Impacts of viscous dissipation and Joule heating on hydromagnetic boundary layer flow of nanofluids over a flat surface subjected to Newtonian heating

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Abstract

Main concern in this analysis is to study the two-dimensional, steady boundary layer flow of viscous, incompressible, electrically conducting nanofluids past a flat plate in the presence of magnetic field with heat being transferred by Newtonian heating way. Influences of viscous dissipation and Joule heating are considered also. Nonlinear partial differential equations are reduced into nonlinear ordinary differential equations by formulating similarity transformations. Numerical solutions of transformed boundary layer equations are clarified by applying the Keller-Box method. Effects of several types of nanofluids and various specified parameters such as solid volume fraction, magnetic parameter, Brinkmann number and local Biot number on velocity and temperature fields have been plotted graphically, while values of surface shear stress and surface heat flux are presented via table. Further, comparison of obtained computational values has been made with earlier published results for non-magnetic case.

Keywords Viscous dissipation · Joule heating · Hydromagnetic boundary layer flow · Nanofluids · Flat surface · Newtonian heating

List of symbols

B_0	Uniform magnetic field strength ($\text{N m}^{-1} \text{A}^{-1}$)
Bi	Local Biot number
Br	Brinkmann number
C_f	Local skin friction coefficient
C_p	Specific heat at constant pressure ($\text{J Kg}^{-1} \text{K}^{-1}$)
f	Dimensionless stream function
h_t	Heat transfer coefficient
M	Magnetic parameter
Nu_x	Local Nusselt number
Pr	Prandtl number
Re_x	Local Reynolds number
T	Temperature of nanofluid (K)
T_w	Hot fluid temperature at surface (K)
T_∞	Ambient fluid temperature (K)
U_∞	Velocity of ambient fluid (m s^{-1})
u	Velocity component parallel to the x-axis (m s^{-1})

v	Velocity component parallel to the y-axis (m s^{-1})
x	Direction along to the plate (m)
y	Direction perpendicular to the plate (m)

Greek symbols

α	Thermal diffusivity ($\text{m}^2 \text{s}^{-1}$)
η	Similarity variable
θ	Dimensionless temperature
κ	Thermal conductivity ($\text{W m}^{-1} \text{K}^{-1}$)
μ	Coefficient of viscosity ($\text{Kg m}^{-1} \text{s}^{-1}$)
ν	Kinematic viscosity ($\text{m}^2 \text{s}^{-1}$)
ρ	Density (Kg m^{-3})
σ_e	Electrical conductivity (S m^{-1})
ϕ	Solid volume fraction
ψ	Stream function ($\text{m}^2 \text{s}^{-1}$)

Superscripts

$'$	Differentiation with respect to η
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Subscripts

f	Base fluid
nf	Nanofluid
s	Nano solid particles

1 Introduction

Along the shear forces action, an effort done through the fluid on adjoining layers is converted into a heat, which is known as viscous dissipation. For higher velocity and viscous flows, the viscous dissipation impact on heat transfer is essential. Viscous dissipation appears in natural convection for different devices and also in powerful gravitational fields. Whereas, a process in which electric current energy is transformed into heat as it flows by resistance is called Joule heating or ohmic heating. Joule heating has a lot of applications in the area of technology and industrial processing. Some examples of applications are electric heaters and fuses, handling of food, electric stoves, electronic cigarette and incandescent light bulb. Initially, El-Amin [1] developed the combined influence of viscous dissipation and Joule heating and observed the convective boundary layer flow situation towards the embedded porous medium. Moreover, numerous authors like as Abo-Eldahab and El-Aziz [2], Jat and Chaudhary [3], Yavari et al. [4] and Das et al. [5] have studied the fluid flow with viscous dissipation and Joule heating influences for some extended effects. In recent years, various explorations have been found by Hayat et al. [6], Hussain et al. [7] and Chaudhary and Choudhary [8] and analyzed the boundary layer flow in the presence of viscous dissipation and Joule heating.

