An accurate solution for stress fields strongly varied across the adhesive thickness in soffit plate-strengthened beams

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Abstract

A closed form solution and a finite difference formulation are developed for an accurate prediction of adhesive shear and normal stresses in soffit plate flexural-strengthened beams. First, three statically admissible stress fields (i.e., longitudinal normal, transverse shear and normal stresses) are derived and expressed in terms of four unknown interfacial shear and normal stresses. Then, a modified complementary strain energy is written in which the energy contributions of the transverse normal stress fields in the beam and in the plate, and the transverse shear stresses in the beam are omitted. Based on an energy variational principle, compatibility equations and corresponding boundary conditions are obtained. A closed form solution and a finite difference formulation are finally developed. Through validations, the adhesive shear and normal stresses near the plate ends predicted by the present study are in excellent agreements with those provided in several experimental and numerical studies. Also, the present study excellently captures stress fields strongly varied across the adhesive thickness, especially the transverse normal stresses. Based on the present theory, a parametric study is conducted to quantify the effects of the plate longitudinal modulus and the thickness on the interfacial shear and normal stresses. The parametric study indicates that the peak normal stresses are significantly higher than the peak shear stresses in the adhesive layer. The present study is applicable to predict the adhesive shear and normal stresses highly concentrated near the plate ends in plate-strengthened beams with general applied loads and force boundary conditions.

Keywords Complementary energy · Variational principle · Finite difference · Layered beams · Interfacial shear and normal stresses

1 Introduction

In soffit plate-strengthened beams, the adhesive shear and normal stresses near the plate ends are significant [1–3]. At such locations, the geometry of materials in a longitudinal-transverse plane is changed and thus stress flows transferred between the plate and the beam are narrowed, that leads to a stress concentration [4–7]. Because the strength of adhesive layer is small compared to those of the beam

and the plate [2], adhesive failure mode occurring at the plate ends was observed in several experimental studies (e.g., [8–12]). There were many analytical solutions developed in order to predict the high stresses concentrated near the plate ends.

The majority of the analytical solutions were based on the postulation of the continuity of displacement and strain fields at material interfaces. Tounsi and Benyoucef [13] assumed that interfacial stresses were linearly

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proportional to interfacial displacement fields through shear and transverse stiffnesses of the adhesive layer in externally FRP-plated concrete beams. Also, they postulated equal curvatures in the beam and the plate. Zhang and Teng [14] and Smith and Teng [15] adopted the same kinematic assumptions as [13] to develop analytical solutions for plated beams. Guenaneche et al. [16] expressed interfacial stresses through a shear slip in the interface of FRP-strengthened concrete beams. Guenaneche et al. [17] extended their previous study [16] to incorporate the effect of shear deformation. Wong [18] developed an solution through displacement compatibilities for sandwich structures. Narayanamurthy et al. [19] developed an analytical solution considering the shear deformation effects in plated beams. Han et al. [20] proposed a theory in which governing displacement fields were assumed as higher order functions for the analysis of laminated composite structures. Liu and Dawood [21] developed a closed form solution for the interfacial stresses of steel beams strengthened with externally bonded plates using ductile adhesives. Liu et al. [22] presented an analytical solution for interfacial relative slip, shear and peeling stress distributions of an FRP-to-concrete bonded joint in single-shear models. Long et al. [23] developed a closed form solution for the interfacial stresses of beam structures bonded with partially covered plates. However, the solution was restricted to simply supported beams under three-point bending. Also, the adhesive layer were modeled as linear tangential and normal springs and thus a stress field were considered as a constant across the adhesive thickness. Sha and Davison [24] also developed an analytical solution for interfacial stresses in concrete beams strengthened with bonded FRP laminates using composite beam theory and Shan and Su [25] developed an uncoupled closed form solution for adhesive stresses in plated beams based on Timoshenko beam theory. However, both studies [24, 25] again assumed that stress fields were invariant across the adhesive thickness. A characteristic observed in the above analytical studies (e.g., [13–25]) is that they postulated equal curvatures in the beam and the plate, that ignored the important role of the adhesive layer. The postulation led to the invariance of stress fields across the adhesive thickness, that was inconsistent with a finding in numerical studies [8, 14]. The numerical studies [8, 14] conducted in a commercial finite element analyses software showed that the interfacial normal stresses were strongly varied across the adhesive thickness. Also, significant underpredictions of the interfacial normal stresses predicted by the analytical studies [13–25], when compared to the stresses in numerical solutions [8, 14], were reported in [8, 14].

There were several analytical solutions developed based on complementary strain energy variational principles.

Yang et al. [26] developed a Fourier series-based solution for a simply supported plated beam under a uniformly distributed load. Shear stresses were however assumed to vanish at the top and bottom fibers of the faces. Since the interfacial stresses may not vanish at the material boundaries, such an assumption violated infinitesimal stress equilibrium conditions. Yang and Ye [27] developed another Fourier series-based solution for the same structure given in [26]. But the latter solution assumed a constant shear stress across the adhesive thickness. Based on a Fourier series assumption, the theory required a large number of Fourier terms for a converged solution. Wu and Jensen [28, 29] and Wu and Zhao [30] developed closed form solutions for the stress analysis of single-lap bonded joints. Wei et al. [31] conducted an finite element simulation for interfacial stresses of composite adhesively bonded joints. Phe et al. [6] developed a finite element formulation based on a higher order beam theory for the analysis of multi-layered beams. However, the solution [6] may be inapplicable to FRP-strengthened beams in which lengths of the beam and the plate are unequal due to the discontinuity of the longitudinal normal stress field in the beam at the adjoining section between the un-strengthened and the strengthened beam segments.

In the context of lacking a general and accurate solution (as accurate as numerical solutions [8, 14]) for the high adhesive shear and normal stresses occurring close to the plate ends of the soffit plate-strengthened beams, the present study is going to fill in the gap by developing an advanced theory that can capture the high stresses. The present theory avoids the kinematic assumption of equal curvatures in the beam and the plate, as postulated in several previous studies (e.g., [13–25]). It is able to capture stress fields strongly varied across the adhesive thickness, as neglected in previous studies [13–25, 27, 28]. And it is applicable to general applied loads and force boundary conditions, as not captured by previous studies [6, 26, 27].

The paper is presented in a sequence. First, a detail description of the problem is provided in Sect. 2. Second, the formulation of the present theory is expressed in Sect. 3. In the formulation, four interfacial shear and peeling stress fields at material interfaces are assumed as unknown functions. Three statically admissible stress components of a plane stress state are derived in terms of the unknown functions based on shear stress flow equilibrium conditions. A modified complementary strain energy is then written. Based on a complementary strain energy variational principle, compatibility equations and corresponding boundary conditions are obtained. A closed form solution and a back-up finite difference formulation are developed for the solution of interfacial shear and normal stresses. Third, the formulation of the present theory is validated against available numerical and experimental

studies in Sect. 4. Finally, a parametric study is conducted to investigate the effect of several parameters on the Peak adhesive stresses in Sect. 5.

2 Description of the problem

A three-layer beam is subjected to general end forces $N_1(0)$, $Q_1(0)$, $M_1(0)$, $N_1(L)$, $Q_1(L)$, $M_1(L)$ and a general transverse pressure function $\sigma_0(z)$ applied to the top surface (Fig. 1). Layer 1 (i.e., the beam) possesses a cross-sectional depth of h_1 and transversely variable width of $b(y_1)$ where y_1 is a transverse coordinate from the centroid of Layer 1. At the top and bottom fibers of Layer 1, conditions $b(h_1/2) = b(-h_1/2) = b_1$ are assumed. The cross-sections of Layers 2 and 3 (i.e., the adhesive layer and the plate) both are rectangular with dimensions of $b \times h_2$ and $b \times h_3$, respectively. All materials are linearly elastic orthotropic. Each layer has a longitudinal normal elastic modulus E_{zi} , a transverse normal elastic modulus E_{yi} , a transverse shear modulus G_i and Poisson's ratios μ_{zi}, μ_{yi} where $i = 1, 2, 3$.

