Applications of the local gradient elasticity to the description of the size effect of shear modulus

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Abstract

The classical theory of elasticity does not describe the experimentally observed phenomenon of change in the mechanical characteristics of the material in micro/nano-scale structures. In the present paper, it is shown that a local gradient theory of elasticity describes the size effect of the shear modulus in thin strips. The mentioned theory is based on taking account of non-convective and non-diffusive mass flux associated with the changes in the material microstructure. In the paper, the governing differential equations of the local gradient mechanics and related boundary conditions are obtained by implementing a variational principle. Based on these equations, the stationary stress–strain state of an infinite isotropic strip whose surfaces are subjected to shear displacements is studied. It is shown that in thin strips, the displacement profile deviates significantly from the linear law predicted by the classical theory. Using the solution to this boundary value problem, a formula that describes the size effect of shear modulus is proposed. Such a study is important for modeling micro- and nano-mechanical structures that are widely used in modern technologies.

Keywords Local gradient elasticity · Shear deformation · Thin strip · Shear modulus · Size effect

1 Introduction

The experimental studies of the last few decades show that the mechanical behavior of ultra-thin structures cannot be described in terms of deformation mechanisms of bulk solids. The size dependency behavior of elastic materials in ultra-thin structures was experimentally demonstrated and discussed by many authors (see, for example, [1–5]). In order to describe the specific behavior of thin films, fibers, bars, and strips, the modified (non-classical) theories of elastic continua were developed [6–11]. These theories can predict the response of small-size structures taking into account the effect of microstructure on the material behavior.

Theoretical studies of the modified theory of elasticity were started by E. and F. Cosserat in 1909 [12]. Their ideas

were being developed since the 1950s [13, 14]. Since the 1960s, the foundations of the micropolar elasticity theory [15, 16], the theory of micromorphic materials [15–17], and the theory of multipolar continua [18] have been formulated. Mathematical models of micromorphic, micropolar and multipolar media are known as higher-order continuum theories of elasticity.

Modified theories of elasticity, which include contributions from strain gradients, were developed by Toupin [19], Mindlin and Tiersten [20], Mindlin [21] et al. A continuum theory of elastic materials with long-range cohesive forces was formulated by Kröner [22]. The gradient elasticity model that introduces into constitutive relations the Laplacian of the stress term in addition to the Laplacian of the strain term was developed by Aifantis [23].

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Edelen and Laws [24], Eringer [25], and Eringer and Edelen [26] proposed nonlocal elasticity theory characterized by the functional-type constitutive relations. This theory assumes that the stress at a point is a function of strains at all points in the elastic continuum while the classical theory of elasticity considers that the stress at a point is a function of strains at that point only. In scientific literature, the nonlocal elasticity theories are qualified as 'integral' or 'strongly nonlocal' theories whereas the theories that express the stress as a function of the strain and its gradients are called 'gradient' or 'weakly nonlocal' theories. Gradient-type and integral-type mathematical models of elastic media are sometimes referred to as 'higher-grade' continuum theories of elasticity.

The higher-grade theories made it possible to avoid the stress singularities that appear in the classical elasticity in crack-tip problems and dislocation cores [27]. Non-classical theory of elastic media found various applications in the mechanics of nano-sized solids (see, for example, [28–38]). For instance, the gradient-type stress–strain constitutive relations were used to study the dependence of an effective shear modulus of nanocrystalline materials on the grain size [29]. Size dependence of elastic moduli was analyzed in [5, 29–32, 34, 35, 38]. In [28, 33, 36], the nonlocal elasticity constitutive equations were used to develop modified Euler–Bernoulli, Timoshenko, Reddy, and Levinson elastic beam theories. Various nonlocal beam, plate, and shell theories were analyzed in review papers [8, 9, 37]. The papers [9, 39] present a comprehensive review of the modified couple stress theories and strain gradient models of the mentioned small-scale structures. The modified elastic models for bending, buckling, vibrations, and wave propagation in nanostructures were discussed in [8, 10, 39, 40]. A review on the application of extended continuum theories in modelling and simulation of nanostructures are presented in [7, 35, 41]. The strain and stress gradient models as well as their links with the Eringen nonlocal law were discussed by Aifantis [23, 42, 43].

The other way to consider the surface and size effects for small-scale structures is the local gradient theory of thermoelastic solids proposed by Burak [44] and developed by Kondrat and Hrytsyna [45, 46] and Hrytsyna [47] for elastic and electroelastic media. The principal distinction of the local gradient theory of elasticity from the classical theory and from the above-mentioned non-classical theories of elastic continua consists in taking account of the process of local mass displacement. This process was associated with a mass flux of non-diffusive and non-convective nature. The above mass flux is taken into account in order to describe the effect of material microstructure changes on the mechanical behavior of elastic solids at the nanoscale.

