



Reliability-based optimization of geotechnical design using a constrained optimization technique

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Abstract

Uncertainty and risk are inherent in geotechnical engineering. Reliability-based optimization (RBO) is a methodology that determines the best design solution according to certain design criteria while explicitly considering the effects of uncertainty. RBO ensures a design that is economical as well as reliable in the presence of uncertainties. This paper suggests a direct method for RBO of geotechnical systems by coupling reliability assessment and cost minimization. RBO is proposed as a double-loop (or nested-loop) constrained optimization problem, with cost optimization of geotechnical system through the outer loop repeatedly calling the inner reliability evaluation loop. First-order reliability method is used to compute the reliability index. An algorithm based on constrained optimization is proposed. MATLAB's constrained optimization function, *fmincon*, is used in two loops: the outer loop to minimize the cost of geotechnical system, and the inner loop to minimize the reliability index to a target value while satisfying geotechnical design requirements. The proposed method is employed to obtain the optimal design of a gravity retaining wall, and compared with other methods, and found to be robust, accurate and feasible.

Keywords Reliability-based optimization · Reliability analysis · Constrained optimization · First-order reliability method · Retaining wall

1 Introduction

It is well known that uncertainty and risk are inherent in geotechnical and geological engineering and that the quantification of these uncertainties is rational [1–7]. Traditional deterministic design methods accommodate uncertainties through empirical safety factors. However, these safety factors do not quantitatively measure the safety margin of design and do not account for the influence of different design variables and their uncertainties on overall system performance.

Reliability is the probability of an event occurring or the probability of a positive outcome and can be interpreted as a probability of failure. Calculating reliability gives a

means of assessing the degree of uncertainty involved in geotechnical engineering calculations [6].

From the reliability theory, the chance of unsatisfactory performance can be expressed as $P_f = 1 - \Phi(\beta)$ where P_f is failure probability, β is the reliability index, and Φ is cumulative distribution function (CDF) of a standard normal variable. Phoon and Ching [6] define the reliability index, β , as the number of standard deviations between the most likely value of factor of safety and the critical factor of safety of 1.0. By setting target the reliability index, the chance of unsatisfactory performance can be reduced. For geotechnical systems, the target reliability index is often in the range of 2.0–4.0 [8].

Reliability-based optimization (RBO) is a methodology that determines the best design solution according

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to certain predefined design criteria while explicitly considering the effects of uncertainty [9]. Uncertainties in geotechnical engineering come from loads, geotechnical properties and calculation models [10, 11]. RBO constitutes a powerful methodology for solving geotechnical-related design problems because it results in a geotechnical solution optimized according to predefined criteria (e.g. construction costs, benefits, etc.) while accounting for unavoidable uncertainties and satisfying geotechnical design requirements.

RBO is emerging as a design based on safety-under-uncertainty. A number of studies have been carried out for the Reliability-Based Optimization of geotechnical systems in recent past. Valdebenito and Schuëller [9] carried a detailed literature review on reliability-based optimization. Wang and Kulhawy [12] and Wang [13] used partial factors derived from reliability theory to optimize the design of shallow foundations. The partial factors needed calibration for their method. The inverse reliability approach has been used to demonstrate the optimum design of cantilever sheet pile walls [14] and anchored sheet pile walls [15]. Ching and Hsu [16] proposed a methodology to transform reliability limit-state constraints into deterministic limit-state constraints. Ching [17] presented an equivalence between reliability and factor of safety. Zhang et al. [8] presented an indirect method on the basis of the Mean First-Order Reliability Method (MFORM) to optimize the geotechnical system. Langford and Diederchs [18] developed a reliability-based design of a composite tunnel lining by combining a modified Rosenbluth Point Estimate Method (PEM), First Order Reliability Method (FORM), Monte Carlo Sampling method, and finite element analysis. Gong et al. [19] proposed and applied a fuzzy set-based Robust Geotechnical Design (RGD) method for the design of shield-driven tunnels. Zhao et al. [20] employed an Artificial Bee Colony (ABC) algorithm for reliability-based optimization retaining walls and spread footings. Zevgolits et al. [21] proposed a probabilistic geotechnical design optimization framework for large open pit excavations. Zhao et al. [22] employed least square support vector machine (LSSVM) and artificial bee colony (ABC) algorithm for reliability-based support optimization of rockbolt reinforcement around tunnels. Santos et al. [23] used FORM based ant colony optimization (ACO) algorithm for reliability-based design optimization of geosynthetic-reinforced soil walls. Ji et al. [24] used inverse FORM approach for reliability-based design in geotechnical engineering. Raviteja and Basha [25] presented a target reliability-based design optimization (TRBDO) approach of V-shaped anchor trenches for municipal solid waste (MSW) landfills.

Although considerable research work has been reported in the field of RBO of geotechnical systems, the application is still limited. One limitation has been

the difficulty of direct coupling between reliability assessment and cost minimization. This paper suggests a FORM-based direct method for RBO of geotechnical systems. In the proposed method, RBO is implemented by coupling reliability assessment and cost minimization. RBO is proposed as a double-loop (or nested-loop) constrained optimization problem, with cost optimization of the geotechnical system through the outer loop repeatedly calling the inner reliability evaluation loop. In the outer loop, the objective is the minimization of cost function while the constraint is a target reliability index; whereas, in the inner loop, the objective is the minimization of reliability index while constraints are geotechnical design requirements. The proposed method is used to illustrate the optimal design of retaining wall. MATLAB's constraint optimization function, `fmincon` [26], is used to implement this algorithm. The interior-point algorithm is used in the outer loop to search the design variables in global search space. The interior-point algorithm is used in the inner loop to calculate the reliability index.

