Research Article

Aligned magnetic field and Navier slip effects on free convective radiative flow of nanofluids with imbedded carbon nanotubes: a Lie group analysis

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Received: 19 December 2019 / Accepted: 17 June 2020 / Published online: 25 June 2020
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Abstract

The significance of aligned magnetic flux on free convective radiative nanofluid along a stretching surface with thermal energy storage is investigated in this paper. The problem has been modelled mathematically under Newtonian slip effect. The Prandtl boundary-layer expressions are framed using Lie group analysis and treated numerically. By employing the shooting method and the Runge–Kutta fourth-order technique, a solution to the dimensionless equations is obtained. The influence of pertinent dimensionless variables on the velocity, temperature, shear stress and heat transfer rate are scrutinized via graphical plots. The results of this study show that the velocity of the nanofluid is lower and wall shear flux is higher for greater aligned magnetic flux. More energy is transferred from the fluid system when multi-walled carbon nanotubes are immersed in the base liquid. The velocity of the nanofluid is decreased with increased slip parameter. Also, the temperature rises and heat flux is reduced with enhanced thermal energy storage/generation. The numerical solutions obtained in the present work are compared to existing literature, and they are found to be in a good agreement.

Keywords Aligned magnetic flux · Carbon nanotubes · Lie Group analysis · Navier slip · Radiation · Thermal energy storage

List of symbols

B_0	Uniform magnetic flux density (T)
g	Gravity
N	Slip parameter
$f(\eta)$	Dimensionless velocity
k_f	Thermal conductivity
k^*	Coefficient of mean absorption
Pr	Prandtl number
q_w	Wall heat flux
q_r	Radiative heat flux
R	Radiation parameter
Re_x	Local Reynolds number
$\bar{u}$	Component of velocity parallel to x direction (m/s)
$\bar{u}_w$	Stretching velocity (m/s)

$\bar{v}_w$	Suction velocity (m/s)
$\bar{v}$	Component of velocity parallel to y direction (m/s)
$\bar{T}$	Local temperature of the nanofluid (K)
T	Dimensionless temperature
T_w	Wall temperature
T_∞	Ambient temperature
Q_0	Volumetric rate of heat generation

Greek symbols

α	Thermal diffusivity of the nanofluid ($\text{m}^2 \text{s}^{-1}$)
σ^*	Stefan–Boltzmann constant
σ	Electrical conductivity (S/m)
ρ	Density of the nanofluid (kg/m^3)
μ	Dynamic viscosity of the nanofluid (Pa s)
ν	Kinematic viscosity of the nanofluid ($\text{m}^2 \text{s}^{-1}$)

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SN Applied Sciences (2020) 2:1283 | <https://doi.org/10.1007/s42452-020-3105-5>

ξ	Inclination angle of the magnetic field
θ	Dimensionless temperature
ϕ	Nanoparticle volume fraction
η	Similarity variable

Subscripts

f	Base fluid (water)
nf	Nanofluid
CNT	Carbon nanotubes

Abbreviations

MHD	Magnetohydrodynamics
MWCNT	Multi-walled CNTs (carbon nanotubes)
SWCNT	Single-walled CNTs (carbon nanotubes)

1 Introduction

In order to achieve enhancement of thermal conductivity in heat transfer systems, the suspension of nanometer-sized particles (nanoparticles) in a suitable base liquid is a major advancement in the thermodynamics field. Studying these fluid suspensions, termed as nanofluids, has been a prominent subject of investigation in recent times due to their importance in many industrial and engineering applications. In particular, miniaturizing the system is playing a prominent role today; for this reason, nanotechnology is a popular and booming area of research. In the Fullerene family, the best allotrope is the carbon nanotube (CNT), which has a long and hollow chemical structure. The CNT wall comprises atom-dense carbon sheets, called graphene. Two types of CNTs are available; namely, multi-walled carbon nanotubes (MWCNTs) and single-walled carbon nanotubes (SWCNTs). Their thermophysical properties are reliant on the alignment of the graphene sheet inside the tube and this causes them to behave like a metal or a semiconductor; the investigations regarding this have started in early 90 s. Waqar et al. [1] presented the analysis of MHD heat transfer on CNT-water nanofluid past a static or moving wedge. Haq et al. [2] examined MHD flows of CNTs along a stretching surface with water, ethylene glycol (EG) and engine oil as base fluids. Mahantesh et al. [3] investigated the CNTs immersed MHD nanofluids along with Marangonic effect. Later, he extended his work by incorporating varying heat source [4]. Khan et al. [5] explored the impact of heat transfer past a moving wedge. Reddy et al. [6] studied the MHD flow past a non-linear stretching sheet. Obula Reddy et al. [7] worked on the analysis of single wall and multi wall CNTs over a vertical cone. Ramzan et al. [8] analyzed the entropy generation past a vertical cone with radiation. Job et al. [9] investigated an unsteady MHD flow of natural convection water-based alumina or single wall carbon nanotubes within a wavy trapezoidal enclosed space in the presence