The local gradient mathematical models were successfully used to describe size effects and near-surface inhomogeneity of mechanical fields [46, 48], high-frequency dispersion of longitudinal elastic waves [45], etc. In paper [49] it was proved that the extended theory of local gradient elasticity is capable of predicting SH surface waves in an isotropic and homogeneous half-space. Note that the classical elasticity is not capable of predicting this type of surface waves in a homogeneous and isotropic half-space despite the laboratory confirmation of its existence [50].

The objective of this study is to prove that the extended theory of the local gradient elasticity is capable of predicting the size dependence of mechanical characteristics of a material in small-scale structures. In this study, linear governing equations of the local gradient mechanics and the associated boundary conditions are derived by variational principle.

The paper content is as follows. The theoretical background for the local gradient continuum mechanics is presented in Sect. 2. The constitutive relations and governing equations of the local gradient elasticity are outlined within this section. In Sect. 3, linear constitutive equations of the local gradient elasticity and the Aifantis gradient theory of elasticity [23] are compared. In Sect. 4, the variational principle is used to derive the governing equations and the corresponding boundary conditions of the linear local gradient mechanics. In Sect. 5, the shear of an infinite elastic strip is considered within the local gradient theory. The numerical analysis of the obtained solution is carried out in Sect. 6. In this section, an effective shear modulus is computed directly from the analytical solution to the formulated boundary-value problem. Conclusions are drawn in the final Sect. 7.

2 Theoretical background for the local gradient theory of elasticity

We consider an elastic body that is under the effect of external forces inducing mechanical processes in it. To describe a more complex material behavior, we use the relations of the local gradient theory of elasticity that takes into account the changes in the material microstructure alongside the mechanical deformation. This theory relates the above microstructure changes to the process of the local mass displacement [46, 47].

According to the extended linear theory of elastic media with local mass displacement, the density of internal energy u is the function of strain tensor $\hat{\mathbf{e}} = \{\mathbf{e}_{ij}\}$, the density of an induced mass $\hat{\rho}_m = \{\rho_{ij}^m\}$, and the specific third-order tensor of the local mass displacement $\hat{\pi}_m^{(3)} = \{\pi_{ijk}\}$ [47]. Note that here and in what follows, italic

symbols are associated with scalar quantities, bold-face letters stand for vectors, and bold-face symbols with caps denote tensors. Within this theory, the modified Gibbs relation looks as follows [47]:

$$du = \frac{1}{\rho_0} \sigma_{ij} de_{ij} + \mu'_{ij} d\rho_{ij}^m - \mu'_{ij,k} d\pi_{kij}. \quad (1)$$

Here, $\hat{\sigma} = \{\sigma_{ij}\}$ represents the stress tensor, ρ_0 is the mass density, $\hat{\mu}'_\pi = \{\mu'_{ij}\}$ is the modified chemical potential, $\hat{\mu}'_\pi = \hat{\mu}_\pi - \hat{\mu}$, where $\hat{\mu} = \{\mu_{ij}\}$ is the second-order tensor of chemical potential originally introduced by Pidstryhach [51], and $\hat{\mu}_\pi = \{\mu_{ij}^\pi\}$ is the second-order tensor whose components characterize the effect of material microstructure changes (the local mass displacement) on the internal energy [46, 47]. Note that summation is implied over the repeated indices, while the index after a comma denotes differentiation with respect to the corresponding spatial variables.

We recall that in terms of the internal energy function u , the linear theory of local gradient elasticity requires the following constitutive equations:

$$\sigma_{ij} = \rho_0 \frac{\partial u}{\partial e_{ij}}, \mu'_{ij} = \frac{\partial u}{\partial \rho_{ij}^m}, \mu'_{ij,k} = -\frac{\partial u}{\partial \pi_{kij}}. \quad (2)$$

It should be remarked that within the local gradient elasticity, additional pairs of conjugate variables, namely (μ'_{ij}, ρ_{ij}^m) and $(\mu'_{ij,k}, \pi_{kij})$, appear in the space of constitutive parameters alongside the classical parameters (σ_{ij}, e_{ij}) . The above additional parameters are related to the local mass displacement.