2 Hasofer–Lind index and FORM algorithm

The stability of an engineering system, owing to the inherent uncertainty of input variables, is measured by the reliability index. The Hasofer-Lind index is widely used as a reliability index [27], the matrix formulation of which is,

$$\beta = \min_{x \in F} \sqrt{(X - \mu)^T C^{-1} (X - \mu)} \quad (1)$$

where X is a vector that represents a set of random variables x_i , μ is a vector of mean values, C is the covariance matrix, and F is the failure domain. The superscripts " T " and " -1 " denote transpose and inverse respectively. The Hasofer-Lind index can be regarded as the shortest distance in directional standard deviation units from the mean value point of the random variables to the boundary of the limit state surface.

Low and Tang [28–30] presented an alternative formulation of the Hasofer-Lind index, Eq. 1, based on the perspective of an expanding ellipsoid in the original space of the basic random variables and expressed the index as

$$\beta = \min_{x \in F} \sqrt{\left[\frac{x_i - \mu_i}{\sigma_i} \right]^T R^{-1} \left[\frac{x_i - \mu_i}{\sigma_i} \right]} \quad (2)$$

where R is the correlation matrix, σ_i is the standard deviation of random variable x_i .

For correlated non-normal parameters, Low and Tang [30] modified Eq. 2 as the following

$$\beta = \min_{x \in F} \sqrt{\left[\frac{x_i - \mu_i^N}{\sigma_i^N} \right]^T R^{-1} \left[\frac{x_i - \mu_i^N}{\sigma_i^N} \right]} \tag{3}$$

where μ_i^N and σ_i^N are the equivalent normal mean and equivalent normal standard deviation, respectively, for random variable x_i and can be calculated by the Rackwitz and Fiessler [31] transformation. The computation of β is regarded as that of finding the smallest equivalent hyper-ellipsoid (centered at the equivalent normal mean-value μ^N and with equivalent standard deviation σ^N) that is tangent to the limit state surface.

Low and Tang [32] presented an efficient alternative to Eq. 3 to calculate the reliability index β by varying dimensionless number n_i as the following equation

$$\beta = \min_{x \in F} \sqrt{\mathbf{n}^T R^{-1} \mathbf{n}} \tag{4}$$

where \mathbf{n} , termed as an equivalent standard normal vector, is a column vector of n_i . Low and Tang [32] have shown that random variable x_i can be obtained from n_i by Eq. 5a for normal distribution and by Eq. 5b for the lognormal distribution.

$$x_i = \mu_i + n_i \sigma_i \tag{5a}$$

$$x_i = \exp[\lambda + \zeta n_i], \quad \zeta = \sqrt{\ln\left[1 + \left(\frac{\sigma_i}{\mu_i}\right)^2\right]}, \quad \lambda = \ln \mu_i - 0.5\zeta^2 \tag{5b}$$

The reliability index, in this study, is calculated using Eq. 4 as proposed by Low and Tang [32].

3 MATLAB constrained optimization function, fmincon

The constrained minimization [26] is the technique of finding a vector \mathbf{x} that is the local minimum to a scalar function $f(\mathbf{x})$ subject to constraints on the allowable \mathbf{x} :

$$\min_x f(\mathbf{x}) \quad \text{such that} \quad \begin{cases} c(\mathbf{x}) \leq 0 \\ c_{eq}(\mathbf{x}) = 0 \\ A \cdot \mathbf{x} \leq b \\ A_{eq} \cdot \mathbf{x} = b_{eq} \\ lb \leq \mathbf{x} \leq ub \end{cases} \tag{6}$$

where b and b_{eq} are vectors, A and A_{eq} are matrices, $c(\mathbf{x})$ and $c_{eq}(\mathbf{x})$ are functions that return vectors, lb and ub are lower and upper bounds, respectively and $f(\mathbf{x})$ is a function that returns a scalar. In MATLAB's Optimization toolbox, fmincon function is used for constrained optimization. MATLAB uses different nonlinear methods including Trust Region

Reflective algorithm, Sequential Quadratic Programming (SQP) algorithm, and Interior Point algorithm.

4 Reliability-based optimization using a constrained optimization technique

In order to minimize the cost of a geotechnical system while satisfying a minimum value of the reliability index, termed as target reliability index β_T , MATLAB constrained optimization function, fmincon, is used. It is a two-level (or double loop) optimization problem. The fmincon function is used in two loops; the outer loop to minimize the cost of the geotechnical system and the inner loop to minimize the reliability index β to a target reliability index value β_T while satisfying geotechnical design requirements (Fig. 1).

