Variation of fraction in FOPID controller for vibration control of Euler–Bernoulli beam

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Abstract

Fractional-order derivatives and integrals have gained serious attention by the researchers in past few decades. It has been observed that fractional calculus is more suitable in modeling real life systems. In this paper, Fractional-Order PID (FOPID) controller is studied to suppress the vibration of cantilever beam modeled by Euler–Bernoulli model. Finite difference method is applied to analyze the behavior of the system at different fractional orders of PID controller, and the governing equations are simulated using MATLAB. The response of the system is evaluated at different fractional-order control points, under constant PID parameters (K_p , K_i , and K_d). Furthermore, the variation of frequency in transient response and 5% settling time is observed. Simulation results reveal that, though system with fractional order controller is more flexible now, the classical model of PID controller can also be properly tuned through PID constants to achieve the resultant response of the system. So, due to more simplified modelling, the classical model has an edge over the fractional order in the vibration control of cantilever beam.

Keywords Vibration control · Fractional order control · FOPID · Beams · Beam control

1 Introduction

In recent decades, fractional-order systems have gained much attention from scholars because of the non-local properties of fractional-integrals and derivatives [1]. Researchers have demonstrated that many real life phenomena such as viscoelasticity and diffusion etc. can be more accurately modelled using fractional calculus [2]. Exploitation of fractional calculus over classical one affects the response of the system and gives better flexibility [3] to tune for desired response. Fractional calculus has been studied extensively and used for modelling, prediction and control [4] of many phenomena.

The PID controllers have long been used in dynamic control systems and are replaced by many new approaches

today, some of which are given in [5]. However, the concept of Fractional order PID controller has recently been investigated. Furthermore, PID controllers being simple in nature, are ideal to analyze the behavior of its complicated fractional order form against the classical controller form. The extension of PID controllers from the classical to fractional order translates into more robust control design of the system and better flexible in tuning the frequency response and the settling time of systems [6]. The tuning methods for fractional order controllers are reported in [7–9].

The Euler Bernoulli cantilever beam model is a simplified mathematical model of linear elastic theory for determining deflection of the beam in space and time domain. Due to its simple nature, it finds its diverse applications

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in modeling aircraft wings, bridges, and towers etc. The model is also used to analyze the dynamic behavior of long robotic arms, and solar panels attached on satellites. The vibrations in these structures may cause detrimental effects which also includes fatigue failure, and unbalancing of the system, so vibration suppression is necessary to reduce the harmful effects of vibrations. It is, therefore, important to study the vibration suppression using fractional order controllers.

The previous work done in the vibration control domain include, experimentally verifying, the successful reduction of settling time and amplitude of the system using optimized parameters incorporated Fraction order PID controlled system [10]. In [11], authors have proposed a reduced model to ease the calculations for active vibration control of cantilever beam using PID controller. In [12], the author use cubic velocity feedback control law for the vibration suppression of varying orientation beam. The robust sliding mode control is employed for the beam vibration control in [13]. In [14] Authors, demonstrate vibration control of Timoshenko smart structures by incorporating output feedback based sliding mode control. Boundary control approach has been acquired for the vibration control of flexible beam in reference [15], with application in aircrafts wings.

In [16] it has been shown experimentally that vibration suppression of cantilever beam can be achieved even with variable mass characteristics by using fractional controller with suitable actuating patches. In [17] continued fractional expansion of different fractional order has been used to design the closed loop controller for the vibration suppression of smart beam. In [18] the authors have obtained a set of stabilizing parameters of fractional order PID and PI controllers for stabilizing fractional-order systems. In [19], a tuning method for the fractional-order proportional and derivative controller for a 2nd order plant systems has been proposed. Vibration control and robustness of the fractional-order PID system with different fractional-orders of viscoelastic material have been studied in [20].

However, it is not yet studied, that how the response of Fractional Order PID Controller (FOPID) controlled system varies when the fractional order of controller is changed in cantilever beam. Therefore the contribution of this paper includes, studying, how the frequency and settling time response of the system changes when fraction order of the controllers is varied, and using pure numerical method approach to analyse the behaviour of the PDE system with fractional order controlled. From the effects observed, proper tuning of the classical model of PID controller is done to imitate the effects of the response resulted from the variation of fraction in FOPID controller. The response of the system with tuned classical model of the controller

is observed to be in good agreement to imitate the effects of FOPID controlled response. Consequently, in such cases, fractional order controllers can be avoided for their complex rigorous mathematical nature and simplified classical PID controllers can be opt. In future we may generalize this concept for other dominant controllers such as sliding mode and adaptive types.

