Solving for the random component time-fractional partial differential equations with the new Sumudu transform iterative method

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Received: 2 November 2019 / Accepted: 30 March 2020 / Published online: 22 May 2020
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Abstract

The new Sumudu transform iterative method is implemented to get the approximate solutions of random component time-fractional partial differential equations with Caputo derivative. The parameters and the initial conditions of the random component time-fractional partial differential equations are analyzed with Gamma distribution. The expected values and variances of these solutions are calculated, and the graphs of the expected values and variances are plotted in Maple software. The results for the random component time-fractional partial differential equations with Gamma distribution are examined to investigate effects of this distribution on results. The numerical experiments indicate that this method is very effective.

Keywords Sumudu transform · Gamma distribution · Caputo fractional derivative · Mittag–Leffler function

1 Introduction

Fractional calculus is a considerably important topic in several scientific areas [1–7]. Methodologies of fractional calculus have been widely used in the modeling of many real matters in applied mathematics.

Specifically, fractional partial differential equations (FPDEs) describe certain many phenomena in various areas of science and engineering such as damping laws, diffusion equations, heat transfer modeling, electrostatics, electrodynamics, fluid flow, elasticity and many others [8–18].

In the literature, there are very few studies on random fractional partial differential equations (RFPDEs). Random fractional partial differential equations (RFPDEs) are defined as fractional partial differential equations with random inputs that can be a random variable or a stochastic process. These equations have a great importance in many

applications in areas such as engineering, biology, physics, mathematics and many other applied sciences.

It is almost impossible to solve random nonlinear fractional partial differential equations analytically. So various numerical methods and approximation schemes for RFPDEs, ordinary differential equations and fractional partial differential equations have been developed. There are many numerical methods and approximation schemes as adomian decomposition method (ADM) [19], homotopy perturbation method [20], collocation method [21–26], Galerkin-type method [27], differential transformation method (DTM) [28], variational iteration method (VIM) [29], fractional variational iteration method (FVIM) [30], random finite difference scheme [31, 32] and many other methods. The key motivation of writing this paper is to analyze random nonlinear fractional partial differential equations using the new Sumudu transform iterative method (NSTIM) which is the reliable computational method.

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This paper studies random component time-fractional partial differential equations and to solve numerically by the new Sumudu transform iterative method (NSTIM). Wang and Liu [33] established this new Sumudu transform iterative method (NSTIM). They successfully applied this method to obtain the approximate analytical solutions of time-fractional Cauchy reaction–diffusion equations.

In the literature, there are very few research work and articles on the power series transformation such as Sumudu transform. The Sumudu transform method (STM), proposed by Watugala, was applied to solve engineering problems [34]. The method was applied to partial differential equations by Weerakoon [35]. Then, Weerakoon found the inverse formula of this transform [36]. Demiray et al. [37] used the Sumudu transform method (STM) to get exact solutions of fractional differential equations. Kumar and Daftardar-Gejji [38] extended Sumudu transform iterative method (STIM) to solve a variety of time and space FPDEs and systems of FPDEs. Rahman and Ahmad applied fuzzy Sumudu transform (FST) for solving linear fuzzy fractional differential equations (FFDEs) by Caputo fuzzy fractional derivative [39]. Prakash et al. [40] suggested a new iterative Sumudu transform method (NISTM) to solve numerically nonlinear time fractional Zakharov–Kuznetsov equations.

In this paper, approximate solutions of two random component time-fractional partial differential equations are obtained by the new Sumudu transform iterative method (NSTIM) and variational iteration method. Projected technique is used for error analysis. In addition, approximate solutions obtained with two methods and exact solutions are shown in comparison tables. The difference of this work from Prakash et al. [40] and Wang and Liu [33] studies is to examine the random component fractional partial differential equations.

The aim of this study is to present the application of the new Sumudu transform iterative method for obtaining the approximate analytical solutions of random component time-fractional partial differential equations with Caputo derivative and for calculating the expected values and variances of these solutions. It is observed that the numerical solutions obtained by NSTIM for RFPDEs are almost similar to exact solutions for RFPDEs. Tables show that absolute errors are negligible. It is observed that NSTIM is superior than VIM for RFPDEs.