To adjust the structure of boundary layer, an efficient technique as magnetohydrodynamic (MHD) principle has been used. MHD is an observation of electrically conducting fluids in a magnetic field, which depends on the induced magnetic field strength. Some examples of magneto fluids are liquid of metals, salt water, plasmas and electrolytes, and few applications of MHD effect such as boundary layer control in aerodynamics, geothermal energy, bearing, MHD generators and sensors, crystal growth, pumps, electromagnetic castings and plasma studies have significant scope in the areas of engineering and technologies. Alfven [9] pioneered the analysis of MHD field and analyzed the electromagnetic hydrodynamic waves existence. After that Ganesan and Palani [10] executed the study of unsteady MHD flow towards an inclined plate with natural convection. Moreover, Jat and Chaudhary [11], Butt and Ali [12], Imtiaz et al. [13], Chaudhary and Choudhary [14], Aydin and Selvitopi [15], Rehman et al. [16, 17] and Jha and Malgwi [18] have inscribed some articles on heat and mass transfer in MHD flow against to various situations.

The conventional ordinary fluids specifically oil, toluene, water and ethylene glycol have lower thermal conductivity and heat transfer rate capability. But increasing demand of advanced technology in electronic devices miniaturization have requirement of heat transfer medium behaving like a liquid, which has greater heat transfer efficiency to develop the thermal characteristics. This type of medium is known as nanofluid. Nanofluids have two types of materials namely base fluid and ultrafine nanoparticles with the diameter size 1–100 nm. Some frequent nanoparticles made by metallic are silicon, titanium, copper, aluminum and silver. Nanofluids have drawn researchers awareness in the fields of engineering technology and science as a wide range of industrial applications such as food, drinks, dyes and toners, airplane engine, bio-chemical dispensation, microchip technology, pharmacological, dynamism, aerospace, remedial apparatus and devices. Choi [19] essentially discussed the concept of nanofluids for the enlargement of the thermal conductivity of fluids. Further, Buoyancy-driven heat transfer increment in a two-dimensional enclosure by using nanofluids established by Khanafer et al. [20]. Latterly, Chein and Chuang [21], Yang and Lai [22], Mital [23], Ibrahim and Makinde [24], Makinde et al. [25], Chaudhary and Kanika [26], Rehman et al. [27], Kandasamy et al. [28], Ma et al. [29] and Sheikholeslami et al. [30] have inspected the numerous numerical and analytical explorations for the improvement of nanofluids heat transfer.

Newtonian heating is a way of heat transfer into the conventional fluid from the boundary surface with specific heat capacity. That type of composition appears in the setup of convection flows and heat introduces through solar radiation. The Newtonian heating impact is found if heat flux from the wall is proportional to the local surface temperature. This heating process utilized some practical base applications, particularly thermal energy storage, conjugate heat transport around fines, petroleum industry, nuclear turbines and heat exchanger. Merkin [31] illustrated the first analysis in the area of Newtonian heating and applied the effect of Newtonian heating on natural convection boundary layer flow via a vertical plate. Until, Lesnic et al. [32] established the free convection boundary-layer flow above a nearly horizontal surface in a porous medium with Newtonian heating. Further, an extensive literature on the behavior of Newtonian heating on boundary layer flow have been discussed by Makinde [33], Akbar and Khan [34], Hayat et al. [35], Chaudhary et al. [36] and Kamran and Wiwatanapataphee [37].

Keller-Box method is a very versatile modeling scheme. It is comparatively easy to understand and implemented. The Keller-Box technique works well for the two-dimensional model with simplified system geometries. This method is used directly to the differential form of the governing

equations. The details of Keller-Box method can be found in the book by Vajravelu and Prasad [38].

Above mentioned literature pointed out that Newtonian heating effect on MHD boundary layer flow of electrically conducting fluid as water containing nanoparticles specifically silver (Ag), copper (Cu), titanium dioxide (TiO₂) and alumina (Al₂O₃) is not examined yet. So the main objective of the present study is to extend the investigation of Makinde [39] with the impacts of viscous dissipation and Joule heating. Subsequently, the behaviors of nanofluids and considering parameters on the velocity, temperature, local skin friction coefficient and the local Nusselt number are given by plotted or tabulated values and discussed in detail.