This paper can be summarized as follows. Section 2 gives some preliminaries about fractional calculus, and discrete formulae to evaluate the approximate values of fractional-order integral and derivatives. In Sect. 3, the beam model for Euler Bernoulli model is discussed along with finite difference approach to give numerical solution of the system. Section 4 discusses the controller for the system, fractional-order PID controller is introduced, and equations of actuator are written, based on which an assumption is applied, and actuator response is coupled with system equation to achieve the response of the system. Section 5 gives the MATLAB simulation results, and the effects of fractional-order is observed on the response. Section 6 concludes the paper.

2 Fractional order derivatives

Fractional calculus involves generalization of integer-order derivatives and integrals. Although the order of the integration can be a complex number [7], but in this paper, we shall focus on positive real number of the order lesser than 1. Operator for the fractional-order derivative is represented as $D_{a,t}^\alpha$, given by the following equation where α is the fractional order, k is a dummy variable, and a and t are the limits of the operation.

$$D_{a,t}^\alpha = \frac{d^\alpha}{dt^\alpha} \text{ for } \alpha > 0 \quad (1)$$

$$D_{a,t}^\alpha = \int_a^t dk^{-\alpha} \text{ for } \alpha < 0 \quad (2)$$

Different researchers have presented different definitions of fractional-order derivatives and integral. In this paper, we shall use the Riemann Liouville derivatives and integrals. The left hand sided Riemann Liouville derivative of a function $u(t)$ at time t of any order $\alpha > 0$ is defined as [21].

$$D_{a,t}^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-k)^{m-\alpha-1} u(k) dk \quad (3)$$

where Γ is the standard gamma function, and m is a positive integer such that $m - 1 < \alpha < m$. If the order is negative, then the following definition of Reiman Liouville is opted in this paper. The Riemann Liouville derivative operator is defined as for $\alpha < 0$

$$D_{a,t}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{u(k) dk}{(t-k)^{1-\alpha}} \quad (4)$$

In this paper we shall focus on PID controllers that will be involving fraction order $-1 < \alpha < 1$.

The approximate numerical solutions can be obtained by reformulating the fractional derivatives and integrals as infinite integral solutions to integer-order ordinary differential equations [21]. For a function $u(t)$ defined in some time interval $[0, T]$, if we divide the time domain in N parts with uniform step length then the step size will be $\Delta t = \frac{T}{N}$, and the i th grid point or instant is obtained by $t_i = i\Delta T$ for $i = 0, 1, 2, \dots, N$. The value of function at the i th instant is represented as $u_i = u(t_i)$. In order to approximate the Riemann Liouville derivative at an instant t_i , one of the most popular method is called L1 Scheme. The numerical approximation of the derivative of order $0 < \alpha < 1$ by L1 scheme gives the following results [22, 23].

$$D_{0,t_n}^{\alpha} = \frac{u(0)t_n^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{k=0}^{n-1} b_{n-k-1} [u_{(k+1)} - u_{(k)}] \quad (5)$$

where,

$$b_i = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} [(i+1)^{1-\alpha} - i^{1-\alpha}] \quad (6)$$

With the order of error given by $C\Delta t^{2-\alpha}$, where C is dependent on α and the function $u(t)$ [22]. The numerical approximation to compute the weighted fractional integral of any order over the interval $[0, t_n]$ can be obtained as [21].

$$D_{0,t_n}^{-\alpha} = \sum_{k=0}^{n-1} c_{n-k-1} (\theta u_k + (1-\theta)u_{k+1}) \quad (7)$$

where,

$$c_i = \frac{\Delta t^{\alpha}}{\Gamma(\alpha+1)} [(i+1)^{\alpha} - i^{\alpha}] \quad (8)$$

and θ is the weight. In this paper we will be using $\theta = 1/2$ (also known as midpoint formula). These formulae will be used to compute the value of fractional order PID controller output, which will in turn affect the response of the beam vibration suppression.