Through the Legendre transformations $f = u + \mu'_{ij,k} \pi_{kij}$ and $H = u + \rho_{ij}^m \mu'_{ij} + \mu'_{ij,k} \pi_{kij}$, the new energy density functions $f = f(e_{ij}, \rho_{ij}^m, \mu'_{ij,k})$ and $H = H(e_{ij}, \mu'_{ij}, \mu'_{ij,k})$ can be introduced. By choosing the energy density function f as the basic potential and using Eq. (1), the generalized Gibbs relation and appropriate constitutive equations read

$$df = \frac{1}{\rho_0} \sigma_{ij} de_{ij} + \mu'_{ij} d\rho_{ij}^m + \pi_{kij} d\mu'_{ij,k}, \quad (3)$$

$$\sigma_{ij} = \rho_0 \frac{\partial f}{\partial e_{ij}}, \mu'_{ij} = \frac{\partial f}{\partial \rho_{ij}^m}, \pi_{kij} = \frac{\partial f}{\partial \mu'_{ij,k}}. \quad (4)$$

If we choose the energy density function H as the basic function, instead of the set of Eqs. (3) and (4), we have the following relations:

$$dH = \frac{1}{\rho_0} \sigma_{ij} de_{ij} - \rho_{ij}^m d\mu'_{ij} + \pi_{kij} d\mu'_{ij,k}, \quad (5)$$

$$\sigma_{ij} = \rho_0 \frac{\partial H}{\partial e_{ij}}, \rho_{ij}^m = -\frac{\partial H}{\partial \mu'_{ij}}, \pi_{kij} = \frac{\partial H}{\partial \mu'_{ij,k}}. \quad (6)$$

For stationary approximation, the mechanical behavior of a linear local gradient elastic continuum is describes by system of coupled equations that includes:

- field equations (the equilibrium equation and the induced mass balance equation)

$$\sigma_{ij,j} + \rho_0 F_i = 0, \quad (7)$$

$$\pi_{kij,k} + \rho_{ij}^m = 0, \quad (8)$$

in which $\mathbf{F} = (F_1, F_2, F_3)$ denotes body force;

- gradient-type constitutive relation (2) [either (4) or (6)];
- strain–displacement relations

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad (9)$$

in which $\mathbf{u} = (u_1, u_2, u_3)$ is the vector of the elastic displacement.

Note that, in addition to the classical equilibrium Eq. (7), the governing set of relations includes a new balance-type Eq. (8) to take the effect of changes of material microstructure (i.e., the local mass displacement) into account.

Constitutive Eqs. (2), (4), (6) of the local gradient elasticity should be written in the explicit form. For isotropic materials, constitutive Eqs. (4) read as [46]:

$$\sigma_{ij} = 2Ge_{ij} - 2G_{pe}\rho_{ij}^m + \left[\left(K - \frac{2}{3}G \right) e - \left(K_{pe} - \frac{2}{3}G_{pe} \right) \rho_m \right] \delta_{ij}, \quad (10)$$

$$\tilde{\mu}'_{ij} = 2G_{pe}\rho_{ij}^m - \frac{2}{\rho_0}G_{pe}e_{ij} + \left[\left(K_p - \frac{2}{3}G_p \right) \rho_m - \frac{1}{\rho_0} \left(K_{pe} - \frac{2}{3}G_{pe} \right) e \right] \delta_{ij}, \quad (11)$$

$$\begin{aligned} \pi_{ijk} = & -\varsigma_1 \tilde{\mu}'_{ll,i} \delta_{jk} - \varsigma_2 \left(\tilde{\mu}'_{ll,j} \delta_{ik} + \tilde{\mu}'_{ll,k} \delta_{ij} \right) - \varsigma_3 \tilde{\mu}'_{ll,i} \delta_{jk} \\ & - \varsigma_4 \left(\tilde{\mu}'_{ll,i} \delta_{ik} + \tilde{\mu}'_{ll,k} \delta_{ij} \right) - \varsigma_5 \tilde{\mu}'_{kj,i} - \varsigma_6 \left(\tilde{\mu}'_{ki,j} + \tilde{\mu}'_{ij,k} \right). \end{aligned} \quad (12)$$

Here, $e = e_{ii}$, $\rho_m = \rho_{ii}^m$, $\tilde{\mu}'_{ij} = \mu'_{ij} - \mu'_{ij}{}^0$, $\mu'_{ij}{}^0$ is the modified chemical potential of an infinite medium, δ_{ij} denotes the Kronecker delta, and $K, G, K_p, G_p, K_{pe}, G_{pe}, \varsigma_i$, $i = 1, 6$, are material constants. Note that the local gradient theory of elasticity involves additional material higher-order constants $K_p, G_p, K_{pe}, G_{pe}, \varsigma_i$, $i = 1, 6$, compared to the bulk constitutive parameters K and G .