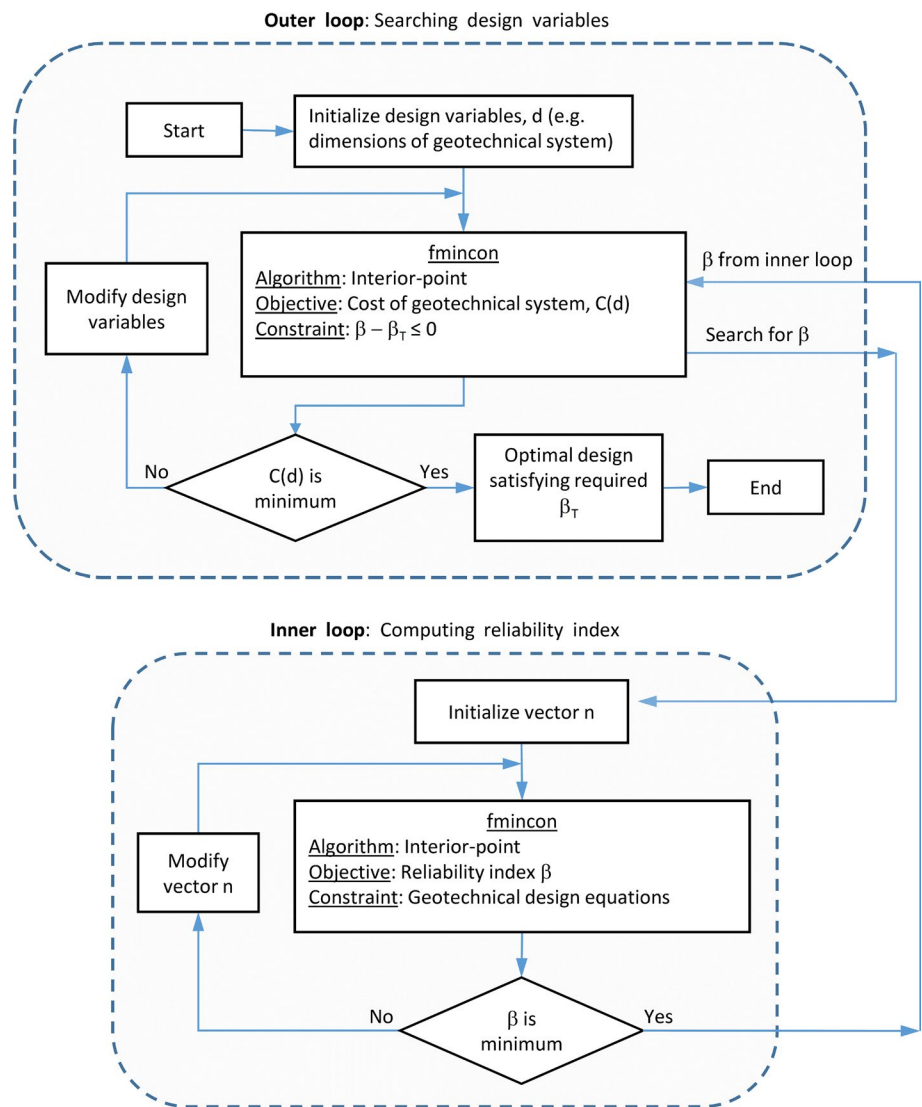
Calculation sequence to minimize the cost function of the geotechnical system, referred to as an outer loop, is summarized as follows:

- Step 1: Initialize design variables (e.g. dimension of the geotechnical system).
- Step 2: Run the interior-point algorithm with MATLAB fmincon function. The objective is the minimization of the cost of the geotechnical system. The constraint is the target reliability index β_T . The calculated β should be less than or equal to β_T . For the calculation of β , the control is passed to the inner loop where the β is calculated (Eq. 4) and control is passed back.
- Step 3: If the cost of the geotechnical system is minimized while satisfying the target beta constraint (i.e. $\beta - \beta_T \leq 0$), optimal design requirements are satisfied.
- Step 4: If the cost of the geotechnical system is not minimum, design variables are modified and the process is repeated from step 2.

Calculation sequence to obtain the reliability index β , referred to as the inner loop, is summarized as follows:

- Step 1: Initialize vector \mathbf{n} .
- Step 2: Run the interior-point algorithm with MATLAB fmincon function. The objective is the minimization of the reliability index β . The constraints are geotechnical design equations (e.g. factor of safety against overturning, sliding, bearing capacity, and eccentricity for a gravity retaining wall).
- Step 3: Calculate β using Eq. 4. If the β is minimum while satisfying the geotechnical design equations, pass control to the outer loop.
- Step 4: If the β is not minimum, vector \mathbf{n} is modified and the process is repeated from Step 2.

Fig. 1 Flowchart of reliability-based optimization algorithm



The flowchart of the above-mentioned calculation sequence is shown in Fig. 1.

5 Illustrative example: retaining wall design

Gravity retaining wall design is adopted to illustrate the application RBO. Four failure modes are considered: (1) overturning failure, (2) sliding failure, (3) bearing capacity failure and, (4) eccentricity failure. Each failure mode is a design constraint. The goal is to obtain a design with minimum wall area (i.e. cost function) satisfying both target reliability requirements in addition to all design requirements. Gravity retaining wall design, adopted from the literature [8, 20], is based on Knappett and Craig [33]. A 6 m high concrete gravity retaining wall, as shown in Fig. 2 is considered.

The active force F_a , based on Coulomb's earth pressure theory [33] is given as

$$F_a = 0.5K_a\gamma_s H^2 \tag{7}$$

where K_a is coulomb's active earth pressure coefficient. Based on Fig. 2, the following geometric relationships exist:

$$\theta = a \tan\left(\frac{b_3}{H}\right) \tag{8}$$

$$\alpha = \delta_w + \theta \tag{9}$$

$$B = b_1 + b_2 + b_3 \tag{10}$$

The forces and moment arms involved in the overturning limit state are listed in Table 1.