of viscous and Joule heating. Job and Gunakala [10] studied the influence of SWCNT's and Au-water based nanoparticles in mixed convection boundary layer flow within a grooved channel.

The research problem of the stretching sheet is of great importance and is applicable to various concrete situations involving metallurgical processes and polymer extrusion phenomena. Applications of these phenomena include continuous stretching, annealing, rolling, tinning of copper wires, manufacturing of polymer sheets, cooling of long metallic plates in a cooling path, the boundary layer flow along a liquid film during condensation processes [11, 12]. Khan and Pop [13] conducted an investigation on the boundary layer flow along a stretching sheet. Musa et al. [14] discussed the chemical reaction and thermal radiation impact on time-dependent nanoflow with varying fluid properties. Akbar et al. [15] explored the slip impact on MHD nanoflow along an extended sheet. Irfan et al. [16] worked on exponentially stretching sheet with MHD and variable fluid properties. Waini et al. [17] studied on a porous extending/spread out sheet. Rida et al. [18] investigated the combined effects of heat source/sink on radiative MHD nanoliquid. Poornima et al. [19] examined the influence of MHD on mixed convection temperature-dependent viscosity on the flow of a cylindrically-shaped copper water based nanofluid over a moving vertical surface. The hydromagnetic convective flow of a nanofluid over an exponential stretching surface with viscous dissipation effects was examined by Sreenivasulu and Poornima [20].

Magnetohydrodynamic (MHD) flows and slip phenomena receive considerable attention from researchers because of their distinct and widespread contribution to industrial and manufacturing processes. In general, the no-slip condition produces erroneous results and is no longer valid [21–28]; this is particularly true when studying the cooling of microelectromechanical systems (MEMS). Further, the role of radiation in heat transportation has wider involvement in distinct branches of thermal engineering and industrial processes like gas turbines, aero-space vehicles, aircraft propulsion processes, missiles, nuclear power (thermal) plants and satellites [29–35].

In moving fluids, the investigation of heat source/sink is essential for the understanding of chemically-reacting systems and in cases where dissociation of the fluid occurs; in these situations, the temperature gradients produced within the fluid are significant. Possible heat sources influence heat and mass transfer phenomena in semiconductor wafers, etc. A Lie-group analysis on stagnation point flow of nanofluid past a stretching porous sheet with radiation and heat generation/absorption was presented by Sreenivasulu and Bhaskar Reddy [36]. Sreenivasulu and Bhaskar Reddy [37] extended their study by including the effects of chemical

reaction. Burgers nanofluid with heat source/sink along a stretching sheet was investigated by Khan and Khan [38]. Very recently heat source/sink effect on MHD flow along a vertical stretching sheet was presented by Ibrahim et al. [39]. Konda et al. [40] investigated the effect of heat generation/absorption and melting heat on MHD Williamson nanofluid past a porous sheet.

To the authors' comprehension, the majority of the literature on nanoliquids considered MHD flows with heat sources/sinks. However, this paper is concerned with the aligned magnetic field impact on Navier slip flow of nanofluid imbedded carbon nanotubes along a stretching surface with thermal energy storage/heat source. Moreover, the similarity variables are obtained through Lie group analysis, and the comparisons of flow and heat transfer performance between SWCNTs and MWCNTs had not previously been considered. This is currently a booming topic as it has substantial medical and industrial applications such as nanofluidic pumps and micro-fluidic machinery. Thus, in our present work, we numerically examine the behavior of aligned MHD flow of nanofluid with immersed CNTs under the influence of radiation and Navier slip effects. The impact of pertinent physical controlled variables on velocity, friction, temperature factor (Nusselt number) and the streamlines are portrayed and analysed numerically.