Introducing Eqs. (10)–(12) into the balance Eqs. (7) and (8), one can find the following governing differential equations:

$$Gu_{i,II} + \left(K + \frac{G}{3}\right)u_{i,II} - 2G_{pe}\rho_{ij}^m - \left(K_{pe} - \frac{2}{3}G_{pe}\right)\rho_{mj}\delta_{ij} + \rho_0 F_i = 0, \quad (1 - b_1\delta)\sigma_{ij} = 2G(1 - b_3\delta)e_{ij} + \left(K - \frac{2}{3}G\right)(1 - b_2\delta)e\delta_{ij}, \quad (13) \quad (18)$$

$$\begin{aligned} & \rho_{ij}^m - \left[3\zeta_1 K_p + (\zeta_5 + \zeta_3)\left(K_p - \frac{2}{3}G_p\right)\right]\rho_{m,II}\delta_{ij} - 2\left[3\zeta_2 K_p + (\zeta_4 + \zeta_6)\left(K_p - \frac{2}{3}G_p\right)\right]\rho_{m,ij} \\ & - 2G_p\left[(\zeta_4 + \zeta_6)\left(\rho_{ij,II}^m + \rho_{i,IIj}^m\right) + \zeta_3\rho_{pl,Ip}^m\delta_{ij} + \zeta_5\rho_{ij,II}^m\right] + (\zeta_4 + \zeta_5 + \zeta_6)\frac{G_{pe}}{\rho_0}(u_{j,III} + u_{i,III}) \\ & + \frac{1}{\rho_0}\left[3\zeta_1 K_{pe} + 2\zeta_3 G_{pe} + (\zeta_5 + \zeta_3)\left(K_{pe} - \frac{2}{3}G_{pe}\right)\right]u_{i,IIp}\delta_{ij} \\ & + \frac{2}{\rho_0}\left[3\zeta_2 K_{pe} + (\zeta_4 + \zeta_6)\left(K_{pe} + \frac{G_{pe}}{3}\right)\right]u_{i,IIj} = 0. \end{aligned} \quad (14)$$

3 Comparison to the constitutive equations of the Aifantis gradient theory

To remove the singularity of solutions to the problem with cracks and dislocations, Aifantis [23] modified the classical Hooke's law and adopted the constitutive equation of the following forms:

$$(1 - c_1^g\delta)\hat{\sigma} = (1 - c_2^g\delta)\left[2G\hat{e} + \left(K - \frac{2}{3}G\right)e\hat{I}\right], \quad (15)$$

where c_1^g and c_2^g are constant gradient coefficients [23], δ is the Laplacian. Aifantis called this theory the "Laplacian-based gradient elasticity theory" [23, 31]. Note that assuming $c_2^g = 0$ and $c_1^g = (ae_0)^2$, the constitutive relation of the nonlocal elasticity [52] can be obtained

$$[1 - (ae_0)^2\delta]\hat{\sigma} = 2G\hat{e} + \left(K - \frac{2}{3}G\right)e\hat{I}$$

from formula (15). Here, a is an internal characteristic length (for example, lattice parameter), and e_0 is a material constant [52].

Let us show that constitutive Eqs. (10)–(12) of the local gradient elasticity contain Eq. (15) as a partial case. To this end, in formula (12), for simplicity we assumed:

$$\zeta_2 = \zeta_3 = \zeta_4 = \zeta_6 = 0, \quad \zeta_1 = -\frac{\zeta_5}{3K_p}\left(K_p - \frac{2}{3}G_p\right). \quad (16)$$

In this case, the governing Eq. (14) is simplified and can be written as follows:

$$\delta\rho_{ij}^m - \frac{1}{2G_p\zeta_5}\rho_{ij}^m = \frac{G_{pe}}{\rho_0 G_p}\delta e_{ij} + \frac{1}{3\rho_0}\left(\frac{K_{pe}}{K_p} - \frac{G_{pe}}{G_p}\right)\delta e\delta_{ij}. \quad (17)$$

Applying an operator $\mathbf{L} = \delta - 1 \setminus (2G_p\zeta_5)$ to Eq. (10) and using Eq. (17), we can eliminate functions ρ_{ij}^m from constitutive equation. As a result, after some algebra, we obtain the following constitutive relation to the stress components:

where $b_1 = 2G_p\zeta_5$, $b_2 = 2G_p\zeta_5(1 - \bar{\gamma})$, $b_3 = 2G_p\zeta_5(1 - \gamma)$, $\bar{\gamma} = \rho_0^{-1}\left(K - \frac{2}{3}G\right)^{-1}\left(\frac{K_{pe}^2}{K_p} - \frac{2}{3}\frac{G_{pe}^2}{G_p}\right)$, $\gamma = G_{pe}^2 \setminus (\rho_0 G G_p)$.