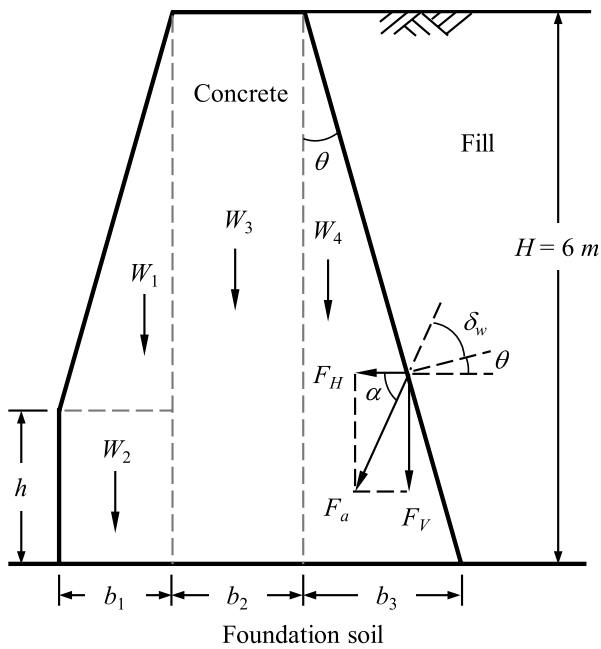


Fig. 2 Reliability-based design of gravity retaining wall (based on Zhang et al. [8])

Table 1 Forces and moment arms for rotational failure of retaining wall

Force	Moment arm
$W_1 = 0.5b_1(H - h)\gamma_c$	$a_1 = 2b_1/3$
$W_2 = b_1h\gamma_c$	$a_2 = b_1/3$
$W_3 = b_2h\gamma_c$	$a_3 = b_1 + 0.5b_2$
$W_4 = b_3H\gamma_c/2$	$a_4 = b_1 + b_2 + b_3/3$
$F_H = F_a \cos(\alpha)$	$a_{F_H} = H/3$
$F_V = F_a \sin(\alpha)$	$a_{F_V} = b_1 + b_2 + 2b_3/3$

5.1 Limit state function for overturning failure

Based on the forces and moment arms shown in Table 1, the resisting moment M_R and overturning moment M_O are given in the following equations:

$$M_R = \sum_{i=1}^4 W_i a_i + F_V a_{F_V} \tag{11}$$

$$M_O = F_H a_{F_H} \tag{12}$$

The limit state function for overturning failure is

$$g_1(\mathbf{x}) = M_R - M_O \tag{13}$$

5.2 Limit state function for sliding failure

Considering the force equilibrium in the vertical direction, the resultant base reaction R_V is

$$R_V = \sum_{i=1}^4 W_i + F_V \tag{14}$$

The limit state function for sliding failure is

$$g_2(\mathbf{x}) = R_V \tan(\delta_b) + Bc_b - F_H \tag{15}$$

5.3 Limit state functions of bearing capacity and eccentricity failure

Considering the moment equilibrium at the toe, the moment arm of the resultant base reaction is given by

$$a_{R_V} = (M_R - M_O) / R_V \tag{16}$$

The eccentricity of base reaction is given by

$$e = 0.5B - a_{R_V} \tag{17}$$

The maximum base pressure is given by

$$q_{\max} = \frac{R_V}{B} \left(1 + \frac{6e}{B} \right) \tag{18}$$

The limit state functions for bearing capacity and eccentricity are given by Eqs. 19 and 20 respectively.

$$g_3(\mathbf{x}) = q_u - \frac{R_V}{B} \left(1 + \frac{6e}{B} \right) \tag{19}$$

$$g_4(\mathbf{x}) = 1 - \frac{6e}{B} \tag{20}$$

Limit state functions for overturning (Eq. 13), sliding (Eq. 15), bearing capacity (Eq. 19), and eccentricity (Eq. 20) are the geotechnical design requirements.

5.4 Uncertainties of input variables

Six uncertain variables are considered: the friction angle of backfill material (ϕ_b); the friction angle between the backfill material and the retaining wall (δ_w); the unit weight of backfill material (γ_b); the cohesion along the interface between the wall base and the founding soil (c_b); the friction angle along the interface between the wall base and the founding soil (δ_b), and the maximum allowable bearing pressure of the foundation (q_u). The distribution and statistics of these variables are adopted from [8] and summarized in Table 2. The correlation matrix assumed for

Table 2 Distribution and statistics of uncertain variables for retaining wall design

Variable	Distribution	Mean	SD
γ_f (kN/m ³)	Normal	18.5	1
c_b (kN/m ²)	Lognormal	20	5
δ_b (°)	Lognormal	26	3
q_u (kN/m ²)	Lognormal	350	70
δ_w (°)	Lognormal	24	3
ϕ_f (°)	Lognormal	34	4

these variables is also adopted from [8] and summarized in Table 3.

5.5 Procedure for design optimization

Suppose the objective is to minimize the area of a retaining wall (i.e. the cost function) by changing the design variables $\mathbf{d} = \{b_1, b_2, b_3, h\}$. The target failure probability for retaining wall is in the range of 0.01–0.0001 [34, 35]. In this example, the target reliability index $\beta_T \geq 3.0$ is set for all failure modes which corresponds to the failure probability of 0.0013. The design space is considered as $0 < b_1 < 3$ m; $0.5 \text{ m} < b_2 < 3$ m, $0 < b_3 < 3$ m, $0.4 \text{ m} < h < 6$ m. The reliability-based optimization problem can be written as

$$\min C(\mathbf{d}) = b_1 h + 0.5(H - h)b_1 + b_2 H + 0.5b_3 H \quad (21)$$

subject to $\beta_i(\mathbf{d}) \geq 3 \quad i = 1, 2, 3, 4$

$$0 < b_1 < 3$$

$$0.5 < b_2 < 3$$

$$0 < b_3 < 3$$

$$0.5 < h < 6$$

where $C(\mathbf{d})$ is area of retaining wall and $\beta_1, \beta_2, \beta_3, \beta_4$ and are the reliability constraints for the limit states of overturning, sliding, bearing capacity and eccentricity respectively.