3 Mathematical modeling of beam

The Euler–Bernoulli model of the cantilever beam is opted for the mathematical modeling of the system, which is based on the assumption that shear deformation and rotational effects of Inertial are neglected.

The partial differential equation for the vertical distance $w(x, t)$, as shown in Fig. 1, at space coordinate x and time t can be derived as [25].

$$EI \frac{\partial^4 w(x, t)}{\partial x^4} + \rho A \frac{\partial^2 w(x, t)}{\partial t^2} = f(x, t) \quad (9)$$

where E is modulus of elasticity, I is area moment of inertia, ρ is the density of material, A is the cross sectional area of the beam, and $f(x, t)$ is the linear force acting downward per unit length on the beam. The $f(x, y)$ represents all external forces acting on the beam including resistance and disturbing forces. In this paper we deal with negligible resistance environment, and random disturbing force is assumed to be absent, the force density $f(x, y)$ only incorporate actuation force applied by piezoelectric patches. If an axial tensile force T is applied then the following equation can be derived [24, 25]

$$EI \frac{\partial^4 w(x, t)}{\partial x^4} + \rho A \frac{\partial^2 w(x, t)}{\partial t^2} + T \frac{\partial^2 w(x, t)}{\partial x^2} = f(x, t) \quad (10)$$

Fig. 1 Diagram for cantilever beam [24]

Consider the following boundary conditions.

No motion at fixed end and rigid support respectively yields,

$$w(0, t) = 0 \quad (11)$$

$$\frac{\partial w(0, t)}{\partial x} = 0 \quad (12)$$

Free end, and Attached tip mass gives following boundary conditions respectively at $x = L$, refer [24].

$$\frac{\partial^2 w}{\partial x^2} = 0 \quad (13)$$

$$-EI \frac{\partial^3 w(L, t)}{\partial x^3} + M \frac{\partial^2 w(L, t)}{\partial t^2} + T \frac{\partial w(L, t)}{\partial x} = 0 \quad (14)$$

where M is mass attached at the end.

There are analytic solutions available to satisfy the Euler Bernoulli beam [25]. In this paper, we will use finite difference method to solve the system with fractional order PID controller, so in order to find the response of the system we will first convert the PDE of Euler Bernoulli equation into Finite difference equation. Using the definition of finite difference method, the solution of the above PDE can be approximated. If we divide the length of the beam into N_x parts then step size $\Delta x = \frac{L}{N_x}$ and the position of $(i + 1)^{th}$ element from the fixed position can be represented as $x_i = (i + 1)\Delta x$ and $w(x_i, t_k) = w_k^i$. Using the finite difference method and doing some basic algebra, the finite difference equation of the beam is derived as shown in Eq. (15) for approximating the response of system at (i) th time instant, along the whole beam i.e. at $(x_0, x_1, x_3, \dots, x_{N_x})$.

$$\begin{bmatrix} w_i^2 & w_{i-1}^2 & w_{i-2}^2 \\ w_i^3 & w_{i-1}^3 & w_{i-2}^3 \\ \vdots & \vdots & \vdots \\ w_i^{N_x-2} & w_{i-1}^{N_x-2} & w_{i-2}^{N_x-2} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{-IE(\Delta t)^2}{\rho A(\Delta x)^4} \begin{bmatrix} w_{i-1}^0 - 4w_{i-1}^1 + 6w_{i-1}^2 - 4w_{i-1}^3 + w_{i-1}^4 \\ w_{i-1}^1 - 4w_{i-1}^2 + 6w_{i-1}^3 - 4w_{i-1}^4 + w_{i-1}^5 \\ \vdots \\ w_{i-1}^{N_x-4} - 4w_{i-1}^{N_x-3} + 6w_{i-1}^{N_x-2} - 4w_{i-1}^{N_x-1} + w_{i-1}^{N_x} \end{bmatrix} + \frac{-T(\Delta t)^2}{\rho A(\Delta x)^2} \begin{bmatrix} w_{i-1}^1 - 2w_{i-1}^2 + w_{i-1}^3 \\ w_{i-1}^2 - 2w_{i-1}^3 + w_{i-1}^4 \\ \vdots \\ w_{i-1}^{N_x-3} - 2w_{i-1}^{N_x-2} + w_{i-1}^{N_x-1} \end{bmatrix} + \frac{1}{\rho A} \begin{bmatrix} f_{i-1}^3 \\ f_{i-1}^4 \\ \vdots \\ f_{i-1}^{N_x-2} \end{bmatrix} \quad (15)$$

The values of w_j^0 , w_j^1 , $w_j^{N_x-1}$, and $w_j^{N_x}$ for $0 < j < N_t$ are evaluated from the four boundary conditions, and

values of w_0^k and w_1^k for $0 < k < N_x$ are evaluated using the two initial conditions of the system.

