Optimum simple and complex power-law channels

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Received: 26 December 2019 / Accepted: 9 July 2020 / Published online: 19 July 2020
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Abstract

This study develops solutions for the hydraulically and most hydraulically efficient power-law sections (simple and complex). For this, the dimensionless objective function is defined as the wetted perimeter to the square root of the flow area. The objective function is then minimized. This study uses a simple procedure to establish the optimum dimensions. In comparison with the Lagrange multiplier method, this approach without any constraints is very simple for implementation. The optimization model of the simple power-law section has two decision variables (dimensionless depth and exponent) while the optimization model of the complex power-law section has three decision variables (dimensionless depth, dimensionless width and exponent). Optimizing all decision variables results in the most hydraulically efficient section (best hydraulic section), while optimizing some of the decision variables results in the hydraulically efficient section. This study presents a novel graphical comparison between simple and complex optimum power-law sections and shows that the dimensionless decision variables are constant for the most hydraulically efficient sections while are not constant for the hydraulically efficient sections. The characteristics of the hydraulically efficient sections are presented in tabular forms and explicit equations. These solutions are easy to use for optimum design of the power-law sections.

Keywords Power-law section · Irrigation channels · Best hydraulic section · Hydraulically efficient section · Optimization

1 Introduction

From economic view point, the cross-section shape of an open channel is an important factor. The channel cross-sections can be optimized to maximize the flow discharge under the same flow area. When optimum channel sections are applicable, the channel construction cost will be reduced. A channel section with the least area is efficient because of having the least volume of excavation and minimum lining area [1]. Open channel sections can be divided into two families. Simple sections include sections that consist of only two elements (two identical sides), while complex sections consist of at least three elements (two identical sides and a horizontal bottom). Both simple and complex channel sections can be optimized to maximize the flow discharge. A channel section having the smallest wetted perimeter for a given flow area is known

as the best hydraulic section [2] or the most hydraulically efficient section. The cross section of a channel can be optimized to obtain the optimum section of the channel which commonly known as the most hydraulically or hydraulically efficient section. If all decision variables of a channel section are optimized to obtain the smallest wetted perimeter for a given flow area, this section will be the best hydraulic section or the most hydraulically efficient section and when only some of the variables of a channel section are optimized then this section will be the hydraulically efficient section.

The simple semi-circular section has the largest discharge capacity for a given section area and consequently is the best hydraulic section among all possible channel shapes. It is desirable to develop and design other open channel sections with the smallest deviation from the semi-circular section. The most hydraulically

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efficient sections are usually used as a guide or starting point in designing the open channel. Vatankhah [3] presented a general solution for semi-regular polygon sections as the most hydraulically efficient sections (the best hydraulic sections). Abdulrahman [4] presented a best hydraulic section with a trapezoidal section at the bottom and a rectangular section at the top. This section is the special case of the general solution presented by [3]. Babaeyan-Koopaei et al. [5] proposed a parabolic bottomed triangular shape section using the Lagrange multiplier optimization method. However, when comparing the optimum sections, there is little difference between round-bottom triangular and sharp-cornered trapezoidal sections.

The side slope of the semi-circular section at the water surface is vertical which will cause construction difficulties [3]. This issue can be eliminated by choosing other hydraulically efficient channel sections such as power-law section. Optimization procedure allows the analysis of sophisticated types of irrigation open channels such as power-law and curved channel sections. Large parabolic open channels have been constructed in Spain and Pakistan [6]. Since the technology for constructing curved channel sections already exists, the construction of the power-law channel section is easy and possible. A power-law channel section can model a wide range of man-made and natural channel shapes including rectangular, cubic parabolic, parabolic, semi-cubic parabolic and triangular sections. The generic results of a power-law channel can be also applied to its special forms. Generally unlined channels (natural rivers) and irrigation furrows tend to approximate a stable power-law shape after a long period [2]. The side slope of the power-law section at the water surface is not vertical and thus this section eliminates the construction difficulties when the side slopes are restricted. This section can be designed as the simple (with no horizontal bottom) or complex (with a horizontal bottom) sections with the smallest deviation from the semi-circular section.

The hydraulically efficient power-law channels with specified exponent s , have been partially studied and solved by some researchers. For example, several special cases of this versatile section have been studied in the literature as optimum channel sections. These special sections are: semi-cubic section [7] with $s = 5/3$, cubic section [8] with $s = 4/3$, one-third parabolic section [9] $s = 13/10$ and parabolic section [10–14] with $s = 3/2$. The exponent s is not a decision variable in these optimization studies and thus these sections are the hydraulically efficient power-law sections and are not the most hydraulically efficient ones. All these separate studies can be combined in a unified optimization model of a general power-law section to find the hydraulically efficient power-law sections as shown in this study.

Due to the specific mathematical form of the wetted perimeter, the optimum power-law section has been developed in the literature for a specific exponent s . Computation of the wetted perimeter of the power-law sections results in the hypergeometric functions [15–17]. The hypergeometric functions can be computed using commercial mathematical software such as Matlab, MathCad and Maple. Anwar and de Vries [6] presented two approximations for the wetted perimeter of the broad and narrow sections using series expansion. Anwar and Clarke [18] adopted a similar approximation for the power-law section, but when $s = 2$ and water surface width to flow depth is equal to unity, the solution is discontinuous. Exact solutions of the wetted perimeter are available for specific values [19] of the exponent s . Hussein [20] approximated the wetted perimeter of a power-law channel for entire range of the exponent with a relative error up to 2.3%. A solution of the wetted perimeter for $1.5 < s < 2$ has been also developed by Vatankhah [17], but other accurate solutions that are applicable for a wider range of the exponent are still required.

Recently, Han and Easa [21] and Chen et al. [22] presented optimum hydraulic power-law sections using the Lagrange multiplier optimization method. In practice, sometimes open channel sections are constrained by side slope (when slope stability and geotechnical conditions are important), but, the optimum hydraulic section for a given side slope has not been solved by the authors. Analytical solutions for the wetted perimeter of the power-law sections are developed by Han and Easa [21] and Chen et al. [22] using the Gaussian hypergeometric functions. They also proposed exact optimization model using the Gaussian hypergeometric functions. For the exponent $s = 2$, the denominator of Gaussian hypergeometric function proposed for the wetted perimeter approaches zero, thus the solution is discontinuous and the Gaussian hypergeometric functions proposed for the wetted perimeter computation and optimization model are inaccurate near $s = 2$ (an ill-posed problem). The hypergeometric functions near $s = 2$ are not accurate to deduce the exact and accurate solutions for the most hydraulically and hydraulically efficient power-law sections. In this study, the wetted perimeter and other related integral equations are computed using the accurate numerical integration methods such as Romberg method (with a tolerance of 10^{-12}). The Romberg integration method involves creating trapezoids under the curve and bisecting the width of the trapezoids until the approximation stabilizes.

The rigorous Lagrange multiplier method is widely used for determining the dimensions of optimum channel sections by considering the Manning formula as a constraint [7–9, 21, 22]. This method is particularly useful in studying and solving the constrained optimization problems.

Defining the dimensionless objective function as the wetted perimeter to the square root of the flow area [15], the objective function can be minimized without considering any constraint such as the Manning formula. Unconstrained optimization problems minimize the objective function with no restrictions on their values. The simple formulation and implementation of this method can be considered as its advantage. This alternative method can be easily adopted to establish the optimum dimensions of the most hydraulically and hydraulically efficient power-law sections and other channel sections.

This study presents different research contributions to the power-law sections. First, optimum dimensions are optioned using simple unconstrained optimization models (this approach can be simply applied for other channel sections). Second, optimum dimensions are presented in tabular forms or accurate regression-based relations for practical applications. Third, a new simple approximation of the wetted perimeter for a wide range of $s \in [1.2, 2]$ is developed which is useful for normal flow depth computation of optimum sections by Manning formula. Besides, the efficiency of the optimum sections are quantified and ranked as: semi-circle shaped > best power-law $\cong$ cubic > parabolic > semi-cubic > triangular = rectangular. The following sections present the characteristics of the simple and complex power-law channels along with their most hydraulically and hydraulically efficient sections. Comparison of the most hydraulically

and hydraulically efficient power-law sections is then presented followed by application examples and conclusions.