In a general case: $b_2 \neq b_3$. Thus, according to Eq. (18), the effect of the local mass displacement on stresses will be quantitatively different for 'compression-tension' deformations and for 'shear' deformations. This means that formula (18) is more general than the constitutive Eq. (15) predicted by the Aifantis gradient theory. Equations (18) and (15) will have the same structure if $b_2 = b_3$, that is, $GG_p K_{pe}^2 = KK_p G_{pe}^2$. Note that assuming $b_2 = b_3 = c_2^g$ and $b_1 = c_1^g$, the constitutive Eq. (18) can be reduced to the Aifantis constitutive Eq. (15).

4 Variational approach

In [46, 47], the continuum-thermodynamic approach is utilized to obtain the fundamental set of equations of the local gradient elasticity. In this section, the variational principle is used to derive the governing equations and the corresponding boundary conditions of the linear local gradient elasticity. The present study starts with the form of the strain energy density H , postulated by formula (5).

Assume that energy density potential H exists for the elastic continuum introduced in Sect. 2. To take the changes in the material microstructure into account, we assume the potential H to be a C^2 -continuous function of the strain tensor e_{ij} , modified chemical potential $\tilde{\mu}_{ij}^{\prime\pi}$, and space gradient of chemical potential $\tilde{\mu}_{ij,k}^{\prime\pi}$, that is, $H = H(e_{ij}, \tilde{\mu}_{ij}^{\prime\pi}, \tilde{\mu}_{ij,k}^{\prime\pi})$. Let us also assume that constitutive relations are defined by formulae (6). For elastic solids that occupy the region (V) bounded by a smooth surface (Σ) , the variational principle can be expressed by the following formula:

$$\delta \left(\int_{(V)} H(e_{ij}, \tilde{\mu}_{ij}^{\prime\pi}, \tilde{\mu}_{ij,k}^{\prime\pi}) dV - W \right) = 0, \quad (19)$$

where W represents the work done by external forces:

$$W = \int_{(V)} F_i u_i dV + \int_{(\Sigma)} \left(\frac{1}{\rho_0} \sigma_i^* u_i + n_k \pi_{kij}^* \tilde{\mu}_{ij}^{\prime\pi} \right) d\Sigma. \quad (20)$$

Here n_i are the components of unit outward normal vector to the body surface (Σ) , σ_i^* and π_{kij}^* are prescribed functions on (Σ) . Note that the second term in the surface integral is associated with the surface energy of deformation [46].

Let us show that variational expression (19) generates the governing equations and natural boundary conditions of the local gradient elasticity. The first variation of the work done by external forces, W , is defined as:

$$\delta W = \int_{(V)} F_i \delta u_i dV + \int_{(\Sigma)} \left(\frac{1}{\rho_0} \sigma_i^* \delta u_i + n_k \pi_{kij}^* \delta \tilde{\mu}_{ij}^{\prime\pi} \right) d\Sigma. \quad (21)$$

In view of Eqs. (5), and (6), we can write:

$$\begin{aligned} \delta \int_{(V)} H(e_{ij}, \tilde{\mu}_{ij}^{\prime\pi}, \tilde{\mu}_{ij,k}^{\prime\pi}) dV &= \int_{(V)} \left(\frac{\partial H}{\partial e_{ij}} \delta e_{ij} + \frac{\partial H}{\partial \tilde{\mu}_{ij}^{\prime\pi}} \delta \tilde{\mu}_{ij}^{\prime\pi} + \frac{\partial H}{\partial \tilde{\mu}_{ij,k}^{\prime\pi}} \delta \tilde{\mu}_{ij,k}^{\prime\pi} \right) dV \\ &= \int_{(V)} \left(\frac{1}{\rho_0} \sigma_{ij} \delta e_{ij} - \rho_{ij}^m \delta \tilde{\mu}_{ij}^{\prime\pi} + \pi_{kij} \delta \tilde{\mu}_{ij,k}^{\prime\pi} \right) dV. \end{aligned}$$