2 Basic definitions

A few basic definitions and properties of fractional calculus, Sumudu transform and Gamma distribution are presented in this section.

Definition 2.1 [33, 41] A real function $f(x)$, $x > 0$, is said to be in the space C_μ , $\mu \in \mathbb{R}$ if there exists a real number p , ($p > \mu$), such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 2.2 [1, 33, 42] The Riemann–Liouville fractional integral operator of order $a \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$ is given in the following

$$I^a f(x) = \begin{cases} \frac{1}{\Gamma(a)} \int_0^x (x-t)^{a-1} f(t) dt, & a > 0, x > 0, \\ I^0 f(x) = f(x), & a = 0 \end{cases} \quad (1)$$

where $\Gamma(\cdot)$ is the Gamma function.

Some features of the operator I^a that will be used in this study are given below [33]: For $f \in C_\mu$, $\mu, \gamma \geq -1$, $\alpha, \beta \geq 0$,

1. $I^\alpha I^\beta f(x) = I^\beta I^\alpha f(x) = I^{\alpha+\beta} f(x)$,
2. $I^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

Definition 2.3 [1, 7, 33, 42] The Caputo fractional derivative of $f(x)$ is defined by

$$D^a f(x) = I^{a-n} D^n f(x) = \frac{1}{\Gamma(n-a)} \int_0^x (x-t)^{n-a-1} f^{(n)}(t) dt \quad (2)$$

where $n-1 < a \leq n$, $n \in \mathbb{N}$, $0, f \in C_{-1}^n$.

Some properties of the operator D^a that will be used in this study are given below [33]:

3. $D^a I^a f(x) = f(x)$,
4. $I^a D^a f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}$, $x > 0$.

Definition 2.4 [33, 43] For $a > 0$, the Mittag–Leffler function E_a is given as follows

$$E_a(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(na+1)}. \quad (3)$$

Definition 2.5 [33, 43] The Sumudu transform on the set of functions $A = \left\{ f(t) | \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{t}{\tau_1}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}$ is given as follows

$$S[f(t)] = \int_0^{\infty} e^{-vt} f(vt) dt, \quad v \in (\tau_1, \tau_2). \quad (4)$$

Definition 2.6 [33] The Sumudu transform of the Caputo fractional derivative with respect to x is given below

$$S[D_x^{na} u(x, t)] = v^{-na} S[u(x, t)] - \sum_{k=0}^{n-1} v^{(-na+k)} u^{(k)}(0, t), \quad n-1 < na \leq n. \quad (5)$$

Definition 2.7 [44] If the probability density function of a random variable X has the following form, this random variable has the Gamma distribution and is called a Gamma random variable. For $x, \alpha, \beta > 0$,

$$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} \quad (6)$$

If the random variable X has a Gamma distribution with parameters α and β , the expected value and variance of the random variable X are given as follows

$$E(X) = \alpha\beta, \quad \text{Var}(X) = \alpha\beta^2. \quad (7)$$

In this study, Gamma distribution in the examples is chosen as Gamma ($\alpha = 3, \beta = 2$).

Theorem 2.1 [40] A real number $0 < k < 1$ satisfies $u_{m+1}(x, t) \leq ku_m(x, t)$ for all values of m . Furthermore, if the truncated series $\sum_{m=0}^l u_m(x, t)$ is used as an approximate solution $u(x, t)$, then maximum absolute truncated error is obtained by

$$\left\| u(x, t) - \sum_{m=0}^l u_m(x, t) \right\| \leq \frac{k^{l+1}}{(1-k)} \|u_0(x, t)\|.$$

3 The new Sumudu transform iterative method

Consider the following equation with the initial condition

$$\begin{cases} D_t^{na} u(x, t) + Lu(x, t) + Ru(x, t) = g(x, t), \\ n-1 < na \leq n, \\ u(x, 0) = h(x). \end{cases} \quad (8)$$

where D_t^{na} is the Caputo fractional derivative operator, $D_t^{na} = \frac{\partial^{na}}{\partial t^{na}}$, $g(x, t)$ is a continuous function, and L and R are linear and nonlinear operators, respectively [33].