2 Power-law channel section

A simple open channel section with symmetrical power-law sides is represented by:

$$A = ry^s \quad (1)$$

where y is the depth of the cross section and A is the cross section area corresponding to the depth y . Practical cross sections for power-law channels are in range of $1 \leq s \leq 2$. For $s > 2$, the power-law sections are concave downward (a pair of bird-wing) which are not of practical interest in open channel hydraulics. The dimension of r is $[L^{2-s}]$ (L = length) and the exponent s is dimensionless. The parameters r and s should be determined by optimization.

The top width, T , of the channel for the depth of y is equal to:

$$T = 2x = rsy^{s-1} \quad (2)$$

Since the channel section is symmetrical, the half top width of the water surface, $T/2$, is equal to x coordinate. As shown in Fig. 1 using Eq. (2), r and s are parameters for which channel section takes different shapes. For example, for $s = 1, 4/3, 3/2, 5/3$ and 2 , the function represents rectangular, cubic, parabolic, semi-cubic and triangular sections,

Fig. 1 Power-law sections for different values of exponent s

respectively. Open channel sections can be divided into two families. Simple sections consist of only two elements, while the complex sections consist of at least three elements. According to this definition, Eq. (2) does not result in a simple channel shape for $s = 1$ (rectangular section) as shown in Fig. 1.

The geometry of the power-law section is shown in Fig. 2 for two cases. Case (a) shows a simple power-law channel while case (b) shows a complex power-law channel. In Fig. 2, T is the water surface width at the water flow depth h and b is the bottom width.

3 Geometric properties of channels with power-law section

3.1 Simple power-law section ($b = 0$)

For simple power-law channels, as shown in Fig. 2a, the side slope at the water surface, $1/Z$, can be computed as:

$$\frac{1}{Z} = \left. \frac{dy}{dx} \right|_{y=h} = \frac{2}{rs(s-1)h^{s-2}} \quad (3)$$

in which h is the water flow depth.

Equation (3) can be rewritten as:

$$Z = \frac{1}{2}rs(s-1)h^{s-2} \quad (4)$$

Applying $A = rh^s$ into Eq. (4) yields:

$$Z = \frac{1}{2}As(s-1)h^{-2} = \frac{1}{2}s(s-1)h_*^{-2} \quad (5)$$

in which $h_* = h/A^{1/2}$ is dimensionless flow depth and A is the flow area for the water flow depth h .

The wetted perimeter P for the water flow depth h can be derived by integration of the arc length as:

$$P = 2 \int_0^h \sqrt{1 + (dx/dy)^2} dy \quad (6)$$

Substituting dx/dy from Eq. (2) into Eq. (6) one gets:

$$P = 2h \int_0^1 \sqrt{1 + [rs(s-1)h_*^{-2}t^{s-2}/2]^2} dt \quad (7)$$

Equation (7) is not valid for $s = 1$ which results in a complex section with three distinct elements.

The shape factor, P_* , defined as the ratio of the wetted perimeter to the square root of the flow area and computed as:

$$P_* = \frac{P}{\sqrt{A}} = 2h_* \int_0^1 \sqrt{1 + [s(s-1)h_*^{-2}t^{s-2}/2]^2} dt \quad (8)$$

in which $P_* = P/A^{1/2}$, $h_* = h/A^{1/2}$.

Eliminating h_* from Eqs. (5) and (8) yields:

$$P_* = \frac{P}{\sqrt{A}} = 2\sqrt{\frac{s(s-1)}{2Z}} \int_0^1 \sqrt{1 + Z^2 t^{2s-4}} dt \quad (9)$$

Equation (8) has two independent variables (s, h_*), while Eq. (9) has two independent variables (s, Z).

A channel section having the least wetted perimeter for a given flow area has the maximum conveyance (flow discharge) and is known as the best hydraulic section or the most hydraulically efficient section. The best hydraulic section can be equivalently regarded as the section having the minimum value of the shape factor, P_* , or maximum value of the hydraulic radius. For the ideal semi-circle section, the shape factor is equal to $P_* = (2\pi)^{1/2} = 2.5066$. The shape factor of a channel section can be scaled with the corresponding value for a semi-circle. The scaled shape factor is defined as $P_{**} = P/(2\pi A)^{1/2}$. The semi-circle has the least scaled shape factor ($P_{**} = 1$). The best hydraulic

Fig. 2 Geometry of the power-law section **a** simple power-law channel and **b** complex power-law channel

sections should have a scaled shape factor sufficiently close to unity.

3.2 Complex power-law section ($b \neq 0$)

The flow area of a complex power-law section with the horizontal bottom width of $b \neq 0$ (Fig. 2b), is computed as:

$$A = bh + rh^s \quad (10)$$

The wetted perimeter P for the water flow depth h in a complex power-law channel section is also expressed as:

$$P = b + 2 \int_0^h \sqrt{1 + (dx/dy)^2} dy \quad (11)$$

Substituting dx/dy from Eq. (2) into Eq. (11) one gets:

$$P = b + 2h \int_0^1 \sqrt{1 + [rs(s-1)h^{s-2}t^{s-2}/2]^2} dt \quad (12)$$

Applying $A-bh=rh^s$ into Eq. (12) yields:

$$P = b + 2h \int_0^1 \sqrt{1 + [s(s-1)(A-bh)h^{-2}t^{s-2}/2]^2} dt \quad (13)$$

Using Eq. (13), the shape factor, P_* , for the power-law channel sections can be written in dimensionless form as:

$$P_* = \frac{P}{\sqrt{A}} = b_* + 2h_* \int_0^1 \sqrt{1 + [s(s-1)(h_*^{-2} - b_*h_*^{-1})t^{s-2}/2]^2} dt \quad (14)$$

in which the dimensionless bottom width is $b_* = b/A^{1/2}$.

For the complex power-law channel sections, the side slope at the water surface, Z (1 vertical to Z horizontal), can be computed by applying $A-bh=rh^s$ into Eq. (4) as:

$$Z = \frac{1}{2}s(s-1)(A-bh)h^{-2} = \frac{1}{2}s(s-1)(1-b_*h_*)h_*^{-2} \quad (15)$$

For $b_*=0$, Eqs. (14) and (15) reduce to the simple power-law section [Eqs. (8) and (5)].

In this study, the dimensionless parameters are defined to develop the general solution for the optimum hydraulic power-law sections. The most difficult task with a power-law section is the calculation of the wetted perimeter which ends up the hypergeometric functions that cannot be analytically expressed by transcendental functions. Approximate solutions for the wetted perimeter in range of $s \in [1.5, 2]$ have been developed [17]. The approximate solution of the wetted perimeter presented by Han and Easa [21] for the range of $1.167 < s < 2$ is only accurate for the shape factor $a=0.4$ (they defined power-law section by $y=ax^k$, where $k=(s-1)^{-1}$). Their solution is not accurate for other shape factors (for $a=0.5$ and water surface width of 2 m, the relative error reaches up to 6.8% for s near 1.167).

In this study, using the method of undetermined coefficients [17], a new approximation for a wider range of $s \in [1.2, 2]$ is developed as follows:

$$\begin{aligned} \frac{P}{2h} = & 0.00065 \sqrt{1 + \frac{s^2(s-1)^2}{4h_*^4} \times 0.00006^{2s-4}} \\ & + 0.1214 \sqrt{1 + \frac{s^2(s-1)^2}{4h_*^4} \times 0.03137^{2s-4}} \\ & + 0.8755 \sqrt{1 + \frac{s^2(s-1)^2}{4h_*^4} \times 0.5148^{2s-4}} \end{aligned} \quad (16)$$

Equation (16) is valid over the ranges of $h_* \in [0.01, 10]$ and $s \in [1.2, 2]$, and has a maximum relative error less than 0.6%. Eq. (16) is an approximate solution for the wetted perimeter and is not suitable to deduce the exact and accurate solutions for the optimum hydraulic sections of the power-law channels. In this study for deducing the accurate solutions for the optimum hydraulic sections, the wetted perimeter is computed using the exact numerical integration methods. After determining the optimum values of h_* and s , the wetted perimeter can be accurately computed using the exact numerical integration methods or can be estimated using simple Eq. (16) with a maximum relative error less than 0.6%. In this research all computations in optimization procedure are exact and accurate based on the exact numerical integration methods with a tolerance of 10^{-12} .