4 FOPID controller design

The PID controller uses the general control law of the form,

$$O = K_p e + K_i \int_0^t e(x, t) dt + K_d \frac{de}{dt} \quad (16)$$

where O is the output of the system, K_p , K_i and K_d are the PID constants and e is the error. The error in this case is $w(x, t)$ at x and time t , since the vibrations are to be suppressed so input of the system is zero deflection. The fractional-order model of the system can be obtained by varying the order of derivatives in equation. So the fractional order PID controller can be defined as

$$O_f = K_p e + K_i D_{0,t}^\beta e(x, t) + K_d D_{0,t}^\alpha e(x, t) \quad (17)$$

where $\alpha > 0$ and $\beta < 0$ are the orders of the systems. In this paper, we will be considering the effects of fractional order controller in which $\beta = -\alpha$. Next section shows the actuation of piezo electric materials.

5 Controller actuation

Piezoelectric sensors and actuators are considered to be bonded on beam along its whole length. The governing equations for the strain (therefore stresses) in piezoelectric material under the electric field E_i , are given by [26].

$$\varepsilon_x = S_{ij}^D \sigma_j + g_{mi} D_m \quad (18)$$

$$E_i = g_{mi}\sigma_i + \beta_{ik}^\sigma D_k \quad (19)$$

where i, j, m and k refer to perpendicular directions in the coordinate system, $\mathbf{E}$ is electric field, $\boldsymbol{\sigma}$ is the stress vector, $\boldsymbol{\varepsilon}$ is the strain vector, $\mathbf{S}$ is compliance coefficient matrix, $\mathbf{D}$ is electric displacement vector, $\boldsymbol{\beta}$ is imprimitivity component, $\mathbf{g}$ is piezoelectric constants matrix.

When the electric field direction is applied along the polarization vector [26], then the above system of Eqs. (18) and (19), are reduced to

$$\varepsilon = CV \quad (20)$$

where C represents a constant that depend on thickness and properties of piezoelectric material, and V is voltage applied. If we consider the double sided collocated piezoelectric actuator (as in Fig. 2) laminated on entire beam, then by applying suitable assumptions, the equation of the bending moment M created in beam is derived to be [26];

$$M = IE\alpha \quad (21)$$

where I is the moment of inertia of beam and E is the Young's modulus of the beam and α is a constant that is proportional to strain of the system, using above equations following equation can be derived [14].

$$M = kV \quad (22)$$

where k is a constant that depends on piezoelectric and beam system, and the moment generated can be used to obtain the force applied by the piezoelectric material [14]. This force intensity then, is assumed as the output of the piezoelectric controller on the beam i.e. $(f_j^i = O_j^i)$. In actual environment, motion is detected using sensors, an external circuit and signal generator can be used to generate the signals that obey control laws accordingly. These signals are applied to the piezoelectric actuators and thus used to control the vibration of the Euler–Bernoulli beam. Therefore, using the Eq. (5), (7) for the numerical approximate values to estimate derivatives and putting these

values in Eq. (17), the output of the piezoelectric produced force intensity can be written as

$$f_j^i = K_p w_j^i + \frac{K_d w_0^i t_i^{-\alpha}}{\Gamma(1-\alpha)} + K_d \sum_{k=0}^{i-1} b_{i-k-1} [w_{j+1}^i - w_j^i] + K_i \sum_{k=0}^{i-1} c_{i-k-1} (\theta w_j^i + (1-\theta)w_{j+1}^i) \quad (23)$$

where the function u is replaced by displacement w . The dynamic response of the beam system with actuation force intensity is obtained by substituting Eq. (23) in to Eq. (15), and repeating the process for $1 < i < N_t + 1$ using MATLAB, the grid for w_i^j is obtained. These points are plotted in MATLAB at their respective coordinates, and simulation is achieved.