If Sumudu transform is implemented to both sides of Eq. (8), it can be found that

$$S[D_t^{na} u(x, t) + Lu(x, t) + Ru(x, t)] = S[g(x, t)]. \quad (9)$$

The following equation is obtained with the feature of the Sumudu transform [33]

$$S[u(x, t)] - v^{na} \sum_{k=0}^{n-1} u^{(k)}(x, 0) + v^{na} S[Lu(x, t) + Ru(x, t) - g(x, t)] = 0. \quad (10)$$

From Eq. (10), it can be found that

$$S[u(x, t)] = v^{na} \sum_{k=0}^{n-1} u^{(k)}(x, 0) - v^{na} S[Lu(x, t) + Ru(x, t) - g(x, t)]. \quad (11)$$

If the inverse Sumudu transform is applied to Eqs. (11) and (12) is obtained

$$u(x, t) = S^{-1} \left[v^{na} \sum_{k=0}^{n-1} u^{(k)}(x, 0) \right] - S^{-1} [v^{na} S[Lu(x, t) + Ru(x, t) - g(x, t)]]. \quad (12)$$

Wang and Liu [33] assumed that the following equalities are hold:

$$\begin{cases} f(x, t) = S^{-1} \left[v^{na} \sum_{k=0}^{n-1} u^{(k)}(x, 0) + v^{na} S[g(x, t)] \right], \\ N(u(x, t)) = -S^{-1} [v^{na} S[Ru(x, t)]], \\ K(u(x, t)) = -S^{-1} [v^{na} S[Lu(x, t)]]. \end{cases}$$

So Eq. (12) becomes Eq. (13):

$$u(x, t) = f(x, t) + K(u(x, t)) + N(u(x, t)), \quad (13)$$

where f is a known function, K is a linear operator of u and N is a nonlinear operator of u . The solution of Eq. (13) is given by the series form as follows [33]

$$u(x, t) = \sum_{i=0}^{\infty} u_i(x, t). \quad (14)$$

Since K is a linear operator, the following equation can be written as

$$K\left(\sum_{i=0}^{\infty} u_i\right) = \sum_{i=0}^{\infty} K(u_i). \quad (15)$$

The nonlinear operator N can be written as [33, 45]

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=0}^{\infty} \left[N\left(\sum_{j=0}^i u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right) \right]. \quad (16)$$

Thus, Eq. (13) is given as [33]

$$\sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} K(u_i) + N(u_0) + \sum_{i=0}^{\infty} \left[N\left(\sum_{j=0}^i u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right) \right]. \quad (17)$$

If the following recurrence is defined

$$\begin{cases} u_0 = f, \\ u_1 = K(u_0) + N(u_0), \\ u_{m+1} = K(u_m) + N(u_0 + \dots + u_m) - N(u_0 + u_1 + \dots + u_{m-1}). \end{cases} \quad (18)$$

the Eq. (19) is obtained [33]

$$(u_1 + \dots + u_{m+1}) = K(u_0 + \dots + u_m) + N(u_0 + \dots + u_m). \quad (19)$$

Thus, Eq. (20) is obtained as

$$\sum_{i=0}^{\infty} u_i = f + K\left(\sum_{i=0}^{\infty} u_i\right) + N\left(\sum_{i=0}^{\infty} u_i\right). \quad (20)$$

The m -term approximate solution of Eq. (13) is obtained as

$$u = u_0 + u_1 + u_2 + u_3 + \dots + u_{m-1}. \quad (21)$$

In this work, we obtain the approximate analytical solutions of random component time-fractional partial differential equations with the new Sumudu transform iterative method. It is illustrated in the numerical experiments.

4 Numerical experiments

Example 1 Consider the RFPDE

$$\begin{cases} u_t^\alpha(x, t) + u(x, t)u_x(x, t) + u(x, t)(1 + u(x, t)) = 0, \\ 0 < \alpha \leq 1, \\ u(x, 0) = Ae^{-x}. \end{cases} \quad (22)$$

where A is a Gamma distributed random variable with parameters $\alpha = 3$ and $\beta = 2$, i.e., $A \sim G(\alpha = 3, \beta = 2)$.