As shown in next sections, the best hydraulic section of the power-law channels and hydraulically efficient power-law channels can be accurately determined using the shape factor equation without any need for Lagrange multiplier technique and the Gaussian hypergeometric function which is inaccurate near $s=2$ (an ill-posed problem).

4 Best hydraulic and hydraulically efficient simple power-law section

4.1 Simple power-law section with P_* as a function of s and h_*

For determining the optimum hydraulic sections of the simple power-law channels ($b=0$), the partial derivatives of P_* should be determined. The partial derivatives of P_* with respect to s and h_* are computed using Eq. (8) as the following:

$$\frac{\partial P_*}{\partial s} = \frac{s(s-1)}{2h_*^3} \int_0^1 \frac{[2s + (s^2-s)\ln(t)-1]t^{2s-4}}{\sqrt{1 + [s(s-1)h_*^{-2}t^{s-2}/2]^2}} dt \quad (17)$$

$$\frac{\partial P_*}{\partial h_*} = 2 \int_0^1 \sqrt{1 + [s(s-1)h_*^{-2}t^{s-2}/2]^2} dt - \frac{s^2(s-1)^2}{h_*^4} \int_0^1 \frac{t^{2s-4}}{\sqrt{1 + [s(s-1)h_*^{-2}t^{s-2}/2]^2}} dt \quad (18)$$

4.1.1 Best hydraulic section (s and h_* as variable parameters)

For determining the best hydraulic section of the simple power-law channels, the system of equations $\partial P_*/\partial s = 0$ and $\partial P_*/\partial h_* = 0$ should be solved. Solving this system of equations numerically [Eqs. (17) and (18)], yields the only feasible solution for s and h_* ($1 < s \leq 2$ and $h_* > 0$) as $h_* = 0.7814$ and $s = 1.299$. These results are the same as those obtained for the power-law section by Kacimov [15]. This extremum point provides the global minimum wetted perimeter (maximum discharge) among all possible power-law section shapes. The dimensionless flow depth ($h_* = 0.7814$) and exponent ($s = 1.299$) of the best hydraulic power-law section are the most important design parameters for the simple power-law section. Based on these optimum values and using Eq. (5), one gets $Z = 0.318$. Also based on these optimum values, one gets $r = 1.638 h^{0.701}$, $A = 1.638 h^2$, $P = 3.231 h$ and $T = 2.127 h$. Substituting optimum values of $A = 1.638 h^2$ and $P = 3.231 h$ into Manning's formula, the normal depth can be simply computed. Similarly, the critical flow depth can be computed for this section using $A = 1.638 h^2$ and $T = 2.127 h$.

4.1.2 Hydraulically efficient sections for a given exponent s (h_* as a variable parameters)

For determining the hydraulically efficient power-law sections for a given exponent, s , equation $\partial P_*/\partial h_* = 0$ should be solved numerically. The general governing equation for this case is:

$$2 \int_0^1 \sqrt{1 + [s(s-1)h_*^{-2}t^{s-2}/2]^2} dt - \frac{s^2(s-1)^2}{h_*^4} \int_0^1 \frac{t^{2s-4}}{\sqrt{1 + [s(s-1)h_*^{-2}t^{s-2}/2]^2}} dt = 0 \quad (19)$$

By solving Eq. (19), the optimum hydraulic sections of power-law channels can be determined for any given exponent, s , in practical range of (1, 2]. The exact solution of h_* for any value of s is numerically computed (with a tolerance of 10^{-12}) and reported in Table 1. Other characteristics of the optimum hydraulic sections are also presented in Table 1. It is worth noting that the results of Table 1 are the same as those obtained for semi-cubic section [7] with

$s = 5/3$, cubic section [8] with $s = 4/3$, one-third parabolic section [9] $s = 13/10$ and parabolic section [12] with $s = 3/2$.

Using curve fitting method ($s \in (1, 2]$, step = 0.001 is used for developing an accurate relation between h_* and s), h_* can be expressed in terms of s as:

$$h_* = 2.657s^{0.176} + 0.049s^{-20} - 2 \quad (20)$$

The graphical representation between h_* and s of Eq. (19) is shown in Fig. 3. The curve has a minimum at $s = 1.03$. In the figure, the relative error of proposed Eq. (20) is also presented. As noted, the relative error ranges from -0.22 to $+0.22\%$, which clearly shows that the proposed approximation for h_* is very accurate for practical purposes.

Figure 4 shows different dimensionless parameters of the optimum power-law sections based on Eq. (19) and Table 1. There is a minimum point for $P/A^{0.5}$ at $s = 1.299$ and $h_* = 0.7814$ that corresponds to the best hydraulic power-law section which has the largest discharge (smallest wetted perimeter) for a given flow area.

The optimum values of $P_{**} = P/(2\pi A)^{0.5}$ for different values of s are shown in Fig. 5. The relationship between P_{**} and s shows that the scaled wetted perimeter of the optimum hydraulic section decreases as s increases from 1 to 1.299 and increases as s increases from 1.299 to 2. There is a minimum point that corresponds to the best hydraulic power-law section ($s = 1.299$ and $h_* = 0.7814$) which has the smallest scaled wetted perimeter or largest discharge for a given flow area. Other optimum hydraulic sections (rectangular, cubic, parabolic, semi-cubic and triangular sections) are also shown on the figure. As noted, the best hydraulic power-law section ($s = 1.299$) and cubic section ($s = 4/3$) have approximately the same efficiency. The scaled wetted perimeters of semi-circular, best power-law, rectangular, cubic, parabolic, semi-cubic and triangular sections are respectively as $P_{**} = 1, 1.0073, 1.1284, 1.0078, 1.0218, 1.0486$ and 1.1284 . The difference in the scaled wetted perimeters of the optimum power-law sections with the semi-circle shaped section are: best power-law (0.73%), rectangular (12.84%), cubic (0.78%), parabolic (2.18%), semi-cubic (4.86%) and triangular (12.84%). The comparison of the results shows that, for a given flow area, there is a small difference between the best power-law, cubic and parabolic sections. The efficiency of the sections can be ranked as: semi-circle shaped $>$ best power-law $\cong$ cubic $>$ parabolic $>$ semi-cubic $>$ triangular ($Z = 1$) = rectangular ($Z = 0$). The semi-circular section has a vertical side slope at the top. However, the proposed sections (best power-law, cubic and parabolic sections) will eliminate the construction difficulties in relation to the vertical side slope with practically no increase in the scaled wetted perimeter.

Table 1 Optimum hydraulic quantities of the simple power-law sections for given values of exponent s

