Electroosmotic natural convection flow in a vertical microchannel with asymmetric heat fluxes

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Abstract

This article inspects the role of wall asymmetric heat fluxes and buoyant force on steady natural convection flow in a vertical microchannel with electrokinetic effect. The governing Poisson–Boltzmann, momentum and energy equations representing the electric potential, velocity profile and temperature distribution in the microchannel are obtained for the current investigation. Using the Debye–Hückel linearization, exact solutions for electric potential, velocity and temperature are found by method of undetermined coefficients and direct integration. In the course of graphical simulations, result shows that the skin-friction and mass flow rate in the microchannel would have been overestimated by approximately 37% and 30% respectively if the slip-condition at the microchannel walls for current investigation had not been taken into consideration regardless of the magnitude of buoyant force. In addition, the inclusion of buoyancy parameter acts as an enhancer on velocity profile and mass flow rate.

Keywords Asymmetric heat flux · Electroosmosis · Microchannel · Natural convection

List of symbols

C_p, C_v	Specific heats at constant pressure and constant volume, respectively
E'_x	Electric field strength
E_x	Dimensionless streaming potential
e	Electron charge
g	Acceleration due to gravity
Gr	Grashof number
G_3	Dimensionless parameter
D_h	Hydraulic diameter ($= 2h$)
k_b	Boltzmann constant
n_0	Bulk-liquid ion density
q_1, q_2	Imposed heat fluxes at $y = 0$ and $y = h$ respectively
r_q	Ratio of heat flux
T	Dimensional temperature
T_0	Ambient fluid temperature
T_1	Temperature at the wall $y = 0$
T_2	Temperature at the wall $y = h$
$\hat{T}$	Absolute temperature

u'	Dimensional axial velocity
U	Dimensionless axial velocity
$\vec{v}$	Vectorial velocity profile
h	Channel width
x, y	Axial and transverse coordinates respectively
X	Dimensionless axial coordinate
Y	Dimensionless transverse coordinate
z^+, z^-	Valence of the positive and negative ions
z'	Valence number of ions in the solution

Greek letters

α'	Dimensional velocity-slip length
α	Dimensionless velocity-slip length
β'	Dimensional temperature-jump length
β_t	Dimensionless temperature-jump length
β	Thermal expansion coefficient
ϵ'	Fluid permittivity
ξ_1, ξ_2	"Dimensional zeta-potential"
ξ_1^*, ξ_2^*	Dimensionless zeta-potential

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Γ	"Magnitude of electrokinetic effect (dimensionless)"
ϕ	"Electrostatic potential"
Φ	"Externally imposed electrostatic potential"
κ	"Debye–Hückel parameter"
λ	"Mean free path"
λ_D	"Debye length"
ψ'	"Dimensional EDL potential"
ψ	"Dimensionless EDL potential"
k	"Fluid thermal conductivity"
μ	"Fluid dynamic viscosity"
ρ	"Fluid density"
σ_0	Current density
τ	Skin-friction
ν	Kinematic viscosity
ρ_0	Fluid density
ρ_e	Charge density
θ	Dimensionless temperature
θ^*	Dimensionless temperature with temperature jump

Subscripts

0	Inlet properties of fluid at the entrance region
1	Value on the wall $y = 0$
2	Value on the wall $y = h$
e	Purely electroosmotic flow
m	Bulk value
$s1$	Fluid property on the wall $y = 0$
$s2$	Fluid property on the wall $y = h$

1 Introduction

The analysis of flow formation and heat transfer in microgeometries continues to gain significant attention, owing to its applications in medical field (drug delivery to cancer patients), biochemistry, mechanical engineering (design of micropumps) and cooling phenomenon of micro-electro-mechanical-system (MEMS). It is well understood from literature [1–3] that flow formation and heat transfer in microchannel are significantly different from those of macrochannel, owing to the fact that the classical no-slip condition at the walls becomes an impractical effect. Detailed advances in studies of gas and liquid flows in microchannel is presented by Liu et al. [1]. The magnitude of rarefaction of gas flow in microchannel is described by the dimensionless number called Knudsen number (Kn) which is defined as the ratio of the mean free path to the characteristic length of the flow. Tsien [4] classified the rarefaction in gases into three flow regimes, which are dictated by the magnitude of Kn. They are; "the continuum flow regime" ($\text{Kn} \leq 0.001$), "the near continuum flow regime" (slip-flow) ($0.001 < \text{Kn} < 0.1$), transition flow regime ($0.1 \leq \text{Kn} \leq 10$)

and "the free molecular flow regime" ($\text{Kn} > 10$). This classification is not valid for liquid flow in microchannel. To investigate liquid flow formation in microchannel, the flow can be considered as continuum, so that the macroscopic continuum theory of flow formation and heat transfer is valid together with Navier velocity-slip and the temperature-jump conditions at the walls.

Natural convection flow in microchannel has potential applications in cooling high power density microchips, due to the microsize of the cooling device which enhances the capacity to remove greater amount of heat than the large scale cooling systems. Jha and Oni [2] obtained an exact solution for natural convection flow in a vertical microchannel with electromagnetic effect. They established that heat transfer can be enhanced when heat is conveyed from the walls of the microchannel to the fluid relative to the case when heat is transferred from fluid to the walls. Weng and Cheng [5] in other work investigated the natural convection in an open-ended vertical parallel plate micro channel with isothermally heated inside wall. They concluded that the impact of rarefaction and fluid wall interaction enhances volume flow rate and reduces heat transfer. Other related articles on natural convection flow in a vertical channel/microchannel can be found in [6–12].