If Sumudu transform is applied to Eq. (22) and the differential feature of Sumudu transform is used, Eq. (23) is found

$$S[u] = u(x, 0) + v^\alpha S[-uu_x - u - u^2]. \quad (23)$$

If the inverse Sumudu transform is applied to Eqs. (23) and (24) is obtained

$$u(x, t) = S^{-1}[Ae^{-x}] + S^{-1}[v^\alpha S[-uu_x - u - u^2]]. \quad (24)$$

From Eq. (24), it is obtained as

$$u(x, t) = Ae^{-x} + S^{-1}[v^\alpha S[-uu_x - u - u^2]]. \quad (25)$$

For NSTIM, Eq. (26) holds:

$$\begin{cases} u_0 = Ae^{-x}, \\ K[u(x, t)] = S^{-1}[v^\alpha S[-uu_x - u - u^2]]. \end{cases} \quad (26)$$

By iteration, the results are obtained as follows

$$\begin{aligned} u_0 &= Ae^{-x}, \\ u_1 &= \frac{-Ae^{-x}t^\alpha}{\Gamma(1+\alpha)}, \\ u_2 &= \frac{Ae^{-x}t^{2\alpha}}{\Gamma(1+2\alpha)}, \\ u_3 &= \frac{-Ae^{-x}t^{3\alpha}}{\Gamma(1+3\alpha)}, \\ &\vdots \\ u_n &= \frac{(-1)^n Ae^{-x}t^{n\alpha}}{\Gamma(1+n\alpha)} = \frac{Ae^{-x}(-t)^{n\alpha}}{\Gamma(1+n\alpha)}. \end{aligned}$$

Thus, the approximate solution of Eq. (22) is obtained as the following

$$\begin{aligned} u_a(x, t) &= Ae^{-x} - \frac{Ae^{-x}t^\alpha}{\Gamma(1+\alpha)} + \frac{Ae^{-x}t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{Ae^{-x}t^{3\alpha}}{\Gamma(1+3\alpha)} + \dots + \frac{Ae^{-x}(-t)^{n\alpha}}{\Gamma(1+n\alpha)} \\ &= Ae^{-x} \left(1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \dots + \frac{(-t)^{n\alpha}}{\Gamma(1+n\alpha)} \right) \\ &= Ae^{-x} E_\alpha[(-t)^\alpha]. \end{aligned} \quad (27)$$

The form $u_a(x, t) = Ae^{-x-t}$ is the approximate solution of the Eq. (22) for $\alpha = 1$. Also, this form is the exact solution of the Eq. (22) for $\alpha = 1$.

In Table 1, we chose $\alpha = 1$, because for $\alpha = 1$, it allows us to compare the the exact solution between approximate solution obtained with the fifth-order NSTIM and the VIM solution. $A=6$ is randomly chosen. So the approximate solution from fifth-order NSTIM indicates that it is closer to the exact solution than the approximate solution from VIM for a random A in Example 1. It provided a comparison of the fifth-order NSTIM and VIM solution for $\alpha = 0.9$ and $A=6$ in Table 2.

It is observed from Table 1 that the numerical solution obtained by NSTIM is almost similar to exact solution. Table 3 shows that absolute error is negligible, because the absolute errors in Table 1 of the article of Sakar and Ergören are bigger than our absolute errors [46]. Table 3 shows the comparison of absolute error between approximate solutions acquired from different methods and exact solution different values of x and t . It is observed from Tables 1, 2 and 3 that NSTIM is superior than VIM.

Let $A \sim G(\alpha = 3, \beta = 2)$.

So the approximated values of A and A^2 are obtained as follows

$$E(A) = 3.2 = 6, E(A^2) = 3.4(1 + 3) = 48.$$

The approximated value of Eq. (27) is given by

Table 1 Comparison of the exact solution, approximate solution obtained with the fifth-order NSTIM and the VIM solution for $\alpha = 1$ and $A = 6$