3 Formulation

3.1 Development of statically admissible stress fields

A free-body diagram of an infinitesimal segment dz is presented in Fig. 2 in which $\tau_1(z)$ and $\sigma_1(z)$ are the interfacial shear and normal stresses at the interface of layers 1 and 2, while $\tau_2(z)$ and $\sigma_2(z)$ are the interfacial shear and normal stresses at the interface of layers 2 and 3. Three transverse Y_i -coordinates where $i = 1, 2, 3$ are selected for each material layer in which coordinate origins are coincident with the layer centroids. The axial resultant force, transverse shear force, and transverse bending moment in each layer at z - and $(z + dz)$ - sections are denoted as N_i, Q_i, M_i and $N_i + dN_i, Q_i + dQ_i, M_i + dM_i$ respectively (where $i = 1, 2, 3$). By writing static force equilibrium conditions in the infinitesimal segments (i.e., $\sum F_x = 0$, $\sum F_y = 0$, $\sum M_x = 0$), the incremental resultant forces can be expressed as

Fig. 2 Infinitesimal segments of the 3 separated layers

$$dN_1(z) = b\tau_1(z)dz \quad (1a)$$

$$dQ_1(z) = [b\sigma_0(z) - b\sigma_1(z)]dz; \quad (1b)$$

$$dM_1(z) = [Q_1(z) + (bh_1/2)\tau_1(z)]dz; \quad (1c)$$

$$dN_2 = b[-\tau_1(z) + \tau_2(z)]dz; \quad (1d)$$

$$dQ_2 = b[\sigma_1(z) - \sigma_2(z)]dz; \quad (1e)$$

$$dM_2 = \{Q_2(z) + (bh_2/2)[\tau_1(z) + \tau_2(z)]\}dz; \quad (1f)$$

$$dN_3 = -b\tau_2(z)dz; \quad (1g)$$

$$dQ_3 = b\sigma_2(z)dz; \quad (1h)$$

$$dM_3 = [Q_3(z) + (bh_3/2)\tau_2(z)]dz; \quad (1i)$$

Fig. 1 Beam profile and cross-section

From Eqs. (1a–1i), by taking integrals of z from 0 to z , the expressions of the stress resultants at z - cross section are obtained as

$$N_1(z) = N_1(0) + \int_0^z b\tau_1(z)dz; \quad (2a)$$

$$Q_1(z) = Q_1(0) + b_1 \int_0^z \sigma_0(z)dz - b \int_0^z \sigma_1(z)dz; \quad (2b)$$

$$M_1(z) = M_1(0) + \int_0^z \left[Q_1(0) + b_1 \int_0^z \sigma_0(z)dz \right] dz - b \int_0^z \int_0^z \sigma_1(z)dzdz + \frac{bh_1}{2} \int_0^z \tau_1(z)dz; \quad (2c)$$

$$N_2(z) = N_2(0) - b \int_0^z \tau_1(z)dz + b \int_0^z \tau_2(z)dz; \quad (2d)$$

$$Q_2(z) = Q_2(0) + b \int_0^z \sigma_1(z)dz - b \int_0^z \sigma_2(z)dz; \quad (2e)$$

$$M_2(z) = M_2(0) + \int_0^z Q_2(0)dz + b \int_0^z \int_0^z [\sigma_1(z) - \sigma_2(z)]dzdz + \frac{bh_2}{2} \int_0^z [\tau_1(z) + \tau_2(z)]dz; \quad (2f)$$

$$N_3(z) = N_3(0) - b \int_0^z \tau_2(z)dz; \quad (2g)$$

$$Q_3(z) = Q_3(0) + b \int_0^z \sigma_2(z)dz; \quad (2h)$$

$$M_3(z) = M_3(0) + \int_0^z Q_3(0)dz + b \int_0^z \int_0^z \sigma_2(z)dzdz + \frac{bh_3}{2} \int_0^z \tau_2(z)dz; \quad (2i)$$

The longitudinal normal stress profile across the thickness of each material layer is assumed as a straight line [26, 27], i.e.,

$$\sigma_{zi}(y, z) = \frac{N_i(z)}{A_i} - \frac{M_i(z)}{I_i} y_i \quad (3)$$

where $i = 1, 2, 3$; A_i is the cross-sectional area, and I_i is the sectional moment of inertia about X_i -axis of the i th layer; y_i is the transverse coordinate ranged from $-h_i/2$ to $h_i/2$ (Fig. 1).

From the axial resultant forces and the bending moments in Eq. (2a–2i), by substituting into Eq. (3), the

expressions of the longitudinal normal stresses in each layer in terms of the interfacial shear and normal stresses can be obtained as

$$\sigma_{z1}(y_1, z) = F_{z1}(y_1, z) + \left(-\frac{bh_1 y_1}{2I_1} + \frac{b}{A_1} \right) \int_0^z \tau_1(z)dz + \frac{by_1}{I_1} \int_0^z \int_0^z \sigma_1(z)dzdz \quad (4a)$$

$$\sigma_{z2}(y_2, z) = F_{z2}(y_2, z) - \left(\frac{A_2 y_2}{2I_2} + \frac{b}{A_2} \right) \int_0^z \tau_1(z)dz + \left(\frac{-A_2 y_2}{2I_2} + \frac{b}{A_2} \right) \int_0^z \tau_2(z)dz - \frac{by_2}{I_2} \int_0^z \int_0^z [\sigma_2(z) - \sigma_1(z)]dzdz \quad (4b)$$

$$\sigma_{z3}(y_3, z) = F_{z3}(y_3, z) - \left(\frac{b}{A_3} + \frac{A_3 y_3}{2I_3} \right) \int_0^z \tau_2(z)dz - \frac{by_3}{I_3} \int_0^z \int_0^z \sigma_2(z)dzdz \quad (4c)$$

in which $F_{z1}(y_1, z)$, $F_{z2}(y_2, z)$, $F_{z3}(y_3, z)$ are specified functions and defined in Eq. (9). For the stresses in Eq. (4a–4c) to be statically admissible, following shear flow equilibrium conditions [6] must be satisfied.

$$\frac{\partial}{\partial y_i} [b(y_i) \tau_i(y_i, z)] + \frac{\partial}{\partial z} [b(y_i) \sigma_{zi}(y_i, z)] = b(y_i) p_z(y_i, z) \quad (5a)$$

$$\frac{\partial}{\partial y_i} [b(y_i) \sigma_{yi}(y_i, z)] + \frac{\partial}{\partial z} [b(y_i) \tau_i(y_i, z)] = b(y_i) p_y(y_i, z) \quad (5b)$$

where $b(y_i)$ is the cross-sectional width at y_i - coordinate (with $i = 1, 2, 3$). In the present study, body forces $p_z(y_i, z)$ and $p_y(y_i, z)$ are assumed to vanish. From Eq. (4a), by substituting into Eqs. (5a–5b) and setting $i = 1$, the transverse shear and normal stress fields in the first layer can be obtained. Similarly, from Eq. (4b), by substituting into Eqs. (5a–5b) and setting $i = 2$ and from Eq. (4c), by substituting into Eqs. (5a–5b) and setting $i = 3$, one obtains the expressions for the transverse shear and normal stress fields in the second and third layers. The statically admissible stresses fields are thus derived as

$$\sigma_{zi}(y_i, z) = \mathbf{f}_{zi}(y_i)_{1 \times 4}^T \mathbf{H}(\mathbf{z})_{4 \times 1} + F_{zi}(y_i, z) \quad (6a)$$

$$\tau_i(y_i, z) = \mathbf{f}_{\tau i}(y_i)_{1 \times 4}^T \mathbf{H}'(\mathbf{z})_{4 \times 1} + F_{\tau i}(y_i, z) \quad (6b)$$

$$\sigma_{yi}(y_i, z) = \mathbf{f}_{yi}(y_i)_{1 \times 4}^T \mathbf{H}''(\mathbf{z})_{4 \times 1} + F_{yi}(y_i, z) \quad (6c)$$

where i is the index of a material layer ($i = 1, 2, 3$). $\mathbf{H}(\mathbf{z})$ is a vector of stress integrals and defined as

$$\langle \mathbf{H}(\mathbf{z}) \rangle_{1 \times 4}^T = \langle A(z) \ B(z) \ C(z) \ D(z) \rangle \quad (7)$$

where $A(z) = \int_0^z \tau_1(z) dz$; $B(z) = \int_0^z \int_0^z \sigma_1(z) dz dz$; $C(z) = \int_0^z \tau_2(z) dz$; and $D(z) = \int_0^z \int_0^z \sigma_2(z) dz dz$; and vectors of the transverse coordinates and the sectional properties are defined as