MATLAB’s constrained optimization function, *fmincon*, is used for the double loop optimization problem. In the outer loop, the objective is the minimization of the area of the retaining wall (Eq. 21) while the constraint is $\beta_i \leq \beta_T$. In

the inner loop, the objective is the minimization of the reliability index β_i (Eq. 4) while the constraints are corresponding limit state functions of overturning failure (Eq. 13), sliding failure (Eq. 15), bearing capacity failure (Eq. 19) and eccentricity (Eq. 20) respectively.

5.6 Results and discussion

The optimization process is started with initial design values as $b_1 = 1.6$ m, $b_2 = 1$ m, $b_3 = 0.6$ m, and $h = 1$ m. The proposed method presented in Sect. 4 is implemented in MATLAB R2018a on a laptop equipped with an Intel Core i7-7500 CPU running at 2.70 GHz. The average computational time is less than 31 s. After running the optimization, the results summary is shown in Fig. 3 and Table 4. Four optimal scenarios listed in Table 4 are discussed below.

Overturning failure: Optimal 1 is the optimized solution when the design constraint is limit state function of overturning failure (Eq. 13) with the reliability constraint β_1 minimized to the target reliability index value β_T . Figure 4a shows the convergence of cost function and reliability constraint β_1 . From the calculated values of design variables (i.e. retaining wall dimensions), reliability constraints β_2, β_3 and β_4 corresponding to limit state functions of sliding failure, bearing capacity failure and eccentricity,

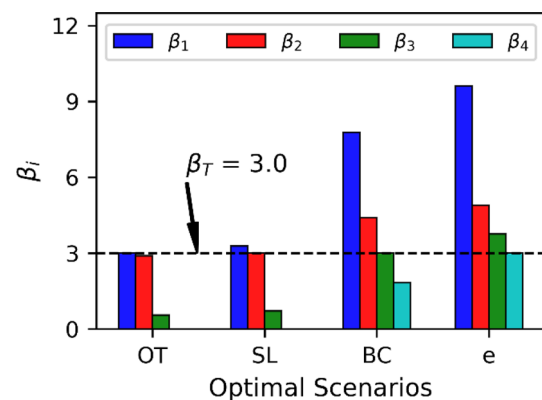


Fig. 3 Calculated values of reliability constraints β_i for optimal scenarios of overturning (OT), sliding (SL), bearing capacity (BC) and eccentricity (e) failures, respectively, for the target reliability constraint $\beta_T \geq 3.0$

Table 3 Correlation matrix of uncertain variables for retaining wall design

	γ_f	c_b	δ_b	q_u	δ_w	ϕ_f
γ_f	1	0	0	0	0	0
c_b	0	1	-0.4	0.4	0	0
δ_b	0	-0.4	1	0.4	0	0
q_u	0	0.4	0.4	1	0	0
δ_w	0	0	0	0	1	0.8
ϕ_f	0	0	0	0	0.8	1

respectively are then calculated. Figure 4b shows the convergence of these reliability constraints when these are determined using Eq. 4. It is noted that only β_1 corresponding to overturning failure is equal to β_T whereas the rest of reliability constraints are less than β_T . Thus optimal 1 does not satisfy all reliability constraints, and thus design requirements.

Sliding failure: Optimal 2 is the optimized solution when the design constraint is limit state function of sliding failure (Eq. 15) with the reliability constraint β_2 minimized to

the target reliability β_T . The convergence of cost function and reliability constraint β_2 is shown in Fig. 5a. The rest of reliability constraints β_1, β_3 and β_4 are calculated from the optimized solution. The convergence of β_1, β_3 and β_4 are shown in Fig. 5b. Only β_1 and β_2 are above β_T . Thus Optimal 2 does not satisfy bearing capacity and eccentricity requirements.

Bearing capacity failure: Optimal 3 is the optimized solution when the design constraint is limit state function of bearing capacity failure (Eq. 19) with the reliability

Table 4 Optimization results for target reliability constraint $\beta_T \geq 3.0$

Optimal scenarios	Design Constraint	d					Reliability constraint				Remarks
		b_1 (m)	b_2 (m)	b_3 (m)	h (m)	$C(d)$ (m ²)	β_1^a	β_2^b	β_3^c	β_4^d	
Optimal 1	Overturning	1.224	0.5	0	0.4	6.917	3.000	2.900	0.548	0.000	All $\beta_i \neq \beta_T$
Optimal 2	Sliding	1.262	0.5	0	0.4	7.038	3.287	3.000	0.718	0.000	All $\beta_i \neq \beta_T$
Optimal 3	Bearing Capacity	1.838	0.5	0	0.4	8.883	7.772	4.400	3.000	1.828	All $\beta_i \neq \beta_T$
Optimal 4	Eccentricity	2.065	0.5	0	0.4	9.607	9.615	4.895	3.767	3.000	All $\beta_i \geq \beta_T$