x	t	Exact sol.	NSTIM	VIM
0.5	0.2	2.97951181	2.97951150	2.98413084
0.5	0.4	2.43941795	2.43939838	2.47464509
0.5	0.6	1.99722650	1.99700947	2.11072669
0.5	0.8	1.63519075	1.63400330	1.89237565
0.5	1.0	1.33878096	1.33436745	1.81959197
1.0	0.2	1.80716527	1.80716508	1.80996685
1.0	0.4	1.47958178	1.47956991	1.50094812
1.0	0.6	1.21137910	1.21124857	1.28022045
1.0	0.8	0.99179332	0.99107310	1.14778385
1.0	1.0	0.81201169	0.80933477	1.10363832
1.5	0.2	1.09610114	1.09610102	1.09780038
1.5	0.4	0.89741171	0.89740451	0.91037105
1.5	0.6	0.73473856	0.73465872	0.77649295
1.5	0.8	0.60155306	0.60111622	0.69616609
1.5	1.0	0.49250999	0.49088635	0.66939048

Table 2 Comparison of the fifth-order NSTIM and VIM solution for $\alpha = 0.9$ and $A = 6$.

x	t	NSTIM	VIM
0.5	0.2	2.85953155	2.84840575
0.5	0.4	2.33297833	2.28819073
0.5	0.6	1.93196722	1.90982510
0.5	0.8	1.61509200	1.70165945
0.5	1.0	1.35590765	1.65793613
1.0	0.2	1.73439356	1.72764542
1.0	0.4	1.41502288	1.38785783
1.0	0.6	1.17179735	1.15836748
1.0	0.8	0.97960282	1.03210863
1.0	1.0	0.82239956	1.00558910
1.5	0.2	1.05196287	1.04786991
1.5	0.4	0.85825476	0.84177832
1.5	0.6	0.71073102	0.70258539
1.5	0.8	0.59415914	0.62600552
1.5	1.0	0.49881054	0.60992062

$$E[u_a(x, t)] = E[A]e^{-x}E_a[(-t)^\alpha] = 6e^{-x}E_a[(-t)^\alpha].$$

The graphs of the approximated values of the Eq. (27) for different α values are plotted in Maple software as follows (Figs. 1, 2, 3, 4).

For $A \sim G(\alpha = 3, \beta = 2)$, the variance of Eq. (27) is calculated as follows

$$\begin{aligned} V[u_a(x, t)] &= E[u^2(x, t)] - (E[u(x, t)])^2 \\ &= E[A^2]e^{-2x}(E_a[(-t)^\alpha])^2 - (E[A])^2e^{-2x}(E_a[(-t)^\alpha])^2 \\ &= 48e^{-2x}(E_a[(-t)^\alpha])^2 - 36e^{-2x}(E_a[(-t)^\alpha])^2 \\ &= 12e^{-2x}(E_a[(-t)^\alpha])^2. \end{aligned}$$

The graphs of the variances of the Eq. (27) for different α values are plotted in Maple software as follows (Figs. 5, 6, 7, 8)

Example 2 Consider the RFPDE

$$\begin{cases} u_t^\alpha(x, t) + u(x, t)u_x(x, t) + u(x, t)(1 + u(x, t)) \ln a = 0, \\ 0 < \alpha \leq 1, \\ u(x, 0) = Aa^{-x}. \end{cases} \quad (28)$$

where A is a Gamma distributed random variable with parameters $\alpha = 3$ and $\beta = 2$, i.e., $A \sim G(\alpha = 3, \beta = 2)$.

If Sumudu transform is applied to Eq. (28) and the differential feature of Sumudu transform is used, Eq. (29) is found

$$S[u] = u(x, 0) + v^\alpha S[-uu_x - u \ln a - u^2 \ln a]. \quad (29)$$

If the inverse Sumudu transform is applied to Eqs. (29) and (30) is obtained

$$u(x, t) = S^{-1}[Aa^{-x}] + S^{-1}[v^\alpha S[-uu_x - u \ln a - u^2 \ln a]]. \quad (30)$$

From Eq. (30), it is obtained as

$$u(x, t) = Aa^{-x} + S^{-1}[v^\alpha S[-uu_x - u \ln a - u^2 \ln a]]. \quad (31)$$

For NSTIM, Eq. (32) holds:

$$\begin{cases} u_0 = Aa^{-x}, \\ K[u(x, t)] = S^{-1}[v^\alpha S[-uu_x - u \ln a - u^2 \ln a]]. \end{cases} \quad (32)$$