2 Mathematical analysis

We consider a steady two-dimensional boundary layer flow of nanofluid along a vertical stretching sheet. The flow of nanofluid with imbedded carbon nanotubes is considered to be incompressible and laminar. The presence of an aligned magnetic field and thermal radiation are assumed, and thermal energy storage and internal heat generation are discussed in this paper. The flow is taken along the $\bar{x}$ -axis (upward direction) and the sheet is elongated. The $\bar{y}$ -axis is taken as being perpendicular to the sheet and inclined magnetic field is applied as shown in the flow diagram Fig. 1. The fluid temperature along the stretching surface is a constant value T_w , whereas the ambient temperature of the fluid is T_∞ ($T_w > T_\infty$).

Based on the description of the problem, the equations that govern the Prandtl boundary layer flow is given as follows:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \quad (1)$$

$$\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = \frac{1}{\rho_{nf}} \left[\mu_{nf} \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + (\rho\beta)_{nf} g (\bar{T} - T_\infty) - \sigma_{nf} B_0^2 \bar{u} \sin^2 \xi \right] \quad (2)$$

Fig. 1 Physical model and coordinate system

$$\bar{u} \frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{T}}{\partial \bar{y}} = \alpha_{nf} \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} - \frac{1}{(\rho C_p)_{nf}} \frac{\partial q_r}{\partial \bar{y}} + \frac{Q_0}{(\rho C_p)_{nf}} (\bar{T} - T_\infty) \quad (3)$$

The corresponding conditions at the stretching surface and far from this surface are:

$$\begin{aligned} \bar{u} &= \bar{u}_w(\bar{x}) + N \frac{\partial \bar{u}}{\partial \bar{y}}, \quad \bar{v} = \bar{v}_w, \quad \bar{T} = T_w \text{ at } \bar{y} = 0 \\ \bar{u} &\rightarrow 0, \quad \bar{T} \rightarrow T_\infty \text{ as } \bar{y} \rightarrow \infty \end{aligned} \quad (4)$$

The radiative heat flux within the nanofluid is described (using the Rosseland approximation) by the following equation:

$$q_r = -\frac{4}{3} \frac{\sigma^*}{k^*} \frac{\partial \bar{T}^4}{\partial \bar{y}} \quad (5)$$

The Rosseland model is applicable to optically dense media, where radiation travels only a short distance before dispersed or immersed. When material has a great extinction coefficient, it can be treated as optically thick. We have taken a fluid model in which the energy transfer through radiation within optically dense nanofluid with CNTs exists before the heat is dispersed, so, radiative heat transfer is taken into account. By employing the Rosseland's approximation, the temperature variations inside the fluid are sufficiently small; thus Eq. (5) is linearized by extending T^4 into the Taylor's series expansion about T_∞ which, upon ignoring higher-order terms yields the equation

$$\bar{T}^4 \cong 4T_\infty^3 \bar{T} - 3T_\infty^4 \quad (6)$$

Consequently, Eq. (3) becomes

$$\bar{u} \frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{T}}{\partial \bar{y}} = \alpha_{nf} \left(1 + \frac{4}{3} R \right) \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} + \frac{Q_0}{(\rho C_p)_{nf}} (\bar{T} - T_\infty) \quad (7)$$

The thermophysical properties of CNT nanofluids are given by the following analytical models.

1. Viscosity

$$\mu_{nf} = \frac{\mu_f}{(1 - \phi)^{2.5}} \quad (8)$$

2. Density

$$\rho_{nf} = (1 - \phi) \rho_f + \phi \rho_{CNT} \quad (9)$$

3. Heat capacity

$$(\rho C_p)_{nf} = (1 - \phi) (\rho C_p)_f + \phi (\rho C_p)_{CNT} \quad (10)$$

4. Thermal expansion coefficient

$$(\rho \beta)_{nf} = (1 - \phi) (\rho \beta)_f + \phi (\rho \beta)_{CNT} \quad (11)$$

5. Electrical conductivity

$$\sigma_{nf} = \sigma_f \left[1 + \frac{3\phi(\sigma_s/\sigma_f - 1)}{\sigma_s/\sigma_f - 2 - (\sigma_s/\sigma_f - 1)\phi} \right] \quad (12)$$