^a β_1 is the calculated value of reliability index for overturning limit state (Eq. 13)

^b β_2 is the calculated value of reliability index for sliding limit state (Eq. 15)

^c β_3 is the calculated value of reliability index for bearing capacity limit state (Eq. 19)

^d β_4 is the calculated value of reliability index for eccentricity limit state (Eq. 20)

Fig. 4 **a** Convergence of objective and reliability constraint β_1 for overturning failure, **b** determination of other reliability constraints from the optimized solution

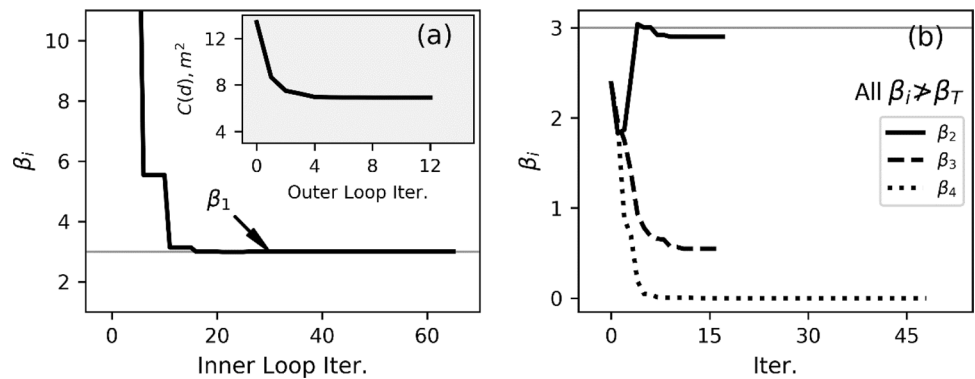


Fig. 5 **a** Convergence of objective and reliability constraint β_2 for sliding failure, **b** determination of other reliability constraints from the optimized solution

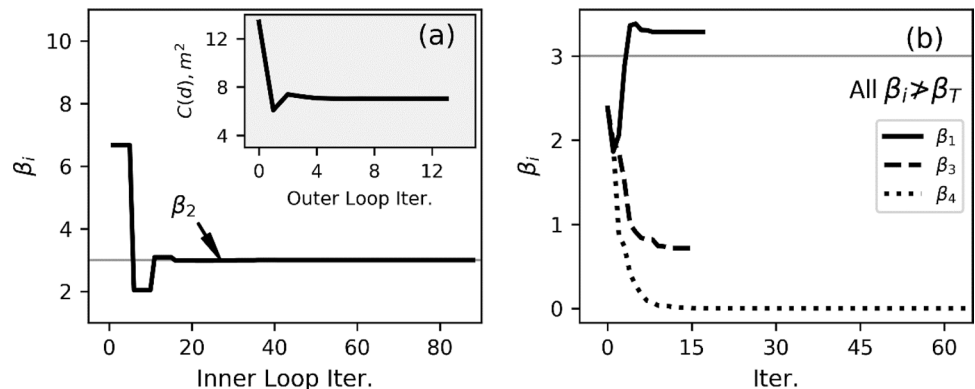


Fig. 6 **a** Convergence of objective and reliability constraint β_3 for bearing capacity failure, **b** determination of other reliability constraints from the optimized solution

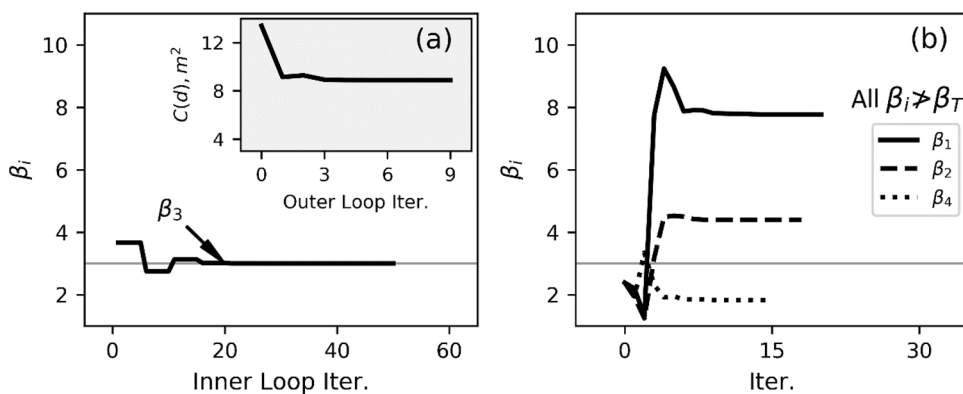
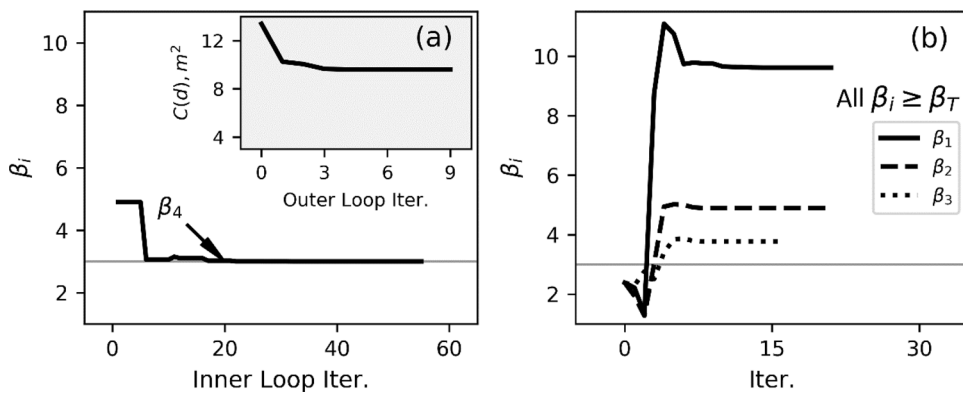