By iteration, the results are obtained as follows

$$\begin{aligned} u_0 &= Aa^{-x}, \\ u_1 &= \frac{-Aa^{-x}t^\alpha \ln a}{\Gamma(1 + \alpha)}, \\ u_2 &= \frac{Aa^{-x}t^{2\alpha} \ln^2 a}{\Gamma(1 + 2\alpha)}, \\ u_3 &= \frac{-Aa^{-x}t^{3\alpha} \ln^3 a}{\Gamma(1 + 3\alpha)}, \\ &\vdots \\ u_n &= \frac{Aa^{-x}(-\ln at^\alpha)^n}{\Gamma(1 + n\alpha)}. \end{aligned}$$

Table 3 Comparison between absolute error when $\alpha = 1$ and $A = 6$, for Example 1

	x	t					
		0.00	0.02	0.04	0.06	0.08	0.10
NSTIM	0.00	0.00000	0.00000000	0.00000000	0.00000003	0.00000016	0.00000049
VIM	0.00	0.00000	0.00000790	0.00006300	0.00021000	0.00050000	0.00097000
NSTIM	0.02	0.00000	0.00000000	0.00000000	0.00000003	0.00000015	0.00000048
VIM	0.02	0.00000	0.00000780	0.00006200	0.00020000	0.00049000	0.00095000
NSTIM	0.04	0.00000	0.00000000	0.00000000	0.00000003	0.00000015	0.00000047
VIM	0.04	0.00000	0.00000760	0.00006000	0.00020000	0.00048000	0.00093000
NSTIM	0.06	0.00000	0.00000000	0.00000000	0.00000003	0.00000015	0.00000046
VIM	0.06	0.00000	0.00000740	0.00005900	0.00020000	0.00047000	0.00091000
NSTIM	0.08	0.00000	0.00000000	0.00000000	0.00000003	0.00000014	0.00000045
VIM	0.08	0.00000	0.00000730	0.00005800	0.00019000	0.00046000	0.00090000
NSTIM	0.10	0.00000	0.00000000	0.00000000	0.00000003	0.00000014	0.00000044
VIM	0.10	0.00000	0.00000720	0.00005700	0.00019000	0.00045000	0.00088000

Fig. 1 For $\alpha = 0.9$, time-dependent variation of approximated value of the Eq. (22) **Fig. 3** For $\alpha = 0.7$, time-dependent variation of approximated value of the Eq. (22) **Fig. 2** For $\alpha = 0.8$, time-dependent variation of approximated value of the Eq. (22) **Fig. 4** For $\alpha = 0.6$, time-dependent variation of approximated value of the Eq. (22)

Fig. 5 For $\alpha = 0.9$, time-dependent variation of variance of the Eq. (22)

Fig. 7 For $\alpha = 0.7$, time-dependent variation of variance of the Eq. (22)

Fig. 6 For $\alpha = 0.8$, time-dependent variation of variance of the Eq. (22)

Fig. 8 For $\alpha = 0.6$, time-dependent variation of variance of the Eq. (22)

Thus, the approximate solution of Eq. (28) is obtained as the following

$$\begin{aligned} u_a(x, t) &= Aa^{-x} - \frac{Aa^{-x}t^\alpha \ln a}{\Gamma(1+\alpha)} + \frac{Aa^{-x}t^{2\alpha} \ln^2 a}{\Gamma(1+2\alpha)} - \frac{Aa^{-x}t^{3\alpha} \ln^3 a}{\Gamma(1+3\alpha)} \\ &\quad + \dots + \frac{Aa^{-x}(-\ln at^\alpha)^n}{\Gamma(1+n\alpha)} \\ &= Aa^{-x} \left(1 - \frac{t^\alpha \ln a}{\Gamma(1+\alpha)} + \frac{t^{2\alpha} \ln^2 a}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha} \ln^3 a}{\Gamma(1+3\alpha)} + \dots + \frac{(-\ln at^\alpha)^n}{\Gamma(1+n\alpha)} \right) \\ &= Aa^{-x} E_\alpha [-\ln at^\alpha]. \end{aligned}$$

The form $u_a(x, t) = Aa^{-x-t}$ is the approximate solution of the Eq. (28) for $\alpha = 1$. Also, this form is the exact solution of the Eq. (28) for $\alpha = 1$. (33)

In Table 4, we chose $a = 1$ and $a = 2$, because for $a = 1$, it allows us to compare the the exact solution between approximate solution obtained with the fifth-order NSTIM and the VIM solution. $A = 3$ is randomly chosen. So the approximate solution from fifth-order NSTIM indicates that it is closer to the exact solution than the approximate solution from VIM for a random A in Example 2. It provided a comparison of the fifth-order NSTIM and VIM solution for $\alpha = 0.9$, $a = 2$ and $A = 3$ in Table 5.