Row	s	h_*	Z	A/h^2	T/h	P/h	$P/A^{0.5}$	Row	s	h_*	Z	A/h^2	T/h	P/h	$P/A^{0.5}$
1	1.01	0.700282	0.010298	2.039171	2.059563	3.960606	2.773542	52	1.51	0.857693	0.523424	1.359367	2.052644	2.989945	2.564453
2	1.02	0.698148	0.020927	2.051657	2.092690	3.922427	2.738436	53	1.52	0.861109	0.532967	1.348601	2.049874	2.982003	2.567830
3	1.03	0.697640	0.031744	2.054649	2.116288	3.885451	2.710645	54	1.53	0.864504	0.542504	1.338030	2.047186	2.974321	2.571313
4	1.04	0.698110	0.042679	2.051881	2.133957	3.849655	2.687483	55	1.54	0.867877	0.552037	1.327649	2.044580	2.966894	2.574900
5	1.05	0.699256	0.053686	2.045163	2.147421	3.815010	2.667667	56	1.55	0.871228	0.561565	1.317456	2.042056	2.959715	2.578588
6	1.06	0.700901	0.064731	2.035575	2.157709	3.781483	2.650444	57	1.56	0.874557	0.571092	1.307445	2.039615	2.952780	2.582375
7	1.07	0.702930	0.075793	2.023836	2.165505	3.749040	2.635314	58	1.57	0.877864	0.580618	1.297615	2.037255	2.946083	2.586260
8	1.08	0.705265	0.086852	2.010459	2.171295	3.717646	2.621926	59	1.58	0.881148	0.590143	1.287960	2.034977	2.939620	2.590240
9	1.09	0.707846	0.097895	1.995823	2.175447	3.687267	2.610019	60	1.59	0.884409	0.599670	1.278479	2.032781	2.933386	2.594313
10	1.10	0.710630	0.108912	1.980220	2.178242	3.657867	2.599389	61	1.60	0.887648	0.609200	1.269166	2.030666	2.927376	2.598479
11	1.11	0.713580	0.119895	1.963877	2.179903	3.629412	2.589877	62	1.61	0.890864	0.618733	1.260020	2.028632	2.921585	2.602734
12	1.12	0.716671	0.130837	1.946976	2.180613	3.601867	2.581353	63	1.62	0.894057	0.628270	1.251036	2.026678	2.916010	2.607079
13	1.13	0.719879	0.141734	1.929662	2.180518	3.575201	2.573711	64	1.63	0.897227	0.637813	1.242211	2.024805	2.910647	2.611510
14	1.14	0.723185	0.152582	1.912056	2.179744	3.549381	2.566861	65	1.64	0.900374	0.647363	1.233543	2.023010	2.905490	2.616028
15	1.15	0.726576	0.163379	1.894254	2.178392	3.524377	2.560727	66	1.65	0.903498	0.656921	1.225027	2.021294	2.900538	2.620630
16	1.16	0.730036	0.174124	1.876337	2.176551	3.500160	2.555244	67	1.66	0.906599	0.666486	1.216660	2.019656	2.895784	2.625316
17	1.17	0.733557	0.184815	1.858372	2.174295	3.476701	2.550357	68	1.67	0.909678	0.676062	1.208440	2.018095	2.891227	2.630084
18	1.18	0.737127	0.195452	1.840412	2.171687	3.453972	2.546016	69	1.68	0.912733	0.685647	1.200363	2.016610	2.886862	2.634934
19	1.19	0.740740	0.206034	1.822506	2.168782	3.431948	2.542179	70	1.69	0.915765	0.695244	1.192426	2.015200	2.882685	2.639863
20	1.20	0.744387	0.216563	1.804690	2.165628	3.410603	2.538808	71	1.70	0.918775	0.704853	1.184626	2.013865	2.878694	2.644873
21	1.21	0.748063	0.227038	1.786997	2.162266	3.389915	2.535870	72	1.71	0.921763	0.714474	1.176961	2.012602	2.874884	2.649961
22	1.22	0.751762	0.237460	1.769452	2.158732	3.369859	2.533334	73	1.72	0.924727	0.724108	1.169426	2.011412	2.871253	2.655126
23	1.23	0.755481	0.247831	1.752078	2.155056	3.350415	2.531173	74	1.73	0.927670	0.733757	1.162019	2.010293	2.867797	2.660368
24	1.24	0.759213	0.258152	1.734893	2.151267	3.331561	2.529365	75	1.74	0.930590	0.743420	1.154738	2.009244	2.864513	2.665687
25	1.25	0.762956	0.268423	1.717910	2.147388	3.313277	2.527886	76	1.75	0.933488	0.753098	1.147578	2.008262	2.861398	2.671081
26	1.26	0.766707	0.278647	1.701143	2.143440	3.295544	2.526718	77	1.76	0.936365	0.762792	1.140539	2.007348	2.858448	2.676549
27	1.27	0.770463	0.288825	1.684600	2.139443	3.278345	2.525842	78	1.77	0.939220	0.772502	1.133615	2.006499	2.855660	2.682092
28	1.28	0.774220	0.298958	1.668290	2.135411	3.261661	2.525242	79	1.78	0.942053	0.782228	1.126805	2.005714	2.853031	2.687708
29	1.29	0.777976	0.309047	1.652217	2.131360	3.245477	2.524904	80	1.79	0.944866	0.791971	1.120106	2.004990	2.850558	2.693396
30	1.30	0.781355	0.318093	1.637960	2.127710	3.231325	2.524812	81	1.80	0.947659	0.801731	1.113515	2.004327	2.848238	2.699157
31	1.30	0.781730	0.319095	1.636387	2.127303	3.229776	2.524814	82	1.81	0.950431	0.811508	1.107029	2.003723	2.846066	2.704989
32	1.31	0.785480	0.329104	1.620802	2.123250	3.214543	2.524959	83	1.82	0.953183	0.821302	1.100646	2.003175	2.844041	2.710892
33	1.32	0.789223	0.339074	1.605464	2.119212	3.199763	2.525327	84	1.83	0.955916	0.831113	1.094361	2.002681	2.842159	2.716865
34	1.33	0.792958	0.349008	1.590374	2.115197	3.185424	2.525909	85	1.84	0.958630	0.840941	1.088174	2.002240	2.840416	2.722907
35	1.34	0.796684	0.358906	1.575533	2.111214	3.171512	2.526694	86	1.85	0.961325	0.850785	1.082080	2.001848	2.838809	2.729019
36	1.35	0.800400	0.368772	1.560940	2.107269	3.158014	2.527673	87	1.86	0.964003	0.860647	1.076077	2.001504	2.837335	2.735199
37	1.36	0.804103	0.378606	1.546594	2.103368	3.144918	2.528839	88	1.87	0.966663	0.870524	1.070162	2.001204	2.835989	2.741446

Table 1 (continued)

Row	s	h_*	Z	A/h^2	T/h	P/h	$P/A^{0.5}$	Row	s	h_*	Z	A/h^2	T/h	P/h	$P/A^{0.5}$
38	1.37	0.807794	0.388410	1.532493	2.099516	3.132212	2.530183	89	1.88	0.969307	0.880416	1.064333	2.000946	2.834768	2.747760
39	1.38	0.811471	0.398187	1.518637	2.095719	3.119886	2.531697	90	1.89	0.971934	0.890324	1.058586	2.000728	2.833669	2.754140
40	1.39	0.815133	0.407936	1.505022	2.091980	3.107929	2.533377	91	1.90	0.974547	0.900245	1.052919	2.000545	2.832688	2.760586
41	1.40	0.818780	0.417661	1.491645	2.088303	3.096332	2.535214	92	1.91	0.977144	0.910180	1.047328	2.000396	2.831819	2.767096
42	1.41	0.822410	0.427362	1.478505	2.084692	3.085083	2.537204	93	1.92	0.979728	0.920127	1.041811	2.000277	2.831061	2.773670
43	1.42	0.826024	0.437041	1.465598	2.081149	3.074174	2.539341	94	1.93	0.982299	0.930086	1.036365	2.000185	2.830408	2.780306
44	1.43	0.829620	0.446701	1.452921	2.077677	3.063597	2.541620	95	1.94	0.984857	0.940054	1.030987	2.000115	2.829856	2.787005
45	1.44	0.833197	0.456341	1.440470	2.074277	3.053342	2.544036	96	1.95	0.987405	0.950031	1.025675	2.000066	2.829401	2.793764
46	1.45	0.836756	0.465964	1.428243	2.070953	3.043401	2.546585	97	1.96	0.989941	0.960016	1.020425	2.000034	2.829039	2.800582
47	1.46	0.840296	0.475572	1.416236	2.067704	3.033767	2.549262	98	1.97	0.992468	0.970007	1.015236	2.000014	2.828765	2.807459
48	1.47	0.843816	0.485165	1.404444	2.064533	3.024432	2.552064	99	1.98	0.994986	0.980002	1.010103	2.000004	2.828574	2.814393
49	1.48	0.847316	0.494746	1.392865	2.061440	3.015389	2.554988	100	1.99	0.997497	0.990000	1.005025	2.000001	2.828463	2.821383
50	1.49	0.850796	0.504315	1.381495	2.058428	3.006632	2.558030	101	2.00	1.000000	1.000000	1.000000	2.000000	2.828427	2.828427
51	1.50	0.854255	0.513874	1.370330	2.055495	2.998152	2.561186	102

Fig. 3 Graphical representation between h_* and s , along with relative error of proposed Eq. (20) for direct computation of h_* in terms of s (optimum simple power-law section for a given exponent s)

A comparison between the semi-circle shaped and best power-law sections is presented in Fig. 6. Comparing the sections shows that there is minute difference between them. The relative difference in the scaled wetted perimeter is 0.73%.