Since δ is a symmetric tensor and the operation of variation commutes with the operation of differentiation, the variation of the potential H can be expressed as follows:

$$\delta \int_{(V)} H dV = \int_{(V)} \left[\frac{1}{\rho_0} (\sigma_{ij} \delta u_j)_{,i} + (\pi_{kij} \delta \tilde{\mu}_{ij}^{\prime\pi})_{,k} - \frac{1}{\rho_0} \sigma_{ij,i} \delta u_j - (\rho_{ij}^m + \pi_{kij,k}) \delta \tilde{\mu}_{ij}^{\prime\pi} \right] dV. \quad (22)$$

Note that in relation (22) the formulae

$$\sigma_{ij} \delta u_{ij} = (\sigma_{ij} \delta u_i)_j - \sigma_{ij,j} \delta u_i, \quad (23)$$

$$\pi_{kij} \delta \tilde{\mu}_{ij,k}^{\prime\pi} = (\pi_{kij} \delta \tilde{\mu}_{ij}^{\prime\pi})_{,k} - \pi_{kij,k} \delta \tilde{\mu}_{ij}^{\prime\pi} \quad (24)$$

are taking into account.

Inserting (21) and (22) into (19), applying the divergence theorem and rearranging the terms in the surface and volume integrals, one finally arrives at the variational equation of the form:

Fig. 1 Shear of an infinite strip

$$\begin{aligned} &\int_{(V)} \left[\frac{1}{\rho_0} (\sigma_{ij,j} + \rho_0 F_i) \delta u_i + (\rho_{ij}^m + \pi_{kij,k}) \delta \tilde{\mu}_{ij}^{\prime\pi} \right] dV \\ &- \int_{(\Sigma)} \left(\frac{1}{\rho_0} (\sigma_i^n - \sigma_i^*) \delta u_i + n_k (\pi_{kij} - \pi_{kij}^*) \delta \tilde{\mu}_{ij}^{\prime\pi} \right) d\Sigma = 0. \end{aligned} \quad (25)$$

Here, $\sigma_i^n = \sigma_{ik} n_k$ ($i = 1, 2, 3$) are the components of the stress vector.

Since Eq. (25) holds for any variations of the admissible state, one obtains the equilibrium Eq. (7), the induced mass balance Eq. (8) and the associated boundary conditions that can be expressed as follows:

$$\text{traction boundary condition } (\sigma_i^n - \sigma_i^*) \delta u_i = 0, \quad \forall \mathbf{r} \in (\Sigma), \quad (26)$$

local mass displacement boundary condition n_k

$$(\pi_{kij} - \pi_{kij}^*) \delta \tilde{\mu}_{ij}^{\prime\pi} = 0, \quad \forall \mathbf{r} \in (\Sigma). \quad (27)$$

To illustrate the effectiveness of the local gradient elasticity, a simple boundary-value problem of shear of an infinite elastic layer is discussed in the next section.

5 Problem formulation

The shear of an infinite elastic layer (strip) $|x| \leq l$ ($-\infty < y < +\infty$, $-\infty < z < +\infty$) of thickness $2l$ is considered as described in Fig. 1. The strip surfaces $x = -l$ and $x = l$ are subjected to the shear displacements $(0, -u_0, 0)$ and $(0, u_0, 0)$. Let us assume that the body is in contact with a vacuum. For the sake of simplicity, suppose that the effect of the surface energy of deformation on the internal energy of the body can be neglected. In this case, we can write boundary conditions as follows:

$$u_y|_{x=-l} = -u_0, \quad u_y|_{x=l} = u_0, \quad \tilde{\mu}_{xy}'|_{x=\pm l} = 0. \quad (28)$$

Assume also that the displacement field is in the form $\mathbf{u} = (0, u_y(x), 0)$. For this case, there is only one nonzero strain component, e_{xy} , that is:

$$e_{xy} = u_{y,x}/2. \quad (29)$$

For the one-dimensional problem, Eqs. (7) and (8) can be written as:

$$\sigma_{xy,x} = 0, \quad \rho_{xy}^m = -\pi_{xxy,x}. \quad (30)$$

Based on constitutive Eqs. (10)–(12), the following relations are obtained for the stress σ_{xy} , modified chemical potential $\tilde{\mu}_{xy}'$, and local mass displacement π_{xxy} :

$$\sigma_{xy}(x) = Gu_{y,x} - 2G_{pe}\rho_{xy}^m = G(1 - \gamma)u_{y,x} - \frac{G_{pe}}{G_p}\tilde{\mu}_{xy}', \quad (31)$$

$$\tilde{\mu}_{xy}'(x) = 2G_p\rho_{xy}^m - \frac{G_{pe}}{\rho_0}u_{y,x}, \quad (32)$$

$$\pi_{xxy}(x) = \pi_{xyx}(x) = -(\zeta_4 + \zeta_5 + \zeta_6)\tilde{\mu}_{xy,x}', \quad (33)$$

where $\gamma = G_{pe}^2/(\rho_0 GG_p)$ is a nondimensional coupling factor between the mechanical fields and local mass displacement. Note that from the positive-definiteness of the energy density function f follows that coefficient G_p is positive defined. Thus, the coefficient γ should be positive defined as well.