6. Thermal conductivity:

At a small scale, proposed nanofluid models are applicable to circular or elliptic particles, which actually lag in exhibiting the spatial distribution properties. Xue [41] suggested the following model for CNTs, which are large axial ratio ellipsoids (Table 1):

$$\kappa_{nf} = \kappa_f \frac{1 - \phi + 2\phi \frac{\kappa_{CNT} - \kappa_f}{\kappa_{CNT} + \kappa_f} \ln \frac{\kappa_{CNT} + \kappa_f}{2\kappa_f}}{1 - \phi + 2\phi \frac{\kappa_f}{\kappa_{CNT} - \kappa_f} \ln \frac{\kappa_{CNT} + \kappa_f}{2\kappa_f}} \quad (13)$$

We introduce the below similarity (dimensionless) variables:

Table 1 Thermo-physical properties of SWCNT, MWCNT and the base liquid

Thermo physical properties	Base liquid	Types of nanoparticles	
	Water	SWCNT	MWCNT
ρ (kg/m ³)	997	2600	1600
C_p (J/kg K)	4179	425	796
k (W/m K)	0.613	6600	3000
$\beta \times 10^{-5}$ (K ⁻¹)	21	0.33	0.72
σ (Ω ⁻¹ m ⁻¹)	0.05	4.8×10^7	3.3×10^6

$$\begin{aligned} x &= \frac{\bar{x}}{L}, \quad y = Gr^{1/4} \frac{\bar{y}}{L}, \quad u = Gr^{-1/2} \frac{L \rho_f}{\mu_f} \bar{u}, \quad v = Gr^{-1/4} \frac{L \rho_f}{\mu_f} \bar{v}, \quad T = \frac{\bar{T} - T_\infty}{T_w - T_\infty}, \\ \alpha_{nf} &= \kappa_{nf} / (\rho C_p)_{nf}, \quad \bar{u}_w(\bar{x}) = \frac{v_f Gr^{1/2}}{L^2} \bar{x}, \quad R = 16\sigma^* T_\infty^3 / 3k_f k^*, \quad S = -\frac{\rho_f L \bar{v}_w}{\mu_f Gr^{1/4}}, \\ Gr &= \rho_f^2 g(\rho \beta)_f (T_w - T_\infty) L^3 / \mu_f^2, \quad Pr = \mu_f / (\rho_f \alpha_f), \quad M = \sigma_f B_0^2 L^2 / (\mu_f Gr^{1/2}), \\ \gamma &= \frac{Gr^{1/4} N}{L}, \quad Re_x = \bar{x} \bar{u}_w / \nu_f \end{aligned} \quad (14)$$

Equations (1), (2) and (7) acquire the subsequent non-dimensional form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (15)$$

$$\begin{aligned} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{\rho_f}{(1 - \phi) \rho_f + \phi \rho_{CNT}} \\ &\left\{ \frac{1}{(1 - \phi)^{2.5}} \frac{\partial^2 u}{\partial y^2} + Gr [1 - \phi + \phi(\rho \beta)_{CNT} / (\rho \beta)_f] T - \sigma_{nf} / \sigma_f M u \sin^2 \xi \right\} \end{aligned} \quad (16)$$

$$\begin{aligned} u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} &= \frac{4R + 3}{3Pr} \frac{(\rho C_p)_f}{(1 - \phi) (\rho C_p)_f + \phi (\rho C_p)_{CNT}} \frac{\kappa_{nf}}{\kappa_f} \frac{\partial^2 T}{\partial y^2} \\ &+ \frac{\lambda (\rho C_p)_f}{(1 - \phi) (\rho C_p)_f + \phi (\rho C_p)_{CNT}} T \end{aligned} \quad (17)$$

The associated boundary conditions are:

$$\begin{aligned} u &= x + \gamma \frac{\partial u}{\partial y}, \quad v = -S, \quad T = 1 \text{ at } y = 0 \\ u &\rightarrow 0, \quad T \rightarrow 0 \text{ as } y \rightarrow \infty \end{aligned} \quad (18)$$

