequency $\omega_{gc} = 3.1$ rad/sec and phase margin $\phi_{pm} = 80$ are the given design specifications of the system. For these criteria, (23) and (24) are solved using 'fmincon' optimization function of MATLAB. The controller parameters determined are

$$\begin{aligned} K_1 &= 6.356, \quad K_2 = 21.231, \quad K_3 = 0.0025, \\ w &= 0.192, \quad z = 0.853 \end{aligned} \quad (49)$$

The FOPID controller for the specified design criteria is obtained as

$$\tilde{C}(s) = 6.356 + \frac{21.231}{s^{0.192}} + 0.0025s^{0.853} \quad (50)$$

The step responses of feedback system of Fig. 5 having HOIP (i.e. (37)) and controller (i.e. (50)) are depicted in Fig. 6. In Fig. 6, LL denotes lower limits of HOIP and UL represents upper limits of HOIP. The Bode plots of $\hat{C}(s)\hat{G}(s)$, given in Fig. 7, offers the stable response. In this case also, LL and UL denote, respectively, lower limits and upper limits. From Figs. 6 and 7, it is clearly visible that the application of FOPID controller in cascade with HOIP assures the stable responses. Therefore, conclusion can be made that proposed technique has utility in designing FOPID controller for interval systems using reduced-order modeling.

7 Conclusion

This article proposed the design of fractional-order proportional-integral-derivative (FOPID) controller for higher order continuous interval system on the basis of reduced-order modeling. For controller design, higher order interval plant (HOIP) is first converted into reduced-order interval plant (ROIP). Multipoint Padé approximation interlaced with Routh table is used for deriving ROIP. After obtaining the ROIP, it is utilized for FOPID controller design. Phase margin and gain cross-over frequency are the basis of design of FOPID controller. Thus designed FOPID controller is also implemented on higher order interval plant. The results showed that FOPID controller working efficiently when implemented on HOIP. The key feature of this article is that it is addressing the design of control using reduced-order modelling. However, as far as limitations of work proposed are considered, ROIP has to be obtained first for HOIP before proceeding to controller design. The future work of this contribution lies in design of controller for MIMO continuous and discrete interval systems.

Table 5 Routh like for $\beta(s)$ and $E(s)$

$[2.545, 4.523] * [1, 2]$	$[2.545, 4.523] * [1, 3] + [0.562, 1.777] * [1, 2]$	$[2.545, 4.523] * [2, 4] + [0.562, 1.777] * [1, 2]$	$[2.545, 4.523] * [2, 4] + [0.562, 1.777] * [1, 2]$
1	-3	0	0
$[2.545, 4.523] * [1, 3] + [0.562, 1.777] * [2, 4]$	$[2.545, 4.523] * [2, 4] + [0.562, 1.777] * [1, 2]$	$[2.545, 4.523] * [2, 4] + [0.562, 1.777] * [1, 2]$	

Fig. 6 Step responses of feed-back control having controller and HOIP

Fig. 7 Bode plot of $\tilde{C}(s)\tilde{G}(s)$

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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