It was discussed in [1] that the major factors affecting liquid flow in microchannels are geometry, hydrophilic or hydrophobic of the surface condition and electro-kinetic transport process induced by electric double layer (EDL). An EDL phenomenon is induced when a liquid containing a few number of ions, the electrostatic charges on the non-conducting solid surface attracts the counter ions in the liquid flow. Many investigations have been carried out to understand flow formation and heat transfer in different geometries with electrokinetic effect. Electrokinetic occurrence is applicable in medical field for cardiopulmonary recovery, improvement or recharging of battery cells, laboratory experimentations, engineering and industrial applications such as the removal of contaminants in soil and imposing electric ion on flow formation [13]. The process involves the passing of low-voltage direct current electric field across the boundary of a fluid. The earliest discoveries are found in the works of Reus [14] and Probst [15] where they observed that the "like charges repel while the unlike charges attract when a liquid containing small number of ions is brought in contact with a solid boundary". The EDL model which relates the electric field and flows parameters for electrokinetics effect was formulated by Helmholtz [16], Patankar and Hu [17] obtained a numerical solution of microfluidic injection using electroosmotic forces through the intersection of two channels. Due to the complexity of the solution of electrokinetic effect on flow formation and heat transfer, Debye and Hückel [18]

presented a linear approximation which is applicable for low electric potential. Later, Yang and Li [19] employed the Debye–Hückel linearization to develop a numerical algorithm for electrokinetically-driven Newtonian liquid flows. They established that for a liquid solution of low ionic concentration and a solid surface of high zeta potential, the liquid flow in rectangular microchannel is clearly controlled by the presence of the EDL field and hence leaves from the flow characteristics described by classical fluid mechanics. Recent investigations in the study of combined electrokinetic and pressure gradient effects on flow formation in channel and microchannel can be found in [20–24].

Generally, flows in vertical geometries are often been affected by force of gravity. Moreover, the unequal wall heat fluxes at the microchannel walls leads to density difference which induced natural convection flow in the microchannel. This phenomenon has widely been neglected by researchers in electroosmotic flow. Recently, Khan et al. [25] employed the homotopy analysis method (HAM) to investigate the mixed convection in a vertical channel in the presence of electrical double layers and found that the flow formation is slightly dependent on Debye–Hückel parameter. Their work considered the case of prescribed electric field strength, which is not a valid assumption in discussing the physics of flow formation.

The use of exact solutions for mathematical equations play an important role in the proper understanding and qualitative features of many phenomena in natural science. It also allows researchers to design and run simulations to ascertain the effect of various governing parameters and minimize the need to continually run experiments.

Despite the contributions in electroosmotic flow, to the best of authors' knowledge, no work has been carried out to analytically investigate the combined role of buoyancy and electroosmotic force on natural convection flow formation in a vertical microchannel with asymmetric wall heat fluxes. This work has significant applications in microfluidic and enhancement of micro pumps whenever the application of pressure is readily unachievable. Therefore, the aim of this article is to investigate the impact of asymmetric heat fluxes on electroosmotic natural convection flow in a vertical microchannel. The novelty of this present investigation is the development of exact solution for flow formation of electroosmotic natural convection flow in a vertical microchannel with unequal wall heat fluxes. Such solution can help to predict which pertinent parameters results to enhancement of mass flow rate and minimization of flow drag. Section 1 presented the introduction, while mathematical analysis, results and conclusion are respectively given in Sects. 2, 3 and 4.

2 Problem formulation and mathematical analysis

Consider a steady flow of viscous incompressible fluid in a vertical channel subjected to unequal wall heat fluxes and unequal wall zeta-potential. The flow is induced as a result of asymmetric wall heat fluxes (which leads to density difference and hence natural convection) as well as the electric body force generated by the EDL. Following above assumptions, the governing equations for electric potential, flow formation and temperature distribution respectively in the microchannel are the Poisson–Boltzmann equation, Navier–Stokes equation and the energy equation as:

2.1 Electric potential

Let us consider a fluid phase enclosing positive and negative ions in contact with a planar positively or negatively charged surface, then, an EDL field will be set up. Suppose the surfaces bear unequal electrostatic potential ξ_1 and ξ_2 at the microchannel surfaces $y = 0$, $y = h$ respectively, which reduces as one proceeds into the fluid. The charge distribution in the EDL follows Boltzmann distribution, therefore, the ion convection effects are insignificant. The electrical potential distribution is obtained from Poisson–Boltzmann equation as:

$$\nabla^2 \phi = -\frac{\rho_e}{\epsilon'} \quad (1)$$

where the potential, ϕ is due to combination of externally imposed field Φ (which satisfies the Laplace equation $\nabla^2 \Phi = 0$) and EDL potential ψ' , ($\phi = \psi' + \Phi$), ρ_e is the net volume charge density of symmetric electrolyte defined as (Yang et al. [26], Mala and Li [27]):

$$\rho_e = (n^+ - n^-)z'e = -2n_0z'e \sinh\left(\frac{z'e\psi'}{k_b\hat{T}}\right) \quad (2)$$

where

$$n^+ = n_0 e^{\left(-\frac{z'e\psi'}{k_b\hat{T}}\right)}, n^- = n_0 e^{\left(\frac{z'e\psi'}{k_b\hat{T}}\right)}, \nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \quad (3)$$

By considering the case of the fully developed region, the exterior potential gradient is along axial direction only, therefore, $\Phi = \Phi(x)$ and $\psi' = \psi'(y)$. So that;

$$\frac{\partial^2 \psi'}{\partial y^2} = \frac{2n_0z'e}{\epsilon_0\epsilon'} \sinh\left(\frac{z'e\psi'}{k_b\hat{T}}\right) \quad (4)$$

subject to the boundary conditions:

$$\psi'(0) = \xi_1, \psi'(h) = \xi_2, \quad (5)$$

Introducing the following dimensionless variable;

$$Y = \frac{y}{D_h}, \psi = \frac{z'e\psi'}{k_b\hat{T}}, U = \frac{uD_h}{\nu}, E_x = \frac{E'_xD_h}{\xi_1}, \theta = \frac{(T - T_{s1})k}{q_2D_h} \quad (6)$$

Substituting Eq. (6) into Eqs. (4) and (5) and considering the case when the electric potential is small relative to the thermal energy of the ions ($z'e\psi' \ll k_b\hat{T}$), so that the Debye–Hückel linearization is obtained by truncating the Taylor's series after the first term is valid and Eqs. (4) and (5) become:

$$\frac{d^2\psi}{dY^2} - \kappa^2\psi = 0. \quad (7)$$

$$\psi(0) = \xi_1^*, \psi(0.5) = \xi_2^* \quad (8)$$

where ξ_1^* and ξ_2^* are the dimensionless wall zeta-potentials at $Y = 0$ and $Y = 0.5$ respectively.

The solution of (7) along with boundary conditions (8) is given as:

$$\psi(Y) = \frac{(\xi_2^* - \xi_1^*e^{-\kappa/2})e^{\kappa Y} + (\xi_1^*e^{\kappa/2} - \xi_2^*)e^{-\kappa Y}}{2 \sinh(\kappa/2)} \quad (9)$$

The above Eq. (9) describes the electrostatic potential at any point in the microchannel.

