Square grid deformation in wedge indentation: a new mathematical approach

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Abstract

The square grid deformation approach is generally employed to forming processes solved with the aid of slip-line field in which a ductile body is supposed to be made with arrays of square grids of finite sizes before the deformation process. After the application of the load, as the deformation occurs, square grids are deformed. The square grid deformation has three utilities—(1) visualization of actual deformation process, (2) prediction of trajectory at any point and (3) computation of strain. The theory of square grid deformation mathematics was originally developed by Hill considering the linear formation of the lip. The present work develops a new mathematical approach for deformation of square grids considering a parabolic lip formation when a rigid wedge is gradually indented to a ductile block of semi-infinite size.

Keywords Slip line field · Square grid deformation · Numerical technique · New mathematical approach

1 Introduction

Hill et al. [1] developed a mathematical formulation of slip-line field for wedge indentation to a ductile material when coefficient of friction was taken to be zero. The works reported that there might be formation of lip around the wedge or sinking of material. The work also presented deformation of square grids for wedge indentation. The work of Grunzweiz et al. [2] presented the modified analysis considering friction in between wedge and deforming material. The results of this theoretical work were depicted in a tabular form for different wedge angles varying from 10° to 90° for different co-efficient of friction. The theoretical results also verified with experimental results. The work of Haddow and Danyluk [3] presented a detailed mathematical formulation and its experimental validation for a problem of indentation of smooth rigid wedge to a semi-infinite medium. The work of Haddow et al. is very noteworthy as this work reported that for an obtuse

wedge angle of the indenter, the periphery of the slip line meet at a point below the apex where 45° line to the undeformed plane cuts the centre line of the wedge, rather than meeting at the vertex of the wedge.

The initial researchers, in this field, considered the lip around the wedge to be linear. Mulhearn [4] was the first to determine experimentally using interferometric method that the lips were not linear in his experiment with a pyramid shaped Vickers's indenter having semi-apex angle of 68° on various materials. The author reported different types of lip profiles according to the types of materials indented and almost all the profiles were of non-linear in nature. Schmalanga and Hartmaier [5] reported that the shape of pile up material was found to be non-linear in nature. Moharrami and Bull [6] worked on the comparison of nano-indentation pile-up in bulk materials and thin films by conducting experiments using atomic force microscopy and scanning electron microscope, that showed pile up of material followed a non-linear nature.

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Chen et al. [7] discussed the effect of loading rate on pile up of material for indentation to a polycrystalline copper in their work. The indentation for this work was of nanometer-order range. The experimental analyses showed the non-linear nature of pile up. The theoretical and experimental works of Biswas et al. [8–10] showed that the actual lip profile in wedge indentation is not linear but a parabolic one. Biswas et al. [10] have drawn numerically generated slip line field for this wedge indentation problem to ductile materials. The parabolic nature of the lips was also verified by finite element analysis and validated by experimental works [11].

As the lip profile is parabolic in nature, the drawing of deformed square grid is difficult as per theory developed by Hill [1]. Hence, a new numerical square grid deformation algorithm is attempted in this work.

2 Square grid deformation

The present authors have drawn numerically generated slip line field for wedge indentation which has been shown in Fig. 1. Symmetric conical indenter is considered in this

work, and one symmetric half part of the conical indentation is shown in the figure where FO stands for the height of the conical indentation with point O indicating the apex of the cone. Detailed description has been given in the work by Biswas et al. [8–10]. Figure 1 clearly shows that the lip EC around the wedge EOF is parabolic in nature.

Figure 2 shows the original work of Hill et al. [1] for a smooth rigid wedge indented to a semi-infinite ductile block. The figure shows the slip-line field with α -line and β -lines. The generated slip line field has three distinct zones ACE zone; ADE called centered fan zone and the third zone ADB. The work by present authors as depicted in Fig. 1 has three distinctive zones. Only point in which the present study differs with the study of Hill et al. [1] lies in the consideration of parabolic profile instead of the linear one as presumed by Hill et al.

Figure 3 [1] shows the unit diagram for the indentation when wedge is indented to a depth of unity i.e. OB is equal to 1. The figure shows the trajectory of a particle if wedge is indented from any depth to a higher depth.

The deformation of square grid in such an enhanced indentation, as shown in Fig. 4 indicates that within

Fig. 1 The generated slip-line field for straight lip for 60° wedge angle

Fig. 3 Trajectory of a particle in wedge indentation to a semi-infinite block in unit diagram [1]

Fig. 2 Wedge indentation to semi-infinite block [1]

Fig. 4 Square grid deformation in wedge indentation to a semi-infinite block [1]

the periphery of slip-lines, each square grid has been deformed.