2 Flow analysis

Consider an analysis of two-dimensional, steady boundary layer flow of viscous, electrically conducting nanofluids over a flat surface with Newtonian heating. It is assumed that there are thermal equilibrium and no-slip condition between base fluid water and nanoparticles like Ag, Cu, TiO₂ and Al₂O₃. A coordinate system (x, y) is chosen such that x - and y -directions are along to the plate and perpendicular to it respectively, and the flow is taken place at $y \geq 0$ as shown in Fig. 1. The hot fluid temperature T_w at the surface is surmised, while the velocity and temperature of the ambient nanofluid are taken constant value U_∞ and T_∞ respectively. A uniform magnetic field of strength B_0 is applied parallel to the y -axis. For small value of the magnetic Reynolds number, the impact of an induced magnetic field is considered negligible. After that, the influences of viscous dissipation and Joule heating are considered. Under these assumptions, the basic governing equations are defined as (Bansal [40])

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu_{nf} \frac{\partial^2 u}{\partial y^2} - \frac{(\sigma_e)_{nf} B_0^2}{\rho_{nf}} (u - U_\infty) \quad (2)$$

Fig. 1 Flow configuration

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_{nf} \frac{\partial^2 T}{\partial y^2} + \frac{\mu_{nf}}{(\rho C_p)_{nf}} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{(\sigma_e)_{nf} B_0^2}{(\rho C_p)_{nf}} (u - U_\infty)^2 \quad (3)$$

with the corresponding boundary conditions

$$y = 0: \quad u = 0, \quad v = 0, \quad -\kappa_f \frac{\partial T}{\partial y} = h_t (T_w - T) \quad (4)$$

$$y \rightarrow \infty: \quad u \rightarrow U_\infty, \quad T \rightarrow T_\infty$$

where subscripts nf and f indicate the thermophysical properties of the nanofluid and the base fluid respectively, u and v are the velocity components parallel to the x - and y -axes respectively, $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity, μ is the coefficient of viscosity, ρ is the density, σ_e is the electrical conductivity, T is the temperature of nanofluid, $\alpha = \frac{\kappa}{\rho C_p}$ is the thermal diffusivity, κ is the thermal conductivity, C_p is the specific heat at constant pressure and h_t is the heat transfer coefficient.

Further, the nanofluid's thermophysical properties namely coefficient of viscosity, density, electrical conductivity, thermal conductivity and heat capacitance proceed from Mohyud-Din et al. [41] are given by

$$\frac{\mu_{nf}}{\mu_f} = \frac{1}{(1 - \phi)^{5/2}} \quad (5)$$

$$\frac{\rho_{nf}}{\rho_f} = 1 - \phi + \phi \frac{\rho_s}{\rho_f} \quad (6)$$

$$\frac{(\sigma_e)_{nf}}{(\sigma_e)_f} = \frac{2(\sigma_e)_f + (\sigma_e)_s - 2\phi[(\sigma_e)_f - (\sigma_e)_s]}{2(\sigma_e)_f + (\sigma_e)_s + \phi[(\sigma_e)_f - (\sigma_e)_s]} \quad (7)$$

$$\frac{\kappa_{nf}}{\kappa_f} = \frac{2\kappa_f + \kappa_s - 2\phi(\kappa_f - \kappa_s)}{2\kappa_f + \kappa_s + \phi(\kappa_f - \kappa_s)} \quad (8)$$

$$\frac{(\rho C_p)_{nf}}{(\rho C_p)_f} = 1 - \phi + \phi \frac{(\rho C_p)_s}{(\rho C_p)_f} \quad (9)$$

where subscript s represents the physical characteristics for nano solid particles and ϕ is the solid volume fraction. Moreover, Table 1 (Su and Zheng [42]) given the values for the above mentioned physical properties of conventional fluid and nanoparticles.

3 Transformed problem

Similarity variables are imported by Makinde [39] as follows

Table 1 Thermophysical resources of water and nanoparticles

Properties	Water	Ag	Cu	TiO ₂	Al ₂ O ₃
κ (W m ⁻¹ K ⁻¹)	0.613	429	400	8.9538	40
ρ (Kg m ⁻³)	997.1	10500	8933	4250	3970
C_p (J Kg ⁻¹ K ⁻¹)	4179	235	385	686.2	765
σ_e (S m ⁻¹)	0.05	6.3×10^7	5.96×10^7	0.24×10^7	3.69×10^7

$$\psi = (v_f U_\infty x)^{1/2} f(\eta), \quad \eta = \left(\frac{U_\infty}{v_f x} \right)^{1/2} y, \quad (10)$$

$$T = T_\infty + (T_w - T_\infty) \theta(\eta)$$

where $\psi(x, y)$ is the stream function, which is utilized in the usual manner as $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$ and symmetrically satisfied the continuity Eq. (1), $f(\eta)$ is the dimensionless stream function, η is the similarity variable and $\theta(\eta)$ is the dimensionless temperature.