The following equations governing the static behavior of the strip are found from Eq. (30) by utilizing constitutive relations (31)–(33):

$$G(1 - \gamma)u_{y,xx} - \frac{G_{pe}}{G_p}\tilde{\mu}_{xy,x}' = 0, \quad (34)$$

$$\tilde{\mu}_{xy,xx}' - \Lambda^2\tilde{\mu}_{xy}' = \Lambda^2\frac{G_{pe}}{\rho_0}u_{y,x} \quad (35)$$

in order to find functions u_y and $\tilde{\mu}_{xy}'$. Here, $\Lambda^2 = 1/[2G_p(\zeta_4 + \zeta_5 + \zeta_6)]$, and $l_* = 1/\Lambda$ is the material length scale parameter that can be identified from molecular dynamics approaches, lattice dynamics, experimental tests or from other techniques. It is worth to note that such a parameter is absent in the classical theory and it appears only within the higher-grade theories of elasticity.

6 Problem solution and its analysis

Combining Eqs. (34) and (35), the following displacement-equation of equilibrium is derived

$$(u_{y,xx} - \tilde{\Lambda}^2 u_y)_{,xx} = 0, \quad (36)$$

where $\tilde{\Lambda}^2 = \Lambda^2/(1 - \gamma)$. Within the local gradient theory of elasticity, the mechanical behavior of the strip is governed by the fourth-order differential Eq. (36). The solution to Eq. (36) that satisfied the boundary conditions (28) is as follows:

$$u_y(x) = u_0 \left\{ \frac{x}{l} + \frac{\gamma [\text{sh}(\tilde{\Lambda}x) - \text{sh}(\tilde{\Lambda}l)x/l]}{\tilde{\Lambda}l(1 - \gamma)\text{ch}(\tilde{\Lambda}l) + \gamma \text{sh}(\tilde{\Lambda}l)} \right\}. \quad (37)$$

For the non-vanishing stress component σ_{xy} , the following expression is obtained:

$$\sigma_{xy} = G \frac{u_0}{l} \left[1 - \frac{\gamma}{\Lambda l \sqrt{1 - \gamma} \text{cth}(\Lambda l / \sqrt{1 - \gamma}) + \gamma} \right]. \quad (38)$$

By assuming that the coupling factor γ is equal to zero, $\gamma = 0$, non-classical solution (37) and (38) reduce to the classical one. Note that according to the classical theory of elasticity, the displacement u_y is a linear function of space coordinate x , that is, $u_y = u_0 x/l$. This displacement causes uniform shear deformation, $e_{xy} = u_0/2l$, and uniform shear stress, $\sigma_{xy} = G^{bulk}u_0/l$. Within classical theory, the shear stress is proportional to the shear modulus G^{bulk} , which does not depend on the strip thickness.

The profiles of displacement component u_y for different thicknesses of strip (i.e., $l = 3l_*$, $l = 8l_*$ and $l = 40l_*$) were plotted in Fig. 2. The dotted line in Fig. 2 corresponds to the classical solution. This figure shows that in thick strips ($l = 40l_*$), the displacement profile is close to a linear

Fig. 2 Profiles of the normalized displacement u_y/u_0 for material parameter $\gamma=0.8$ and different strip thicknesses

displacement profile predicted by the classical theory of an elastic continuum. In thick strips, the deviation of displacements from the linear law is only in the vicinity of the strip surfaces $x = \pm l$. In thin strips, the displacement profile deviates significantly from the linear law. Note that a similar result was previously obtained by Cordero et al. [34] who used the strain gradient theory to study the stress–strain behavior of an infinite isotropic strip subjected to a shear deformation.