3 The mathematical approach to square grid deformation

A new mathematical approach is developed to find out the deformation field analytically considering the lip parabolic.

If the radius vector of a point is $\vec{r} = \hat{i}x + \hat{j}y$, where x and y are the co-ordinates of any point in actual space when indenter indents up to a depth of c as shown in Fig. 2, then in unit diagram, the radius vector of the point and the velocity will be $\vec{p} = (\vec{r}/c)$ and $\vec{v} = (d\vec{r}/dc)$ respectively. The stress and velocity of the point will be the function of $(\vec{r}/c)$. The actual diagram, hence, is scaled down by the ratio ($c : 1$).

The point in the unit diagram, corresponding to the actual point in the square grid moves with respect to time towards a focus having positional velocity vector is $\vec{v}$ following the equation [1].

$$\vec{v} - \vec{p} = c \frac{d\vec{r}}{dc} \quad (1)$$

The focal distance $f(s)$ for the displacement (s) of a point along the trajectory from the point on the unit diagram defined by the positional vector $\vec{p}_{U0}$, where $\vec{p}_{U0} = (\vec{p}_0/c_0)$ can be evaluated as [1]:

$$\log \left(\frac{c}{c_0} \right) = \int_0^s \frac{ds}{f(s)} \quad (1A)$$

In which, final focal distance is $f(s) = (d - s)$ and d is the distance of the point p_0 from the appropriate focus H or K as shown in Fig. 3.

Thus the ratio of c and c_0 becomes equal to the ratio of d and $(d - s)$ and the final position (s_f) can be evaluated using Eq. (2) following the definite integration in the limit $[0, s_f]$.

However, the above approach cannot be applied when the lip profile is considered a parabolic one. In the present study a differential approach is adopted.

Substitution of $\vec{v}$ and $\vec{p}$ in terms of $\vec{r}$ in Eq. (1), the following two differential equations result,

$$c \frac{d^2x}{dc^2} - (1 + c) \frac{dx}{dc} + x = 0 \quad (2)$$

$$c \frac{d^2y}{dc^2} - (1 + c) \frac{dy}{dc} + y = 0 \quad (3)$$

Equations (2) and (3) are similar and hence x and y will have the same solution. The solution technique of y is only described below.

The Eq. (3) can be re-written as,

$$\frac{d^2y}{dc^2} - \frac{(1 + c)}{c} \frac{dy}{dc} + \frac{y}{c} = 0 \quad (4)$$

If y is considered as function as given below,

$$y = c^m (a_0 + a_1c + a_2c^2 + a_3c^3 + \dots) \quad (5)$$

$$\text{Then, } \frac{dy}{dc} = ma_0c^{m-1} + (m+1)a_1c^m + (m+2)a_2c^{m+1} + \dots \quad (6)$$

$$\text{and } \frac{d^2y}{dc^2} = m(m-1)a_0c^{m-2} + (m+1)ma_1c^{m-1} + (m+2)(m+1)a_2c^m + \dots \quad (7)$$

Substituting the values from Eqs. (5), (6) and (7) in Eq. (4) and equating the co-efficient of c^{m-1} , the following equation of m results when $a_0 \neq 0$,

$$m^2 - 2m = 0 \quad \text{i.e. } m = 0 \text{ or } 2$$

Similarly, equating the co-efficient of c^m ,

$$m(m+1)a_1 - ma_0 - (m+1)a_1 + a_0 = 0 \quad \text{Or, } a_1 = \frac{a_0}{(1+m)}$$

Again, equating the co-efficient of c^{m+1} and rearranging, a_2 is obtained as,

$$a_2 = \frac{a_1}{(m+2)} \text{ Or, } a_2 = \frac{a_0}{(m+1)(m+2)}$$

Successively balancing the co-efficient of c^{m+2} results in,

$$a_3 = \frac{a_0}{(m+1)(m+2)(m+3)}$$

When, $m = 0$, a_1, a_2, a_3 , will be $a_1 = a_0$,

$$a_2 = \frac{a_0}{1.2}, \quad a_3 = \frac{a_0}{1.2.3}, \dots$$

$$\text{And hence, } y_1 = a_0 \left\{ 1 + c + \frac{c^2}{1.2} + \frac{c^3}{1.2.3} + \frac{c^4}{1.2.3.4} + \dots \right\} \quad (8)$$

$$\text{Again when, } m = 2, \quad a_1 = \frac{a_0}{3}, \quad a_2 = \frac{a_0}{3.4}, \quad a_3 = \frac{a_0}{3.4.5}, \dots$$

$$\text{And thereby, } y_2 = a_0 c^2 \left\{ 1 + \frac{c}{3} + \frac{c^2}{3.4} + \frac{c^3}{3.4.5} + \frac{c^4}{3.4.5.6} + \dots \right\} \quad (9)$$

Following the same approach it can be shown that x will have two solutions x_1 and x_2 having the similar expressions of y_1 and y_2 respectively. Therefore, the final expressions of x and y can be expressed as,

$$\begin{cases} x = A_{11}x_1 + A_{12}x_2 \\ y = A_{21}y_1 + A_{22}y_2 \end{cases} \quad (10)$$

where A_{11} , A_{12} , A_{21} and A_{22} are arbitrary constants to be determined.