4.1.3 Hydraulically efficient sections for a given dimensionless flow depth h_* (s as a variable parameters)

For determining the hydraulically efficient power-law sections for a given dimensionless flow depth, h_* , equation $\partial P_*/\partial s = 0$ should be solved numerically. The general governing equation for this case is:

Fig. 4 Different dimensionless parameters of the optimum simple power-law sections (hydraulically efficient sections for a given exponent s)

Fig. 5 Scaled wetted perimeter of the simple optimum power-law sections in terms of exponent s (hydraulically efficient sections for a given exponent s)

$$\frac{s(s-1)}{2h_*^3} \int_0^1 \frac{[2s + (s^2 - s) \ln(t) - 1]t^{2s-4}}{\sqrt{1 + [s(s-1)h_*^{-2}t^{s-2}/2]^2}} dt = 0 \quad (21)$$

By solving Eq. (21), the optimum hydraulic sections of power-law channels can be determined for any given dimensionless flow depth. The exact solution of s for any value of h_* is numerically computed with a tolerance of 10^{-12} . According to numerical results, Eq. (21) has feasible solutions only for $s \in [1.299, 1.765]$. Using curve fitting

Fig. 6 Comparison between the semi-circle shaped section and best hydraulic power-law section

method to fit the data of h_* and s , the exponent s can be computed in terms of dimensionless flow depth h_* as:

$$s = \frac{1.05h_*^{0.2} + 2.6h_*^{-1.1}}{1 + 1.5h_*^{-1.92}} \quad (22)$$

The graphical representation between s and h_* of Eq. (21) is shown in Fig. 7. In the figure, the relative error of proposed Eq. (22) is also presented. The relative error of Eq. (22) ranges from -0.53 to $+0.53\%$, which clearly shows that the proposed approximation for s is very accurate for practical purposes.

4.2 Simple power-law section with P_* as a function of s and Z

In this case, the partial derivatives of P_* should be determined. The partial derivatives of P_* with respect to s and Z can be computed using Eq. (9) as the following:

$$\begin{aligned} \frac{\partial P_*}{\partial s} = & Z\sqrt{2Zs(s-1)} \int_0^1 \frac{\ln(t)t^{2s-4}}{\sqrt{1 + Z^2t^{2s-4}}} dt \\ & + \frac{2s-1}{\sqrt{2Zs(s-1)}} \int_0^1 \frac{t^{2s-4}}{\sqrt{1 + Z^2t^{2s-4}}} dt \end{aligned} \quad (23)$$

$$\begin{aligned} \frac{\partial P_*}{\partial Z} = & \sqrt{2Zs(s-1)} \int_0^1 \frac{t^{2s-4}}{\sqrt{1 + Z^2t^{2s-4}}} dt \\ & - \frac{1}{Z} \sqrt{\frac{s(s-1)}{2Z}} \int_0^1 \frac{t^{2s-4}}{\sqrt{1 + Z^2t^{2s-4}}} dt \end{aligned} \quad (24)$$

Fig. 7 Graphical representation between s and h_* , along with relative error of proposed Eq. (21) for direct computation of s in terms of h_* (optimum simple power-law section for a given exponent s)

4.2.1 Best hydraulic section (s and Z as variable parameters)

For determining the best hydraulic section of the simple power-law channels the system of equations $\partial P_*/\partial s = 0$ and $\partial P_*/\partial Z = 0$ should be solved. Solving this system of equations numerically yields the only and feasible solution for s and Z ($1 < s \leq 2$ and $Z > 0$) as $s = 1.299$ and $Z = 0.318$. This solution is the same as that solution when P_* was considered as a function of s and h_* .

4.2.2 Hydraulically efficient sections for a given exponent s (Z as variable parameter)

For determining the hydraulically efficient power-law sections for a given exponent, s , equation $\partial P_*/\partial Z = 0$ should be solved. The general governing equation for this case is:

$$\sqrt{2Zs(s-1)} \int_0^1 \frac{t^{2s-4}}{\sqrt{1+Z^2t^{2s-4}}} dt - \frac{1}{Z} \sqrt{\frac{s(s-1)}{2Z}} \int_0^1 \sqrt{1+Z^2t^{2s-4}} dt = 0 \quad (25)$$

Substituting Z from Eq. (5) into Eq. (25), yields Eq. (19) thus proposed Eq. (20) is also valid for this case. Using Eqs. (5) and (20) one gets:

$$Z = \frac{s(s-1)}{2(2.657s^{0.176} + 0.049s^{-20} - 2)^2} \quad (26)$$

For rectangular section $s = 1$ and thus $Z = 0$.

4.2.3 Hydraulically efficient sections for a given inverse side slope at the water surface Z (s as variable parameter)

For determining the hydraulically efficient power-law sections for a given inverse side slope at the water surface, Z , equation $\partial P_*/\partial s = 0$ should be solved numerically. The general governing equation for this case will be as:

$$Z\sqrt{2Zs(s-1)} \int_0^1 \frac{\ln(t)t^{2s-4}}{\sqrt{1+Z^2t^{2s-4}}} dt + \frac{2s-1}{\sqrt{2Zs(s-1)}} \int_0^1 \sqrt{1+Z^2t^{2s-4}} dt = 0 \quad (27)$$

By solving Eq. (27), the optimum hydraulic sections of power-law channels can be determined for any given inverse side slope at the water surface, Z , in practical range of $s \in (1, 2]$. The exact solution of s for any value of Z is computed numerically (with a tolerance of 10^{-12}).

Using curve fitting method to fit the data, the exponent s can be expressed in terms of s as:

$$s = 1 - 0.56Z^{1.411} + 1.28Z \quad (28)$$

The graphical representation between s and Z of Eq. (27) is shown in Fig. 8. The relative error of proposed Eq. (28) is also presented in the figure. The relative error ranges from -0.29 to $+0.27\%$, which shows that the proposed approximation for s is very accurate for practical purposes.

Fig. 8 Graphical representation between s and Z , along with relative error of proposed Eq. (28) for direct computation of s in terms of Z (optimum simple power-law section for a given Z)

5 Best hydraulic and hydraulically efficient complex power-law section

5.1 Complex power-law section (P_* as a function of s , b_* and h_*)

For determining the best hydraulic section of the complex power-law channels the system of equations $\partial P_*/\partial s = 0$, $\partial P_*/\partial h_* = 0$ and $\partial P_*/\partial b_* = 0$ should be solved while for determining the hydraulically efficient power-law sections for a given exponent, s , the system of equations $\partial P_*/\partial h_* = 0$ and $\partial P_*/\partial s = 0$ should be solved.

The partial derivatives of P_* with respect to s , h_* and b_* can be obtained using Eq. (14) for a complex power-law channel ($b \neq 0$) as:

$$\frac{\partial P_*}{\partial s} = \frac{s(s-1)(h_*^{-2} - b_* h_*^{-1})^2 h_*}{2} \int_0^1 \frac{[2s + (s^2 - s) \ln(t) - 1] t^{2s-4}}{\sqrt{1 + [s(s-1)(h_*^{-2} - b_* h_*^{-1}) t^{s-2}/2]^2}} dt \quad (29)$$

$$\frac{\partial P_*}{\partial h_*} = 2 \int_0^1 \sqrt{1 + [s(s-1)(h_*^{-2} - b_* h_*^{-1}) t^{s-2}/2]^2} dt - \frac{s^2(s-1)^2}{2} (2h_*^{-3} - b_* h_*^{-2})(h_*^{-1} - b_*) \int_0^1 \frac{t^{2s-4}}{\sqrt{1 + [s(s-1)(h_*^{-2} - b_* h_*^{-1}) t^{s-2}/2]^2}} dt \quad (30)$$

$$\frac{\partial P_*}{\partial b_*} = 1 - \frac{s^2(s-1)^2}{2} (h_*^{-2} - b_* h_*^{-1}) \int_0^1 \frac{t^{2s-4}}{\sqrt{1 + [s(s-1)(h_*^{-2} - b_* h_*^{-1}) t^{s-2}/2]^2}} dt \quad (31)$$

5.1.1 Best hydraulic section (s , b_* and h_* as variable parameters)

Solving the system of equations $\partial P_*/\partial s = 0$, $\partial P_*/\partial h_* = 0$ and $\partial P_*/\partial b_* = 0$ [Eqs. (29)–(31)] numerically yields the only and feasible solution for s , h_* and b_* ($1 < s \leq 2$ and $h_* > 0$) as:

$$s = 1.299, \quad b_* = 0, \quad h_* = \frac{h}{\sqrt{bh + rh^s}} = r^{-0.5} h^{1-0.5s} = 0.7814 \quad (32)$$

Extremum point presented by Eq. (32) provides the global minimum wetted perimeter (maximum discharge) among all possible power-law section shapes. The exponent and dimensionless depth and width of the best hydraulic power-law section are $s = 1.299$, $h_* = 0.7814$ and $b_* = 0$, respectively, which are the most important design parameters for the power-law sections. Based on these optimum values and using Eq. (15), one gets $Z = 0.318$. Also based on these optimum values one gets $r = 1.638 h^{0.701}$, $A = 1.638 h^2$, $P = 3.231 h$ and $T = 2.127 h$.

