Exact and limiting solutions of fluid flow for axially oscillating cylindrical pipe and annulus

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Abstract

The Navier–Stokes equations have been solved to derive the expressions of the velocity distributions for two cases: (1) oscillatory flows inside and outside of an axially oscillating cylindrical pipe, and (2) oscillatory flow inside an axially oscillating cylindrical annulus. In both the cases, in addition to the exact expressions for the velocity profiles, particular emphasis has been given for the determination of approximate velocity distributions for the high frequency and low frequency or quasi-static limits. It is shown that, for sufficiently large value of an appropriate frequency parameter, the velocity distribution inside the axially or longitudinally oscillating cylindrical annulus can be approximated as a superposition of the velocity distribution inside an axially oscillating cylindrical pipe of radius $\bar{R}_o$ and the velocity distribution outside an axially oscillating cylindrical pipe of radius $\bar{R}_i$, where $\bar{R}_i$ and $\bar{R}_o$ are the inner and outer radii of the axially oscillating annulus, respectively.

Keywords Exact solutions · Navier–Stokes · Stokes-layer · Oscillatory flow · Cylindrical annulus

1 Introduction

The Navier–Stokes equations, the governing equations of fluid motions, is a set of non-linear partial differential equations. Thus, exact solutions of fluid flows could be obtained only for some special cases. In the present study, the focus is on the analytical solutions of oscillatory or time-periodic flows which is a subset of the general unsteady flows. The oscillatory flows can be driven either by the time-periodic pressure gradient or by the time-periodic motion of the flow-boundaries. For the case of oscillatory flows driven by the time-periodic pressure gradient, the exact solutions have been derived for fully developed flows of Newtonian fluids in different straight conduits of constant cross-section, such as, plane channel [1], pipes with circular [1–7], elliptic [8, 9], rectangular [10–13] and triangular [14] cross-sections, and cylindrical annulus [15]. The reviews of exact solutions of Navier–Stokes equations for unsteady flows

of Newtonian fluids can be found in the works of Wang [4] and Riley and Drazin [16]. Recently, exact solutions are also obtained for the flows of non-Newtonian fluids, driven by the time-periodic pressure gradient, in circular pipe [17, 18].

In contrast to the oscillatory flows driven by the time-periodic pressure-gradient, the literature on the oscillatory flows of Newtonian fluid driven by the time-periodic motion of the flow-boundary are relatively less. However, the pioneering study in the field of oscillatory flow happens to be a time-periodic flow driven by the oscillatory motion of an infinite flat plate in its own plane, inside an infinite layer of fluid. The problem was first studied by Stokes [19] and later by Rayleigh [20]; it is also referred to as ‘Stokes’ second problem’ [21]. The time-periodic motion of the plate induces an oscillatory motion in the surrounding fluid. The amplitude of the oscillation decreases exponentially with the perpendicular distance away from the plane of the plate. So, there is a boundary layer of fluid, called the

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'Stokes layer', beyond which the velocity of fluid is negligibly small. The classical 'Stokes' second problem' has been revisited by Erdogan [22] and Fetecau et al. [23], and generalized exact solutions have been obtained which are valid for small as well as large time.

The Stokes layer on a flat plate is treated as a paradigm of oscillatory flows, for which closed form exact solution of the Navier–Stokes equations is known. All though it is an idealized flow, proper understanding of Stokes layer is important to shed light for more complicated unsteady flows, which have characteristics of Stokes layer near to the flow-boundaries. Further, it is to be noted that, because of its practical applications, a number of studies are also reported on the 'Stokes' second problem' with different types of non-Newtonian fluids [24–26].

As in the case of Stokes layers around an oscillating flat plate, a cylindrical pipe of infinite length oscillating longitudinally along its axis should also possess boundary layers of fluid near to the wall of the cylinder beyond which the effect of viscosity can be neglected; these boundary layers formed on both the sides of the wall of the oscillating cylinder may be seen as the cylindrical analogue of the classical Stokes layer formed over and below an oscillating flat plate. However, in spite of the similarities between planar and cylindrical Stokes layers, it may be anticipated that, due to the curvature of cylindrical geometry, there ought to be differences in characteristics between the Stokes layers formed inside and outside the axially oscillating cylindrical pipe, which are not present between the Stokes layers formed over and below an oscillating flat plate.

The cylindrical geometries with internal, external or annular time-dependent flows have numerous engineering applications in the oil, petrochemical, power-generation, and chemical industries [27]. Thus, from the theoretical standpoint, as discussed in the above paragraph, as well as from the point of view of practical applications, a detailed analysis of cylindrical analogue of Stokes layer is important. However, after an extensive literature survey, it has been found that the literature on oscillatory flows of Newtonian fluids driven by the time-periodic axial motion of the cylindrical pipe or annulus is relatively scarce. The most probable reason of lack of separate study on cylindrical Stokes layer may be that the mathematical expression of velocity profile and the flow features are anticipated to be analogous to the flows inside the cylindrical pipe and annulus driven by time-periodic pressure-gradient, for which, as already discussed, an extensive amount of studies are available in the literature.

However, in spite of certain similarities in flow-features between the time-dependent flow inside a pipe driven by the oscillatory pressure-gradient and the time-dependent flow inside a pipe driven by oscillatory axial motion of the cylindrical wall, it has been observed that the two cases

are not analogous. For the pressure-gradient driven flow, the amplitude factor appearing in the expression of axial velocity depends on the frequency of imposed pulsation whereas the amplitude factor is independent of imposed frequency in case of flow driven by the motion of the pipe. The same argument holds true for the time-dependent flows inside a cylindrical annulus driven by time-periodic pressure-gradient and motion of the walls, respectively. To further emphasize the aforementioned point, it is to be noted that, due to the differences in amplitude factor appearing in the resulting expressions of axial velocity, in the quasi-static limit, the features of pressure-gradient driven flows are found to be entirely different than the features of wall-motion driven flows.

In the context of present work on cylindrical Stokes layer, it is to be mentioned that studies are reported on the 'Stokes' second problem' with different non-Newtonian fluids in cylindrical geometries; for instances, readers may refer to the works of Rajagopal [28], Rajagopal and Bhatnagar [29], Fetecau et al. [17, 30], and the references there in. The exact solutions for some of the oscillatory flows of Newtonian fluids, presented in this paper, may be obtained as special cases of the more general results associated with the non-Newtonian fluids, reported by the aforementioned studies. But, for the different limiting cases, the differences between the characteristics of time-periodic flows driven by the oscillatory pressure-gradient and oscillatory motion of the boundary are not clearly highlighted in the literature of non-Newtonian fluid mechanics as well. Similarly, the characteristic differences between the internal and external time-periodic flows in cylindrical geometries, attributed due to the difference in curvatures, are not studied in a unified manner, especially for the limiting cases presented in this study.