6 Simulation results

The values of the parameters used in simulations are obtained from Table 1

The Fig. 3 shows free vibrations of the beam without any controller, while the figure Fig. 4 shows the vibrations

Table 1 Values of parameters used in simulations

Parameter	Value
Length, L	1 m
Flexure Rigidity, EI	5 Nm
Linear Density, ρA	1 kg/m
Tip mass, M	5 kg
K_p	10
K_i	10
K_d	20

Fig. 2 Identical collocated piezoelectric actuators attached on beam [26]

Fig. 3 Displacement profile of beam without control

Fig. 4 Displacement profile of beam with 0.5 fractional order PID control

Fig. 5 Displacement profile of beam with 1st order PID control

when the fractional order PID control of order 0.5 is applied. The figure Fig. 5 shows the behavior of the system under 1st order normal PID controller. In Figs. 3, 4, and 5 the values of K_p and K_i are set to 10, and K_d is set to 20.

The Fig. 6 shows a comparison of the tip response when the order of derivatives and integrals are varied, while keeping all the other parameters same. The values of K_p , K_i and K_d are kept same as in Figs. 3, 4 and 5.

The graph of the tip response for different values of fractional order α are observed and data is collected for each graph. The figure Fig. 7 shows the frequency observed as average of first two oscillations of the transient response against the fractional order of PID controller. Figure 8 shows the settling time of the system to achieve 95% of the displacement reduction versus the fractional order α .

The effects of the variation in order of FOPID can be observed from Figs. 7 and 8. From this we can figure out that the fractional order of PID controller have a considerable effect on the transient response of the systems,

therefore fractional order PID controller give us flexibility to response alongside with K_p , K_i and K_d parameters. As a result, the system can be tuned for a desired response with additional parameter α , however the desired effect can also be obtained by tuning integer-order PID parameters as can be depicted from Fig. 9, which shows a tuned integer order response to match the effect caused the variation of the fractional order.

7 Discussion

Fractional Calculus has emerged as a hot topic in research in some recent decades. It has been observed that fractional calculus is more suitable in modeling real life systems. In control field, introducing fractional calculus yields better flexibility for the process control. However, the mathematical modeling of the system with fractional orders are quite complex. In this paper, the response of the system with classical model of the controller have been shown to be in good agreement to imitate the effects of FOPID controlled response by proper tuning of the simple PID Controller. So in such case, simplified version of controller has an edge over the fractional order PID controller. There has been significant work done in the domain of motion control by FOPID controller as in [17, 18, 20]. In Reference [20] and [17], the authors exploit the fractional order PID controller for the vibration suppression of viscoelastic beam and smart beam respectively, the fractional order controller can be seen to be effective and flexible. However results of the systems were focused on only the variation of the fractional orders, this paper on the other hand shows the good enough similar response can be acquired by simple classical model, resulting in relief from complex fractional ordered modeled systems. Therefore, for future, we may generalize this concept for other controllers such as sliding mode and adaptive controllers, and compare their response variation when parameters of their classical models are tuned to imitate the response of their fractional model.

8 Conclusion

In this paper, the effect of variation fraction in FOPID Controller for vibration control of Euler Bernoulli system has been studied, it has been shown that classical PID controllers for vibration control of Euler–Bernoulli beam can be tuned to get good enough imitation of response obtained as FOPID controller is used. Therefore, in such cases, complex fractional order Controllers

Fig. 6 The comparison of tip response under same condition with order 1, 0.75, and 0.5

Fig. 7 Variation of Settling time with Fractional order

Fig. 8 Variation of Settling time with Fractional order

can be avoided for their complex rigorous mathematical nature and simplified classical PID controllers can be opt. The effects of variation of fractional order in FOPID controller is analyzed for the transient response of the system using MATLAB simulations. Though this

paper is designed only for PID controllers, in future we may generalize this concept and extensive research is required for other dominant controllers such as sliding mode and adaptive, to see if their response of fractional model can be imitated by classical model by tuning. This may result in relieving from complex nature of fractional order models in dynamic control field.

Fig. 9 Tip response of the system with tuned integer order PID controller to match the effect of 0.75 fractional order PID controller

Compliance with ethical standards

Conflicts of interest The authors declare that they have no conflict of interests.

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