It is observed from Table 4 that the approximate solution obtained by NSTIM is almost similar to exact solution. Table 6 shows that absolute error is negligible, because the absolute errors in Table 1 of the article of Sakar and Ergören are bigger than our absolute errors [46]. Table 6 shows the comparison of absolute error between approximate solutions acquired from different methods and exact solution different values of x and t . It is observed from Table 4, 5 and 6 that NSTIM is superior than VIM.

Let $A \sim G(\alpha = 3, \beta = 2)$.

Table 4 Comparison of the exact solution, approximate solution obtained with the fifth-order NSTIM and the VIM solution for $\alpha = 1$, $A = 3$ and $a = 2$

x	t	Exact sol.	NSTIM	VIM
0.5	0.2	1.84671662	1.84671659	1.84762679
0.5	0.4	1.60766019	1.60765890	1.61470103
0.5	0.6	1.39954948	1.39953510	1.42254306
0.5	0.8	1.21837859	1.21829928	1.27115289
0.5	1.0	1.06066017	1.06036316	1.16053050
1.0	0.2	1.30582584	1.30582583	1.30646943
1.0	0.4	1.13678742	1.13678651	1.14176605
1.0	0.6	0.98963093	0.98962076	1.00588985
1.0	0.8	0.86152376	0.86146768	0.89884083
1.0	1.0	0.75000000	0.74978998	0.82061898
1.5	0.2	0.92335830	0.92335829	0.92381339
1.5	0.4	0.80383009	0.80382945	0.80735051
1.5	0.6	0.69977474	0.69976755	0.71127153
1.5	0.8	0.60918929	0.60914963	0.63557644
1.5	1.0	0.53033008	0.53018158	0.58026525

Table 5 Comparison of the fifth-order NSTIM and VIM solution for $\alpha = 0.9$, $A = 3$ and $a = 2$

x	t	NSTIM	VIM
0.5	0.2	1.79362121	1.79278800
0.5	0.4	1.55491049	1.53936543
0.5	0.6	1.36085738	1.34137016
0.5	0.8	1.19876559	1.19409532
0.5	1.0	1.06113562	1.09521457
1.0	0.2	1.26828172	1.26769255
1.0	0.4	1.09948775	1.08849573
1.0	0.6	0.96227148	0.94849193
1.0	0.8	0.84765527	0.84435290
1.0	1.0	0.75033619	0.77443365
1.5	0.2	0.89681060	0.89639400
1.5	0.4	0.77745524	0.76968271
1.5	0.6	0.68042869	0.67068508
1.5	0.8	0.59938279	0.59704766
1.5	1.0	0.53056781	0.54760728

So the approximated values of A and A^2 are obtained as follows

$$E(A) = 3.2 = 6, E(A^2) = 3.4(1 + 3) = 48.$$

The approximated value of Eq. (33) is given by

$$E[u_a(x, t)] = E[A]a^{-x}E_a[-\ln at^\alpha] = 6a^{-x}E_a[-\ln at^\alpha].$$

The graphs of the approximated values of the Eq. (33) for different α values and $a = 5$ are plotted in Maple software as follows (Figs. 9, 10, 11, 12)

For $A \sim G(\alpha = 3, \beta = 2)$, the variance of Eq. (33) is calculated as follows

$$\begin{aligned} V[u_a(x, t)] &= E[u^2(x, t)] - (E[u(x, t)])^2 \\ &= E[A^2]a^{-2x}(E_a[-\ln at^\alpha])^2 - (E[A])^2a^{-2x}(E_a[-\ln at^\alpha])^2 \\ &= 48a^{-2x}(E_a[-\ln at^\alpha])^2 - 36a^{-2x}(E_a[-\ln at^\alpha])^2 \\ &= 12a^{-2x}(E_a[-\ln at^\alpha])^2. \end{aligned}$$