Based on the above discussion, it is evident that the understanding of the fundamental flow-physics of oscillatory flows in the cylindrical geometries is incomplete, especially for the limiting cases discussed above. Thus, to address the aforementioned literature gap, the time-periodic flows of Newtonian fluids with cylindrical Stokes-layers, generated by the oscillatory flow-boundaries, deserve an exclusive investigation, which is the motivation of the present work. In this study, at first exact expressions for the velocity distributions have been presented for flows inside and outside of an axially oscillating cylindrical pipe of infinite extent. Thereafter, exact solution is derived for the flow inside an axially oscillating cylindrical annulus of infinite extent. In both the cases, the limiting expressions have been obtained for high frequency and low frequency or quasi-static limits. Finally, it is shown that for the high frequency limit, the oscillatory flow inside the annulus is a superposition of oscillatory flow inside an oscillating pipe of radius equal to the outer radius of the annulus and

oscillatory flow outside an oscillating pipe of radius equal to the inner radius of the annulus.

2 The governing equations and the solution of the problems

Let consider a cylinder of infinite length oscillating along its axis in an infinite layer of quiescent fluid. The time-periodic motion of the cylinder will induce oscillatory flows inside and outside the cylinder. The configuration of the problem and the cylindrical polar coordinate system $(\bar{r}, \bar{\theta}, \bar{z})$ used in the analysis are shown in Fig. 1. The axis of the cylinder is aligned with z-axis of the cylindrical polar coordinate system. The thickness of the wall is neglected. The time-periodic motion of the wall of the cylinder can be described as

$$\bar{v}_z(\bar{R}, \bar{t}) = \bar{V}_0 \cos \bar{\omega} \bar{t}, \quad (1)$$

where $\bar{V}_0$ and $\bar{\omega}$ denote the dimensional amplitude and frequency of oscillation of the cylinder, respectively. In this paper, all dimensional variables are represented with a bar above.

2.1 General solution of the governing differential equation

The surrounding fluid is considered to be Newtonian, and is at rest. The viscosity and density of the fluid are constant. Since the length of the cylinder is infinite in z-direction, the flow can be treated to be parallel to the z-axis, i.e., the fluid velocity vector can be expressed as $\bar{V} = (0, 0, \bar{v}_z)$, where only non-zero velocity component is in z-direction. Under the aforementioned assumptions, the Navier–Stokes equations reduce to the following non-dimensional partial differential equation for $v_z(r, t)$:

$$(\beta_c)^2 \frac{\partial v_z}{\partial t} = \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r}, \quad (2)$$

where the non-dimensional variables are introduced as

Fig. 1 Geometry and coordinate system for flow around an axially oscillating cylinder

$$t = \bar{\omega} \bar{t}, r = \bar{r}/\bar{R}, \text{ and } v_z = \bar{v}_z/\bar{V}_0. \quad (3)$$

The non-dimensional frequency parameter, the Womersley number, is defined as

$$\beta_c = \bar{R} \sqrt{\left(\frac{\bar{\omega}}{\bar{\nu}}\right)}, \quad (4)$$

where $\bar{\nu}$ is the kinematic viscosity of the fluid.

The subscript 'c' is added to distinguish it from the similar non-dimensional frequency parameter β used in the studies of stability of time-periodic flows in the plane channel by Davis [31] and von Kerczek [32], where $\beta = \bar{h} \sqrt{\left(\frac{\bar{\omega}}{2\bar{\nu}}\right)}$, $\bar{h}$ is the half-width of the plane channel. The Womersley number β_c , as expressed in Eq. (4), can be seen as a measure of ratio of radius of the cylinder, $\bar{R}$, to the thickness of the Stokes layer formed near to the oscillating boundary, which is of the order of $\sqrt{(\bar{\nu}/\bar{\omega})}$.

Since the coefficients of the partial differential Eq. (2) are independent of time t , using the method of separation of variables, solution of (2) can be expressed as

$$\Phi(r, t) = F(r)e^{it}, \quad (5a)$$

where the complex axial velocity $\Phi(r, t)$ and the complex amplitude function $F(r)$ are introduced to take advantage of representation in terms of complex variables. However, the actual velocity, $v_z(r, t)$, of the flow will be given by the real part of the above expression (5a), i.e.

$$v_z(r, t) = \text{Real}\{\Phi(r, t)\} = \text{Real}\{F(r)e^{it}\}. \quad (5b)$$

Substitution of the solution of the form (5a) in the partial differential Eq. (2) leads to the following ordinary differential equation for $F(r)$:

$$\frac{d^2 F}{dr^2} + \frac{1}{r} \frac{dF}{dr} - i(\beta_c)^2 F = 0. \quad (6)$$

The second order homogeneous ordinary differential Eq. (6) is in the form of modified Bessel's equation of zeroth order. So, the general solution of (6) can be expressed as

$$F(r) = c_1 I_0\left(i^{\frac{1}{2}} \beta_c r\right) + c_2 K_0\left(i^{\frac{1}{2}} \beta_c r\right), \text{ for } 0 \leq r \leq \infty, \quad (7)$$

where I_0 and K_0 are the modified Bessel functions of first and second kinds of zeroth order, respectively, and c_1 and c_2 are the arbitrary constants.

2.2 Velocity profile inside the cylinder ($0 \leq r \leq 1$)

Considering the no-slip condition on the oscillating wall at $r = 1$, and regularity condition (velocity remains bounded)

at $r=0$, the expression of the complex amplitude function $F(r)$ for flow inside the oscillating cylinder is derived as

$$F(r; \beta_c) = I_0\left(i^{\frac{1}{2}} \beta_c r\right) / I_0\left(i^{\frac{1}{2}} \beta_c\right), \text{ for } 0 \leq r \leq 1. \quad (8a)$$

Following the Eq. (5b), the actual temporal and spatial variation of the velocity of fluid inside the oscillating cylinder is given by

$$v_z(r, t; \beta_c) = \text{Real} \left[\frac{I_0\left(i^{\frac{1}{2}} \beta_c r\right)}{I_0\left(i^{\frac{1}{2}} \beta_c\right)} e^{it} \right], \text{ for } 0 \leq r \leq 1. \quad (8b)$$

The velocity profile inside a longitudinally oscillating cylinder has reported by Fetecau et al. [17], for a generalized Burgers fluid. The expression (8a, 8b) may be obtained from the results of Fetecau et al. [17], by setting the material constants, appearing in the expression of the generalized Burgers fluid, to zero.