Using the similarity variables Eq. (10), the boundary layer Eqs. (2) and (3) with the associated boundary conditions Eq. (4) can be replaced as

$$f''' + \frac{1}{2}(1 - \phi)^{5/2} \left(1 - \phi + \phi \frac{\rho_s}{\rho_f} \right) f f'' - \frac{(\sigma_e)_{nf}}{(\sigma_e)_f} (1 - \phi)^{5/2} M(f' - 1) = 0 \quad (11)$$

$$\frac{\kappa_{nf}}{\kappa_f} \theta'' + \frac{1}{2} \left[1 - \phi + \phi \frac{(\rho C_p)_s}{(\rho C_p)_f} \right] Pr f \theta' + Br \left[\frac{1}{(1 - \phi)^{5/2}} f'^2 + \frac{(\sigma_e)_{nf}}{(\sigma_e)_f} M(f' - 1)^2 \right] = 0 \quad (12)$$

subject to the relevant boundary conditions

$$\eta = 0: f = 0, \quad f' = 0, \quad \theta' = Bi(\theta - 1) \\ \eta \rightarrow \infty: f' \rightarrow 1, \quad \theta \rightarrow 0 \quad (13)$$

where prime ($'$) denotes the differentiation with respect to η , $M = \frac{(\sigma_e)_f B_0^2 v_f Re_x}{\rho_f U_\infty^2}$ is the magnetic parameter, $Re_x = \frac{U_\infty x}{v_f}$ is the local Reynolds number, $Pr = \frac{v_f}{\alpha_f}$ is the Prandtl number, $Br = \frac{\mu_f U_\infty^2}{\kappa_f (T_w - T_\infty)}$ is the Brinkmann number and $Bi = \frac{h_f v_f}{\kappa_f U_\infty} Re_x^{1/2}$ is the local Biot number.

4 Declaration of curiosity

The local skin friction coefficient C_f and the local Nusselt number Nu_x are the physical quantities of primary interest, which are expressed as

$$C_f = \frac{\mu_{nf} \left(\frac{\partial u}{\partial y} \right)_{y=0}}{\frac{\rho_f U_\infty^2}{2}}, \quad Nu_x = -\frac{\kappa_{nf} x \left(\frac{\partial T}{\partial y} \right)_{y=0}}{\kappa_f (T_w - T_\infty)} \quad (14)$$

By using the non-dimensional variables Eq. (10), the physical quantities Eq. (14) can be defined as

$$Re_x^{1/2} C_f = \frac{2}{(1 - \phi)^{5/2}} f''(0), \quad \frac{1}{Re_x^{1/2}} Nu_x = -\frac{\kappa_{nf}}{\kappa_f} \theta'(0) \quad (15)$$

5 Solution methodology

The system of the nonlinear ordinary differential Eqs. (11) and (12) along with the associated boundary conditions Eq. (13) is solved numerically with a finite difference scheme as Keller-Box method (Kumar and Sood [43]). For the computational procedure, the suitable finite value of the far field boundary condition as $\eta \rightarrow \infty = 6$ is assumed.

5.1 Scheme of implicit finite difference

Introducing

$$f' = p \quad (16)$$

$$p' = q \quad (17)$$

$$\theta' = s \quad (18)$$

so the Eqs. (11) and (12) can be given as

$$q' + \frac{1}{2}(1 - \phi)^{5/2} \left(1 - \phi + \phi \frac{\rho_s}{\rho_f} \right) f q - \frac{(\sigma_e)_{nf}}{(\sigma_e)_f} (1 - \phi)^{5/2} M(p - 1) = 0 \quad (19)$$

$$\frac{\kappa_{nf}}{\kappa_f} s' + \frac{1}{2} \left[1 - \phi + \phi \frac{(\rho C_p)_s}{(\rho C_p)_f} \right] Pr f s + Br \left[\frac{1}{(1 - \phi)^{5/2}} q^2 + \frac{(\sigma_e)_{nf}}{(\sigma_e)_f} M(p - 1)^2 \right] = 0 \quad (20)$$

along with the corresponding boundary conditions Eq. (13) become as follows

$$\eta = 0: f = 0, \quad p = 0, \quad s = Bi(\theta - 1) \\ \eta = 6: p \rightarrow 1, \quad \theta \rightarrow 0 \quad (21)$$