$$\begin{aligned} \mathbf{f}_{z1}(y_1)_{1 \times 4}^T &= b \left\langle \left(-\frac{1}{A_1} + \frac{h_1 y_1}{2I_1} \right) \left(\frac{y_1}{I_1} \right) \ 0 \ 0 \right\rangle; \\ \mathbf{f}_{\tau 1}(y_1)_{1 \times 4}^T &= \frac{b}{b(y_1)} \left\langle \int_{y_1}^{h_1/2} \left(-\frac{1}{A_1} + \frac{h_1 y_1}{2I_1} \right) b(y_1) dy_1 \ \int_{y_1}^{h_1/2} \frac{1}{I_1} y_1 b(y_1) dy_1 \ 0 \ 0 \right\rangle; \\ \mathbf{f}_{y1}(y_1)_{1 \times 4}^T &= \frac{b}{b(y_1)} \left\langle \int_{y_1}^{h_1/2} \int_{y_1}^{h_1/2} \left(-\frac{1}{A_1} + \frac{h_1 y_1}{2I_1} \right) b(y_1) dy_1 dy_1 \ \int_{y_1}^{h_1/2} \int_{y_1}^{h_1/2} \frac{y_1}{I_1} b(y_1) dy_1 dy_1 \ 0 \ 0 \right\rangle; \\ \mathbf{f}_{z2}(y_2)_{1 \times 4}^T &= \left\langle \left(\frac{1}{h_2} + \frac{6y_2}{h_2^2} \right) \left(-\frac{12y_2}{h_2^3} \right) \left(-\frac{1}{h_2} + \frac{6y_2}{h_2^2} \right) \frac{12y_2}{h_2^3} \right\rangle; \\ \mathbf{f}_{\tau 2}(y_2)_{1 \times 4}^T &= \left\langle -1 + \int_{y_2}^{h_2/2} \left(\frac{1}{h_2} + \frac{6y_2}{h_2^2} \right) dy_2 \ -\int_{y_2}^{h_2/2} \frac{12y_2}{h_2^3} dy_2 \ \int_{y_2}^{h_2/2} \left(-\frac{1}{h_2} + \frac{6y_2}{h_2^2} \right) dy_2 \ \int_{y_2}^{h_2/2} \frac{12y_2}{h_2^3} dy_2 \right\rangle; \\ \mathbf{f}_{y2}(y_2)_{1 \times 4}^T &= \left\langle -\int_{y_2}^{h_2/2} dy_2 + \int_{y_2}^{h_2/2} \int_{y_2}^{h_2/2} \left(\frac{1}{h_2} + \frac{6y_2}{h_2^2} \right) dy_2 dy_2 \ 1 - \int_{y_2}^{h_2/2} \int_{y_2}^{h_2/2} \frac{12y_2}{h_2^3} dy_2 dy_2 \ \int_{y_2}^{h_2/2} \int_{y_2}^{h_2/2} \left(-\frac{1}{h_2} + \frac{6y_2}{h_2^2} \right) dy_2 dy_2 \ \int_{y_2}^{h_2/2} \int_{y_2}^{h_2/2} \frac{12y_2}{h_2^3} dy_2 dy_2 \right\rangle; \\ \mathbf{f}_{z3}(y_3)_{1 \times 4}^T &= \left\langle 0 \ 0 \ \left(\frac{1}{h_3} + \frac{6y_3}{h_3^2} \right) \left(-\frac{12y_3}{h_3^3} \right) \right\rangle; \quad \mathbf{f}_{\tau 3}(y_3)_{1 \times 4}^T = \frac{1}{b} \left\langle 0 \ 0 \ -1 + \int_{y_3}^{h_3/2} \left(\frac{1}{h_3} + \frac{6y_3}{h_3^2} \right) dy_3 \ -\int_{y_3}^{h_3/2} \frac{12y_3}{h_3^3} dy_3 \right\rangle; \\ \mathbf{f}_{y3}(y_3)_{1 \times 4}^T &= \left\langle 0 \ 0 \ -\int_{y_3}^{h_3/2} dy_3 + \int_{y_3}^{h_3/2} \int_{y_3}^{h_3/2} \left(\frac{1}{h_3} + \frac{6y_3}{h_3^2} \right) dy_3 dy_3 \ 1 - \int_{y_3}^{h_3/2} \int_{y_3}^{h_3/2} \frac{12y_3}{h_3^3} dy_3 dy_3 \right\rangle \end{aligned} \quad (8)$$

while specified stress functions $F_{zi}(y_i, z)$, $F_{\tau i}(y_i, z)$, and $F_{yi}(y_i, z)$ are defined as:

$$\begin{aligned} F_{z1}(y_1, z) &= N_1(0)/A_1 - \left[M_1(0) + \int_0^z Q_1(0) dz + b_1 \int_0^z \int_0^z \sigma_0(z) dz dz \right] y_1 / l_1; \\ F_{\tau 1}(y_1, z) &= -\frac{1}{b(y_1)} \int_{y_1}^{h_1/2} y_1 b(y_1) \left\{ \left[Q_1(0) + b_1 \int_0^z \bar{\sigma}_0(z) dz \right] / l_1 \right\} dy_1; \\ F_{y1}(y_1, z) &= \frac{1}{b(y_1)} \left[b - \frac{b}{l_1} \int_{y_1}^{h_1/2} \int_{y_1}^{h_1/2} y_1 b(y_1) dy_1 dy_1 \right] \bar{\sigma}_0(z); \\ F_{z2}(y_2, z) &= F_{\tau 2}(y_2, z) = F_{y2}(y_2, z) = F_{y3}(y_3, z) = F_{z3}(y_3, z) = F_{\tau 3}(y_3, z) = 0; \end{aligned} \quad (9)$$

3.2 Compatibility equations and boundary conditions

The strain energy of the system, denoted as U^* , is normally contributed by the longitudinal normal, transverse normal, and transverse shear stress components in 3 layers of the composite beam [6]. However, the present study neglects the contributions of the transverse normal stresses in the beam and the plate (i.e., Layers 1, 3) and the transverse shear in the beam (i.e., Layer 1), because their energy contributions are found to provide a underprediction solution for the interfacial shear and peeling stresses. The expression for the strain energy of the system in the present study thus takes the form

$$\begin{aligned}
U^* = & \frac{1}{2} \sum_{i=1}^3 \int_L \int_{A_i} (\sigma_{zi} \varepsilon_{zi}) dA_i dz \\
& + \frac{1}{2} \int_L \int_{A_2} (\tau_2 \gamma_2 + \sigma_{y2} \varepsilon_{y2}) dA_2 dz \\
& + \frac{1}{2} \int_L \int_{A_3} (\tau_3 \gamma_3) dA_3 dz
\end{aligned} \quad (10)$$

in which the following constitutive material models are assumed:

$$\begin{aligned}
\varepsilon_{z1} &= \sigma_{z1}/E_{z1}; \quad \varepsilon_{z2} = (\sigma_{z2} - \mu_{z2}\sigma_{y2})/E_{z2}; \quad \varepsilon_{z3} = \sigma_{z3}/E_{z3}; \quad \gamma_2 = \tau_2/G_2; \\
\varepsilon_{y2} &= (\sigma_{y2} - \mu_{y2}\sigma_{z2})/E_{y2}; \quad \gamma_3 = \tau_3/G_3;
\end{aligned} \quad (11)$$

From Eqs. (11), by substituting into Eq. (10), the expression for the complementary strain energy is obtained as

$$U^* = \sum_{i=1}^3 \frac{1}{2} \int_L \int_{A_i} \left\{ \frac{1}{E_{zi}} \sigma_{zi}^2 \right\} dA_i dz + \frac{1}{2} \int_L \int_{A_2} \left\{ \frac{1}{G_2} \tau_2^2 + \frac{1}{E_{y2}} \sigma_{y2}^2 - \left(\frac{\mu_{z2}}{E_{z2}} + \frac{\mu_{y2}}{E_{y2}} \right) \sigma_{z2} \sigma_{y2} \right\} dA_2 dz + \frac{1}{2} \int_L \int_{A_3} \frac{1}{G_3} \tau_3^2 dA_3 dz \quad (12)$$

From Eqs. (6a–6c), by substituting into Eq. (12) and noting the conditions in Eq. (9), the expression for the complementary strain energy of the system is obtained as

$$\begin{aligned}
U^* = & \int_L \mathbf{H}(\mathbf{z})_{1 \times 4}^T \left\{ \sum_{i=1}^3 \frac{1}{2E_{zi}} \int_{A_i} [\mathbf{f}_{zi}(y_i)_{4 \times 1} \mathbf{f}_{zi}(y_i)_{1 \times 4}^T] dA_i \right\} \mathbf{H}(\mathbf{z})_{4 \times 1} dz + \int_L \mathbf{H}(\mathbf{z})_{1 \times 4}^T \left\{ \frac{1}{E_{z1}} \int_{A_1} [\mathbf{f}_{z1}(y_1)_{4 \times 1} F_{z1}(y_1, z)] dA_1 \right\} dz \\
& + \frac{1}{2E_{z1}} \int_L \int_{A_1} \{F_{z1}^2(y_1, z)\} dA_1 dz + \int_L \mathbf{H}'(\mathbf{z})_{1 \times 4}^T \left\{ \sum_{i=2}^3 \frac{1}{2G_i} \int_{A_i} [\mathbf{f}_{ti}(y_i)_{4 \times 1} \mathbf{f}_{ti}(y_i)_{1 \times 4}^T] dA_i \right\} \mathbf{H}'(\mathbf{z})_{4 \times 1} dz \\
& + \int_L \mathbf{H}''(\mathbf{z})_{1 \times 4}^T \left\{ \frac{1}{2E_{y2}} \int_{A_2} [\mathbf{f}_{y2}(y_2)_{4 \times 1} \mathbf{f}_{y2}(y_2)_{1 \times 4}^T] dA_2 \right\} \mathbf{H}''(\mathbf{z})_{4 \times 1} dz \\
& - \int_L \mathbf{H}(\mathbf{z})_{1 \times 4}^T \left\{ \left(\frac{\mu_{z2}}{2E_{z2}} + \frac{\mu_{y2}}{2E_{y2}} \right) \int_{A_2} [\mathbf{f}_{z2}(y_2)_{4 \times 1} \mathbf{f}_{y2}(y_2)_{1 \times 4}^T] dA_2 \right\} \mathbf{H}''(\mathbf{z})_{4 \times 1} dz
\end{aligned} \quad (13)$$