We define the stream function ψ by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

so that the equation of continuity is satisfied and the remaining equations take the form:

$$\begin{aligned} \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} &= \frac{\rho_f}{(1 - \phi) \rho_f + \phi \rho_{CNT}} \frac{1}{(1 - \phi)^{2.5}} \frac{\partial^3 \psi}{\partial y^3} \\ &+ \frac{\rho_f}{(1 - \phi) \rho_f + \phi \rho_{CNT}} \\ &\left\{ Gr [1 - \phi + \phi(\rho \beta)_{CNT} / (\rho \beta)_f] T - \sigma_{nf} / \sigma_f M \frac{\partial \psi}{\partial y} \sin^2 \xi \right\} \end{aligned} \quad (19)$$

$$\frac{\partial \psi}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial y} = \frac{\lambda(\rho C_p)_f}{(1-\phi)(\rho C_p)_f + \phi(\rho C_p)_{CNT}} T + \frac{4R+3}{3Pr} \frac{(\rho C_p)_f}{(1-\phi)(\rho C_p)_f + \phi(\rho C_p)_{CNT}} \frac{\kappa_{nf}}{\kappa_f} \frac{\partial^2 T}{\partial y^2} \quad (20)$$

and the boundary conditions become

$$\frac{\partial \psi}{\partial y} = x + \gamma \frac{\partial^2 \psi}{\partial y^2}, \quad \frac{\partial \psi}{\partial x} = S, \quad T = 1 \text{ at } y = 0 \quad (21)$$

$$\frac{\partial \psi}{\partial y} \rightarrow 0, \quad T \rightarrow 0 \text{ as } y \rightarrow \infty$$

Equations (19) and (20) along with (21) are a highly non-linear and coupled system of partial differential equations. To convert them into ordinary differential equations, we introduce the scaling Lie-group Γ of transformations defined by [42, 43]:

$$\Gamma: \quad x^* = xe^{\epsilon\alpha_1}, \quad y^* = ye^{\epsilon\alpha_2}, \quad \psi^* = \psi e^{\epsilon\alpha_3}, \quad T^* = Te^{\epsilon\alpha_4} \quad (22)$$

Substituting Eq. (22) in Eqs. (19) and (20), we have

$$e^{\epsilon(\alpha_1+2\alpha_2-2\alpha_3)} \left(\frac{\partial \psi^*}{\partial y^*} \frac{\partial^2 \psi^*}{\partial x^* \partial y^*} - \frac{\partial \psi^*}{\partial x^*} \frac{\partial^2 \psi^*}{\partial y^{*2}} \right) = \frac{\rho_f}{(1-\phi)\rho_f + \phi\rho_{CNT}} \frac{1}{(1-\phi)^{2.5}} e^{\epsilon(3\alpha_2-\alpha_3)} \frac{\partial^3 \psi^*}{\partial y^{*3}} + \frac{\rho_f}{(1-\phi)\rho_f + \phi\rho_{CNT}} \left\{ Gr[1-\phi + \phi(\rho\beta)_{CNT}/(\rho\beta)_f] e^{-\epsilon\alpha_4} T^* - e^{\epsilon(\alpha_2-\alpha_3)} \sigma_{nf}/\sigma_f M \frac{\partial \psi^*}{\partial y^*} \sin^2 \xi \right\} \quad (23)$$

$$e^{\epsilon(\alpha_1+\alpha_2-\alpha_3-\alpha_4)} \left(\frac{\partial \psi^*}{\partial y^*} \frac{\partial T^*}{\partial x^*} - \frac{\partial \psi^*}{\partial x^*} \frac{\partial T^*}{\partial y^*} \right) = \frac{4R+3}{3Pr} \frac{(\rho C_p)_f}{(1-\phi)(\rho C_p)_f + \phi(\rho C_p)_{CNT}} e^{\epsilon(2\alpha_2-\alpha_4)} \frac{\kappa_{nf}}{\kappa_f} \frac{\partial^2 T^*}{\partial y^{*2}} + \frac{\lambda(\rho C_p)_f}{(1-\phi)(\rho C_p)_f + \phi(\rho C_p)_{CNT}} e^{-\epsilon\alpha_4} T^* \quad (24)$$

Since Eqs. (23) and (24) are invariant under the Lie-group Γ , the relations between the parameters are as follows:

$$\alpha_1 + 2\alpha_2 - 2\alpha_3 = 3\alpha_2 - \alpha_3 = -\alpha_4 = \alpha_2 - \alpha_3 \quad (25)$$

$$\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 2\alpha_2 - \alpha_4 = -\alpha_4$$

From these relations, we get $\alpha_3 = \alpha_4 = \alpha_1$ and $\alpha_2 = 0$, which reduces Γ to a single-parameter group of transformations

$$x^* = xe^{\epsilon\alpha_1}, \quad y^* = y, \quad \psi^* = \psi e^{\epsilon\alpha_1}, \quad T^* = Te^{\epsilon\alpha_1} \quad (26)$$