The rectangular grid $X\eta$ -plane and net points are defined by Fig. 2, such as

$$\begin{aligned} x_0 = 0, x_i = x_{i-1} + k_i, i = 1, 2, 3, \dots I \\ \eta_0 = 0, \eta_j = \eta_{j-1} + h_j, j = 1, 2, 3, \dots J \end{aligned} \quad (22)$$

$$f_0 = 0, p_0 = 0, s_0 = Bi(\theta_0 - 1), p_J \rightarrow 1, \theta_J \rightarrow 0 \quad (28)$$

By using centered difference derivatives, the finite difference form of the Eqs. (16) to (20) for the mid point $(x_i, \eta_{j-1/2})$ of the segment QR are defined as

$$f_j - f_{j-1} - \frac{1}{2}h_j(p_j + p_{j-1}) = 0 \quad (23)$$

$$p_j - p_{j-1} - \frac{1}{2}h_j(q_j + q_{j-1}) = 0 \quad (24)$$

$$\theta_j - \theta_{j-1} - \frac{1}{2}h_j(s_j + s_{j-1}) = 0 \quad (25)$$

$$\begin{aligned} q_j - q_{j-1} + \frac{1}{8}h_j(1 - \phi)^{5/2} \left(1 - \phi + \phi \frac{\rho_s}{\rho_f} \right) \\ \times (f_j + f_{j-1})(q_j + q_{j-1}) \\ - \frac{1}{2}h_j \frac{(\sigma_e)_{nf}}{(\sigma_e)_f} (1 - \phi)^{5/2} M(p_j + p_{j-1} - 1) = 0 \end{aligned} \quad (26)$$

$$\begin{aligned} \frac{\kappa_{nf}}{\kappa_f} (s_j - s_{j-1}) + \frac{1}{8}h_j \left[1 - \phi + \phi \frac{(\rho C_p)_s}{(\rho C_p)_f} \right] \\ \times Pr(f_j + f_{j-1})(s_j + s_{j-1}) \\ + \frac{1}{4}h_j Br \left[\frac{1}{(1 - \phi)^{5/2}} (q_j + q_{j-1})^2 \right. \\ \left. + \frac{(\sigma_e)_{nf}}{(\sigma_e)_f} M(p_j + p_{j-1} - 1)^2 \right] = 0 \end{aligned} \quad (27)$$

Equations (23) to (27) are exhibited for $j = 1, 2, 3, \dots J - 1$ and from the appropriate boundary conditions Eq. (21) it is prominent that

Fig. 2 Domain schematic representation

5.2 Newton's method

A nonlinear system is converted into a linear system by applying the Newton's method. So the following iterations are established

$$\begin{aligned} f_j^{(i+1)} &= f_j^{(i)} + \delta f_j^{(i)}, p_j^{(i+1)} = p_j^{(i)} + \delta p_j^{(i)}, \\ q_j^{(i+1)} &= q_j^{(i)} + \delta q_j^{(i)}, \\ \theta_j^{(i+1)} &= \theta_j^{(i)} + \delta \theta_j^{(i)}, s_j^{(i+1)} = s_j^{(i)} + \delta s_j^{(i)} \end{aligned} \quad (29)$$

Substituting the above considered expressions into the Eqs. (23) to (27) and then neglect the quadratic and higher order terms in $\delta f_j^{(i)}, \delta p_j^{(i)}, \delta q_j^{(i)}, \delta \theta_j^{(i)}$ and $\delta s_j^{(i)}$. Thus above scheme yields a tridiagonal model as follows

$$\delta f_j - \delta f_{j-1} - \frac{1}{2}h_j(\delta p_j + \delta p_{j-1}) = (r_1)_{j-\frac{1}{2}} \quad (30)$$

$$\delta p_j - \delta p_{j-1} - \frac{1}{2}h_j(\delta q_j + \delta q_{j-1}) = (r_2)_{j-\frac{1}{2}} \quad (31)$$