Method of solution In this work, the square grid deformation is obtained by solving the differential Eqs. (2) and (3). The deformed co-ordinates (x, y) are defined by Eq. (10). First, the slip-line field is generated in the plastic deformed zones, considering small square grids for indentation depth c_1 and c_2 and the (x, y) co-ordinates of end points of square grids are evaluated, which are then put in Eq. (10) to obtain the values of A_{11} , A_{12} , A_{21} and A_{22} . Equation (10) thus becomes a function of c_3 .

4 Results obtained

The formation of slip-line network for wedge indentation has been discussed in the works of Biswas et al. [8–10]. The specialty of the work of Biswas et al. was that they proposed the lip profile of bulged out material around the wedge to be parabolic. The wedge indentation problem to a ductile material was initiated by Hill et al. [1] in 1947.

After that a lot of works was done in this field. Hill considered lip profile to be linear. Afterwards, some authors like Mulhern [4], Schmalanga et al. [5] and Chen et al. [7] depicted that the lip around the wedge might not be linear. Square grid deformation is studied in plastic flow problem to observe the strains and locus of any point in the plastically deforming field. For this present problem, Hill et al. [1] suggested square grid deformation approach in considering lip to be linear and also for frictionless surfaces, but this algorithm is not valid for the present work as it considers the lip to be non-linear. Hence, a new mathematical development is tried to be generated in this work so that parabolic lip can be taken care of.

Figure 5 shows the square grids and the generated slip line field with the semi-apex angle of the wedge of 10° . From Eq. (10), it is observed that when $x(c) = A_{11}x_1(c) + A_{12}x_2(c)$, the constants are two, and the coordinate is a function of indentation depth c . Hence, for solving each x and y equation, two sets of equations are to be solved for each indentation depth c . Now, to overcome this difficulty, the square grids are approximated to slip-line field. The corners of square grids are matched with the intersecting points in the slip-line field. This has been done with a searching algorithm such that the distance of intersecting any point of a square grid from all point of intersection of slip-line network is minimized. The nearest point of slip-line network is searched and the nearest coordinate of the point in slip-line field is assigned to nearest point in square grid. Thus, a modified or transformed square grid is obtained for a depth of indentation c_1 . The modified square is shown in Fig. 6. The figure clearly shows that the square grids are modified accordingly. However, another equation is required to be formed. Hence, the indentation of wedge is further increased from 0.1 micron to 0.1001 micron such that the deformation process is not much affected with this increment in the indentation process.

Figure 7 shows modified square grids for $c_2 = 0.1005$ micron indentation. Now, two equations for x can be formed, are given by

$$x_{c1} = A_{11}x_1^{c1} + A_{12}x_2^{c1} \quad \text{and} \quad x_{c2} = A_{11}x_1^{c2} + A_{12}x_2^{c2}.$$

It is evident that the constants A_{11} and A_{12} can be solved from these equations and a new equation can be formed for other depth of indentation for c_3 , the unknown depth. Thus, for c_3 indentation, the coordinates for x_{c3} can be easily found from the equation $x_{c3} = A_{11}x_1^{c3} + A_{12}x_2^{c3}$. Similarly, this process can be repeated for y -coordinates also. Hence, the all coordinates of new position in square grids are completely known. The deformation of modified square grids for new indentation depth $c_3 = 0.101$ has been shown in Fig. 8. The figure shows that the grids are severely deformed. A comparison of square grid deformation for

Fig. 5 Square grid and slip-line field network for 10° semi-wedge angle with 0.1 micron indentation

Fig. 6 Transformed square grid for 10° semi-wedge angle with 0.1 micron indentation

indentation for depth 0.1, 0.1005 and 0.101 is shown in Fig. 9. The blue lines corresponds to modified square grid at depth of indentation 0.1 and 0.1001 micron and the violet lines represent the deformed square grid after further indentation to a depth 0.101.