For a better understanding of the problem, next, we discuss the asymptotic forms of the solution (8) for different limiting cases. For the derivation of limiting solutions, following asymptotic forms of I_0 are used (refer to Abramowitz and Stegun [33] for the details of different asymptotic forms of the modified Bessel functions).

$$I_0(x) \approx 1, \text{ as } x \rightarrow 0, \quad (9a)$$

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}, \quad \left(|\arg(x)| < \frac{\pi}{2} \right), \text{ as } x \rightarrow \infty. \quad (9b)$$

$$v_z(r, t; \beta_c) = \left(\frac{1}{r} \right)^{\frac{1}{2}} e^{-\left\{ \frac{1}{\sqrt{2}} \beta_c (1-r) \right\}} \cos \left\{ t - \frac{1}{\sqrt{2}} \beta_c (1-r) \right\}, \text{ for } 0 < r \leq 1. \quad (12b)$$

2.2.1 Limiting solution for $\beta_c \rightarrow 0$

For small values of Womersley number, with $\beta_c \rightarrow 0$, using the series expansion of I_0 [33], expression (8a) for the amplitude function $F(r)$ is approximated as

$$F(r) = \frac{1 + \frac{i(\beta_c r)^2}{4} + \dots}{1 + \frac{i(\beta_c)^2}{4} + \dots}, \quad 0 \leq r \leq 1. \quad (10)$$

For the limit $\beta_c \rightarrow 0$, the above expression of $F(r)$ further simplifies to

$$F(r) \approx 1. \quad (11a)$$

The same result can be obtained directly using the asymptotic form of I_0 , for $\beta_c \rightarrow 0$, given by Eq. (9a).

So, for small β_c , according to Eqs. (5b) and (11a), the expression of axial velocity reduces to

$$v_z(r, t) = \cos t. \quad (11b)$$

Thus, in the limit $\beta_c \rightarrow 0$, the flow inside the axially oscillating cylinder becomes entirely inviscid, and the cylinder and the fluid inside it oscillate time-periodically as a whole like a rigid body. It is interesting to note that, in the quasi-static limit, for the case of flow driven by the time-periodic pressure-gradient, the velocity profile of flow becomes identical to the velocity profile of quasi-steady Poiseuille flow in phase with slowly varying time-periodic pressure-gradient [7, 21], which is completely different than the quasi-static solution (11) of the case studied in this investigation.

2.2.2 Limiting solution $\beta_c \rightarrow \infty$

It has been discussed that Womersley number is a measure of the ratio of radius of the cylinder, $\bar{R}$, to thickness of the Stokes layer. Consequently, in the limit $\beta_c \rightarrow \infty$, thickness of the Stokes layer formed near to the walls of the oscillating cylinder is quite small compared to the radius of the cylinder.

For the limit $\beta_c \rightarrow \infty$, using the asymptotic form of the modified Bessel function I_0 , given by Eq. (9b), the expressions of complex amplitude, $F(r)$, and axial velocity, $v_z(r, t)$, are approximated as

$$F(r; \beta_c) = \left(\frac{1}{r} \right)^{\frac{1}{2}} e^{-\left\{ \frac{1}{\sqrt{2}} (1+i) \beta_c (1-r) \right\}}, \text{ for } 0 < r \leq 1, \quad (12a)$$

From Eqs. (12a, 12b), it follows that, for a fixed r , the amplitude of $v_z(r, t)$ decreases exponentially with the increase of β_c . As a result, for large Womersley number β_c , fluid velocity diminishes to zero within a short distance from the wall, where $r \approx 1$. Thus, the curvature factor $(1/r)^{\frac{1}{2}}$ in Eqs. (12a, 12b) can be neglected, and the expressions of $F(r)$ and $v_z(r, t)$, for $\beta_c \rightarrow \infty$, further simplify to

$$F(r; \beta_c) = e^{-\left\{ \frac{1}{\sqrt{2}} (1+i) \beta_c (1-r) \right\}}, \text{ for } 0 < r \leq 1, \quad (13a)$$

$$v_z(r, t; \beta_c) = e^{-\left\{ \frac{1}{\sqrt{2}} \beta_c (1-r) \right\}} \cos \left\{ t - \frac{1}{\sqrt{2}} \beta_c (1-r) \right\}, \text{ for } 0 < r \leq 1. \quad (13b)$$

However, it has been found, as discussed in Sect. 3.1.1, that if the curvature factor, $(1/r)^{\frac{1}{2}}$, is retained, then the limiting expression (12b) of $v_z(r, t)$ serves as a good

approximation of the exact expression (8b) even for the moderate values of β_c as well.

The expression (13b) for $v_z(r, t)$ can be written in a more compact form, if the radial distance away from the wall of the cylinder is scaled with the factor $\sqrt{(2\nu/\omega)}$. So, with the introduction of a new dimensionless radial coordinate y_i , the limiting expression of $v_z(r, t)$, for large β_c , is expressed as

$$v_z(r, t; \beta_c) = e^{-y_i} \cos(t - y_i), \text{ where } y_i = \frac{(R - r)}{\sqrt{(2\nu/\omega)}} = \frac{1}{\sqrt{2}} \beta_c (1 - r). \quad (13c)$$

It should be noted that the above expression of $v_z(r, t)$ is exactly same as the flow over an oscillating flat plate. The scaling factor $\sqrt{(2\nu/\omega)}$ in the definition of y_i is chosen, because of the fact that thickness of the Stokes layer is equal to $\sqrt{(2\nu/\omega)}$ for flow over an oscillating flat plate [19, 34].

Since in the limit $\beta_c \rightarrow \infty$, the thickness of the Stokes layer is quite small compared to the radius of the cylinder, the effect of curvature of the cylindrical geometry may be neglected. As a result, as seen from the Eq. (13c), the temporal and spatial variations of fluid velocity inside the oscillating cylinder becomes identical to the case of classical Stokes layer over an oscillating flat plate.

2.3 Velocity distribution for flow outside the cylinder ($1 \leq r < \infty$)

Evaluating the arbitrary constants c_1 and c_2 of Eq. (7) for the flow outside the oscillating cylinder, with $1 \leq r < \infty$, the expression of the complex amplitude function $F(r)$ is derived as

$$F(r; \beta_c) = K_0 \left(i^{\frac{1}{2}} \beta_c r \right) / K_0 \left(i^{\frac{1}{2}} \beta_c \right). \quad (14a)$$

From the Eq. (5b), it follows that the actual temporal and spatial variation of the fluid velocity is given by

$$v_z(r, t; \beta_c) = \text{Real} \left[\frac{K_0 \left(i^{\frac{1}{2}} \beta_c r \right)}{K_0 \left(i^{\frac{1}{2}} \beta_c \right)} e^{it} \right], \text{ for } 1 \leq r \leq \infty. \quad (14b)$$