5.1.2 Hydraulically efficient sections for a given exponent s (b_* and h_* as variable parameters)

For determining the hydraulically efficient complex power-law sections for a given exponent, s , the system of equations $\partial P_*/\partial h_* = 0$ and $\partial P_*/\partial b_* = 0$ should be solved numerically. The general governing equations for this case are:

$$2 \int_0^1 \sqrt{1 + [s(s-1)(h_*^{-2} - b_* h_*^{-1})t^{s-2}/2]^2} dt - \frac{s^2(s-1)^2}{2} (2h_*^{-3} - b_* h_*^{-2})(h_*^{-1} - b_*) \int_0^1 \frac{t^{2s-4}}{\sqrt{1 + [s(s-1)(h_*^{-2} - b_* h_*^{-1})t^{s-2}/2]^2}} dt = 0 \quad (33)$$

$$1 - \frac{s^2(s-1)^2}{2} (h_*^{-2} - b_* h_*^{-1}) \int_0^1 \frac{t^{2s-4}}{\sqrt{1 + [s(s-1)(h_*^{-2} - b_* h_*^{-1})t^{s-2}/2]^2}} dt = 0 \quad (34)$$

By solving Eqs. (33) and (34), the optimum hydraulic sections of the complex power-law channels can be determined for any given exponent, s , in practical range of (1, 2]. It is interesting that for $s < 1.327$, the dimensionless width b_* will be negative which is not feasible for practical applications. For $1 < s < 1.327$, b_* should be set to zero for a feasible solution and thus for this range of exponent the optimum hydraulic sections are as reported in Table 1 (optimum simple power-law sections). The exact solutions of h_* and b_* for any value of $s > 1.327$ are computed numerically (with a tolerance of 10^{-12}) and reported in Table 2. Other characteristics of the optimum complex power-law sections [$T/h = sh_*^{-2} + b_* h_*^{-1}(1-s)$] are also presented in Table 2. It is noteworthy that the results of Table 2 are the same as those obtained for complex parabolic section [13, 14] with $s = 3/2$ and complex cubic section [7] with $s = 4/3$.

Using curve fitting method h_* can be expressed in terms of s in range of $1.33 \leq s \leq 2$ as:

$$h_* = 0.8 - 0.0028s^{3.86} \quad (35)$$

The graphical representation between the fitted numerical values of h_* and s is shown in Fig. 9. In the figure, the relative error of proposed Eq. (35) compared to exact numerical values is also presented. The relative error ranges from -0.069 to $+0.065\%$, which clearly shows that the proposed approximation for h_* is very accurate for practical purposes.

Similarly b_* can be expressed in terms of s in range of $1.33 \leq s \leq 2$ as:

$$b_* = 1.245 \sinh(0.314s^{1.1086} - 3.69s^{-7.6}) \quad (36)$$

The graphical representation between the fitted numerical values of b_* and s is shown in Fig. 10. In the figure, the relative error of proposed Eq. (36) compared to exact numerical values is also presented. As noted, the relative error ranges from -0.27 to $+0.28\%$, which clearly shows that the proposed approximation for b_* is very accurate.

The optimum values of P_{**} in terms of s for both simple and complex power-law sections are shown in Fig. 11. For $s > 1.327$ the complex power-law section with a horizontal bottom $b \neq 0$ is more efficient than the simple power-law section with no horizontal bottom $b = 0$.

The scaled wetted perimeters of complex sections with parabolic sides ($s = 1.5$), semi-cubic sides ($s = 5/3$) and linear sides ($s = 2$, most efficient trapezoidal section with $Z = 3^{-0.5}$) are respectively as $P_{**} = 1.014$, 1.023 , and 1.05 . Thus the difference in the scaled wetted perimeters of the optimum complex power-law sections with the semi-circle shaped section will be as: parabolic sides (1.4%), semi-cubic sides (2.3%) and trapezoidal Section (5%). The proposed complex sections (sections with parabolic sides and sections with semi-cubic sides) will eliminate the construction difficulties in relation to the vertical side slope with practically no increase in the scaled wetted perimeter.

5.1.3 Hydraulically efficient sections for a given dimensionless bottom width b_* (s and h_* as variable parameters)

For determining the hydraulically efficient complex power-law sections for a given dimensionless bottom width, b_* , the system of equations $\partial P_*/\partial s = 0$ and $\partial P_*/\partial h_* = 0$ should be solved numerically. The general governing equations for this case are:

$$\frac{s(s-1)(h_*^{-2} - b_* h_*^{-1})^2 h_*}{2} \int_0^1 \frac{[2s + (s^2 - s) \ln(t) - 1] t^{2s-4}}{\sqrt{1 + [s(s-1)(h_*^{-2} - b_* h_*^{-1})t^{s-2}/2]^2}} dt = 0 \quad (37)$$

$$2 \int_0^1 \sqrt{1 + [s(s-1)(h_*^{-2} - b_* h_*^{-1})t^{s-2}/2]^2} dt - \frac{s^2(s-1)^2}{2} (2h_*^{-3} - b_* h_*^{-2})(h_*^{-1} - b_*) \int_0^1 \frac{t^{2s-4}}{\sqrt{1 + [s(s-1)(h_*^{-2} - b_* h_*^{-1})t^{s-2}/2]^2}} dt = 0 \quad (38)$$

By solving Eqs. (37) and (38), the optimum hydraulic sections of the complex power-law channels can be determined for any given b_* . This system of equations has feasible solution only in range of $s \in [1.299, 1.765]$. Using curve fitting method to fit the data of b_* and s the exponent s can be expressed in terms of b_* in range of $1.299 \leq s \leq 1.765$ as the following:

$$s = \frac{1.335b_*^{0.006} + 1.41b_*^{1.99}}{1 + 0.742b_*^{2.03}} \quad (39)$$

Table 2 Optimum hydraulic quantities of the complex power-law sections for different values of exponent s