Fig. 7 **a** Convergence of objective and reliability constraint β_4 for eccentricity failure, **b** determination of other reliability constraints from the optimized solution



constraint β_3 minimized to the target reliability β_T . Figure 6a shows the convergence of cost function and reliability constraint β_3 . The rest of reliability constraints β_1 , β_2 and β_4 are calculated from the optimized solution as shown in Fig. 6b. Only β_4 is below β_T . Thus optimal 3 does not satisfy eccentricity requirement.

Eccentricity failure: Optimal 4 is the optimized solution when the design constraint is limit state function of eccentricity failure (Eq. 20) with the reliability constraint β_4 minimized to the target reliability β_T . Figure 7a shows the convergence of cost function and reliability constraint β_4 . Other reliability constraints β_1 , β_2 and β_3 are calculated from the optimized solution as shown in Fig. 7b. All reliability constraints are above β_T .

As seen in Fig. 3 and Table 4, only optimal 4 case satisfies all the reliability constraint requirements (i.e. All $\beta_i \geq \beta_T$). Thus optimal 4 is the optimized solution satisfying the reliability requirement as well as the design requirements. The Convergence of design variables (i.e. retaining wall dimensions) for the optimized solution are shown in Fig. 8. The calculated values of the reliability constraints for initial and optimized design are summarized in Table 5. To check the accuracy of proposed method, the reliability constraints for the optimized solution are also calculated by Microsoft Excel’s built-in optimization routine SOLVER

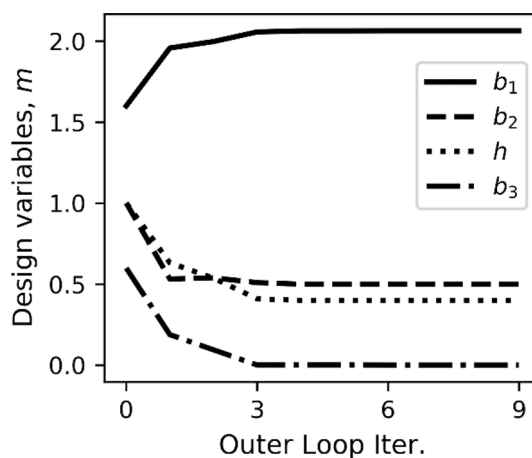


Fig. 8 Convergence of design variables for optimal 4 scenario

as proposed by Low and Tang [32]. GRG nonlinear solving method is used in Excel’s optimization routine. Figure 9 shows the Excel’s spreadsheet results. The reliability constraints calculated by Excel’s optimization routine match with the present study results. Equivalent standard normal vectors, \mathbf{n} , for all optimal scenarios are summarized in Table 6.

Table 5 Summary of optimization results for target reliability constraint $\beta_T \geq 3.0$

	<i>d</i>					Reliability constraint			
	<i>b</i> ₁ (m)	<i>b</i> ₂ (m)	<i>b</i> ₃ (m)	<i>h</i> (m)	<i>C</i> (<i>d</i>) (m ²)	β_1^a	β_2^b	β_3^c	β_4^d
Initial	1.6	1	0.6	1	13.400	17.085	6.667	3.666	4.897
Optimized	2.065	0.5	0	0.4	9.607	9.615	4.895	3.767	3.000

^a β_1 is the calculated value of reliability index for overturning limit state (Eq. 13)

^b β_2 is the calculated value of reliability index for sliding limit state (Eq. 15)

^c β_3 is the calculated value of reliability index for bearing capacity limit state (Eq. 19)

^d β_4 is the calculated value of reliability index for eccentricity limit state (Eq. 20)