It is to be noted that the above expression, associated with the flow of Newtonian fluid, may be obtained as a special case of the general result associated with Oldroyd-B fluid [29]. Similar to the flow inside the cylinder, the limiting solutions for the flow outside the cylinder are also derived using the following asymptotic forms of K_0 [33]:

$$K_0(x) \approx -\ln(x), \text{ as } x \rightarrow 0, \quad (15a)$$

$$K_0(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x}, \left(|\arg(x)| < \frac{3}{2}\pi \right) \text{ as } x \rightarrow \infty. \quad (15b)$$

2.3.1 Limiting solution for $\beta_c \rightarrow 0$

For the case of small Womersley number β_c , using the asymptotic form (15a), the limiting expression of $F(r)$ close to the wall of the cylinder can be derived as

$$F(r; \beta_c) = \frac{\ln \left(i^{\frac{1}{2}} \beta_c r \right)}{\ln \left(i^{\frac{1}{2}} \beta_c \right)}, \text{ for } \beta_c \rightarrow 0 \text{ as } r \text{ is } O(1). \quad (16a)$$

Now, close to the wall of the cylinder, where r is $O(1)$, as $\beta_c \rightarrow 0$, the right hand side of expression (16a) becomes indeterminate of the form (∞/∞) . So, applying L'Hopital's rule, we get

$$F(r) \rightarrow 1, \Phi(r, t) \rightarrow e^{it} \text{ and } v_z(r, t) \rightarrow \cos(t) \text{ as } \beta_c \rightarrow 0. \quad (16b)$$

It is to be emphasized that, for large distance away from the wall, i.e. for $r \gg 1$, even if β_c is small, the argument of K_0 in the numerator of the Eq. (14a), $i^{\frac{1}{2}} \beta_c r$, needs not to be small. So, the limiting expressions (16a) and (16b), for $\beta_c \rightarrow 0$, are valid only near to the wall, where r is $O(1)$.

2.3.2 Limiting solution for $\beta_c \rightarrow \infty$

For the limit $\beta_c \rightarrow \infty$, using the asymptotic form of the modified Bessel function K_0 , given by Eq. (15b), the expression of complex amplitude, $F(r)$, is approximated as

$$F(r; \beta_c) = \left(\frac{1}{r} \right)^{\frac{1}{2}} e^{-\left\{ \frac{1}{\sqrt{2}} (1+i) \beta_c (r-1) \right\}}, \text{ for } 1 \leq r \leq \infty, \quad (17a)$$

and, accordingly, for flow outside the oscillating cylinder, the expression of $v_z(r, t)$ reduces to

$$v_z(r, t; \beta_c) = \left(\frac{1}{r} \right)^{\frac{1}{2}} e^{-\left\{ \frac{1}{\sqrt{2}} \beta_c (r-1) \right\}} \cos \left\{ t - \frac{1}{\sqrt{2}} \beta_c (r-1) \right\}. \quad (17b)$$

As in the case of flow inside the oscillating cylinder, the curvature factor $(1/r)^{\frac{1}{2}}$ in the Eqs. (17a, b) is neglected in the limit $\beta_c \rightarrow \infty$, and the expressions of $F(r)$ and $v_z(r, t)$ are further approximated as

$$F(r; \beta_c) = e^{-\left\{ \frac{1}{\sqrt{2}} (1+i) \beta_c (r-1) \right\}}, \quad (18a)$$

$$v_z(r, t; \beta_c) = e^{-\left\{\frac{1}{\sqrt{2}}\beta_c(r-1)\right\}} \cos\left\{t - \frac{1}{\sqrt{2}}\beta_c(r-1)\right\}. \quad (18b)$$

However, it is observed that if the curvature factor, $(1/r)^{1/2}$, is retained, then the limiting expression (17b) of $v_z(r, t)$, derived for $\beta_c \rightarrow \infty$, remains almost valid even for the moderate values of β_c , as well.

With the introduction of a new dimensionless radial coordinate y_o , the limiting equation of $v_z(r, t)$, for large β_c , is expressed as

$$v_z(r, t; \beta_c) = e^{-y_o} \cos(t - y_o), \text{ where } y_o = \frac{(\bar{r} - \bar{R})}{\sqrt{(2\nu/\bar{\omega})}} = \frac{1}{\sqrt{2}}\beta_c(r-1). \quad (18c)$$

The new dimensionless radial coordinate, y_o , is introduced for flow outside the cylinder, such that, it measures the outward radial distance from the wall with respect to the scaling factor $\sqrt{(2\nu/\bar{\omega})}$. Similar to the case of flow inside the oscillating cylinder, the expression of $v_z(r, t)$ outside the cylinder, for large value of β_c , is same as the classical Stokes layer formed over an oscillating flat plate.

Thus, in the limit $\beta_c \rightarrow \infty$, from the expressions (13) and (18), it can be concluded that the curvature of the axially oscillating cylinder has a negligible effect on the velocity of the surrounding fluid; the temporal and spatial variations of velocity, inside and outside the axially oscillating cylinder, are well-approximated by the spatio-temporal variations of velocity inside the Stokes layers formed on both the sides of a flat plate, time-periodically oscillating in its own plane, inside a quiescent fluid.

denoted by $\bar{R}_i$ and $\bar{R}_o$, respectively. The time-periodic axial motion of both the walls of the annulus can be described in dimensional form by the Eq. (1). The oscillatory motion of the annulus will induce an oscillatory flow inside the annular gap, $\bar{R}_h = (\bar{R}_o - \bar{R}_i)$, between the two cylindrical walls.

To analyze the flow in the oscillating annulus, it is convenient to work with a new non-dimensional radial coordinate η , defined as $\eta = (\bar{r} - \bar{R}_i)/\bar{R}_h$, where $0 \leq \eta \leq 1$ inside the annulus. The hydraulic radius of the annulus is given by $\bar{R}_h = (\bar{R}_o - \bar{R}_i)$. With the introduction of non-dimensional radial coordinate η , time $t = \bar{\omega} \bar{t}$, and axial velocity

$v_z = \bar{v}_z/\bar{V}_o$, the governing differential equation for non-dimensional complex amplitude function $F(\eta)$ is derived as

$$\frac{d^2 F}{d\eta^2} + \frac{1}{(A + \eta)} \frac{dF}{d\eta} - i(\beta_a)^2 F = 0, \text{ for } 0 \leq \eta \leq 1, \quad (19)$$

where non-dimensional axial velocity $v_z(\eta, t)$ is related to the non-dimensional complex amplitude function $F(\eta)$ as

$$v_z(\eta, t) = \text{Real}\{F(\eta)e^{it}\}. \quad (20)$$

The non-dimensional curvature parameter A is defined as $A = \bar{R}_i/\bar{R}_h$, and frequency parameter is introduced as $\beta_a = \bar{R}_h \sqrt{(\bar{\omega}/\nu)}$. The subscript 'a' refers to the cylindrical annulus. The frequency parameter β_a can also be identified as the Womersley number for the cylindrical annulus. The general solution of (19) is given by

$$F(\eta) = c_1 I_0\left\{i^{1/2}\beta_a(A + \eta)\right\} + c_2 K_0\left\{i^{1/2}\beta_a(A + \eta)\right\}, \text{ for } 0 \leq \eta \leq 1. \quad (21)$$