The graphs of the variances of the Eq. (33) for different α values and $a = 5$ are plotted in Maple software as follows (Figs. 13, 14, 15, 16)

5 Results and discussions

It is observed from Tables 1 and 4 that the approximate solutions obtained by NSTIM for RFPDEs is almost similar to exact solutions for RFPDEs. Tables 2 and 5 show the approximate solutions obtained from NSTIM for $\alpha = 0.9$ and the solutions are observed to decrease when x is constant and t is increased. From Tables 2 and 5 show that solutions depend on time fractional derivative. Tables 3 and 6 show that absolute errors are negligible and decrease to 0 as number of iterations increases for different values of fractional order α which shows that this method also works for RFPDEs. Thus, this method is reliable, because absolute errors in Table 1 of the article of Sakar and Ergören are bigger than our absolute errors [46]. Also, the graphs of the approximated values and variances are plotted for different α values with Maple software, and it is observed that the approximated values and variances of the solutions which obtained from NSTIM increase while the α values is decreased. It has been observed that the overall structure of the surface graphics obtained from Maple software differ for Example 1. Similarly, for Example 2, it has been observed that the overall structure of the surface graphics obtained from Maple software differ for Example 2.

Table 6 Comparison between absolute errors when $\alpha = 1, A = 3$ and $a = 2$, for Example 2

	x	t					
		0.00	0.02	0.04	0.06	0.08	0.10
NSTIM	0.00	0.00000	0.00000000	0.00000000	0.00000000	0.00000001	0.00000003
VIM		0.00000	0.00027000	0.00100000	0.00230000	0.00390000	0.00600000
NSTIM	0.02	0.00000	0.00000000	0.00000000	0.00000000	0.00000001	0.00000003
VIM		0.00000	0.00027000	0.00100000	0.00230000	0.00390000	0.00590000
NSTIM	0.04	0.00000	0.00000000	0.00000000	0.00000000	0.00000001	0.00000003
VIM		0.00000	0.00027000	0.00100000	0.00220000	0.00380000	0.00580000
NSTIM	0.06	0.00000	0.00000000	0.00000000	0.00000000	0.00000001	0.00000003
VIM		0.00000	0.00026000	0.00100000	0.00220000	0.00380000	0.00570000
NSTIM	0.08	0.00000	0.00000000	0.00000000	0.00000000	0.00000001	0.00000003
VIM		0.00000	0.00026000	0.00100000	0.00220000	0.00370000	0.00560000
NSTIM	0.10	0.00000	0.00000000	0.00000000	0.00000000	0.00000001	0.00000003
VIM		0.00000	0.00025000	0.00100000	0.00210000	0.00370000	0.00560000

Fig. 9 For $\alpha = 0.9$ and $a = 5$, time-dependent variation of approximated value of the Eq. (28) **Fig. 11** For $\alpha = 0.7$ and $a = 5$, time-dependent variation of approximated value of the Eq. (28) **Fig. 10** For $\alpha = 0.8$ and $a = 5$, time-dependent variation of approximated value of the Eq. (28) **Fig. 12** For $\alpha = 0.6$ and $a = 5$, time-dependent variation of approximated value of the Eq. (28)

Fig. 13 For $\alpha = 0.9$, time-dependent variation of variance of the Eq. (28)

Fig. 15 For $\alpha = 0.7$, time-dependent variation of variance of the Eq. (28)

Fig. 14 For $\alpha = 0.8$, time-dependent variation of variance of the Eq. (28)

Fig. 16 For $\alpha = 0.6$, time-dependent variation of variance of the Eq. (28)

6 Conclusion

In this study, RFPDEs are analyzed by NSTIM method. The approximate analytical solution of RFPDEs has been quickly and successfully obtained. NSTIM is more powerful than VIM as shown in the absolute error comparison in Tables 3 and 6. Thus, it is concluded that NSTIM is reliable, effective and powerful in getting the approximate solutions for different classes of random component linear and nonlinear fractional of ordinary and partial differential equations.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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