Equation (13) is a functional of functions $\mathbf{H}(\mathbf{z})_{4 \times 1}$. Based on a theorem of minimum complementary strain energy [32], the strain energy of the system has a stationary point in a static equilibrium state which corresponds to the

condition that the variation of the complementary strain energy vanish, i.e., $\delta U^* = 0$. From Eq. (13), by evoking the variational operation with respect to vector $\mathbf{H}(\mathbf{z})_{4 \times 1}$ and perform several integrations by parts to turns variations $\delta \mathbf{H}'(\mathbf{z}), \delta \mathbf{H}''(\mathbf{z})$ into $\delta \mathbf{H}(\mathbf{z})$, a set of four compatibility equations of vector $\mathbf{H}(\mathbf{z})_{4 \times 1}$ are obtained as

$$\begin{aligned}
& [\alpha_5]_{4 \times 4} \mathbf{H}''''(\mathbf{z})_{4 \times 1} + [\alpha_{243}]_{4 \times 4} \mathbf{H}''(\mathbf{z})_{4 \times 1} \\
& + [\alpha_1]_{4 \times 4} \mathbf{H}(\mathbf{z})_{4 \times 1} + \{\eta(\mathbf{z})\}_{4 \times 1} = \{\mathbf{0}\}_{4 \times 1}
\end{aligned} \quad (14)$$

and the corresponding 16 boundary conditions at $z = 0, L$ are obtained as

$$\begin{aligned}
& \delta \mathbf{H}(\mathbf{0})_{1 \times 4}^T \{ -[\alpha_5]_{4 \times 4} \mathbf{H}''''(\mathbf{0})_{4 \times 1} + [\alpha_3 - \alpha_4]_{4 \times 4} \mathbf{H}'(\mathbf{0})_{4 \times 1} \} = 0 \\
& \delta \mathbf{H}'(\mathbf{0})_{1 \times 4}^T \{ [\alpha_5]_{4 \times 4} \mathbf{H}''(\mathbf{0})_{4 \times 1} + [\alpha_4]_{4 \times 4} \mathbf{H}(\mathbf{0})_{4 \times 1} \} = 0 \\
& \delta \mathbf{H}(\mathbf{L})_{1 \times 4}^T \{ -[\alpha_5]_{4 \times 4} \mathbf{H}''''(\mathbf{L})_{4 \times 1} + [\alpha_3 - \alpha_4]_{4 \times 4} \mathbf{H}'(\mathbf{L})_{4 \times 1} \} = 0 \\
& \delta \mathbf{H}'(\mathbf{L})_{1 \times 4}^T \{ [\alpha_5]_{4 \times 4} \mathbf{H}''(\mathbf{L})_{4 \times 1} + [\alpha_4]_{4 \times 4} \mathbf{H}(\mathbf{L})_{4 \times 1} \} = 0
\end{aligned} \quad (15)$$

where matrices related to cross-sectional properties are defined as

$$\begin{aligned} [\alpha_1]_{4 \times 4} &= \sum_{i=1}^3 \frac{1}{E_{zi}} \int_{A_i} [\mathbf{f}_{zi}(y_i)_{4 \times 1} \mathbf{f}_{zi}(y_i)_{1 \times 4}^T] dA_i; \quad [\alpha_2]_{4 \times 4} = -\frac{1}{2} \left(\frac{\mu_{z2}}{E_{z2}} + \frac{\mu_{y2}}{E_{y2}} \right) \int_{A_2} [\mathbf{f}_{z2}(y_2)_{4 \times 1} \mathbf{f}_{y2}(y_2)_{1 \times 4}^T] dA_2; \\ [\alpha_3]_{4 \times 4} &= \sum_{i=2}^3 \frac{1}{G_i} \int_{A_i} [\mathbf{f}_{ti}(y_i)_{4 \times 1} \mathbf{f}_{ti}(y_i)_{1 \times 4}^T] dA_i; \quad [\alpha_4]_{4 \times 4} = -\frac{1}{2} \left(\frac{\mu_{z2}}{E_{z2}} + \frac{\mu_{y2}}{E_{y2}} \right) \int_{A_i} [\mathbf{f}_{y2}(y_2)_{4 \times 1} \mathbf{f}_{z2}(y_2)_{1 \times 4}^T] dA_2; \\ [\alpha_5]_{4 \times 4} &= \frac{1}{E_{y2}} \int_{A_2} [\mathbf{f}_{y2}(y_2)_{4 \times 1} \mathbf{f}_{y2}(y_2)_{1 \times 4}^T] dA_2; \quad [\alpha_{243}]_{4 \times 4} = [\alpha_2 + \alpha_4 - \alpha_3]_{4 \times 4}; \end{aligned} \quad (16)$$

and a load-related vector is defined as

$$\{\boldsymbol{\eta}(z)\}_{4 \times 1} = \frac{1}{E_{z1}} \int_{A_1} [\mathbf{f}_{z1}(y_1)_{4 \times 1} F_{z1}(y_1, z)] dA_1 \quad (17)$$

where $F_{z1}(y_1, z)$ has been defined in Eq. (9). In Eqs. (2a–k), by setting $z = 0, L$, vectors $\mathbf{H}(0)_{1 \times 4}^T$, $\mathbf{H}'(0)_{1 \times 4}^T$, $\mathbf{H}(L)_{1 \times 4}^T$, $\mathbf{H}'(L)_{1 \times 4}^T$ vanish because of the satisfactory of the equilibrium conditions. Thus, the boundary conditions in Eq. (15) are then expressed by the following 16 boundary conditions:

$$\mathbf{H}(0)_{1 \times 4}^T = \{\mathbf{0}\}; \quad \mathbf{H}'(0)_{1 \times 4}^T = \{\mathbf{0}\}; \quad \mathbf{H}(L)_{1 \times 4}^T = \{\mathbf{0}\}; \quad \mathbf{H}'(L)_{1 \times 4}^T = \{\mathbf{0}\}; \quad (18)$$

3.3 Closed form solution

A closed form solution of Eq. (14) is assumed to take the form $\mathbf{H}(z)_{4 \times 1} = \mathbf{H}_H(z)_{4 \times 1} + \mathbf{H}_P(z)_{4 \times 1}$ where $\mathbf{H}_H(z)_{4 \times 1}$

$$[\alpha_5]_{4 \times 4} \mathbf{H}''''(z)_{4 \times 1} + [\alpha_{243}]_{4 \times 4} \mathbf{H}''(z)_{4 \times 1} + [\alpha_1]_{4 \times 4} \mathbf{H}(z)_{4 \times 1} = \{\mathbf{0}\}_{4 \times 1} \quad (19)$$

The homogeneous solution is assumed to take the form $\mathbf{H}_H(z)_{4 \times 1} = \sum \mathbf{H}_{Hi}(z)_{4 \times 1}$ where $\mathbf{H}_{Hi}(z)_{4 \times 1} = \{\phi_i\}_{4 \times 1} e^{\lambda_i z} R_i$ in which $\langle \phi_i \rangle_{1 \times 4}^T = \langle 1 \ \phi_{b,i} \ \phi_{c,i} \ \phi_{d,i} \rangle$ is an eigenvector and λ_i is an eigenvalue of Eq. (19). Also, R_i is an unknown constant. By substituting $\mathbf{H}_{Hi}(z)_{4 \times 1} = \{\phi_i\}_{4 \times 1} e^{\lambda_i z} R_i$ into Eq. (19), one obtains

$$[[\alpha_5]_{4 \times 4} \lambda_i^4 + [\alpha_{243}]_{4 \times 4} \lambda_i^2 + [\alpha_1]_{4 \times 4} \lambda_i]_{4 \times 4} \{\phi_i\}_{4 \times 1} R_i e^{\lambda_i z} = \{\mathbf{0}\}_{4 \times 1} \quad (20)$$

For a non-trivial solution, the determinant of coefficient matrix $[\alpha_5]_{4 \times 4} \lambda_i^4 + [\alpha_{243}]_{4 \times 4} \lambda_i^2 + [\alpha_1]_{4 \times 4} \lambda_i$ vanishes. Because the resulting eigenvalue problem is quartic in constant λ_i , we obtain 16 eigen-pairs of $(\lambda_i, \{\phi_i\}_{4 \times 1})$ where $i = 1, 2, \dots, 16$. The resulting solution of the homogeneous solution $\mathbf{H}_H(z)_{4 \times 1} = \sum \mathbf{H}_{Hi}(z)_{4 \times 1}$ is then expressed as

$$\mathbf{H}_H(z)_{4 \times 1} = [\mathbf{f}]_{4 \times 16} [\mathbf{e}(z)]_{16 \times 16} \{\mathbf{R}\}_{16 \times 1} \quad (21)$$

in which

$$[\mathbf{f}]_{4 \times 16} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \phi_{b,1} & \phi_{b,2} & \dots & \phi_{b,16} \\ \phi_{c,1} & \phi_{c,2} & \ddots & \phi_{c,16} \\ \phi_{d,1} & \phi_{d,2} & \dots & \phi_{d,16} \end{bmatrix}; \quad [\mathbf{e}(z)]_{16 \times 16} = \begin{bmatrix} e^{\lambda_1 z} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 z} & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & e^{\lambda_{16} z} \end{bmatrix}; \quad \{\mathbf{R}\}_{16 \times 1} = \begin{Bmatrix} R_1 \\ R_2 \\ \vdots \\ R_{16} \end{Bmatrix} \quad (22)$$

is a homogeneous solution and $\mathbf{H}_P(z)_{4 \times 1}$ is a particular solution.

3.3.1 Homogeneous solution

Homogeneous root of Eq. (14) is obtained by omitting vector $\{\boldsymbol{\eta}(z)\}_{4 \times 1}$, i.e.,

Vector $\{\mathbf{R}\}_{16 \times 1}$ includes 16 unknown constants those will be evaluated by substituting $\mathbf{H}(z)_{4 \times 1}$ into 16 boundary condition equations in Eq. (18).

3.3.2 Particular solution

Particular solution $\mathbf{H}_P(z)_{4 \times 1}$ of Eq. (14) depends on the load-related vector $\{\boldsymbol{\eta}(z)\}_{4 \times 1}$ and thus it depends on the

load conditions given in a particular problem. Once vector $\{\eta(z)\}_{4 \times 1}$ is expanded as a function of z coordinate, the particular solution can be developed. In the present example, three particular solutions corresponding to three typical loading scenarios are developed in the following.