2.2 Equation of fluid motion

Consider a steady, laminar, fully developed flow of viscous incompressible fluid in a vertical microchannel with width h . Flow formation is set-up as a result of combined induced electric body force ($E'_x = -\nabla\Phi$) and buoyancy (induced by temperature difference). The velocity slip as well as temperature jump are also considered at the microchannel walls which are usual assumptions for liquid flow formation and heat transfer in the microchannel.

Applying the Boussinesq approximation defined by:

$$\rho = \rho_0[1 - \beta(T - T_{s1})] \quad (10)$$

By using Eq. (10), the momentum equation governing the flow formation in the vertical microchannel for the current investigation is obtained by incorporating the electric body force (induced by the EDL and applied electric current) to the Navier–Stokes equation in the axial direction only as:

$$\rho_0 \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu \frac{\partial^2 u}{\partial y^2} + \rho_0 g \beta (T - T_{s1}) + \rho_e E'_x, \quad (11)$$

subject to first order velocity-slip boundary conditions [28]:

$$\begin{cases} u(y) = \alpha' \frac{du}{dy}, & \text{at } y = 0 \\ u(y) = -\alpha' \frac{du}{dy}, & \text{at } y = h \end{cases} \quad (12)$$

where α' is the dimensional Navier-slip length which describes the distance beyond the surface where velocity deduces to zero.

Incorporating the following dimensionless parameters:

$$\text{Gr} = \frac{g\beta q_2 \rho_0 D_h^4}{\kappa \mu \nu}, \Gamma = \frac{k_b T \epsilon_0 \epsilon'}{z' e \rho_0 \nu^2}, \text{Pr} = \frac{\nu}{\alpha}, \alpha = \frac{\alpha'}{D_h}, \beta_t = \frac{\beta'}{D_h},$$

$$\kappa = \frac{D_h}{\lambda_D}, \lambda_D = \left[\frac{\epsilon_0 \epsilon' k_b \hat{T}}{2e^2 z'^2 n_0} \right]^{1/2}, \xi_1^* = \frac{z' e \xi_1}{k_b \hat{T}}, \xi_2^* = \frac{z' e \xi_2}{k_b \hat{T}} \quad (13)$$

On substituting (13) into Eqs. (11) and (12) and considering the fully developed region (see Fig. 1), the modified Navier–Stokes equation governing flow formation for this current research in dimensionless form is obtained as:

Fig. 1 Schematic of flow formation

$$\frac{d^2 U}{dY^2} + Gr\theta - E_x \kappa^2 \Gamma \psi = 0, \quad (14)$$

subject to;

$$\begin{cases} U(Y) = \alpha \frac{dU}{dY}, & \text{at } Y = 0 \\ U(Y) = -\alpha \frac{dU}{dY}, & \text{at } Y = 0.5 \end{cases} \quad (15)$$

To solve the equation of motion governing flow formation, the expression for temperature distribution (θ) in the microchannel needs be acquired.

2.3 Energy equation

The solution to the energy equation generates the temperature distributions in the vertical microchannel. Suppose a constant asymmetric heat fluxes q_1 and q_2 are applied at the walls $y = 0$ and $y = h$ respectively, such that $q_1 < q_2$, then a natural convection flow is generated (due to unequal wall heat fluxes, resulting to density difference). Also, the flow is assumed to be thermally fully developed, so that temperature distributions are in axial direction only. Except stated otherwise, all thermo physical parameters are assumed constant. Based on the assumptions above, the energy equation describing temperature distribution in the vertical microchannel is given in dimensional form as:

$$\frac{\partial T}{\partial t'} + u \frac{\partial T}{\partial x} = \frac{k}{\rho_0 C_p} \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho_0 C_p} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{E_x'^2}{\rho_0 C_p \sigma_0}, \quad (16)$$

The second and third terms on the right side of Eq. (16) are respectively the viscous dissipation term and the Joule heating term due to EDL.

Equation (16) is subjected to the following boundary conditions;

$$\begin{cases} \frac{dT}{dy} = -\frac{q_1}{k}, & \text{at } y = 0 \\ \frac{dT}{dy} = \frac{q_2}{k}, & \text{at } y = h \end{cases} \quad (17)$$

For thermally fully-developed flow, $\frac{\partial T}{\partial x}$ is a constant and is given by Barletta [29];

$$\frac{\partial T}{\partial x} = \frac{q_1 + q_2}{\rho C_p u_m W} \quad (18)$$

Since EOF (electroosmotic flow) behaves as a plug flow and considering the case of low electric potential, the viscous dissipation as well as the Joule heating terms in Eq. (7) can be neglected in order to obtain a closed form

solution, so that the energy equation at steady state is given in dimensionless form as:

$$\frac{d^2 \theta}{dY^2} - 2(1 + r_q)U = 0, \quad (19)$$

Subject to the dimensionless conditions;

$$\begin{cases} \frac{d\theta}{dY} = -r_q, & \text{at } Y = 0 \\ \frac{d\theta}{dY} = 1, & \text{at } Y = 0.5 \end{cases} \quad (20)$$

where r_q is the ratio of the wall heat fluxes, i.e. $\left(r_q = \frac{q_1}{q_2}\right)$.

The solution to the energy Eq. (19) subject to (20) requires the solution of (14) subject to (15). These equations are coupled and need to be solved simultaneously.

From Eq. (14),

$$\theta = \frac{1}{Gr} \left(E_x \kappa^2 \Gamma \psi - \frac{d^2 U}{dY^2} \right) \quad (21)$$

So that,

$$\frac{d^2 \theta}{dY^2} = \frac{E_x \kappa^2 \Gamma}{Gr} \frac{d^2 \psi}{dY^2} - \frac{1}{Gr} \frac{d^4 U}{dY^4} \quad (22)$$

By comparing Eqs. (19) and (22), we have;

$$\frac{d^4 U}{dY^4} + 2Gr(1 + r_q)U - E_x \kappa^2 \Gamma \frac{d^2 \psi}{dY^2} = 0 \quad (23)$$

The solution to the auxiliary equation are $\gamma + i\gamma, \gamma - i\gamma, -\gamma + i\gamma, -\gamma - i\gamma$, so that the solution of Eq. (23) is obtained as:

$$\begin{aligned} U(Y) = & e^{\gamma Y} [C_3 \cos(\gamma Y) + C_4 \sin(\gamma Y)] \\ & + e^{-\gamma Y} [C_5 \cos(\gamma Y) + C_6 \sin(\gamma Y)] \\ & + \frac{E_x \kappa^4 \Gamma}{(\kappa^4 + 4\gamma^4)} [C_1 e^{\kappa Y} + C_2 e^{-\kappa Y}] \end{aligned} \quad (24)$$