By expanding the function $e^{\epsilon\alpha_1}$ in powers of ϵ using Taylor series and neglecting terms of order ϵ or higher, Eq. (26) becomes

$$x^* - x = x\alpha_1\epsilon, \quad y^* - y = 0, \quad \psi^* - \psi = \psi\alpha_1\epsilon, \quad T^* - T = T\alpha_1\epsilon \quad (27)$$

The corresponding characteristic equations are

$$\frac{dx}{x\alpha_1} = \frac{dy}{0} = \frac{d\psi}{\psi\alpha_1} = \frac{dT}{T\alpha_1} \quad (28)$$

from which we obtain the similarity transformations

$$\eta = y^*, \quad \psi^* = x^* f(\eta), \quad T^* = x^* \theta(\eta). \quad (29)$$

In view of Eq. (29), Eqs. (23) and (24) reduce to the ordinary differential equations

$$f''' + (1-\phi)^{2.5} \left\{ [1-\phi + \phi(\rho_{CNT}/\rho_f)] [ff'' - (f')^2] + Gr[1-\phi + \phi(\rho\beta)_{CNT}/(\rho\beta)_f] \theta - \sigma_{nf}/\sigma_f M f' \sin^2 \xi \right\} = 0 \quad (30)$$

$$\frac{4R+3}{3Pr} \frac{\kappa_{nf}}{\kappa_f} \theta'' + \lambda\theta + \frac{(1-\phi)(\rho C_p)_f + \phi(\rho C_p)_{CNT}}{(\rho C_p)_f} [f\theta' - f'\theta] = 0 \quad (31)$$

and the corresponding boundary conditions are:

$$f'(0) = 1 + \gamma f''(0), \quad f(0) = S, \quad \theta(0) = 1 \quad (32)$$

$$f' \rightarrow 0, \quad \theta \rightarrow 0 \text{ as } \eta \rightarrow \infty$$

It is necessary to study the engineering quantities such as coefficient of local skin friction C_f and the local Nusselt number Nu_x , which are defined as follows:

$$C_f = \frac{\mu_{nf}}{\rho_f u_w^2} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)_{\bar{y}=0}, \quad Nu_x = \frac{\bar{x} \kappa_{nf}}{\kappa_f (T_w - T_\infty)} \left(-\frac{\partial \bar{T}}{\partial \bar{y}} \right)_{\bar{y}=0} \quad (33)$$

Using Eqs. (14), (29) and (30), the shear stress and rate of heat transfer at the stretching sheet are described as

$$Re_x^{1/2} C_f = \frac{1}{(1-\phi)^{2.5}} f''(0), \quad Re_x^{-1/2} Nu_x = -\frac{\kappa_{nf}}{\kappa_f} \theta'(0) \quad (34)$$

3 Results and discussion

The coupled non-linear ordinary differential Eqs. (30)–(31) and boundary conditions (32) were solved numerically by utilizing the fourth order R-K technique along with the

Fig. 2 Impact of ϕ on velocity

Fig. 4 Impact of ξ on velocity

Fig. 3 Impact of ϕ on temperature

Fig. 5 Impact of ξ on temperature

shooting method. The numerical algorithm was implemented using the Mathematica software. Firstly, the BVP concerning the physical model was transformed into an IVP consisting of five first-order differential equations and appropriate initial conditions. Applying the fourth-order R-K method, this IVP was solved numerically along with the shooting technique until the relative error is less than 10^{-6} . Numerical results were obtained and analysed through graphs (Figs. 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16) and tables.

We validated the algorithm used in this study by making numerical comparisons with the works of Kalidas [44], Swathi Mukhopadhyay [45] and Reddy et al. [46]. Numerical values of $f''(0)$ and $\theta'(0)$ for varying Grashof number Gr

were displayed in Table 2, whereas Nusselt number values for different values of the Prandtl number Pr were shown in Table 3. It was found that the relative error in each of the compared values in Tables 2 and 3 is less than 0.1%. Therefore, we determined that the results obtained using our present numerical algorithm have great consistency with the results of Kalidas [44], Swathi Mukhopadhyay [45] and Reddy et al. [46].

In order to gain some physical insight into the present work, numerical results were computed with the following ranges of values: $0 \leq \phi \leq 0.2$ (for dilute nanoparticle suspensions); $0 \leq \xi \leq \frac{\pi}{2}$ (feasible range for the magnetic field inclination angle). The Prandtl number is chosen to be 6.785, since the base fluid in the present work is water [44].