3.3.3 Loading scenario 1

If the applied traction is a constant, i.e., $\sigma_0(z) = \sigma_0$. From Eq. (9), we have

$$F_{z1}(y_1, z) = -\frac{M_1(0)y_1}{I_1} - \frac{Q_1(0)y_1}{I_1}z - \frac{b_1\sigma_0 y_1}{2I_1}z^2 \quad (23)$$

and thus vector $\{\eta(z)\}_{4 \times 1}$ as defined in Eq. (17) is obtained as

$$\begin{aligned} \{\eta(z)\}_{4 \times 1} = & -\frac{M_1(0)}{E_{z1}I_1} \int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1 \\ & - \frac{Q_1(0)}{E_{z1}I_1} \left[\int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1 \right] z \\ & - \frac{b_1\sigma_0}{2E_{z1}I_1} \left[\int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1 \right] z^2 \end{aligned} \quad (24)$$

Based on the thus derived $\{\eta(z)\}_{4 \times 1}$, we can assume that the particular solution takes the form

$$\mathbf{H}_p(z)_{4 \times 1} = \{\mathbf{c}_1\}_{4 \times 1} + \{\mathbf{c}_2\}_{4 \times 1}z + \{\mathbf{c}_3\}_{4 \times 1}z^2 \quad (25)$$

By substituting the thus assumed $\mathbf{H}_p(z)$ and vector $\{\eta(z)\}$ in Eq. (24) into Eq. (14) and by equating factors associated with z^i where $i = 0, 1, 2$, one obtains the following constant vectors $\{\mathbf{c}_i\}_{4 \times 1}$, $i = 1, 2, 3$

$$\begin{aligned} \{\mathbf{c}_2\}_{4 \times 1} &= [\alpha_1]_{4 \times 4}^{-1} \frac{Q_1(0)}{E_{z1}I_1} \int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1; \quad \{\mathbf{c}_3\}_{4 \times 1} = [\alpha_1]_{4 \times 4}^{-1} \frac{b_1\sigma_0}{2E_{z1}I_1} \int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1 \\ \{\mathbf{c}_1\}_{4 \times 1} &= [\alpha_1]_{4 \times 4}^{-1} \left\{ \frac{M_1(0)}{E_{z1}I_1} \int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1 + 2[\alpha_{243}]_{4 \times 4} \{\mathbf{c}_3\}_{4 \times 1} \right\} \end{aligned} \quad (26)$$

concentrated loads can be converted into an applied traction $\sigma_0(z)$ through Dirac delta functions as

$$\sigma_0(z) = \sum_{i=1}^m \frac{P_i}{b_1} \text{dirac}(z - z_i) \quad (27)$$

From Eq. (27), by substituting into Eq. (9), we have

$$\begin{aligned} F_{z1}(y_1, z) = & -\frac{M_1(0)y_1}{I_1} - \frac{Q_1(0)y_1}{I_1}z \\ & - \frac{y_1}{I_1} \sum_{i=1}^m \int_0^z \int_0^z P_i \text{dirac}(z - z_i) dz dz \end{aligned} \quad (28)$$

and vector $\{\eta(z)\}_{4 \times 1}$ as defined in Eq. (17) is obtained as

$$\begin{aligned} \{\eta(z)\}_{4 \times 1} = & -\frac{M_1(0)}{E_{z1}I_1} \int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1 \\ & - \frac{Q_1(0)}{E_{z1}I_1} \left[\int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1 \right] z \\ & - \left[\sum_{i=1}^m \int_0^z \int_0^z P_i \text{dirac}(z - z_i) dz dz \right] \\ & \frac{1}{E_{z1}I_1} \int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1 \end{aligned} \quad (29)$$

Based on the thus expanded $\{\eta(z)\}_{4 \times 1}$, we can assume that the particular solution takes the form

$$\begin{aligned} \mathbf{H}_p(z)_{4 \times 1} = & \{\mathbf{c}_1\}_{4 \times 1} + \{\mathbf{c}_2\}_{4 \times 1}z + \{\mathbf{c}_3\}_{4 \times 1} \\ & \left[\sum_{i=1}^m \int_0^z \int_0^z P_i \text{dirac}(z - z_i) dz dz \right] \end{aligned} \quad (30)$$

3.3.4 Loading scenario 2

If there are concentrated loads $P_1, P_2, \dots, P_m$ applied to the beam at the corresponding coordinates $z_1, z_2, \dots, z_m$. The

By substituting the thus assumed $\mathbf{H}_p(z)$ and vector $\{\eta(z)\}$ in Eq. (29) into Eq. (14), one obtains the constant vectors $\{\mathbf{c}_i\}_{4 \times 1}, i = 1, 2, 3$ as

$$\begin{aligned} \{\mathbf{c}_1\}_{4 \times 1} &= [\alpha_1]_{4 \times 4}^{-1} \frac{M_1(0)}{I_1 E_{z1}} \int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1 + [\alpha_1]_{4 \times 4}^{-1} [\alpha_{243}]_{4 \times 4} \{\mathbf{c}_3\}_{4 \times 1} \left[\sum_{i=1}^m P_i \text{dirac}(z - z_i) \right] \\ \{\mathbf{c}_2\}_{4 \times 1} &= [\alpha_1]_{4 \times 4}^{-1} \frac{Q_1(0)}{E_{z1} I_1} \left[\int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1 \right]; \quad \{\mathbf{c}_3\}_{4 \times 1} = [\alpha_1]_{4 \times 4}^{-1} \frac{1}{E_{z1} I_1} \left[\int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1 \right] \end{aligned} \quad (31)$$

3.3.5 Loading scenario 3

If there are only end loads $N_1(0), Q_1(0), M_1(0), N_1(L), Q_1(L), M_1(L)$ applied to the beam, by applying a similar procedure as done in Scenario 1, the particular solution in this scenario is obtained in the following

$$\begin{aligned} F_{z1}(y_1, z) &= -\frac{M_1(0)y_1}{I_1} - \frac{Q_1(0)y_1}{I_1} z; \quad \{\eta(z)\}_{4 \times 1} = -\frac{M_1(0)}{E_{z1} I_1} \int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1 - \frac{Q_1(0)}{E_{z1} I_1} \left[\int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1 \right] z \\ \{\mathbf{c}_1\}_{4 \times 1} &= [\alpha_1]_{4 \times 4}^{-1} \left\{ \frac{M_1(0)}{E_{z1} I_1} \int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1 + 2[\alpha_{243}]_{4 \times 4} \{\mathbf{c}_3\}_{4 \times 1} \right\}; \quad \{\mathbf{c}_2\}_{4 \times 1} = [\alpha_1]_{4 \times 4}^{-1} \frac{Q_1(0)}{E_{z1} I_1} \int_{A_1} y_1 \mathbf{f}_{z1}(y_1)_{4 \times 1} dA_1; \\ \mathbf{H}_p(z)_{4 \times 1} &= \{\mathbf{c}_1\}_{4 \times 1} + \{\mathbf{c}_2\}_{4 \times 1} z \end{aligned} \quad (32)$$

In the closed form solution $\mathbf{H}(z)_{4 \times 1} = \mathbf{H}_H(z)_{4 \times 1} + \mathbf{H}_p(z)_{4 \times 1}$, it is required to evaluate vector $\{\mathbf{R}\}_{16 \times 1}$ in Eq. (21) by substituting $\mathbf{H}(z)_{4 \times 1}$ into 16 boundary condition equations in Eq. (18) and performing an matrix inversion or using backslash algorithms. As experienced by the author, the evaluation may meet purely mathematical challenges because the homogeneous solution $\mathbf{H}_H(L)$ includes both extremely large numbers (e.g., $\exp(\lambda_i L)$) and small numbers (e.g., $\exp(-\lambda_i L)$) where L is the strengthened beam length and λ_i is an eigenvalue. Therefore, the present study is going to derive a finite

difference solution to discretize the span into small segments as a back-up solution if any.

3.3.6 A finite difference formulation

The finite difference technique discretizes the compatibility equations and the corresponding boundary conditions into algebraic equations. Functions $\mathbf{H}(z)_{1 \times 4}^T = \langle A(z) \ B(z) \ C(z) \ D(z) \rangle$ with one

dimensional domain $0 \leq z \leq L$ are subdivided into n segments, equidistantly spaced at a distance $\Delta = L/n$. Points are denoted as $i = 1, 2, \dots, (n+1)$ (Fig. 3). The value of $\mathbf{H}(z)_{1 \times 4}^T$ at a given point $z = z_i$ is denoted as $\mathbf{H}(z_i)_{1 \times 4}^T = \langle A(z_i) \ B(z_i) \ C(z_i) \ D(z_i) \rangle$. According to the central finite difference approximation (e.g., LeVeque [33]), the first, second and fourth derivatives of functions $\mathbf{H}(z)_{4 \times 1}$ can be expressed as

$$\begin{aligned} \mathbf{H}'(z_i)_{4 \times 1} &= -\frac{1}{2\Delta} \mathbf{H}(z_{i-1})_{4 \times 1} + \frac{1}{2\Delta} \mathbf{H}(z_{i+1})_{4 \times 1} \\ \mathbf{H}''(z_i)_{4 \times 1} &= \frac{1}{\Delta^2} \mathbf{H}(z_{i-1})_{4 \times 1} - \frac{2}{\Delta^2} \mathbf{H}(z_i)_{4 \times 1} + \frac{1}{\Delta^2} \mathbf{H}(z_{i+1})_{4 \times 1} \\ \mathbf{H}'''(z_i)_{4 \times 1} &= \frac{1}{\Delta^4} \mathbf{H}(z_{i-2})_{4 \times 1} - \frac{4}{\Delta^4} \mathbf{H}(z_{i-1})_{4 \times 1} + \frac{6}{\Delta^4} \mathbf{H}(z_i)_{4 \times 1} \\ &\quad - \frac{4}{\Delta^4} \mathbf{H}(z_{i+1})_{4 \times 1} + \frac{1}{\Delta^4} \mathbf{H}(z_{i+2})_{4 \times 1} \end{aligned} \quad (33)$$

Since the compatibility equations in Eq. (14) include fourth-order derivatives of $\mathbf{H}(z)_{1 \times 4}^T$, functions $A(z), B(z), C(z), D(z)$ are discretized into $(n+1) + 4$ unknown values (Fig. 3). In total, there are $4(n+1) + 16$ unknown discretized function values.