2.4 Flow inside an oscillating annulus

Let consider a cylindrical annulus of infinite length oscillating along its axis in an infinite layer of quiescent fluid as shown in Fig. 2. The inner and outer radii of the annulus are

Fig. 2 Geometry and coordinate system for the flow inside an axially oscillating cylindrical annulus

The no-slip boundary conditions on the inner and the outer oscillating cylindrical walls reduce to

$$F(\eta) = 1 \text{ at } \eta = 0, 1. \quad (22a,b)$$

Accordingly, the expressions of amplitude function $F(\eta)$, subjected to the boundary conditions (22), is derived as

$$F(\eta; A, \beta_a) = \frac{B_1 I_0\{\beta_a(A + \eta)i^{1/2}\} - B_2 K_0\{\beta_a(A + \eta)i^{1/2}\}}{B_1 I_0\{\beta_a A i^{1/2}\} - B_2 K_0\{\beta_a A i^{1/2}\}}, \quad (23a)$$

where

$$B_1(A, \beta_a) = [K_0\{\beta_a(A + 1)i^{1/2}\} - K_0\{\beta_a A i^{1/2}\}], \quad (23b)$$

$$B_2(A, \beta_a) = [I_0\{\beta_a(A + 1)i^{1/2}\} - I_0\{\beta_a A i^{1/2}\}]. \quad (23c)$$

It is to be mentioned that the solution of a similar problem, where oscillatory flow is set-up by the oscillation of outer cylinder while the inner cylinder remains stationary, has also been reported for the generalized Burgers fluid [30].

A comparison of the expression of amplitude function, $F(\eta)$, given by Eq. (23), with the corresponding expression for time-periodic flow inside an annulus driven by the oscillatory pressure gradient, as reported by Tsangaris [15], reveals that, for both the cases, the expressions of amplitude functions may be reduced to identical forms by considering a transformation, $v_z'(\eta, t) = v_z(\eta, t) - \cos(t)$, except for a multiplying factor of $1/(\beta_a)^2$ appearing for the case of flow driven by oscillatory pressure-gradient. Thus, the two cases are not exactly analogous.

2.4.1 Limiting solution for $\beta_a \rightarrow 0$

Similar to the case of flow inside the time-periodically oscillating cylinder, using the asymptotic forms of I_0 and K_0 , and applying L'Hopital's rule, it can be shown that, in the limit $\beta_a \rightarrow 0$,

$$F(\eta) \rightarrow 1, \text{ and } v_z(\eta, t) \rightarrow \cos(t) \quad (24)$$

for axially oscillating annulus as well. Thus, in the quasi-static limit, the annulus and the fluid inside it as a whole oscillate like a rigid body with same amplitude and phase. As discussed for the case of flow inside an axially oscillating cylinder, if the flow is driven by the time-periodic pressure gradient instead of time-periodic motion of the walls of the annulus, then, in the quasi-static limit, the flow behaves like a quasi-steady Poiseuille flow in an annulus [15], which is different than the quasi-static solution described by Eq. (24).

2.4.2 Limiting solution for $\beta_a \rightarrow \infty$

The expression (23) of $F(\eta)$ also simplifies, if we seek the limiting solution for large value of β_a , i.e. if the thickness of Stokes layer near the walls is negligible compared to the annular gap between the two cylinders. Using the asymptotic forms, (9b) and (15b), of the modified Bessel functions it can be shown that, in the limit $\beta_a \rightarrow \infty$,

$$F(\eta; A, \beta_a) = \frac{\sqrt{\frac{A}{A+\eta}} \left[e^{\{i^{1/2}\beta_a(1-\eta)\}} - e^{-\{i^{1/2}\beta_a(1-\eta)\}} \right] + \sqrt{\frac{A+1}{A+\eta}} \left[e^{(i^{1/2}\beta_a\eta)} - e^{-(i^{1/2}\beta_a\eta)} \right]}{e^{(i^{1/2}\beta_a)} - e^{-(i^{1/2}\beta_a)}} \quad (25)$$

Since $\beta_a \rightarrow \infty$ and $0 \leq \eta \leq 1$, the above expression for $F(\eta)$ is further simplified to

$$F(\eta; A, \beta_a) = \sqrt{\frac{A}{A+\eta}} e^{-\{i^{1/2}\beta_a\eta\}} + \sqrt{\frac{A+1}{A+\eta}} e^{\{i^{1/2}\beta_a(\eta-1)\}}. \quad (26)$$

Now, if we consider only the effect of oscillation of inner cylindrical wall on the amplitude function, $F(\eta)$, of the oscillatory flow inside the annulus, then using the expression of amplitude function as derived for flow outside a time-periodically oscillating cylinder, given by (14a), we can write

$$F(\eta; A, \beta_a) \Big|_{\text{inner wall}} = K_0\{i^{1/2}\beta_a(A+\eta)\} / K_0\{i^{1/2}\beta_a A\}. \quad (27)$$

For high frequency limit, $\beta_a \rightarrow \infty$, using the asymptotic form (15b), the Eq. (27) reduces to

$$F(\eta; A, \beta_a) \Big|_{\text{inner wall}} = \sqrt{\frac{A}{A+\eta}} e^{-\left\{i^{1/2}\beta_a\eta\right\}}. \quad (28)$$

Similarly, if only the effect of oscillation of the outer cylindrical wall on the flow inside the annulus is considered, it follows from (8a) that

$$F(\eta; A, \beta_a) \Big|_{\text{outer wall}} = I_0\{i^{1/2}\beta_a(A+\eta)\} / I_0\{i^{1/2}\beta_a(A+1)\}. \quad (29)$$

Again, using the asymptotic form of I_0 for the limit $\beta_a \rightarrow \infty$, given by (9b), the above expression is simplified as

$$F(\eta; A, \beta_a) \Big|_{\text{outer wall}} = \sqrt{\frac{A+1}{A+\eta}} e^{\{i^{1/2}\beta_a(\eta-1)\}}. \quad (30)$$

Upon combining Eqs. (26), (28), and (30), it immediately follows that

$$F(\eta; A, \beta_a) \Big|_{\text{annulus}} = F(\eta; A, \beta_a) \Big|_{\text{inner wall}} + F(\eta; A, \beta_a) \Big|_{\text{outer wall}}, \text{ for } \beta_a \rightarrow \infty. \quad (31)$$

Thus, for large value of Womersley number β_a , the effect of oscillation of walls decays rapidly with the increasing distance from the walls of the annulus, so that the spatial and temporal evolutions of the axial velocity, $v_z(\eta, t)$, inside the annulus can be expressed as a superposition of velocity of flow induced due to the motion of inner cylindrical wall, and

velocity of flow induced due to the motion of outer cylindrical wall, considering the effect of each cylinder separately.