Row	s	h_*	b_*	Z	A/h^2	T/h	P/h	$P/A^{0.5}$	Row	s	h_*	b_*	Z	A/h^2	T/h	P/h	$P/A^{0.5}$
1	1.33	0.791797	0.010350	0.347164	1.595044	2.117095	3.190088	2.525901	35	1.67	0.779658	0.619821	0.475591	1.645096	2.214667	3.290192	2.565226
2	1.34	0.791593	0.044074	0.350854	1.595864	2.119527	3.191728	2.526550	36	1.68	0.779147	0.629622	0.479331	1.647257	2.217890	3.294514	2.566910
3	1.35	0.791380	0.075944	0.354554	1.596723	2.121988	3.193446	2.527230	37	1.69	0.778627	0.639220	0.483056	1.649456	2.221120	3.298911	2.568623
4	1.36	0.791158	0.106117	0.358263	1.597621	2.124479	3.195243	2.527941	38	1.70	0.778100	0.648626	0.486764	1.651692	2.224355	3.303384	2.570363
5	1.37	0.790926	0.134731	0.361981	1.598559	2.126998	3.197118	2.528683	39	1.71	0.777565	0.657850	0.490452	1.653965	2.227594	3.307931	2.572132
6	1.38	0.790684	0.161910	0.365707	1.599536	2.129546	3.199072	2.529455	40	1.72	0.777023	0.666901	0.494120	1.656275	2.230834	3.312551	2.573927
7	1.39	0.790433	0.187768	0.369442	1.600552	2.132123	3.201105	2.530259	41	1.73	0.776473	0.675790	0.497765	1.658622	2.234072	3.317243	2.575750
8	1.40	0.790172	0.212404	0.373184	1.601608	2.134729	3.203216	2.531093	42	1.74	0.775916	0.684524	0.501385	1.661004	2.237308	3.322007	2.577599
9	1.41	0.789902	0.235910	0.376935	1.602704	2.137363	3.205407	2.531959	43	1.75	0.775352	0.693112	0.504977	1.663421	2.240537	3.326841	2.579473
10	1.42	0.789623	0.258367	0.380693	1.603838	2.140025	3.207677	2.532855	44	1.76	0.774781	0.701562	0.508539	1.665872	2.243757	3.331745	2.581374
11	1.43	0.789334	0.279851	0.384458	1.605013	2.142716	3.210025	2.533782	45	1.77	0.774204	0.709882	0.512068	1.668358	2.246966	3.336715	2.583298
12	1.44	0.789036	0.300429	0.388230	1.606227	2.145434	3.212453	2.534740	46	1.78	0.773620	0.718078	0.515563	1.670876	2.250160	3.341752	2.585247
13	1.45	0.788728	0.320163	0.392008	1.607480	2.148181	3.214960	2.535729	47	1.79	0.773031	0.726157	0.519019	1.673426	2.253336	3.346853	2.587220
14	1.46	0.788411	0.339110	0.395792	1.608773	2.150955	3.217547	2.536749	48	1.80	0.772435	0.734126	0.522435	1.676008	2.256491	3.352016	2.589215
15	1.47	0.788084	0.357321	0.399583	1.610106	2.153756	3.220213	2.537799	49	1.81	0.771834	0.741990	0.525806	1.678620	2.259621	3.357240	2.591231
16	1.48	0.787749	0.374845	0.403378	1.611479	2.156584	3.222958	2.538881	50	1.82	0.771227	0.749757	0.529131	1.681261	2.262723	3.362522	2.593269
17	1.49	0.787404	0.391724	0.407178	1.612891	2.159439	3.225782	2.539993	51	1.83	0.770616	0.757430	0.532405	1.683930	2.265794	3.367860	2.595327
18	1.50	0.787049	0.408000	0.410982	1.614343	2.162319	3.228687	2.541136	52	1.84	0.770000	0.765	0.53563	1.6866	2.26883	3.37325	2.5974
19	1.51	0.786686	0.423708	0.414790	1.615835	2.165226	3.231670	2.542310	53	1.85	0.769379	0.7725	0.53879	1.6893	2.27182	3.37869	2.5995
20	1.52	0.786313	0.438885	0.418601	1.617367	2.168157	3.234734	2.543515	54	1.86	0.768755	0.7799	0.54189	1.6921	2.27477	3.38418	2.60161
21	1.53	0.785932	0.453560	0.422414	1.618938	2.171114	3.237877	2.544750	55	1.87	0.768127	0.7873	0.54493	1.6949	2.27768	3.38972	2.60374
22	1.54	0.785541	0.467764	0.426229	1.620550	2.174094	3.241100	2.546016	56	1.88	0.767496	0.7946	0.54791	1.6976	2.28053	3.3953	2.60588
23	1.55	0.785141	0.481525	0.430045	1.622201	2.177098	3.244402	2.547313	57	1.89	0.766862	0.8018	0.55081	1.7005	2.28332	3.40091	2.60803
24	1.56	0.784732	0.494867	0.433862	1.623892	2.180125	3.247784	2.548640	58	1.90	0.766226	0.8089	0.55364	1.7033	2.28606	3.40656	2.6102
25	1.57	0.784314	0.507814	0.437678	1.625622	2.183174	3.251245	2.549998	59	1.91	0.765588	0.816	0.55639	1.7061	2.28873	3.41224	2.61237
26	1.58	0.783888	0.520389	0.441492	1.627393	2.186244	3.254785	2.551386	60	1.92	0.764948	0.8231	0.55907	1.709	2.29134	3.41795	2.61456
27	1.59	0.783452	0.532612	0.445305	1.629203	2.189334	3.258405	2.552804	61	1.93	0.764308	0.83	0.56166	1.7118	2.29387	3.42368	2.61675
28	1.60	0.783008	0.544502	0.449114	1.631052	2.192445	3.262104	2.554253	62	1.94	0.763667	0.837	0.56417	1.7147	2.29633	3.42943	2.61894
29	1.61	0.782555	0.556079	0.452919	1.632941	2.195573	3.265882	2.555732	63	1.95	0.763026	0.8438	0.56659	1.7176	2.29872	3.4352	2.62114
30	1.62	0.782093	0.567357	0.456718	1.634870	2.198719	3.269739	2.557240	64	1.96	0.762385	0.8506	0.56892	1.7205	2.30102	3.44097	2.62335
31	1.63	0.781623	0.578355	0.460511	1.636837	2.201881	3.273674	2.558779	65	1.97	0.761745	0.8574	0.57117	1.7234	2.30324	3.44676	2.62555
32	1.64	0.781144	0.589087	0.464296	1.638844	2.205059	3.277688	2.560347	66	1.98	0.761107	0.8641	0.57332	1.7263	2.30538	3.45254	2.62775
33	1.65	0.780657	0.599566	0.468072	1.640889	2.208250	3.281779	2.561944	67	1.99	0.760470	0.8708	0.57538	1.7292	2.30743	3.45832	2.62995
34	1.66	0.780162	0.609807	0.471838	1.642974	2.211453	3.285947	2.563571	68	2.00	0.759836	0.8774	0.57735	1.7321	2.3094	3.4641	2.63215

Fig. 9 Graphical representation between h_* and s , along with relative error of proposed Eq. (35) for direct computation of h_* in terms of s (optimum complex power-law section for a given exponent s)

The graphical representation between the fitted numerical values of b_* and s is shown in Fig. 12. In the figure, the relative error of proposed Eq. (39) compared to exact numerical values is also presented. The relative error ranges from -0.25 to $+0.26\%$, which clearly shows that the proposed approximation for s is very accurate for practical purposes.

Similarly h_* can be expressed in terms of b_* in range of $1.299 \leq s \leq 1.765$ as:

$$h_* = \frac{0.1b_*^{1.687} + 0.766b_*^{-0.005}}{1 + 0.272b_*^{2.26}} \quad (40)$$

The graphical representation between the fitted numerical values of h_* and s is shown in Fig. 13. In the figure, the relative error of proposed Eq. (40) compared to exact numerical values is also presented. As noted, the relative error ranges

Fig. 10 Graphical representation between b_* and s , along with relative error of proposed Eq. (36) for direct computation of b_* in terms of s (optimum complex power-law section for a given exponent s)

from -0.36 to $+0.35\%$, which clearly shows that the proposed approximation for h_* is very accurate for practical purposes.

5.1.4 Hydraulically efficient sections for a given dimensionless depth h_* (b_* and s as variable parameters)

For determining the hydraulically efficient complex power-law sections for a given dimensionless depth, h_* , the system of equations $\partial P_*/\partial s = 0$ and $\partial P_*/\partial b_* = 0$ should be solved numerically. The general governing equations for this case are:

Fig. 11 Graphical comparison between simple and complex optimum power-law sections; scaled wetted perimeter in terms of exponent s (hydraulically efficient sections for a given exponent s)

Fig. 12 Graphical representation between b^* and s , along with relative error of proposed Eq. (39) for direct computation of s in terms of b^* (optimum complex power-law section for a given b^*)

Fig. 13 Graphical representation between b^* and h^* , along with relative error of proposed Eq. (40) for direct computation of h^* in terms of b^* (optimum complex power-law section for a given b^*)

$$\frac{s(s-1)(h_*^{-2} - b_* h_*^{-1})^2 h_*}{2} \int_0^1 \frac{[2s + (s^2 - s) \ln(t) - 1] t^{2s-4}}{\sqrt{1 + [s(s-1)(h_*^{-2} - b_* h_*^{-1}) t^{s-2}/2]^2}} dt = 0 \quad (41)$$

$$1 - \frac{s^2(s-1)^2}{2} (h_*^{-2} - b_* h_*^{-1}) \int_0^1 \frac{t^{2s-4}}{\sqrt{1 + [s(s-1)(h_*^{-2} - b_* h_*^{-1}) t^{s-2}/2]^2}} dt = 0 \quad (42)$$

In this study, it was not obtained any feasible solution for the system of equations $\partial P_*/\partial s = 0$ and $\partial P_*/\partial b_* = 0$. This can be attributed to the mathematical forms of the governing equations.