Fig. 3 Subdivisions in the present finite difference solution

Fig. 4 A simply supported beam bonded with a soffit plate [8]

To evaluate the $4(n+1) + 16$ unknown discretized values, it is necessary to establish $4(n+1) + 16$ independent equations of the unknowns. This can be done by substituting $\mathbf{H}(\mathbf{z}_i)_{1 \times 4}^T$ and their derivatives in Eq. (33) into the compatibility equations in Eq. (14) and setting $i = 1, 2, \dots, (n+1)$ to yield $4(n+1)$ independent equations. The remaining 16 independent equations are obtained from 16 boundary conditions in Eq. (18), i.e.,

$$\begin{aligned} & \left[\frac{1}{\Delta^4} \alpha_5 \right]_{4 \times 4} \mathbf{H}(\mathbf{z}_{i-2})_{4 \times 1} + \left[\frac{1}{\Delta^2} \alpha_{243} - \frac{4}{\Delta^4} \alpha_5 \right]_{4 \times 4} \mathbf{H}(\mathbf{z}_{i-1})_{4 \times 1} + \left[\alpha_1 - \frac{2}{\Delta^2} \alpha_{243} + \frac{6}{\Delta^4} \alpha_5 \right]_{4 \times 4} \mathbf{H}(\mathbf{z}_i)_{4 \times 1} \\ & + \left[\frac{1}{\Delta^2} \alpha_{243} - \frac{4}{\Delta^4} \alpha_5 \right]_{4 \times 4} \mathbf{H}(\mathbf{z}_{i+1})_{4 \times 1} + \left[\frac{1}{\Delta^4} \alpha_5 \right]_{4 \times 4} \mathbf{H}(\mathbf{z}_{i+2})_{4 \times 1} + \{\boldsymbol{\eta}(\mathbf{z}_i)\}_{4 \times 1} = \{\mathbf{0}\}_{4 \times 1} \\ & \mathbf{H}(\mathbf{z}_1)_{4 \times 1} = \{\mathbf{0}\}_{4 \times 1} \\ & -\mathbf{H}(\mathbf{z}_0)_{4 \times 1} + \mathbf{H}(\mathbf{z}_2)_{4 \times 1} = \{\mathbf{0}\}_{4 \times 1} \\ & \mathbf{H}(\mathbf{z}_{n+1})_{4 \times 1} = \{\mathbf{0}\}_{4 \times 1} \\ & -\mathbf{H}(\mathbf{z}_n)_{4 \times 1} + \mathbf{H}(\mathbf{z}_{n+2})_{4 \times 1} = \{\mathbf{0}\}_{4 \times 1} \end{aligned} \quad (34)$$

When the beam is subjected to a uniformly distributed load, concentrated loads or end forces, vectors $\{\boldsymbol{\eta}(\mathbf{z}_i)\}_{4 \times 1}$ are determined by using vectors $\{\boldsymbol{\eta}(\mathbf{z})\}_{4 \times 1}$ as presented in Eqs. (24), (29) and (32).

4 Validations and examples

4.1 Example 1: validation of the present study against a numerical study

A numerical study on the adhesive shear and normal stresses in a beam bonded with a soffit plate conducted in Teng et al. [8] is re-visited in the present example. The beam is simply supported with a span of 2400 mm, bonded with a 1800 mm-long soffit plate through an adhesive layer, and subjected to a uniformly distributed traction $\sigma_0 = -0.15 \text{ N/mm}^2$ (Fig. 4). The beam cross-section dimensions are $b = 100 \text{ mm}$ and $h_1 = 150 \text{ mm}$, while those of the adhesive layer are $b = 100 \text{ mm}$ and $h_2 = 4.0 \text{ mm}$, and those of the soffit plate are $b = 100 \text{ mm}$ and $h_3 = 4.0 \text{ mm}$. The Young's modulus of the beam, adhesive and the plate materials are $E_{z1} = 20 \text{ GPa}$, $E_{z2} = 2.0 \text{ GPa}$ and $E_{z3} = 200 \text{ GPa}$, respectively. Also, the Poisson's ratio of

these materials are $\mu_{z1} = 0.17$, $\mu_{z2} = 0.25$ and $\mu_{z3} = 0.30$, respectively. It is required to compare the transverse shear and normal stresses in the adhesive layer at the adhesive-beam interface (denoted as BA), at the adhesive-plate (denoted as PA) and at the plane surface at middle thickness of the adhesive layer (denoted as MA) as predicted by the present theory against those of 3-Dimensional Finite Element Analyses (3D FEA solution) under ABAQUS and those of two analytical solutions reported in [15, 17].

4.1.1 Description of the present study

Both closed form solution and finite difference formulation developed in the present study can provide a solution for the adhesive shear and normal stresses. The resultant forces at two ends of the strengthened segment are specified as $Q_1(0) = \sigma_0 b L / 2$; $M_1(0) = \sigma_0 b (L + a) a / 2$. Also, the interfacial shear and normal stresses $\tau_{zy2}(y_2, z)$ at the MA

surface are obtained from Eq. (6b) by setting $y_2 = 0$. The closed form solution is obtained through Eqs. (21) and (25), while the finite difference formulation is obtained through Eq. (34) in which the load vector $\{\boldsymbol{\eta}(\mathbf{z}_i)\}_{4 \times 1}$ is evaluated by using Eq. (24). A mesh study is conducted for the finite difference formulation and it is found that a mesh with spacing distances of $\Delta = 0.1 \text{ mm}$ provides stress convergences (as good as those obtained in the closed form solution).

4.1.2 Description of the 3D FEA solution

A 3D FEA solution is conducted in the present study to replicate the numerical study in [8] with a few modifications. The composite structure [8] was modelled by using 4-node plane stress elements. Near the plate end, the structure was meshed by using square elements with edge sizes of 0.1 mm. In the present study, a similar numerical model is self-written based on ABAQUS to replicate the numerical study in [8] and it is named as 3D FEA solutions. For a more practical simulation, the present ABAQUS model is based on C3D8R brick elements with reduced integration and with element edge sizes of 0.05 mm. The stress predictions at the BA, MA and PA interfaces based on the 3D FEA solution are found to well agree with those in [8],

Fig. 5 Adhesive shear and normal stresses at BA, MA, and PA interfaces as predicted by the 3D FEA solution, Smith and Teng [15], Guananeche et al. [17], and the present study

except 2 peak shear and normal stress values on the BA interface at the plate end (Fig. 5a–d). In the present 3D FEA solution, the peak shear is 8.40 MPa (in magnitude) while the peak normal stress is 32.6 MPa, both of them occurs at coordinate $z = 0$ (Table 1). In the numerical study [8],

the peak shear was 7.97 MPa (in magnitude) while the peak normal stress was 27.1 MPa and they also occurred at $z = 0$. Therefore, a stress singularity may occurs on the BA interface at $z = 0$ in the numerical solutions (as also reported in [8, 14]). Regardless of the stress singularity at

Table 1 Comparison of the peak adhesive stresses and their locations in the adhesive layer between different solutions

Solution	Peak values						Location of the peak stresses					
	Peak shear (MPa)			Peak normal (MPa)			Z coordinate (mm)					
	$\tau_1(z)^a$	τ_{zy2}^b	$\tau_2(z)^c$	$\sigma_1(z)^a$	σ_{y2}^b	$\sigma_2(z)^c$	$\tau_1(z)^a$	τ_{zy2}^b	$\tau_2(z)^c$	$\sigma_1(z)^a$	σ_{y2}^b	$\sigma_2(z)^c$
Smith and Teng [15]	5.98	5.98	5.98	2.43	2.43	2.43	0.0	0.0	0.0	0.0	0.0	0.0
Guananeche et al. [17]	5.86	5.86	5.86	2.01	2.01	2.01	0.0	0.0	0.0	0.0	0.0	0.0
Present study	5.96 (5.89 ^d)	5.69	5.43	31.4	3.90	15.8	0.6 (0.5 ^d)	2.7	5.7	0.0	0.9	0.0
3D FEA	8.40 (5.62 ^d)	5.59	5.15	32.6	3.66	14.4	0.0 (0.5 ^d)	2.7	5.1	0.0	1.2	0.0

^{a,b,c} Magnitude of stresses on the BA interface, MA surface, and PA interface respectively; ^d Stress value of $\tau_1(z)$ at $z = 0.5$ mm

Fig. 6 The simply supported beam bonded with a steel plate [10]

the point, it is observed that stresses at other locations (i.e., the rest of the BA, the MA and PA interfaces) predicted by the 3D FEA solution agree well with those of the numerical solution in [8].