2.4.3 Limiting solution for narrow gap limit, $A \rightarrow \infty$

Using the asymptotic forms (9b) and (15b), the expression of $F(\eta)$, in the limit of narrow gap $A \rightarrow \infty$, is derived such that it is free from the curvature parameter A :

$$F(\eta; \beta_a) = \frac{[e^{i^{1/2}\beta_a(1-\eta)} - e^{-i^{1/2}\beta_a(1-\eta)}] + [e^{i^{1/2}\beta_a\eta} - e^{-i^{1/2}\beta_a\eta}]}{e^{i^{1/2}\beta_a} - e^{-i^{1/2}\beta_a}}. \quad (32a)$$

The above expression can also be given in terms of sine hyperbolic function as

$$F(\eta; \beta_a) = \frac{\sinh\{i^{1/2}\beta_a(1-\eta)\} + \sinh\{i^{1/2}\beta_a\eta\}}{\sinh\{i^{1/2}\beta_a\}}, \text{ for } A \rightarrow \infty. \quad (32b)$$

If we define a new dimensionless radial coordinate, $y_a = 2\eta - 1$, where $-1 \leq y_a \leq 1$, then using the standard trigonometric identities, the expression of $F(\eta)$ is further simplified to show that

$$F(y_a; \beta_a) = \frac{\cosh\left\{\frac{\beta_a}{2\sqrt{2}}(1+i)y_a\right\}}{\cosh\left\{\frac{\beta_a}{2\sqrt{2}}(1+i)\right\}}, \quad \text{for } -1 \leq y_a \leq 1 \quad (33a)$$

So, the expression of the dimensionless axial velocity is given by

$$v_z(y_a, t; \beta_a) = \text{Re} \left[\frac{\cosh\left\{\frac{\beta_a}{2\sqrt{2}}(1+i)y_a\right\}}{\cosh\left\{\frac{\beta_a}{2\sqrt{2}}(1+i)\right\}} e^{it} \right]. \quad (33b)$$

The above expression of axial velocity, $v_z(y_a, t)$, is same as that of the flow inside a plane channel, which is oscillating in its own plane [32]. Thus, in the narrow gap-limit, $A \rightarrow \infty$, the expression of velocity of flow inside an axially oscillating annulus can be well-approximated by the velocity-distribution inside a time-periodically oscillating plane channel.

3 Graphical study of the amplitude function of velocity

In the last section, exact and limiting expressions of amplitude function of axial velocity are obtained for flows around an axially oscillating cylinder and flow inside an axially oscillating annulus. In this section, the amplitude function of axial velocity is studied graphically to understand the physics of the time-periodic flows in details. In the following subsections, at first the case of axially oscillating cylinder is presented; thereafter the case of axially oscillating annulus is discussed.

3.1 Axially oscillating cylinder

The exact variation of the complex amplitude function, $F(r)$, for flow inside an axially oscillating cylinder is expressed by Eq. (8a). Similarly, the exact expression of $F(r)$ for flow outside the axially oscillating cylinder is given by Eq. (14a). The graph of $F(r; \beta_c)$ vs. r , as given by Eqs. (8a) and (14a), is presented in Fig. 3a and b, for different values of the Womersley number, β_c . Figure 3a and b, respectively, show the variations of real and imaginary parts of the complex amplitude function, $F(r)$. In the figures, the interval $0 \leq r \leq 1$ corresponds to the flow inside the oscillating cylinder (refer to Eq. 8a), whereas the interval $r \geq 1$ represents the flow outside the oscillating cylinder (refer to Eq. 14a).

Fig. 3 Variation of real and imaginary parts of the complex amplitude function F as a function of radial coordinate r , given by Eqs. (8a) and (14a), for different values of Womersley number β_c

3.1.1 Results for the limit $\beta_c \rightarrow \infty$

A glance at Fig. 3a and b reveals that, for large β_c , such as $\beta_c = 100$, the curvature of the cylinder has negligible effect, as the graph of $F(r)$ becomes almost symmetric about the vertical line $r = 1$, which represents the wall of the oscillating cylinder; the very fact may also be derived by comparing limiting expression (12a) for flow inside the axially oscillating cylinder and limiting expression (17a) for flow outside the axially oscillating cylinder. For $\beta_c = 100$, the graphs of real and imaginary parts of $F(r)$ also show that the radial variations of $F(r)$ possess the boundary-layer type behaviors in both the sides of the wall of the oscillating cylinder; the effect of viscosity on the amplitude function, $F(r)$, is confined to narrow regions close to the wall of the oscillating cylinder. This boundary layers may be identified as the cylindrical analogue of the classical Stokes layer formed over an oscillating flat plate.

The graphs further indicate that, beyond the Stokes layers formed on both the sides of the wall, radial variation of $F(r)$ becomes almost constant, with $F(r) = 0$. Thus, the graphs of $\beta_c = 100$, corroborate the fact, derived in the limiting expressions of $F(r)$ for $\beta_c \rightarrow \infty$ (refer to Eqs. 13a and 18a), that, for large value of the Womersley number β_c , the flows in the interior and exterior of the axially oscillating infinite cylinder becomes almost similar to the flows around an infinite flat plate which is time-periodically oscillating in its own plane, in a quiescent liquid, known as Stokes' second problem [21].

A comparison between the exact analytical expression and the approximate expressions for high frequency limit $\beta_c \rightarrow \infty$, is presented in Figs. 4 and 5, respectively, for $\beta_c = 100$ and $\beta_c = 10$. It is observed that the high-frequency approximations, without the curvature factor $(1/r)^{1/2}$, given by Eq. (13a) for flow inside the cylinder and Eq. (18a) for flow outside the cylinder, match well with the corresponding exact analytical variation of $F(r)$, given by Eqs. (8a) and (14a), for $\beta_c = 100$, but deviates significantly for $\beta_c = 10$. However, the approximations (12a) and (17a), which include the curvature factor $(1/r)^{1/2}$, remain consistent with the exact variations for $\beta_c = 100$ as well as $\beta_c = 10$. Thus, if the curvature factor, $(1/r)^{1/2}$, is retained, then limiting expressions (12a) and (17a) of $F(r)$, derived for $\beta_c \rightarrow \infty$, remain valid even for moderate values of β_c as well.