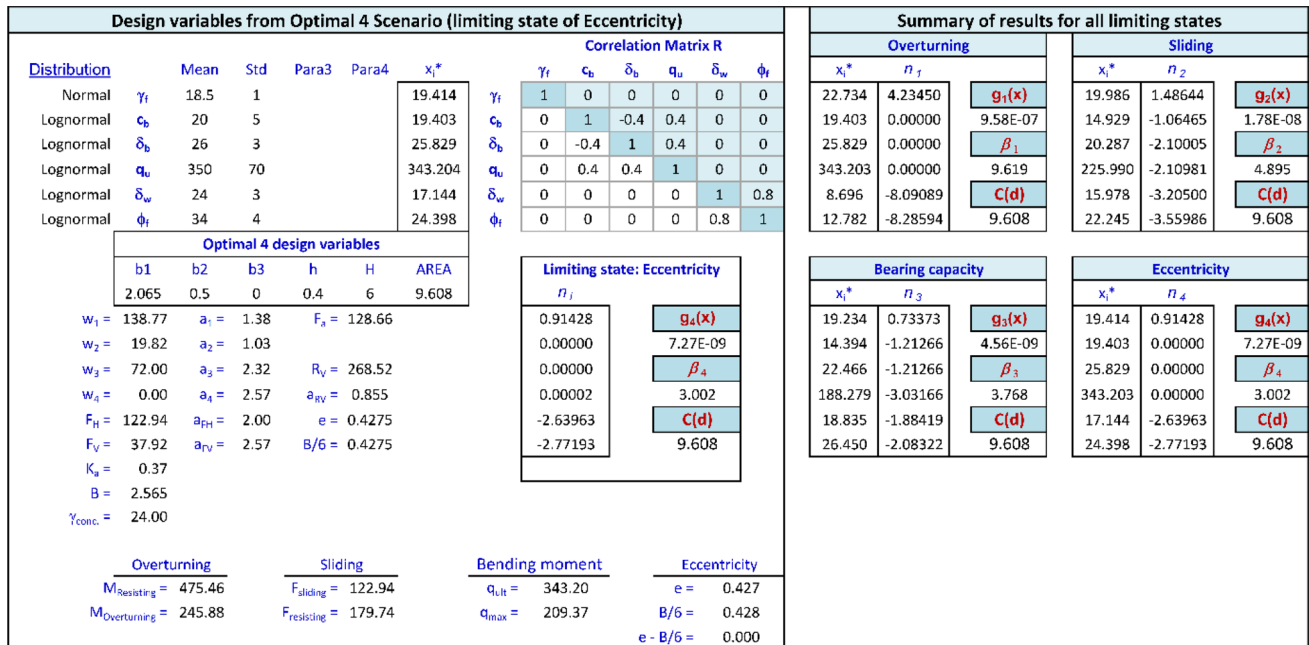


Fig. 9 Verification of Optimal 4 calculations for β_T using EXCEL's solver

Table 6 Equivalent standard normal vector *n* for different optimal scenarios

	β_1	β_2	β_3	β_4	β_1	β_2	β_3	β_4
Optimal 1 design variables				Optimal 2 design variables				
<i>n</i> ₁	0.91091	0.80230	0.10365	0.00000	1.01921	0.83484	0.13638	0.00000
<i>n</i> ₂	0.00000	-0.47681	-0.16355	0.00000	0.00000	-0.50318	-0.21561	0.00000
<i>n</i> ₃	0.00000	-1.23312	-0.16355	0.00000	0.00000	-1.27563	-0.21561	0.00000
<i>n</i> ₄	0.00000	-1.13995	-0.40887	0.00000	0.00000	-1.18587	-0.53904	0.00000
<i>n</i> ₅	-2.63928	-2.01413	-0.30568	0.00000	-2.88881	-2.07790	-0.39820	0.00000
<i>n</i> ₆	-2.76933	-2.23058	-0.34574	0.00000	-3.02650	-2.30158	-0.44984	0.00000
Optimal 3 design variables				Optimal 4 design variables				
<i>n</i> ₁	3.14289	1.30996	0.58348	0.50770	4.23191	1.48605	0.73341	0.91332
<i>n</i> ₂	0.00000	-0.90828	-0.95204	0.00000	0.00000	-1.06431	-1.21211	0.00000
<i>n</i> ₃	0.00000	-1.87936	-0.95204	0.00000	0.00000	-2.09957	-1.21211	0.00000
<i>n</i> ₄	0.00000	-1.85842	-2.38010	0.00000	0.00000	-2.10925	-3.03027	0.00000
<i>n</i> ₅	-6.63972	-2.92364	-1.53572	-1.61369	-8.08771	-3.20440	-1.88346	-2.63737
<i>n</i> ₆	-6.83396	-3.24513	-1.70728	-1.70608	-8.28277	-3.55921	-2.08244	-2.76959

Table 7 Comparison with past studies

Ref.	Method	d					Reliability Constraint			
		b_1 (m)	b_2 (m)	b_3 (m)	h (m)	$C(d)$ (m ²)	β_1	β_2	β_3	β_4
Zhang et al. [8]	Indirect RBO using modified FORM and a mapping function	2.086	0.5	0	0.4	9.675	9.390	4.849	3.849	2.996
Zhao et al. [20]	ABC algorithm and excel solver	2.065	0.5	0	0.4	9.607	9.616	4.893	3.768	3.005
Present study	MATLAB constrained optimization	2.065	0.5	0	0.4	9.607	9.615	4.895	3.767	3.000

The results obtained from the present study are compared, in Table 7, with past studies [8, 20]. The results of the present study match precisely with those reported by Zhao et al. [20]. Zhao et al. [20] used Artificial Bee Colony (ABC) algorithm and Microsoft Excel's SOLVER. Zhang et al. [8] used an indirect method by using a mapping function. The advantage of the proposed method, over others [8, 20], is its direct approach which couples reliability assessment and cost minimization. This study shows that MATLAB's constrained optimization technique can be used in double-loop optimization problems encountered in geotechnical engineering.

6 Conclusions

The findings of the study reported in this paper are summarized as follows:

1. A direct method is proposed for reliability-based optimization (RBO) by coupling reliability assessment and cost optimization. RBO is proposed as a double-loop (or nested-loop) constrained optimization problem, with cost optimization of the geotechnical system through the outer loop repeatedly calling the inner reliability evaluation loop.
2. MATLAB's constrained optimization function, `fmincon`, is used to implement the algorithm. The interior-point algorithm is used in both loops.
3. The optimal design of gravity retaining wall is illustrated using the proposed method. The results of the optimal design match precisely with those reported earlier [8, 20]. The reliability constraints of the optimized solution are verified by calculating them using Microsoft Excel's SOLVER.
4. The advantage of the proposed method is its direct approach which couples reliability assessment and cost minimization.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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