The same procedure is repeated for micro-wedge indentation of a rigid wedge to a semi-infinite block with semi apex angle 20°. Figure 10 shows the generated slip-line field network for 20° semi-apex angle of the wedge with 0.1 micron indentation. Figure 11 shows that the transformed or modified square grid matching with the points in the slip-line field network. Two

equations are to be formed so that the constants can be found out. Hence, Slip-line network is generated for 20° semi-wedge angle of the wedge with 0.1001 micron indentation nearly 1:1000 time smaller increments, so that deformation process is not much affected and corresponding modified square grids are formed as shown in Fig. 12. Now like earlier case, the two constants are found out. The square grid deformation can be found out for a new indentation depth 0.101 micron and has been shown in Fig. 13.

Fig. 7 Transformed square grid for 10° semi-wedge angle with 0.1005 micron indentation

Fig. 8 Deformed square grid deformation for 10° semi-wedge angle with 0.101 micron indentation

5 Discussion on the results observed

As it is discussed in result section that attempts to develop a new mathematical model for square grid deformation was due the formation of parabolic lip in wedge indentation. The similar kind of works was carried out by Shimmin [12] while analyzing the strains in a plastic deformation problem of rolling and drawing through circular dies. The author in that case studied that stream lines generated in plastic deformation as shown in Fig. 14a. Busquet et al. [13] used the work of Shammin to a problem of generating slip-line field

when a rigid cylinder is slid on a ductile material. The authors also calculated deformation of square grid for calculation of strains. The deformed square grid for sliding cylinder problem is shown in Fig. 14b. In a recent work, Rousseau et al. [14] also experimented with square grids and they tallied their results with finite element analysis. The square grid before indentation is shown in Fig. 15a and the deformed square grids after indentation of a wedge are shown in Fig. 15b.

Idea of the present work is adopted from Shammin [12], but it is not identical. The mathematical description is taken from Hill et al. [1]. Hill's method was based on integral approach and present method is based on

Fig. 9 Comparisons of transformed and deformed square grids for 10° semi-wedge angle blue lines indicate before indentation and violet lines represent after indentation

Fig. 10 Square grid and slip-line field network for 20° semi-wedge angle with 0.1 micron indentation

differential approach. Actually, the function $f(s)$ in Eq. (1A) is not known while parabolic lip is considered. Hence, the method elaborated by Hill, cannot be directly applied to the present problem. Slip-line network is actually nothing but flow lines or stream line in the plastic flow field. The problem was how to correlate the square grid points and the Slip-lines. The slip-line points are correlated with the square grid points by a searching algorithm as discussed in results section. The denser slip-line networks are better correlation between the square grid points. The square grids, after correlation i.e. after formation of transformed or modified grids, are of no more exact square shapes. This

can be got rid of by considering denser slip-line field generation but the deviations are not completely vanished. The second difficulty was considering another indentation very close to first indentation nearly 1:1000 ratios, to determine the two constants as discussed in results section. From stress–strain curve as shown in Fig. 16, it is seen that when $c_1 = 0.1$, it is in plastic zone. From Fig. 16, it is clear that $c_1 = 0.1$ and $c_2 = 0.1001$ are quite close points; it would not affect deformation at large.

One point is to be mentioned here is that the c_3 value is generally 3–5% increment of c_1 and the values of indentation depth, c is to be given between 0 to 1 to overcome

Fig. 11 Transformed square grid for 20° semi-wedge angle with 0.1 micron indentation

Fig. 12 Modified square grid deformation for 20° semi-wedge angle with 0.1001 micron indentation

the divergence of the series solutions for x_1, x_2, y_1 and y_2 as shown in Eqs. (8), (9) and (10). Otherwise, the programme shows erroneous results. Once, the coordinates of square grids are known before and after indentation, the strains, both normal and shear, can be found out easily.

6 Conclusion

The mathematical tool developed in this work is versatile in nature and this approach can be applied to any forming process with slip-line field approach. The original method developed by Hill used to find out the exact

equation of the deformation. But sometime this method was practically not suitable as the profile exhibits a non-linear nature. The method developed in this work is purely numerical one. If coordinates of all the points in slip-line field are known before deformation, with the application of this method, the location of the point after deformation can be found out easily. One point is to be noted that the indented value should be restricted within 3–5% of the original indentation depth c which in turn is in between 0 to 1, to avoid the divergence of programme.

Fig. 13 Deformed square grids for 20° semi-wedge angle with 0.101 micron indentation

Fig. 14 Theoretical calculation of square grid and the theoretical results obtained from the method adopted by Busquet et al. [13]

Fig. 15 Square grids before and after indentation by Rouseau et al. [14]

Fig. 16 Stress–strain curve of a mild steel sample by Rousseau et al. [14]

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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