On substituting Eq. (24) into (21), we obtain the temperature distribution at any point in the microchannel as:

$$\begin{aligned} \theta(Y) = & \frac{E_x}{Gr} \left[\kappa^2 \Gamma - \frac{\kappa^6 \Gamma}{(\kappa^4 + 4\gamma^4)} \right] [C_1 e^{\kappa Y} + C_2 e^{-\kappa Y}] \\ & + \frac{2\gamma^2}{Gr} e^{\gamma Y} [C_3 \sin(\gamma Y) - C_4 \cos(\gamma Y)] \\ & + e^{-\gamma Y} [C_6 \cos(\gamma Y) - C_5 \sin(\gamma Y)] \end{aligned} \quad (25)$$

On applying the boundary conditions in Eqs. (15) and (20) on Eqs. (23) and (24), the constants C_3, C_4, C_5 , and C_6 are obtained as follow;

$$\begin{aligned}
C_3 &= \frac{\det \begin{pmatrix} X_4 E_x & X_1 & X_2 & X_3 \\ X_9 E_x & X_6 & X_7 & X_8 \\ E_x X_{12} + XX & -1 & 1 & -1 \\ E_x X_{17} + 1 & X_{14} & X_{15} & X_{16} \end{pmatrix}}{\det \begin{pmatrix} XX_1 & X_1 & X_2 & X_3 \\ X_5 & X_6 & X_7 & X_8 \\ 1 & -1 & 1 & -1 \\ X_{13} & X_{14} & X_{15} & X_{16} \end{pmatrix}}, C_4 = \frac{\det \begin{pmatrix} XX_1 & X_4 E_x & X_2 & X_3 \\ X_5 & X_9 E_x & X_7 & X_8 \\ 1 & E_x X_{12} + XX & 1 & -1 \\ X_{13} & E_x X_{17} + 1 & X_{15} & X_{16} \end{pmatrix}}{\det \begin{pmatrix} XX_1 & X_1 & X_2 & X_3 \\ X_5 & X_6 & X_7 & X_8 \\ 1 & -1 & 1 & -1 \\ X_{13} & X_{14} & X_{15} & X_{16} \end{pmatrix}} \\
C_5 &= \frac{\det \begin{pmatrix} XX_1 & X_1 & X_4 E_x & X_3 \\ X_5 & X_6 & X_9 E_x & X_8 \\ 1 & -1 & E_x X_{12} + XX & -1 \\ X_{13} & X_{14} & E_x X_{17} + 1 & X_{16} \end{pmatrix}}{\det \begin{pmatrix} XX_1 & X_1 & X_2 & X_3 \\ X_5 & X_6 & X_7 & X_8 \\ 1 & -1 & 1 & -1 \\ X_{13} & X_{14} & X_{15} & X_{16} \end{pmatrix}}, C_6 = \frac{\det \begin{pmatrix} XX_1 & X_1 & X_2 & X_4 E_x \\ X_5 & X_6 & X_7 & X_9 E_x \\ 1 & -1 & 1 & E_x X_{12} + XX \\ X_{13} & X_{14} & X_{15} & E_x X_{17} + 1 \end{pmatrix}}{\det \begin{pmatrix} XX_1 & X_1 & X_2 & X_3 \\ X_5 & X_6 & X_7 & X_8 \\ 1 & -1 & 1 & -1 \\ X_{13} & X_{14} & X_{15} & X_{16} \end{pmatrix}}
\end{aligned} \quad (26)$$

where $\det$ denotes the determinant of the matrix, $\gamma = \left[\frac{Gr}{2} (1 + r_q) \right]^{1/4}$ and $X_{i's}, i = 1, 2, 3, \dots$ are constants defined in "Appendix".

Therefore, the temperature distribution in the vertical microchannel with temperature jump is given as (Avci and Aydin [29]):

$$\begin{aligned}
\theta^*(Y) &= \frac{(T - T_1)}{q_2 D_h / k} = \frac{(T - T_{s1} + T_{s1} - T_1)}{q_2 D_h / k} \\
&= \theta(Y) - \beta_t r_q = \frac{E_x}{Gr} \left[\kappa^2 \Gamma - \frac{\kappa^6 \Gamma}{(\kappa^4 + 4\gamma^4)} \right] [C_1 e^{\kappa Y} + C_2 e^{-\kappa Y}] \\
&\quad + \frac{2\gamma^2}{Gr} e^{\gamma Y} [C_3 \sin(\gamma Y) - C_4 \cos(\gamma Y)] \\
&\quad + e^{-\gamma Y} [C_6 \cos(\gamma Y) - C_5 \sin(\gamma Y)] - \beta_t r_q
\end{aligned} \quad (27)$$

The exact solutions of Eqs. (24) and (27) deserve great attention as they respectively describe flow formation and temperature distributions for natural convection flow in a vertical microchannel with electrokinetic effect respectively.

2.4 Streaming potential

The streaming potential is the steady potential which builds up across a capillary in the presence of an applied gradient which is just sufficient to prevent any net current flow (Rice [31]). In most previous researches [21–23, 25], the external electric field E_x has always been prescribed. Meanwhile, it is well-known that the electrical field induced is driven by a hydrostatic force through a

microchannel with charged walls, which results in induced streaming current, defined as [27]:

$$I_s = -2n_0 z e v D_h \int_0^1 U(Y) \psi(Y) dY \quad (28)$$

Similarly, the streaming potential produces a conduction current in the opposite direction, which is given in dimensionless form as:

$$I_c = \frac{E_x \lambda_0 D_h \xi_1}{L} \quad (29)$$

At steady-state, the sum of these currents equals zero, so that:

$$I_s + I_c = 0 \quad (30)$$

Then, the streaming potential in dimensionless form is obtained as:

$$E_x = 2G_3 \int_0^1 U(Y) \psi(Y) dY \quad (31)$$

where $G_3 = \frac{zn_0 e v L}{\lambda_0 \xi_1}$ is a constant.

From Eq. (31), the expression for the electric field strength is given as:

$$E_x = \frac{X_{19} \Delta_5 + X_{20} \Delta_7 + X_{21} \Delta_9 + X_{22} \Delta_{11}}{1 - X_{18} - X_{19} \Delta_6 - X_{20} \Delta_8 - X_{21} \Delta_{10} - X_{22} \Delta_{12}} \quad (32)$$

where $\Delta_{i's}, i = 1, 2, 3, \dots$ are constants defined in "Appendix".