4.1.3 Descriptions of the solutions by Smith and Teng [15] and Guenaneche et al. [17]

Smith and Teng [15] developed a closed form solution for the interfacial stresses based on a non-shear deformable beam theory while Guenaneche et al. [17] recently developed an solution for the stresses based on a shear deformable beam theory. However, adhesive shear and normal

stress fields in [15, 17] were invariant across the adhesive thickness.

4.1.4 Comparisons of stresses between solutions

Figure 5a–f presents the transverse shear and normal stresses on the BA, MA and PA interfaces of the adhesive layer against the z -coordinate, as predicted by the 3D FEA solution, Smith and Teng [15], Guananeche et al. [17], and the present study. As observed in all solutions, both shear and normal stresses are highly concentrated near the plate end. The peak stress values and their locations are shown in Table 1. Among all solutions, the 3D FEA solution is taken as a reference solution to validate other ones.

In Fig. 5a, the shear stresses on the BA interface based on the present study and the solutions [15, 17] are generally in an excellent agreement with those of the 3D FEA solution, exempt the singularity value at coordinate $z = 0$. The peak shear values based on the present study, [15, 17], and 3D FEA solutions are 5.96, 5.98, 5.86 and 8.40 MPa respectively. Thus, the peak shears based on the present study and the solutions [15, 17] are smaller than that of the 3D FEA solution. However, the shear stresses at coordinate $z = 0.5$ mm on the BA interface as predicted by the present study and the 3D FEA solution are respectively 5.89

Fig. 7 Adhesive shear and normal stresses at BA and PA interfaces as predicted by experiment study [10], the 2D FEA solution, Guananeche et al. [17], and the present study

and 5.62 MPa (Table 1), corresponding to a difference of only 4.8%, where the stress based on the present study is higher than that of the 3D FEA stress. More agreements between the 3D FEA solution and the present study are achieved at other distances from $z = 0.5$ mm.

In Fig. 5b, c, it is observed that the shear stresses on the MA and PA interfaces based on the present study and the solutions [15, 17] are generally in an excellent agreement with those of the 3D FEA solution. However, the peak stress values and their locations between the solutions are slightly different. In the 3D FEA solution, the peak shear stresses on the MA and PA interfaces are 5.59 and 5.19 MPa respectively and they occur at coordinates $z = 2.7$ and 5.1 mm (Table 1). In the present study, the peak shear stresses are 5.69 and 5.43 MPa and they occur at coordinates $z = 2.7$ and 5.7 mm. In contrast, the previous solutions [15, 17] only predict a constant stress across the adhesive thickness and the location of their peak stresses again occurs at coordinate $z = 0$. Although the differences of the peak stresses and their locations between the four solutions are slight and maybe negligible, they can reflect that the shear stresses in the adhesive layer are varied across the adhesive thicknesses, as captured by the 3D FEA solution and the present study, but not captured by the previous solutions [15, 17]. The reflection is even more significant for the transverse normal stresses as discussed in the following.

In Fig. 5d, e, and f, the transverse normal stresses as predicted by the present study are observed to excellent agree with those predicted by the 3D FEA solutions. The peak normal stress at the BA interface (Fig. 5d and Table 1) in the 3D FEA solution is 32.6 MPa, while that in the present study is 31.4 MPa, a difference of 3.7%. The location of the peak stresses in both solutions is at coordinate $z = 0$. Also, the peak normal stress at the MA interface (Fig. 5e) in the 3D FEA solution is 3.66 MPa and that in the present study is 3.90 MPa, a difference of 6.1%. The location of the

peak stress in the 3D FEA solution is 0.9 mm while that in the present study is 1.2 mm. Also, the peak normal stress at the PA interface (Fig. 5f) in the 3D FEA solution is 14.4 MPa in magnitude while that in the present study is 15.8 MPa in magnitude, a difference of 9.7%. The location of the peak stresses is again at coordinate $z = 0$. Based on the normal stress distributions, it is observed that the normal stresses significantly vary across the adhesive thickness, as captured in the 3D FEA solution and the present study. In contrast, the stresses as predicted by the solutions [15, 17] are a constant across the thickness and they are only 2.43 and 2.01 MPa, significant smaller than the stresses of the 3D FEA solution and the present study. The present study is developed by using plane stress assumptions while the solutions [15, 17] were based on beam assumptions. Also, the present study avoids an assumption of equal curvatures in the beam and the plate, that was postulated in [15, 17].

4.2 Example 2: validation of the present study against experiment and numerical studies

Jones et al. [10] conducted an experimental study on a beam strengthened with a soffit plate and subjected to two concentrated loads P_1 and P_2 (Fig. 6). The beam cross-section dimensions are $b = 155$ mm and $h_1 = 225$ mm, while those of the adhesive layer are $b = 125$ mm and $h_2 = 1.5$ mm, and those of the soffit plate are $b = 125$ mm and $h_3 = 6.0$ mm. The Young's moduli of beam, the adhesive and the plate are $E_{z1} = 31$ GPa, $E_{z2} = 3.0$ GPa and $E_{z3} = 200$ GPa, respectively. And the Poisson's ratio of these materials are $\mu_{z1} = 0.18$, $\mu_{z2} = 0.35$ and $\mu_{z3} = 0.30$, respectively. The adhesive stresses predicted by the present study are validated against the reported stresses in [10], the stresses in a 2-Dimensional Finite Element Analysis (2D FEA) under ABAQUS and those predicted by Guenaneche et al. [17].

Fig. 8 Effects of the longitudinal modulus and thickness of the soffit plate on the peak shear and normal stresses at the BA interface

4.2.1 Description of the present study

The resultant forces at two ends of the strengthened segment are specified as $Q_1(0) = 30$ kN; $M_1(0) = 1.5$ kNm. The closed form solution is obtained through Eqs. (21) and (30), while the finite difference formulation is obtained through Eq. (34) in which the load vector $\{\eta(z_i)\}_{4 \times 1}$ is evaluated by Eq. (29) and the number of concentrated loads $m = 2$.

4.2.2 Description of the 2D FEA solution

A 2D FEA solution is conducted in the present study to save time run. The composite structure is modelled by using 4-node S4R elements with reduced integration. Near the plate end, the structure was meshed by square elements with element edge sizes of 0.02 mm [8].

4.2.3 Comparison of the interfacial shear and normal stresses on the BA and PA interfaces

Figure 7a–d presents the transverse shear and normal stresses at the BA and PA interfaces of the adhesive layer against the z -coordinate as predicted by the 2D FEA solution, Guenaneche et al. [17], experimental study [10] and the present study. Similar observations as obtained in Example 1 are valid in the present example. Those include: (i) the shear and normal stresses predicted by the present solution are generally in excellent agreements with those of the 2D FEA solution, (ii) the present study captures the normal stresses strongly varied across the adhesive thickness, as also captured by the 2D FEA solution but not captured by Guenaneche et al. [17]. In details, (1) the shear stress on the BA interface based on the 2D FEA solution is 1.97 MPa at $z = 0.0$ mm but it is only 1.23 MPa at $z = 0.11$ mm (Fig. 7a). The shear stress on the BA interface based on the present study is 1.23 MPa at $z = 0.11$ mm and its peak is 1.38 MPa at $z = 0.22$ mm, (2) the peak normal stress on the BA interface based on the 2D FEA solution is 7.57 MPa, that based on the present study is 7.49 MPa but that based on Guenaneche et al. [17] is only 0.70 MPa (Fig. 7b), (3) the peak shear on the PA interface based on the 2D FEA solution is 1.20 MPa while that based on the present study is 1.32 MPa and that based on Guenaneche et al. [17] is 1.27 MPa, and (4) the peak normal stress on the PA interface predicted by the 2D FEA solution is 3.60 MPa in magnitude, while that based on the present study is 3.78 MPa.

As observed in Fig. 7c, the shear stresses on the PA interface predicted by the 2D FEA solution, the Guenaneche et al. [17] and the present study all are in good agreements with the experimental shear stresses reported in [10]. The stresses based experimental work [10] were, however, not

measured and evaluated at locations close to coordinate $z = 0$.

5 Parametric studies

Material and thickness of the soffit plate are often questioned in beams strengthened with a soffit plate to satisfy a specific strengthening requirement. The present parametric study is conducted to investigate the effects of the longitudinal modulus of elasticity and the thickness of the soffit plate on the peak adhesive shear and normal stresses. The structure given in Example 2 is taken as a reference case. Because the material of the soffit plate may be originated from Glass-FRP (with a modulus of about 10–30 GPa), steel (with a modulus of 200 GPa), or Carbon-FRP (with a modulus of about 80–340 GPa), a range of the plate modulus from 10 to 340 GPa are given in the first parametric study. Also, another range of the plate thickness from 1.0 to 9.0 mm are given in the second parametric study. When one parameter is varied, the other ones of the reference case remain unchanged.

5.1 Effect of the plate longitudinal modulus of elasticity

Figure 8a presents the peak values of the interfacial shear and normal stresses against the plate longitudinal modulus of elasticity when the modulus is varied from 10 to 340 GPa. In general, the higher the plate modulus is, the higher the peak shear and normal stresses are. These relationships are not linear. Also, the peak normal stresses are significantly higher than the peak shear stresses. When the modulus is varied from 50 to 340 GPa, the peak normal stresses rapidly increase from 1.68 MPa to 9.11 MPa while the peak shear stresses slightly increase from 0.79 to 1.70 MPa.