5.2 Complex power-law section with P_* as a function of s , h_* and Z

Eliminating b_* from Eqs. (14) and (15) yields:

$$P_* = \frac{1}{h_*} - \frac{2Zh_*}{s(s-1)} + 2h_* \int_0^1 \sqrt{1 + [s(s-1)(h_*^{-2} - 1) + 2Zh_*^2 t^{s-2}/2]^2} dt \quad (43)$$

Following similar procedure as before, four optimum sections can be considered as: best hydraulic section (s , Z and h_* as variable parameters; the result for this section is $s = 1.299$, $Z = 0.318$ and $h_* = 0.7814$ which result in $b_* = 0$), the hydraulically efficient section for a given exponent s (Z and h_* as variable parameters), the hydraulically efficient section for a given side slope Z (s and h_* as variable parameters) and the hydraulically efficient section for a given dimensionless depth h_* (Z and s as variable parameters). These three hydraulically efficient sections can be obtained directly using Eq. (43) or indirectly using previous results of the hydraulically efficient complex power-law sections and using $b_* = 1/h_* - 2Zh_*/[s(s-1)]$ which is obtained by solving Eq. (15) for b_* .

6 Application examples

Example 1

Consider a simple power-law channel with $Q = \text{flow discharge} = 10 \text{ m}^3/\text{s}$, $S = \text{longitudinal slope} = 0.0001$, and $n = \text{Manning coefficient} = 0.015 \text{ (m}^{-1/3}\text{s)}$. According to the stability analysis, Z should be equal or greater than 0.6. Design the hydraulically efficient power-law section for $Z = 0.6$.

Solution

Using Eq. (18) one gets $s = 1.496$. Then from Eq. (5) $h_* = 0.786$ and $A = 1.619 h^2$. Also using Eq. (16) one obtains $P = 3.27 h$. Using Manning formula:

$$Q = \frac{\sqrt{S} A^{5/3}}{n P^{2/3}} \Rightarrow 10 = \frac{\sqrt{0.0001} (1.619 h^2)^{5/3}}{0.015 (3.27 h)^{2/3}}$$

Solving above explicit equation for normal depth yields $h = 2.747 \text{ m}$. Thus $A = 1.619 h^2 = 12.217 \text{ m}^2$ ($r = A/h^5 = 2.695 \text{ m}^{0.504}$), $P = 3.27 h = 8.983 \text{ m}$ and $T = r s h^{s-1} = 6.655 \text{ m}$. The scaled shape factor is also computed as $P_{**} = P/(2\pi A)^{1/2} = 1.025$.

Example 2

Consider a simple power-law channel with $Q = 10 \text{ m}^3/\text{s}$, $S = 0.0001$, and $n = 0.015 \text{ m}^{-1/3}\text{s}$. According to the stability analysis, Z should be equal or greater than 0.3. Design the best hydraulic power-law section.

Solution

For the best hydraulic power-law section; $s = 1.299$, $Z = 0.318$, $r = 1.638 h^{0.701}$, $A = 1.638 h^2$, $P = 3.231 h$ and $T = 2.127 h$.

Using Manning formula:

$$Q = \frac{\sqrt{S} A^{5/3}}{n P^{2/3}} \Rightarrow 10 = \frac{\sqrt{0.0001} (1.638 h^2)^{5/3}}{0.015 (3.231 h)^{2/3}}$$

Solving above explicit equation for normal depth yields $h = 2.719 \text{ m}$. Thus $A = 1.638 h^2 = 12.11 \text{ m}^2$ ($r = A/h^5 = 1.638 h^{0.701} = 3.302 \text{ m}^{0.701}$), $P = 3.231 h = 8.785 \text{ m}$ and $T = 5.783 \text{ m}$. The scaled shape factor is also equal to $P_{**} = P/(2\pi A)^{1/2} = 1.007$.

Example 3

Consider a simple power-law channel with $Q = 10 \text{ m}^3/\text{s}$, $S = 0.0001$, and $n = 0.015 \text{ m}^{-1/3}\text{s}$. Design the optimum hydraulic power-law section for a given exponent of $s = 1.67$ (semi-cubic parabolic section).

Solution

From Table 1 for $s = 1.67$ one gets: $h_* = 0.909678$, $Z = 0.676062$, $A = 1.208440 h^2$, $P = 2.891227 h$, $T = 2.018095 h$ and $P_* = 2.630084$.

Using Manning's formula:

$$Q = \frac{\sqrt{S} A^{5/3}}{n P^{2/3}} \Rightarrow 10 = \frac{\sqrt{0.0001} (1.208h^2)^{5/3}}{0.015 (2.891h)^{2/3}}$$

Solving above explicit equation for normal depth yields $h = 3.198$ m. Thus $A = 12.361$ m² ($r = A/h^2 = 1.774$ m^{0.33}), $P = 9.247$ m and $T = 6.454$ m. The scaled shape factor is also equal to $P_{**} = P/(2\pi A)^{1/2} = 1.049$.

Example 4

Consider a complex power-law channel with $Q = 10$ m³/s, $S = 0.0001$, and $n = 0.015$ m^{-1/3}s. Design the optimum complex power-law section for a given exponent of $s = 1.67$ (section with semi-cubic parabolic sides).

Solution

From Table 2 for $s = 1.67$ one gets: $h_* = 0.779658$, $b_* = 0.619821$, $Z = 0.475591$, $A = 1.645096 h^2$, $P = 3.290192 h$, $T = 2.214667 h$ and $P_* = 2.565226$.

Using Manning formula:

$$Q = \frac{\sqrt{S} A^{5/3}}{n P^{2/3}} \Rightarrow 10 = \frac{\sqrt{0.0001} (1.645h^2)^{5/3}}{0.015 (3.290h)^{2/3}}$$

Solving above equation for normal depth yields $h = 2.724$ m. Thus $A = 12.208$ m² ($r = A/h^2 = 2.290$ m^{0.33}), $P = 8.963$ m, $T = 6.033$ m and $b = 2.166$ m. The scaled shape factor is also equal to $P_{**} = P/(2\pi A)^{1/2} = 1.023$.

A comparison of Examples 3 and 4 shows that the optimum complex power-law section with $b = 2.166$ is 2.6% more efficient than the optimum simple power-law Section ($4.9 - 2.3 = 2.6\%$).

7 Conclusions

When optimum channel sections (sections having minimum wetted perimeter for a given flow area) are applicable, the channel construction cost will be reduced. This paper presents the characteristics of the best hydraulic section and hydraulically efficient sections of the general power-law section for any given exponent $s \in [1.2, 2]$. In this study, using the dimensionless wetted perimeter as an objective function, different optimization models are developed for the simple and complex power-law sections. The objective function with a clear physical meaning is then minimized without applying any constraints. The optimization model of the simple power-law section has two decision variables (h_* and s ; or Z and s) while the optimization model of the complex power-law section has three decision variables (h_* , b_* and s). The relationship between the exponent s and Z for the hydraulically

efficient section is derived which can be used for selecting a suitable exponent for a given side slope when slope stability and geotechnical conditions are important issues. According to the results of this study, the efficiency of the simple sections can be ranked as: semi-circle shaped > best power-law $\cong$ cubic > parabolic > semi-cubic > triangular ($Z = 1$) = rectangular ($Z = 0$). Besides the new results, this study combines all previous separate models in a unified optimization model to find the optimum power-law sections. A novel graphical comparison between simple and complex optimum power-law sections is presented which can be used for design purposes or for preliminary analyses. All optimum results of the hydraulically efficient sections are expressed in tabular forms (Tables 1, 2) or explicit equations which are easy to use for the optimum design of the power-law sections. A new explicit approximation for the wetted perimeter is also developed. This solution is useful for normal flow depth computation of a given optimum sections by Manning formula. This study was limited to open channels with power-law sides. More flexible sections may be introduced by multi-linear or curved (polynomial or multi power-law) side elements to improve flow discharge and maintenance, and ease of construction. It worth noting that the analysis will be more complicated when the freeboard [23] or water width and depth constraints [24] are incorporated in the open channel design process which are subject to future studies.

Compliance with ethical standards

Conflict of interest The author has declared that he has no conflict of interest.

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