2.5 Skin-friction

Using the velocity profile in Eq. (24), the skin-frictions are obtained at the microchannel walls as:

$$\tau_0 = \left. \frac{dU(Y)}{dY} \right|_{Y=0} = C_3\gamma + C_4\gamma - C_5\gamma + C_6\gamma + \frac{E_x k^5 \Gamma}{(\kappa^4 + 4\gamma^4)} [C_1 - C_2] \quad (33)$$

$$\begin{aligned} \tau_1 &= - \left. \frac{dU(Y)}{dY} \right|_{Y=1} \\ &= C_3\gamma e^{\gamma/2} [\sin(\gamma/2) - \cos(\gamma/2)] - C_4\gamma e^{\gamma/2} [\cos(\gamma/2) + \sin(\gamma/2)] \\ &\quad + C_5\gamma e^{-\gamma/2} [\cos(\gamma/2) + \sin(\gamma/2)] - C_6\gamma e^{-\gamma/2} [\cos(\gamma/2) - \sin(\gamma/2)] \\ &\quad - \frac{E_x k^5 \Gamma}{(\kappa^4 + 4\gamma^4)} [C_1 e^{\kappa/2} - C_2 e^{-\kappa/2}] \end{aligned} \quad (34)$$

2.6 Mass flow rate

The expression for the mass flow rate of viscous incompressible fluid in the vertical microchannel with asymmetric wall heat fluxes is given by:

$$\begin{aligned} Q &= \int_0^{1/2} U(Y) dY = \frac{C_3}{2\gamma} [e^{\gamma/2} (\cos(\gamma/2) + \sin(\gamma/2)) - 1] \\ &\quad + \frac{C_4}{2\gamma} [1 - e^{\gamma/2} (\cos(\gamma/2) - \sin(\gamma/2))] \\ &\quad + \frac{C_5}{2\gamma} [1 - e^{-\gamma/2} (\cos(\gamma/2) - \sin(\gamma/2))] \\ &\quad + \frac{C_6}{2\gamma} [1 - e^{-\gamma/2} (\cos(\gamma/2) + \sin(\gamma/2))] \\ &\quad + \frac{E_x k^3 \Gamma}{(\kappa^4 + 4\gamma^4)} [C_1 (e^{\kappa/2} - 1) - C_2 (e^{-\kappa/2} - 1)] \end{aligned} \quad (35)$$

2.7 Special cases

For purely buoyancy driven flow, $\Gamma = 0$, so that velocity profile as well as temperature distribution follows trivially by substituting $\Gamma = 0$ in Eqs. (25) and (27) respectively. On the other hand, for purely electroosmotic flow ($Gr = 0$), flow formation as well as temperature distribution cannot be obtained by substituting $Gr = 0$ in Eqs. (25) and (27).

Fig. 2 Electric field strength for different values of κ , α and r_q

Fig. 3 Electric potential for different values of κ for $\xi_1 = 0.5$, $\xi_2 = 1.0$

Therefore, we shall employ Eq. (14) by substituting $Gr = 0$ as follows:

$$\frac{d^2 U_e}{dY^2} = E_x k^2 \Gamma \psi \quad (36)$$

On solving (36) subject to boundary conditions (15), the flow formation as well as temperature distribution for purely electroosmotic flow is given as:

$$U_e = K_1 Y + K_2 + E_x \Gamma (C_1 e^{\kappa Y} + C_2 e^{-\kappa Y}) \quad (37)$$

$$\theta_e = 2Ra(1 + r_q) \left[\frac{K_1 Y^3}{6} + \frac{K_2 Y^2}{2} + \frac{E_x \Gamma}{\kappa^2} (C_1 e^{\kappa Y} + C_2 e^{-\kappa Y}) \right] + K_3 Y + 1 - \beta_t \text{Kn} r_q \quad (38)$$

Fig. 4 Velocity profiles for different values of α and r_q

Fig. 6 Temperature distribution for different values of β_t and r_q

Fig. 5 Velocity profiles for different values of κ and ξ_2^*

Fig. 7 Temperature distribution for different values of κ and ξ_2^*

where

$$\begin{aligned} K_1 &= \frac{2E_x \Gamma \left\{ \beta_v \text{Kn} [C_2 (e^{-\kappa/2} + 1) - C_1 (e^{\kappa/2} + 1)] + [C_2 (e^{-\kappa/2} - 1) - C_1 (e^{\kappa/2} + 1)] \right\}}{1 + 4\beta_v \text{Kn}} \\ K_2 &= \frac{E_x \Gamma \left\{ [1 + 4\beta_v \text{Kn}] [\beta_v \text{Kn} (C_1 - C_2) - (C_1 + C_2)] + 2 \left\{ (\beta_v \text{Kn})^2 [C_2 (e^{-\kappa/2} + 1) - C_1 (e^{\kappa/2} + 1)] + \beta_v \text{Kn} [C_2 (e^{-\kappa/2} - 1) - C_1 (e^{\kappa/2} + 1)] \right\} \right\}}{1 + 4\beta_v \text{Kn}} \\ K_3 &= \frac{1 - r_q}{4Ra(1 + r_q) \left\{ \frac{K_1}{8} + \frac{K_2}{2} + \frac{E_x}{\kappa} [C_1 (e^{\kappa/2} - 1) - C_1 (e^{-\kappa/2} + 1)] \right\}} \end{aligned} \quad (39)$$

3 Results and discussion

This article is dedicated to inspect the combined impact of buoyancy and wall asymmetric heat fluxes on electroosmotic flow in a vertical microchannel. The solutions

obtained illustrate that the basic governing parameters controlling flow formation and mass flow rate are dimensionless velocity-slip-length (α), dimensionless temperature-jump-length (β_t), Debye-Hückel parameter (κ), ratio of heat fluxes (r_q), Grashof number (Gr) and magnitude

Fig. 8 Skin-friction for different values of κ , α and r_q at $Y=0$

Fig. 9 Skin-friction for different values of κ , α and r_q at $Y=0.5$

of electrokinetic force (Γ). For simplicity, throughout this work $\alpha = \beta_t$. It is desirable to understand the impact of these parameters on mass-flow rate enhancement and reduction of skin-friction. By employing MATLAB software, figures are simulated to explicate the impact of various governing parameters.