Fig. 6 Impact of S on velocity

Fig. 8 Impact of γ on velocity

Fig. 7 Impact of S on Temperature

Fig. 9 Impact of γ on Temperature

Figures 2 and 3 display the influence of nanotube volume fraction ϕ on the fluid velocity and temperature for SWCNTs and MWCNTs at $S=0.2$, $\lambda=0.1$, $\gamma=0.2$, $Gr=1$, $\xi=\pi/6$ and $M=0.5$. The velocity ascends (Fig. 2) for ascending ϕ values due to the enhancement of boundary layer thickness; this result is in agreement with Reddy et al. [46]. It is also seen that the nanofluid flow is faster in MWCNTs than SWCNTs. The nanoliquid temperature (Fig. 3) rises for rising ϕ due to the expansion in the thermal boundary layer thickness. This occurs since enhancing volume fraction of CNTs enhances the thermal conductivity and thus, the thickness of the thermal boundary layer also grows. Furthermore, heat transfer is greater for single-walled nanotubes in comparison to multi-wall nanotubes.

The aligned magnetic field inclination angle ξ influence on the nanofluid velocity and temperature are portrayed in Figs. 4 and 5 when $S=0.2$, $\lambda=0.1$, $\gamma=0.2$, $Gr=1$, $\phi=0.14$ and $M=0.5$. It is noticed in Fig. 4 that the fluid flow velocity is reduced with increasing magnetic field angle of inclination. This occurs since magnetic flux strengthens as aligned angle increases, which enhances the Lorentz force and suppresses the fluid flow. Quite interestingly, it is seen that the temperature of the nanofluid (Fig. 5) is enhanced for an increase in angle of inclination. One can observe that multi-walled nanotubes are more dominant than single wall tubes in the case of velocity whereas the reverse trend is observed for temperature of the nanoliquid.

Fig. 10 Impact of R Temperature

Fig. 11 Impact of λ on Temperature

The effect of suction/injection parameter S ($S > 0$ for suction and $S < 0$ for injection) on velocity and temperature profiles with $\lambda = 0.1$, $\phi = 0.14$, $Gr = 1$, $\xi = \pi/6$, $\gamma = 0.2$ and $M = 0.5$ is depicted in Figs. 6 and 7. For both suction and injection, the velocity (Fig. 6) declines when S increases; this is consistent with the study conducted by Mukhopadhyay [45]. It is also noticed that MWCNTs plays a more dominant role than SWCNTs. In the same way, the temperature of the nanofluid (Fig. 7) decreases with increased values of S . It is evident that the thickness of the thermal boundary layer reduces due to the movement of fluid towards the surface due to suction.

The influence of slip parameter γ on velocity and temperature is displayed in Figs. 8 and 9 respectively.

Fig. 12 Surface skin friction versus volume fraction for different values of ξ

Fig. 13 Surface skin friction versus volume fraction for different values of γ

Escalating slip parameter escalates the slip at the stretching wall since a smaller amount of dispersion occurs inside the fluid, which thereby increases the momentum boundary layer thickness. Thus, increasing the slip parameter enhances the nanofluid velocity within the system. If $\gamma = 0$, then no slip occurs, symbolizing that $f(0) = 1$; i.e., the flow velocity is equal to the stretching surface velocity (Fig. 8). Furthermore, MWCNT exhibits a greater flow velocity in comparison with SWCNT. The fluid temperature (Fig. 9) rises with enhancing γ due to increased thermal boundary layer thickness, and SWCNT has higher temperature values than MWCNT.

Figure 10 represents the impact of thermal radiation on temperature. The fluid temperature (Fig. 10) is enhanced with increased radiation parameter R . It is also evident that

Fig. 14 Surface skin friction versus volume fraction for various S

Fig. 16 Heat transfer rate versus volume fraction for varied R

Fig. 15 Rate of heat transfer versus volume fraction for λ

single wall carbon nanotubes have the dominant character. Thermal variations for improving λ (< 0 for generation and > 0 for thermal energy storage) are depicted in Fig. 11 with parameter values $S = 0.2$, $\gamma = 0.2$, $Gr = 1$, $\xi = \pi/6$, $\phi = 0.14$ and $M = 0.5$. In this figure, we noticed that the nanoliquid temperature rises as the thermal energy storage/heat source parameter λ enhances. For $\lambda < 0$ (source), the energy from the stretching sheet is absorbed by the fluid, which causes an increase in the thermal boundary layer thickness. Therefore, the temperature decreases and the heat energy is released into the fluid for $\lambda > 0$ (sink), which also thickens the thermal boundary layer. Single-walled carbon nanotubes have a greater role in heat transfer from the fluid than multi-walled carbon nanotubes.