Let us use Eqs. (29), (37), and (38) to define the effective stresses. To this end, we calculated the average strain $\bar{e}_{xy}$ and stress $\bar{\sigma}_{xy}$ as follows:

$$\bar{e}_{xy} = \frac{1}{2l} \int_{-l}^l e_{xy} dx, \quad \bar{\sigma}_{xy} = \frac{1}{2l} \int_{-l}^l \sigma_{xy} dx.$$

Within the local gradient theory of elasticity, the average strain and average stress component are:

$$\bar{e}_{xy} = \frac{u_0}{2l}, \quad \bar{\sigma}_{xy} = G \frac{u_0}{l} \left[1 - \frac{\gamma}{\Lambda l \sqrt{1-\gamma} \text{cth}(\Lambda l / \sqrt{1-\gamma}) + \gamma} \right]. \quad (39)$$

Based on the classical Hooke's law, the effective average shear modulus G^{ef} can be defined by the formula $G^{ef} = \bar{\sigma}_{xy} / 2\bar{e}_{xy}$. In view of this expression and Eq. (39), the following formula for effective shear modulus can be obtained

$$G^{ef}(l) = G \left[1 - \frac{\gamma}{\Lambda l \sqrt{1-\gamma} \text{cth}(\Lambda l / \sqrt{1-\gamma}) + \gamma} \right]. \quad (40)$$

Fig. 3 Effect of the dimensionless strip thickness, l/l_* , on the shear modulus normalized by the classical value, G^{ef}/G , for three different positive γ

This formula presents the effect of the local mass displacement (the changes of material microstructure) on the shear modulus. In thin strips, the shear modulus differs from that predicted by the classical theory of elasticity. By observing Eq. (40), it appears that the size effect of shear modulus only occurs if a coupling exists between the mechanical fields and local mass displacement. If $\gamma=0$, the effect of the layer thickness on the shear modulus will be absent. Since the material constant γ is positive, the effective shear modulus in nano-scale strips is smaller than the shear modulus predicted by the classical theory.

In Fig. 3, the dependence of the effective shear modulus on the dimensionless strip thickness, l/l_* , is shown for different values of material constant $\gamma=0.03, 0.09, 0.2$. The horizontal line in Fig. 3 corresponds to the classical solution, G^{bulk} . It can be observed that the size effect of the shear modulus is manifested only in thin strips, the surfaces $x = \pm l$ of which are so close that they can affect each other. The coupling between the mechanical process of deformation and the local mass displacement is found to be essential for small scale structures. For large values of strip thickness l , the effect of the local mass displacement on elastic modulus is negligible and the bulk shear modulus G^{bulk} is retrieved. Note that the size-dependent behavior in micro/nano-scale structures was also experimentally validated and predicted by other non-classical theories of elastic media including the micromorphic continuum, Cosserat continuum, and strain gradient elasticity [34, 38]. Note also that readers interested in studies of the size effect of the shear modulus in micro/nano-scale porous structures and nanopolycrystals can refer to the works [29, 30].

Numerical results showing the variation of the displacement with the material constant γ for strips are presented in Fig. 4. For thin strips, it is found out that as the material coefficient γ increases, the shear modulus reduces.

Fig. 4 Effect of the parameter γ on the shear modulus, G^{ef} , for a different layer thickness

Figures 3 and 4 show that the material microstructure changes can significantly affect the shear modulus in case of the strip thickness being smaller than $40 l_*$.

It should be pointed out that formulae (37)–(40) can be used to evaluate material constants $l_* = 1/\Lambda$ and γ (i.e., the material length scale parameter and coupling factor between the mechanical fields and local mass displacement) from a series of real (or virtual) shear experiments with a number of small-scale specimens of different thickness.

7 Conclusions

The variational formulation and governing equations with boundary conditions are derived for extended local gradient mechanics. The static response of infinite strip on shear deformation is investigated within this theory. The shear modulus for strips is calculated in the framework of the local gradient theory and is compared with that predicted by the classical theory. The comparison of the obtained result demonstrates the insufficiency of the classical theory at the nano-scale. It is shown that the profile of displacement component deviates from the classical profile for the case when the strip thickness approaches the microscopic material length scale parameter. In thin strips, the shear modulus predicted by the local gradient theory differs from that provided by the classical one. This indicates that the local mass displacement (i.e., material microstructure changes) can significantly affect the elasticity module when the strip thickness is much smaller. Therefore, the local gradient theory of elastic continua can be useful for the prediction of mechanical behavior of elastic micro- and nano-scale structures that are widely used in nanotechnologies. The possibility of the evaluation of new material constants associated with the local gradient elasticity is pointed out.

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Compliance with ethical standards

Conflict of interest The author states that there is no conflict of interest.

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