Figure 2 depicts the combined effect of slip-length (α), Debye-Hückel parameter (κ) and ratio of heat fluxes (r_q) on electric field strength (E_x) in the microchannel. It is obvious that electric field strength for slip-flow ($\alpha = \beta_t = 0.1$) is clearly different from that of no-slip flow ($\alpha = \beta_t = 0.0$). In addition, the electric field strength in the microchannel can be boosted by subjecting the microchannel walls to symmetric wall heat flux. A careful look at this figure suggests that the electric field strength is independent on α and r_q as $\kappa \rightarrow \infty$. This is anticipated since κ is inversely proportional to the EDL length.

Fig. 10 Skin-friction for different values of Gr , κ and Γ

Fig. 11 Mass flow rate for different values of κ , α and r_q

Figure 3 shows the impact of κ on EDL potential in the microchannel for the case of asymmetric wall zeta potential. EDL potential is found to decrease with increase in κ throughout the microchannel. Also, the maximum EDL potential is observed at the walls of the microchannel. This is due to the fact that the zeta-potentials are prescribed at the walls.

Figure 4 presents velocity profile of combined buoyancy and electrokinetically driven flow in a vertical microchannel for different values of α and r_q . This figure illustrates that fluid velocity as well as velocity-slip at the walls are amplified with increase in r_q . This is true since increase in velocity slip leads to freedom of fluid molecules, thereby increasing fluid velocity. Also, the peak velocity profile is observed around the center of the microchannel.

Figure 5 describes the effect of asymmetric wall zeta potential on velocity profile for different values of κ . For small value of κ , flow formation is independent of wall

Fig. 12 Mass flow rate for different values of Γ , Gr and κ

asymmetric zeta potential regardless of its magnitude. On the other hand, the role of κ on flow formation is to reduce velocity profile in the microchannel. This could be attributed to the presence of random EDL force around the walls of the microchannel which retards fluid motion.

The impact of asymmetric wall heat flux (r_q) as well as α on temperature distributions in the vertical microchannel is presented in Fig. 6. It is evident from this figure that temperature distribution increases with increase in r_q . It addition, the presence of electrokinetic force distorted the symmetric nature of temperature distributions obtained by Avcin and Aydin [30]. It is important to note that the peak temperature is achieved at the microchannel walls for symmetric heat flux ($r_q = 1.0$) case.

Figure 7 displays temperature distributions in the vertical microchannel as a function of ξ_2^* and κ with slip. Fluid temperature as well as temperature jump is observe to be an increasing function of κ and ξ_2 . This could be explained

Table 1 Numerical values of skin-friction for different values of r_q , α and Gr

r_q	τ_1 at $Gr = 0.1$		τ_1 at $Gr = 10$		% change in τ_1	
	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.0$	$\alpha = 0.1$	% change for $Gr = 0.1$	% change for $Gr = 10$
0.0	0.0847	0.0530	3.3515	2.0110	37.43	40.00
0.1	0.0713	0.0446	3.4386	2.0494	37.45	40.40
0.2	0.0595	0.0372	3.5378	2.0948	37.47	40.47
0.3	0.0489	0.0306	3.6461	2.1453	37.42	41.16
0.4	0.0394	0.0247	3.7611	2.1995	37.31	41.52
0.5	0.0308	0.0193	3.8812	2.2564	37.34	41.82
0.6	0.0230	0.0114	4.0050	2.3151	50.43	42.19
0.7	0.0158	0.0098	4.1314	2.3750	37.97	42.51
0.8	0.0092	0.0057	4.2598	2.4357	38.04	42.82
0.9	0.0031	0.0019	4.3893	2.4968	38.71	43.12
1.0	0.0025	0.0017	4.5194	2.5580	32.00	43.40

Table 2 Numerical values of mass flow rate for different values of r_q , α and Gr

r_q	Q at $Gr = 0.1$		Q at $Gr = 10$		% change in Q	
	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.0$	$\alpha = 0.1$	% change for $Gr = 0.1$	% change for $Gr = 10$
0.0	0.0034	0.0049	0.1372	0.1874	30.61	26.79
0.1	0.0029	0.0041	0.1404	0.1902	29.27	26.18
0.2	0.0025	0.0035	0.1440	0.1936	28.57	25.62
0.3	0.0021	0.0029	0.1479	0.1974	27.59	25.08
0.4	0.0017	0.0024	0.1521	0.2016	29.17	24.55
0.5	0.0014	0.0019	0.1565	0.2060	26.32	24.03
0.6	0.0011	0.0015	0.1611	0.2106	26.67	23.50
0.7	0.0008	0.0011	0.1657	0.2153	27.27	23.04
0.8	0.0006	0.0007	0.1705	0.2201	14.29	22.54
0.9	0.0003	0.0004	0.1752	0.2249	25.00	22.10
1.0	0.0001	0.0001	0.1800	0.2297	0.00	21.64

Table 3 Velocity profile for different values of Gr and α

Y	Gr = 0.1		Gr = 1.0		Gr = 10.0	
	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.0$	$\alpha = 0.1$
0.0	0.0000	0.0017	0.0000	0.0302	0.0000	0.2558
0.1	0.0024	0.0012	0.0383	0.0553	0.3530	0.4602
0.2	0.0062	0.0050	0.0569	0.0677	0.5200	0.5582
0.3	0.0052	0.0034	0.0560	0.0662	0.5136	0.5496
0.4	0.0045	0.0026	0.0360	0.0517	0.3403	0.4399
0.5	0.0000	0.0047	0.0000	0.0274	0.0000	0.2399

due to the fact that increase in κ decreases the EDL length which in turn decreases the EDL around the microchannel walls. It is obvious from this figure that fluid temperature can be improved by subjecting the microchannel walls to symmetric wall heat flux.

Figures 8 and 9 demonstrate the combined impact of κ , α and r_q on skin-friction at the microchannel walls $Y = 0$ and $Y = 0.5$ respectively. These figures reveal that skin-friction increases with increase in r_q and decreases with α . It is interesting to note that skin-friction is weakly dependent on κ for large values of κ . This is due to the fact that as $\kappa \rightarrow \infty$, the EDL length gradually tends to zero and thereby eliminating the electrokinetic effect on skin-friction.