3.1.2 Results for the moderate value of β_c

It is already noted from Fig. 3a and b, that, for $\beta_c = 100$, the graph of the amplitude function, $F(r)$, is almost symmetric about the vertical line $r = 1$, which represents the wall of the oscillating cylinder. However, the figures also show that the graph of $F(r)$ becomes increasingly asymmetric about $r = 1$, as the value of β_c is reduced from $\beta_c = 100$.

Fig. 4 Comparison of limiting expressions of real and imaginary parts of $F(r)$ for $\beta_c \rightarrow \infty$, with the exact analytical expressions, given by Eqs. (8a) and (14a), for $\beta_c = 100$

It implies that the difference in characteristics between the flows inside and outside of the oscillating cylinder increases as the value of β_c is reduced. This is because of the fact that, as the value of β_c is decreased, the thickness of the Stokes layer increases continuously, so that the effect of difference in curvatures between interior and exterior of the oscillating cylinder becomes increasingly significant.

From the graphs of $\beta_c = 100, \beta_c = 10, \beta_c = 5, \beta_c = 2$ and $\beta_c = 1$ in Fig. 3a and b, it is evident that, for the flow outside the oscillating cylinder, where $r \geq 1$, the thickness of the Stokes layer increases continuously as the value of Womersley number, β_c , is reduced progressively. The graphs of $\beta_c = 100$ and $\beta_c = 10$ also suggest that, similar to the case of flow outside the oscillating cylinder, the increase of thickness of the Stokes layer with the decrease of β_c happens for the case of flow inside the cylinder, where $0 \leq r \leq 1$, as well. It is noted that, for the cases of $\beta_c = 100$ and $\beta_c = 10$, the flow inside the cylinder is consisted of two distinct parts: there is a Stokes layer near to the wall

Fig. 5 Comparison of limiting expressions of real and imaginary parts of $F(r)$ for $\beta_c \rightarrow \infty$, with the exact analytical expressions, given by Eqs. (8a) and (14a), for $\beta_c = 10$

where at first $F(r)$ varies rapidly with inward radial distance before it gradually becomes equal to zero at the boundary of the layer and there is a central inviscid core region within which $F(r) = 0$ everywhere.

For the flow inside the axially oscillating cylinder, the graphs of $\beta_c = 10$ and $\beta_c = 5$ in Fig. 3a and b suggest that, as the value of β_c is decreased, for a certain value of the β_c in the range $5 < \beta_c < 10$, Stokes layer thickness increases to such extent that it reaches the center-line of the cylinder. As the value of β_c is reduced further, for the cases of $\beta_c = 5$, $\beta_c = 2$ and $\beta_c = 1$, the flow becomes viscous everywhere inside the oscillating cylinder, which is in contrast to the cases of larger Womersley number, such as $\beta_c = 100$ and $\beta_c = 10$, where the flow is consisted of a viscous region (Stokes layer) near to the wall and an inviscid core near to the axis of the cylinder. Also, it is to be noted that, for $\beta_c = 5$, $\beta_c = 2$ and $\beta_c = 1$, as the flow inside the cylinder becomes entirely viscous the maxima of the graphs of amplitude function, $F(r)$, appears near to the axis of the cylinder, which is again in contrast to the cases of larger

Womersley number, where the maxima of the graphs appears near to wall of the oscillating cylinder.

3.1.3 Results for the limit $\beta_c \rightarrow 0$

It is noted from the graphs of $\beta_c = 0.1$ and $\beta_c = 0.01$ in the Fig. 3a and b that the radial variation of $F(r)$ becomes almost constant inside the cylinder with $F(r) \approx 1$, i.e., the flow inside the cylinder becomes almost inviscid, and the cylinder and the fluid inside it oscillate with same amplitude and phase, like a rigid body, which is also consistent with the limiting expression of $F(r)$ derived in Sect. 2.2.1, for $\beta_c \rightarrow 0$.

Similar to the flow inside the cylinder, it is derived that, in the quasi-static limit, $F(r) \approx 1$ for flow outside the cylinder as well, provided the distance from the wall of the cylinder is not large (see Sect. 2.3.1). However, the graphs of $\beta_c = 0.1$, and $\beta_c = 0.01$, evaluated using the exact analytical expression (14a) of $F(r)$, show that, as the value of Womersley number, β_c , is reduced from 0.1 to 0.01, the graphs of real and imaginary parts of $F(r)$ progressively move towards the limiting result, $F(r) \approx 1$. But, in contrast to the flow inside the cylinder ($F(r) \approx 1$ inside the cylinder for $\beta_c = 0.1$ and $\beta_c = 0.01$), the value of $F(r)$ remains significantly away from the quasi-static approximation $F(r) \approx 1$ in the vicinity of the wall.

Figure 6 presents the comparison between the exact expression (14a) and the quasi-static approximations (16a) of $F(r)$ for flow outside the cylinder, for $\beta_c = 0.01$. It is observed that, near to the wall, the quasi-static approximation (16a) agrees well with the exact variation of $F(r)$, whereas the approximation (16b), $F(r) \approx 1$, is way off from the exact variation. Thus, it can be concluded that, for flow outside the cylinder, the quasi-static limit (16b), $F(r) \approx 1$ near to the wall, may only be realized for extremely small values of β_c , and for moderately small values of β_c (e.g., $\beta_c = 0.1$ or $\beta_c = 0.01$) the quasi-static expression (16a), obtained before applying L'Hopital's rule, represents the radial variation of $F(r)$ more accurately as compared to the final expression $F(r) \approx 1$.

3.2 Axially oscillating cylindrical annulus

The exact radial variation of the complex amplitude function $F(\eta)$ for the oscillating cylindrical annulus, given by Eq. (23), is shown in Figs. 7 and 8, respectively for the curvature parameter $A = 1$ and $A = 0.01$. For the specified value of the curvature parameter, the radial variations of $F(\eta)$ are presented for different values of the Womersley number β_a , varying from $\beta_a = 100$ to $\beta_a = 0.01$. The variations in characteristics of amplitude function $F(\eta)$ as the Womersley number is varied from 100 to 0.01, are summarized in the following subsections.

Fig. 6 Comparison of limiting expression of real and imaginary parts of $F(r)$ for $\beta_c \rightarrow 0$, given by Eq. (16a), with the exact analytical expressions, given by Eq. (14a), for $\beta_c = 0.01$

3.2.1 Results for the limit $\beta_a \rightarrow \infty$

The graphs of $\beta_a = 100$ represent the limiting behavior of $F(\eta)$ for $\beta_a \rightarrow \infty$. A glance at the graphs of real and imaginary parts of $F(\eta)$ indicates that there are formations of narrow Stokes layers near to the walls of the oscillating annulus for $\beta_a = 100$. Within these narrow Stokes layers, at first amplitude function $F(\eta)$ changes rapidly within a short distance from the walls, then it becomes equal to zero at the boundaries of the Stokes layers. Beyond the Stokes layers, $F(\eta)$ remains equal to zero everywhere in the central region of the annulus. So, for large β_a , effect of wall-oscillation is limited to the narrow viscous fluid-layers close to the walls of the annulus, and the rest of the fluid in the core of the annulus remains stationary.