5.2 Effect of the thickness of the soffit plate

Figure 8a presents the peak values of the interfacial shear and peeling stresses against the thickness of the soffit plate when the thickness is varied from 1.0 to 9.0 mm. Again, the higher the plate thickness is, the higher the peak shear and normal stress values are. Also, the peak normal stresses are significantly higher than the peak shear stresses. When the thickness is varied from 1.0 to 9.0 mm, the peak normal stresses rapidly increase from 1.31 MPa to 8.95 MPa while the peak shear stresses slightly increase from 0.56 to 1.65 MPa. Through the parametric studies, it is also observed that the peak normal stresses are significantly higher than peak shear stresses.

6 Conclusions

The present study successfully developed a close from solution and a finite difference formulation for the prediction of the adhesive shear and normal stresses in soffit plate flexural-strengthened beams. The statically admissible stress fields in each materials were developed and expressed in terms of four unknown interfacial shear and normal stresses. A modified complementary strain energy of the system was proposed in which the transverse normal stresses in the beam and the plate and the transverse shear stresses in the beam were neglected in the formulation. Based on a variation principle of the complementary strain energy, a set of compatibility equations and the corresponding boundary conditions were derived. A closed from solution and a finite different formulation were then developed for the compatibility equations. Through examples conducted, key observations in the present study are summarized in the following:

- (1) The adhesive shear and normal stresses near the plate ends predicted by the present study were in excellent agreements with those predicted by the 2- and 3-dimensional finite element analyses solutions under ABAQUS and numerical studies [8, 14]. Also, the shear stresses predicted by the present study were well validated against those of an experimental study [10].
- (2) Based on the FEA solutions, it was observed that the adhesive normal stresses strongly varied across the adhesive thickness (as observed in Examples 1 and 2). The present study excellently captured the stress variance across the thickness of the FEA stresses. In contrast, many available analytical solutions (e.g., [15, 17]) ignored the variances as well as they significantly underpredicted the FEA normal stresses.
- (3) In the examples and the parametric study conducted in the present study, shear and normal stress concentrations in the adhesive layer were observed near the plate ends. In which, the peak value of the adhesive normal stress field were significantly higher than that of the adhesive shear stress field.
- (4) Based on the parametric study, the effects of the modulus of elasticity and the thickness of the soffit plate on the peak interfacial shear and normal stresses were investigated. In general, the higher the plate modulus and/or the plate thickness was, the higher the peak values of the shear and normal stresses were. Also, it was observed that the increasing rate of the peak normal stresses are higher than those of the peak shear stresses.

- (5) The present study is applicable to soffit plate-strengthened beams with isotropic or orthotropic materials under general applied loads and force boundary conditions.
- (6) Based on a personal computer, the computational time for a run of the closed form solution is relatively fast (e.g., 10 s for a job in Examples 1, 2). However, the FEA solutions conducted in the present study involved significant attempts in modelling (e.g., meshing and mesh study) and post-processing (e.g., extracting stress results). Based on the same personal computer, the computational time for a run in the 3D FEA solution in Example 1 was about 112 min while that in the 2D FEA solution in Example 2 was about 26 min. Therefore, the present study may be convenient for parametric studies.

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Availability of data and material Not applicable.

Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

Code availability Matlab codes for my closed form solution and finite difference formulation were written. Also, ABAQUS codes and the solutions of Smith and Teng [15], and Guenaneche et al. [17] were written. All of the codes (supplemental data) are with the manuscript submission as an evidence that trusted data is presented in the study.

References

1. El Damatty A, Abushagur M, Youssef MA (2003) Experimental and analytical investigation of steel beams rehabilitated using GFRP sheets. *J Steel Compos Struct* 3:421–438
2. Miller CT, Chajes JM, Hastings NJ (2001) Strengthening of a steel bridge girder using CFRP plates. *J Bridge Eng* 6:514–522
3. Ramesh B, Eswari S, Sundararajan T (2020) Flexural behavior of glass fiber reinforced polymer (GFRP) laminated hybrid-fiber reinforced concrete beams. *SN Appl Sci* 2:204. <https://doi.org/10.1007/s42452-020-1966-2>
4. Siddika A, Saha K, Mahnud MS et al (2019) Performance and failure analysis of carbon fiber-reinforced polymer (CFRP) strengthened reinforced concrete (RC) beams. *SN Appl Sci* 1:1617. <https://doi.org/10.1007/s42452-019-1675-x>
5. Harries AK, El-Tawil S (2008) Steel-FRP composite structural systems. In: International conference on composite construction in steel and concrete. composite construction in steel and concrete VI, ASCE, Reston, VA 703–716

6. Phe VP, Mohareb M, Fam A (2018) Finite element formulation for the analysis of multilayered beams based on the principle of stationary complementary strain energy. *J Eng Struct* 167:287–307
7. Phe VP, Mohareb M, Fam A (2018) Lateral torsional buckling of steel beams strengthened with GFRP plate. *Thin-Walled Struct* 131:55–75
8. Teng JG, Zhang JW, Smith ST (2002) Interfacial stresses in reinforced concrete beams bonded with a soffit plate: a finite element study. *Constr Build Mater* 16:1–14
9. Vahedian A, Shrestha R, Crews K (2019) Experimental and analytical investigation on CFRP strengthened glulam laminated timber beams: full-scale experiments. *Compos B* 164:377–389
10. Jones KR, Swamy RN, Charif A (1988) Plate separation and anchorage of reinforced concrete beams strengthened by epoxy bonded steel plates. *Struct Eng* 66:85–94
11. Aguilera J, Fam A (2013) Bonded FRP plates for strengthening rectangular hollow steel section T-joints against web buckling induced by transverse compression. *Comp Const ASCE* 17:421–432
12. Linghoff D, Al-Emrani M (2010) Performance of steel beams strengthened with CFRP laminate—Part 1: laboratory tests. *Compos Part B* 41:509–515
13. Tounsi A, Benyoucef S (2007) Interfacial stresses in externally FRP-plated concrete beams. *Int J Adhes Adhes* 27:207–215
14. Zhang L, Teng JG (2010) Finite element prediction of interfacial stresses in structural members bonded with a thin plate. *Eng Struct* 32:459–471
15. Smith ST, Teng JG (2001) Interfacial stresses in plated beams. *Eng Struct* 23:857–871
16. Guenaneche B, Krouar Tounsi BA et al (2010) Elastic analysis of interfacial stresses for the design of a strengthened FRP plate bonded to an RC beam. *Int J Adhes Adhes* 30:636–642
17. Guenaneche B, Krouar B, Tounsi A, Bedia AA (2014) Effect of shear deformation on interfacial stress analysis in plated beams under arbitrary loading. *Int J Adhes Adhes* 48:1–13
18. Wong EH (2015) Interfacial stresses in sandwich structures subjected to temperature and mechanical loads. *Compos Struct* 134:226–236
19. Narayanamurthy V, Chen JF, Cairns J (2016) Improved model for interfacial stresses accounting for the effect of shear deformation in plated beams. *Int J Adhes Adhes* 64:33–47
20. Han JW, Kim JS, Cho M (2017) Generalization of the C0-type zigzag theory for accurate thermomechanical analysis of laminated composites. *Compos B* 122:173–191
21. Liu M, Dawood M (2018) A closed-form solution of the interfacial stresses and strains in steel beams strengthened with externally bonded plates using ductile adhesives. *Eng Struct* 154:66–77
22. Liu S, Yuan H, Wu J (2019) Full-range mechanical behavior study of FRP-to-concrete interface for pull-pull bonded joints. *Compos B* 164:333–344
23. Long H, Wei Y, Liang L (2020) A rigorous analytical solution of interfacial stresses and overall stiffness of beam structures bonded with partially covered plates. *Int J Mech Sci* 167:105284
24. Sha X, Davidson JS (2020) Analysis of interfacial stresses in concrete beams strengthened by externally bonded FRP laminates using composite beam theory. *Compos Struct* 243:112235
25. Shan ZW, Su RKL (2020) Improved uncoupled closed-form solution for adhesive stresses in plated beams based on Timoshenko beam theory. *Int J Adhes Adhes* 96:102472
26. Yang J, Chen JF, Teng JG (2009) Interfacial stress analysis of plated beams under symmetric mechanical and thermal loading. *Constr Build Mater* 23:2973–2987
27. Yang J, Ye J (2010) An improved closed-form solution to interfacial stresses in plated beams using a two-stage approach. *Int J Mech Sci* 52:13–30
28. Wu XF, Jensen RA (2011) Stress function variational method for stress analysis of bonded joints under mechanical and thermal loads. *Int J Eng Sci* 49:279–294
29. Wu XF, Jensen RA (2014) Semianalytic stress-function variational approach for the interfacial stresses in bonded joints. *J Eng Mech* 140:1–11
30. Wu XF, Zhao Y (2013) Stress-function variational method for interfacial stress analysis of adhesively bonded joints. *Int J Solids Struct* 50:4305–4319
31. Wei X, Wang H, Shen HS (2019) A novel approach for the interfacial stress analysis of composite adhesively bonded joints. *Compos Sci Technol* 184:107830
32. Lanczos C (1986) *The variational principles of mechanics*, Dover Books on Physics (Book 4). Dover Publications, New York
33. LeVeque R (2007) *Finite difference methods for ordinary and partial differential equations: steady-state and time-dependent problems*. SIAM, Philadelphia

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