Figure 10 exhibits the significance of Gr and Γ on skin-friction at the wall $Y = 0$ for combined buoyancy and electrokinetically driven flow in a vertical microchannel. It is evident from this figure that the role of Gr is to increase the skin-friction at this wall, while the reverse scenario is found for increase in Γ . This could be attributed to the presence of the buoyancy force which acts as an enhancer on flow formation and therefore increasing skin-friction at this wall. A careful observation of this figure suggests that skin-friction tends to zero in the absence of Gr regardless the magnitude of electrokinetic body force (Γ).

One of the significant advances in microfluidics is the design of micropump; a device that can control and transport small fluid volumes. Having a thorough understanding of mass flow rate in a microchannel can help predict the flow rate in micropumps. Figure 11 presents the mass flow rate as a function of κ , α and r_q in the vertical microchannel. It is

interesting to discover from this figure that the maximum flow rate is found for large EDL (small value of κ) and then gradually decreases with increase in κ . On the other hand, an increase in asymmetric wall heat flux act as an enhancer on mass flow rate around the region with large EDL.

Figure 12 shows the role of magnitude of electrokinetic force (Γ) and Gr on mass flow rate in the vertical microchannel. It is obvious from this figure that flow rate can be boosted by increasing Gr. On the other hand, the role of κ as well as Γ is found to have a destabilizing effect on mass flow rate. This can be attributed to the fact that increasing Gr increases the buoyant force which leads to corresponding increase in temperature difference and hence amplitudes mass flow rate.

Tables 1 and 2 illustrate the role of α , r_q and Gr on skin-friction at the wall $Y = 1$ and mass flow rate respectively. It is found from Table 1 that skin-friction of no-slip-flow ($\alpha = 0.0$) is significantly higher than those at slip regime ($\alpha = 0.1$). This table also shows that the skin-friction of flow formation in the microchannel would have been overestimated by approximately 37% should the slip-condition had not been taken into consideration regardless of the magnitude of buoyancy force. Table 2 on the other hand clarifies the influence of α and Gr on mass flow-rate for different values r_q . It is obvious from this table that the mass flow-rate would have been over computed by up to 30% should the slip-effect had been neglected irrespective of the magnitude of Gr.

Table 3 shows the impact of buoyant force (Gr) and slip-length (α) on velocity profile of natural convection flow in a vertical microchannel with asymmetric heat flux. It is found

Table 4 Numerical comparisons of velocity profile for different values of α

Y	$\alpha = 0.0$		$\alpha = 0.05$		$\alpha = 0.1$	
	Exact	PDEPE	Exact	PDEPE	Exact	PDEPE
0.0	0.0000	0.0000	0.1633	0.1634	0.2558	0.2559
0.1	0.3530	0.3531	0.4214	0.4216	0.4602	0.4604
0.2	0.5200	0.5202	0.5444	0.5445	0.5582	0.5583
0.3	0.5136	0.5136	0.5367	0.5366	0.5496	0.5496
0.4	0.3403	0.3404	0.4040	0.4041	0.4399	0.4398
0.5	0.0000	0.0000	0.1534	0.1534	0.2399	0.2398

that velocity profile for slip flow ($\alpha = 0.1$) are significantly different from those of continuum regime (no-slip flow; $\alpha = 0.0$). Also, it is obvious from this table that the role of Gr is to enhance fluid velocity throughout the microchannel.

Table 4 illustrates a numerical comparison of the present work velocity with those obtained by employing an in built PDE solver called *pdepe* on MATLAB. This method computes initial-boundary value problems for systems of parabolic and elliptic PDEs in the one space variable (y) and time (t). The resulting ordinary differential equations (ODEs) from the discretization of space are integrated to obtain approximate solutions at times specified in *tspan*. The *pdepe* inbuilt function returns values of the solution on a mesh provided in *xmesh* [32]. Before solving the equation, we specified the mesh points (t, x) at which the *pdepe* evaluates the solution. In this current work, we have considered 100 points each over $0 \leq y \leq 0.5$ and $0 \leq t \leq 1.0$. In particular, the cost and accuracy of the solution depends strongly on the length of the vector y . However, the computation is much less sensitive to the values in the vector t . This comparisons give excellent agreement with the exact solution obtained.

4 Conclusion

In this article, exact solutions are obtained to investigate the flow formation and temperature distributions of combined buoyancy and electrokinetically driven flow in a vertical microchannel with asymmetric heat flux. The governing momentum, energy and EDL equations describing flow formation, temperature distributions and electric potential respectively are obtained and solved exactly. The obtained solutions are found to be a function of different physical parameters. In the course of graphical simulation of results, fluid velocity, temperature distribution and mass flow rate increase with increase in Gr. Also, the maximum velocity profile, electric field strength, temperature distribution, skin friction and mass flow rate are obtained when the walls are subjected to symmetric heat flux. In addition, Skin-friction and

mass flow rate would have been overestimated by approximately 37% and 30% respectively should the slip-conditions at the walls had been neglected. The solutions obtained in this current article deserve great attention not just because of their engineering applications (which include enhancement of micro pumps and MEMS cooling), but can also serve as an improvement to previously published results.

Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Appendix

$$\begin{aligned} X_1 &= -\alpha\gamma, & X_2 &= 1 + \beta_v \text{Kn}\gamma, & XXX &= 1 - \alpha\gamma, \\ X_4 &= \frac{E_x \kappa^3 \Gamma}{(\kappa^4 + 4\gamma^4)} [(C_1 - C_2)\alpha\kappa - (C_1 + C_2)] & , \\ X_5 &= e^{\gamma/2} [\cos(\gamma/2) + \alpha\gamma(\cos(\gamma/2) - \sin(\gamma/2))] & , \\ &= e^{\gamma/2} [\sin(\gamma/2) + \alpha\gamma(\cos(\gamma/2) + \sin(\gamma/2))] & , \\ X_7 &= e^{-\gamma/2} [\cos(\gamma/2) - \alpha\gamma(\cos(\gamma/2) + \sin(\gamma/2))] & , \\ X_8 &= e^{-\gamma/2} [\sin(\gamma/2) + \alpha\gamma(\cos(\gamma/2) - \sin(\gamma/2))] & , \\ X_9 &= -\frac{\kappa^4 \Gamma}{(\kappa^4 + 4\gamma^4)} [(C_1 e^{\kappa/2} - C_2 e^{-\kappa/2})\alpha\kappa + C_2(C_1 e^{\kappa/2} + C_2 e^{-\kappa/2})] & , \\ X_{10} &= \frac{\kappa^2 \Gamma}{\text{Gr}} - \frac{\kappa^6 \Gamma}{\text{Gr}(\kappa^4 + 4\gamma^4)}, & X_{12} &= \frac{[C_2 - C_1]\kappa X_{10} \text{Gr}}{2\gamma^3}, \\ X_{13} &= \frac{2\gamma^3}{\text{Gr}} [e^{\gamma/2}(\cos(\gamma/2) + \sin(\gamma/2))] & , \\ X_{14} &= \frac{2\gamma^3}{\text{Gr}} [e^{\gamma/2}(\sin(\gamma/2) - \cos(\gamma/2))] & , \\ X_{15} &= \frac{2\gamma^3}{\text{Gr}} [e^{-\gamma/2}(\sin(\gamma/2) - \cos(\gamma/2))] & , \\ X_{16} &= \frac{2\gamma^3}{\text{Gr}} [e^{\gamma/2}(\sin(\gamma/2) - \cos(\gamma/2))]' & , \\ X_{17} &= \kappa X_{10} (C_2 e^{-\gamma/2} - C_1 e^{\gamma/2}) & , \\ X_{18} &= \frac{2G_3 \kappa^4 \Gamma}{(\kappa^4 + 4\gamma^4)} \left[\frac{C_1^2 (e^\kappa - 1)}{2\kappa} + C_1 C_2 + \frac{C_2^2 (1 - e^{-\kappa})}{2\kappa} \right] & , \end{aligned}$$