A comparison between exact expression (23) and approximate expression (26) of $F(\eta)$, for $\beta_a \rightarrow \infty$, are presented in Figs. 9 and 10, respectively for $A = 1$ and $A = 0.01$. The graphs of real and imaginary parts of complex amplitude function $F(\eta)$ are compared for $\beta_a = 100$ and $\beta_a = 10$. From the Fig. 9 of $A = 1$, it is observed that the limiting approximation of $F(\eta)$, given by Eq. (26), agrees with the

Fig. 7 Exact variation of real and imaginary parts of the complex amplitude function, $F(\eta)$, inside the axially oscillating cylindrical annulus, given by Eq. (23), for different values of Womersley number β_a , for $A = 1$

exact analytical expression (23), for $\beta_a = 100$ as well as $\beta_a = 10$. Similarly, from the Fig. 10 it is observed that, for $A = 0.01$, the high frequency approximation (26) matches well for $\beta_a = 100$, but it deviates from the exact variation (23) for $\beta_a = 10$, near to the inner wall of the annulus. So, the range of Womersley number β_a , for which the high-frequency approximation (26) remains valid, depends on the value of the curvature parameter A , i.e., it depends on the asymmetry of the resultant velocity-profile; for small value of A , with highly asymmetric velocity profile, the high-frequency approximation is only valid for large value of β_a .

3.2.2 Results for the moderate value of β_a

From the plots of $\beta_a = 100$ and $\beta_a = 20$, presented in Figs. 7 and 8, it is observed that, with the decrease of Womersley number β_a , the Stokes layers penetrate further

Fig. 8 Exact variation of real and imaginary parts of the complex amplitude function, $F(\eta)$, inside the axially oscillating cylindrical annulus, given by Eq. (23), for different values of Womersley number β_a , for $A=0.01$

towards the centerline of the annulus, so that the size of the inviscid region, where $F(\eta) = 0$, at the core of the annulus decreases. With further decrease of β_a , Stokes layers extend so far inside that it interact with each other, and flow become entirely viscous throughout the cross-section of the annulus, e.g., see the graphs of $\beta_a = 10$. For $\beta_a = 5$ and $\beta_a = 2$, the thickness of the Stokes layers become of the order of width of the annulus. So, unlike the cases of $\beta_a \geq 10$, the variations of $F(\eta)$ do not possess any Stokes layer near to the wall, instead amplitude function, $F(\eta)$, varies slowly with η , and reaches the maximum value near to the centerline of the annulus.

3.2.3 Results for the limit $\beta_a \rightarrow 0$

The graphs of Womersley number $\beta_a = 0.01$, represent the limiting case of low frequency or quasi-static

Fig. 9 Comparison of limiting expression of real and imaginary parts of $F(\eta)$ for the limit $\beta_a \rightarrow \infty$, given by Eq. (26), with the exact analytical expression, given by Eq. (23), for $A=1$. **a** $\beta_a = 100$, **b** $\beta_a = 10$

approximation ($\beta_a \rightarrow 0$). It is evident from the graphs of $\beta_a = 0.01$, presented in Figs. 7 and 8, that, in the limit $\beta_a \rightarrow 0$, the real and imaginary parts of $F(\eta)$ remains approximately constant throughout the cross-section of the annulus, with $real(F) = 1$ and $imag(F) = 0$. So, the exact analytical variation (23) of $F(\eta)$, for $\beta_a = 0.01$, confirms the fact derived in the quasi-static approximation (24) that, in the limit $\beta_a \rightarrow 0$, the cylindrical annulus and the fluid inside it oscillate like a rigid body with same amplitude and phase.

3.2.4 Effect of curvature parameter, A

The effect of curvature of the annulus on the flow field is governed by the non-dimensional parameter A , defined as $A = R_i/R_h$. As shown in Sect. 2.4.3, the effect of curvature is negligible in the narrow gap limit, $A \rightarrow \infty$, and the

Fig. 10 Comparison of limiting expression of real and imaginary parts of $F(\eta)$ for the limit $\beta_a \rightarrow \infty$, given by Eq. (26), with the exact analytical expression, given by Eq. (23), for $A=0.01$. **a** $\beta_a = 100$, **b** $\beta_a = 10$

flow-features inside the axially oscillating annulus become identical to the case of flow inside an oscillating planar channel. However, for finite value of A , the effect of curvature on the flow cannot be neglected.

A closer look at the graphs of real and imaginary parts of amplitude function, $F(\eta)$, for $A=1$, presented in Fig. 7 reveals that the graphs of real and imaginary parts of $F(\eta)$ are asymmetric with respect to the center line, $\eta = 0.5$, of the annulus. Also, a comparison between the graphs of $A=1$ and $A=0.01$, presented in Figs. 7 and 8 respectively, indicates that the asymmetry of the graphs of $F(\eta)$ increases significantly as value of A is decreased from 1 to 0.01. Thus, it is concluded that, as the value of curvature parameter, A , reduces, the effect of curvature of the cylindrical geometry on the flow inside the axially oscillating annulus becomes progressively more pronounced.

4 Summary

To summarize, in this investigation, exact and limiting expressions for velocity distributions are derived for fully developed time-periodic flows in cylindrical geometry, where the flow is driven by the time-periodic motion of the flow-boundaries. Particular emphasis is given to the determination of the approximate velocity distributions and flow physics for different limiting values of frequency parameter, β_c or β_a , and aspect parameter, A , applicable for the case of cylindrical annulus. For the high frequency limit or large value of Womersley number, the connection between the flows around the axially oscillating cylinder and the flow inside the axially oscillating cylindrical annulus is established analytically. In addition to the above results, the graphical representations of the time-dependent amplitude function illustrate some interesting findings, e.g. the range of validity of different limiting solutions associated with the cylindrical pipe or annulus has been found to vary considerably depending upon the particular limit under consideration. Similarly, it is noted that the intermediate results, obtained while deriving the limiting expressions, are important as it may capture the essential characteristics of the flow for a broader range of values of the parameter as compared to the final results itself. For the case of oscillating cylindrical annulus, in addition to the frequency parameter, β_a , the curvature parameter, A , significantly affects the velocity distribution inside the axially oscillating annulus. It is observed that the velocity profile becomes increasingly asymmetric with the reduction of the value of the aspect parameter, A .

Availability of data and materials The data that support the findings of this study are available from the corresponding author upon request.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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