The effects of various physical parameters on the surface skin friction at the stretching sheet are shown in Figs. 12, 13 and 14. It is noticed in Fig. 12 that the skin friction decreases for larger nanotube volume fraction due to greater viscosity of the nanofluid; this result is in agreement with Kalidas [44]. We found that the skin friction is greater for increased aligned magnetic angle. Moreover, MWCNTs have higher skin friction values than SWCNTs. The same behaviour is observed for escalating slip parameter on the shear stress coefficient (Fig. 13); this is also consistent with the work of Kalidas [44]. From Fig. 14, it is seen that the rate of surface skin friction decays for growing volume fraction. And also it is evident that skin friction is reduced with increased suction/blowing.

Figure 15 reveals that the rate of heat transfer is higher for enhancing volume fraction, which is in agreement with the previous work of Reddy et al. [46]. This phenomenon is quite natural in nanofluid flow phenomena. From Fig. 15, it is clear that the heat transfer rate is decreased for rising thermal energy storage/source. Fascinatingly, SWCNT dominates the MWCNT in the case of heat transfer. The same behaviour is observed for increasing values of radiation parameter on rate of heat transfer, which is observed from Fig. 16.

4 Conclusions

This study focused on investigating the effects of aligned magnetic flux, thermal radiation and heat source/energy storage on a slip flow of water-carbon nanotube nanofluid along a vertical stretching sheet. A comparative study of multi-walled carbon nanotubes (MWCNTs) and

Table 2 Comparison results for $f''(0)$ and $\theta'(0)$ in some reduced cases

Gr	$f''(0)$			$\theta'(0)$		
	Present result	Kalidas [44]	Relative error (%)	Present result	Kalidas [44]	Relative error (%)
0	-1.0014211	-1.001423	1.9×10^{-4}	-2.3219232	-2.321917	2.7×10^{-4}
1	-0.7629796	-0.762983	4.5×10^{-4}	-2.3844349	-2.384484	2.1×10^{-4}

Table 3 Nusselt number for different Prandtl number in some reduced cases

Pr	Present study	Swathi Mukhopadhyay [45]	Relative error (%)	Reddy et al. [46]	Relative error (%)
1	0.954217	0.9547	0.0506	0.95485	0.0663
2	1.471489	1.4714	0.0060	1.47144	0.0033
3	1.869261	1.8691	0.0086	1.86906	0.0108
5	2.500285	2.5001	0.0074	2.50012	0.0066
10	3.660412	3.6603	0.0031	3.66037	0.0011

single-walled carbon nanotubes (SWCNTs) was conducted and based on the computed numerical results, the following conclusions are drawn:

- The nanofluid flow velocity and skin friction along the stretching sheet are higher for MWCNTs than SWCNTs, but heat transfer is more effective with SWCNTs.
- The flow velocity is enhanced by increasing the CNT volume fraction and slip parameter, and reducing the Navier slip parameter, magnetic field inclination angle and suction/injection parameter.
- The temperature of the nanofluid is enhanced by increasing the magnetic field angle of inclination, thermal energy storage/source parameter and radiation parameter, whereas the temperature is reduced by increasing the suction/blowing parameter.
- The skin friction at the stretching sheet is higher with larger aligned magnetic angle, Navier slip parameter, and with smaller nanotube volume fraction and suction/blowing parameter.
- Heat transfer enhancement at the stretching sheet can be achieved by increasing the CNT volume fraction and decreasing the thermal energy storage/source parameter and radiation parameter.

The information contained in this work provides an improved understanding of the factors influencing slip flows of CNT-water nanofluids along a stretching sheet with convective heat transfer and an applied magnetic field. In particular, the results obtained herein regarding the magnetic field inclination angle and thermal energy storage/heat source for the cases of SWCNTs and MWCNTs are important contributions to the existing literature. For future work, we recommend that SWCNT and MWCNT nanofluids be modelled as non-Newtonian fluids with

different types of non-linearly stretching sheets, as well as other geometrical configurations according to the physical nature of the industrial applications.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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