$$\begin{aligned} X_{19} &= 2G_3 \left\{ C_1 \frac{[e^{\frac{\gamma+\kappa}{2}} ((\gamma+\kappa) \cos(\gamma/2) + \gamma \sin(\gamma/2)) - (\gamma+\kappa)]}{(\gamma+\kappa)^2 + \gamma^2} + C_2 \frac{[e^{\frac{\gamma-\kappa}{2}} ((\gamma-\kappa) \cos(\gamma/2) + \gamma \sin(\gamma/2)) - (\gamma-\kappa)]}{(\gamma+\kappa)^2 + \gamma^2} \right\} & , \\ X_{20} &= 2G_3 \left\{ C_1 \frac{[e^{\frac{\gamma+\kappa}{2}} ((\gamma+\kappa) \sin(\gamma/2) + \gamma \sin(\gamma/2)) + \gamma]}{(\gamma+\kappa)^2 + \gamma^2} + C_2 \frac{[\gamma - e^{\frac{\gamma-\kappa}{2}} (\gamma \cos(\gamma/2) - (\gamma-\kappa) \sin(\gamma/2))]}{(\gamma-\kappa)^2 + \gamma^2} \right\} & , \\ X_{21} &= 2G_3 \left\{ C_1 \frac{[e^{\frac{\gamma+\kappa}{2}} (\gamma \sin(\gamma/2) - (\gamma-\kappa) \cos(\gamma/2)) + (\gamma-\kappa)]}{(\gamma-\kappa)^2 + \gamma^2} + C_2 \frac{[(\gamma+\kappa) - e^{-(\frac{\gamma+\kappa}{2})} ((\gamma+\kappa) \cos(\gamma/2) - \gamma \sin(\gamma/2))]}{(\gamma+\kappa)^2 + \gamma^2} \right\} & , \\ X_{22} &= 2G_3 \left\{ C_1 \frac{[\gamma - e^{\frac{\kappa-\gamma}{2}} (\gamma \cos(\gamma/2) + (\gamma-\kappa) \sin(\gamma/2))]}{(\gamma-\kappa)^2 + \gamma^2} + C_2 \frac{[\gamma - e^{-(\frac{\gamma+\kappa}{2})} ((\gamma+\kappa) \sin(\gamma/2) + \gamma \cos(\gamma/2))]}{(\gamma+\kappa)^2 + \gamma^2} \right\} & , \end{aligned}$$

$$\Delta = \det \begin{pmatrix} XX_1 & X_1 & X_2 & X_3 \\ X_5 & X_6 & X_7 & X_8 \\ 1 & -1 & 1 & -1 \\ X_{13} & X_{14} & X_{15} & X_{16} \end{pmatrix},$$

$$\Delta_1 = \det \begin{pmatrix} X_4 E_x & X_1 & X_2 & X_3 \\ X_9 E_x & X_6 & X_7 & X_8 \\ E_x X_{12} + XX & -1 & 1 & -1 \\ E_x X_{17} + 1 & X_{14} & X_{15} & X_{16} \end{pmatrix},$$

$$\Delta_2 = \det \begin{pmatrix} XX_1 & X_4 E_x & X_2 & X_3 \\ X_5 & X_9 E_x & X_7 & X_8 \\ 1 & E_x X_{12} + XX & 1 & -1 \\ X_{13} & E_x X_{17} + 1 & X_{15} & X_{16} \end{pmatrix},$$

$$\Delta_3 = \det \begin{pmatrix} XX_1 & X_1 & X_4 E_x & X_3 \\ X_5 & X_6 & X_9 E_x & X_8 \\ 1 & -1 & E_x X_{12} + XX & -1 \\ X_{13} & X_{14} & E_x X_{17} + 1 & X_{16} \end{pmatrix},$$

$$\Delta_4 = \det \begin{pmatrix} XX_1 & X_1 & X_2 & X_4 E_x \\ X_5 & X_6 & X_7 & X_9 E_x \\ 1 & -1 & 1 & E_x X_{12} + XX \\ X_{13} & X_{14} & X_{15} & E_x X_{17} + 1 \end{pmatrix}, \quad \Delta_6 = \frac{\text{Coefficient of } E_x \text{ in } \Delta_1}{\Delta},$$

$$\Delta_5 = \frac{\text{non - coefficient of } E_x \text{ in } \Delta_1}{\Delta}, \quad \Delta_8 = \frac{\text{Coefficient of } E_x \text{ in } \Delta_2}{\Delta},$$

$$\Delta_7 = \frac{\text{non - coefficient of } E_x \text{ in } \Delta_2}{\Delta}, \quad \Delta_{10} = \frac{\text{Coefficient of } E_x \text{ in } \Delta_3}{\Delta},$$

$$\Delta_9 = \frac{\text{non - coefficient of } E_x \text{ in } \Delta_3}{\Delta}, \quad \Delta_{12} = \frac{\text{Coefficient of } E_x \text{ in } \Delta_4}{\Delta},$$

$$\Delta_{11} = \frac{\text{non - coefficient of } E_x \text{ in } \Delta_4}{\Delta}.$$

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