$$\delta \theta_j - \delta \theta_{j-1} - \frac{1}{2}h_j(\delta s_j + \delta s_{j-1}) = (r_3)_{j-\frac{1}{2}} \quad (32)$$

$$\begin{aligned} (a_1)_{j-\frac{1}{2}} \delta f_j + (a_2)_{j-\frac{1}{2}} \delta f_{j-1} + (a_3)_{j-\frac{1}{2}} \delta p_j \\ + (a_4)_{j-\frac{1}{2}} \delta p_{j-1} + (a_5)_{j-\frac{1}{2}} \delta q_j + (a_6)_{j-\frac{1}{2}} \delta q_{j-1} = (r_4)_{j-\frac{1}{2}} \end{aligned} \quad (33)$$

$$\begin{aligned} (b_1)_{j-\frac{1}{2}} \delta f_j + (b_2)_{j-\frac{1}{2}} \delta f_{j-1} + (b_3)_{j-\frac{1}{2}} \delta p_j \\ + (b_4)_{j-\frac{1}{2}} \delta p_{j-1} + (b_5)_{j-\frac{1}{2}} \delta q_j + (b_6)_{j-\frac{1}{2}} \delta q_{j-1} \\ + (b_7)_{j-\frac{1}{2}} \delta s_j + (b_8)_{j-\frac{1}{2}} \delta s_{j-1} = (r_5)_{j-\frac{1}{2}} \end{aligned} \quad (34)$$

where

$$(a_1)_{j-\frac{1}{2}} = (a_2)_{j-\frac{1}{2}} = \frac{1}{8} h_j (1 - \phi)^{5/2} \left(1 - \phi + \phi \frac{\rho_s}{\rho_f} \right) (q_j + q_{j-1}),$$

$$(a_3)_{j-\frac{1}{2}} = (a_4)_{j-\frac{1}{2}} = -\frac{1}{2} h_j \frac{(\sigma_e)_{nf}}{(\sigma_e)_f} (1 - \phi)^{5/2} M,$$

$$(a_5)_{j-\frac{1}{2}} = 1 + \frac{1}{8} h_j (1 - \phi)^{5/2} \left(1 - \phi + \phi \frac{\rho_s}{\rho_f} \right) (f_j + f_{j-1}),$$

$$(a_6)_{j-\frac{1}{2}} = -1 + \frac{1}{8} h_j (1 - \phi)^{5/2} \left(1 - \phi + \phi \frac{\rho_s}{\rho_f} \right) (f_j + f_{j-1}),$$

$$(b_1)_{j-\frac{1}{2}} = (b_2)_{j-\frac{1}{2}} = \frac{1}{8} h_j \left[1 - \phi + \phi \frac{(\rho C_p)_s}{(\rho C_p)_f} \right] Pr (s_j + s_{j-1}),$$

$$(b_3)_{j-\frac{1}{2}} = (b_4)_{j-\frac{1}{2}} = \frac{1}{4} h_j \frac{(\sigma_e)_{nf}}{(\sigma_e)_f} M Br (p_j + p_{j-1} - 2),$$

$$(b_5)_{j-\frac{1}{2}} = (b_6)_{j-\frac{1}{2}} = \frac{1}{4} h_j \frac{1}{(1 - \phi)^{5/2}} Br (q_j + q_{j-1}),$$

$$(b_7)_{j-\frac{1}{2}} = \frac{\kappa_{nf}}{\kappa_f} + \frac{1}{8} h_j \left[1 - \phi + \phi \frac{(\rho C_p)_s}{(\rho C_p)_f} \right] Pr (f_j + f_{j-1}),$$

$$(b_8)_{j-\frac{1}{2}} = -\frac{\kappa_{nf}}{\kappa_f} + \frac{1}{8} h_j \left[1 - \phi + \phi \frac{(\rho C_p)_s}{(\rho C_p)_f} \right] Pr (f_j + f_{j-1}),$$

$$(r_1)_{j-\frac{1}{2}} = -f_j + f_{j-1} + \frac{1}{2} h_j (p_j + p_{j-1}),$$

$$(r_2)_{j-\frac{1}{2}} = -p_j + p_{j-1} + \frac{1}{2} h_j (q_j + q_{j-1}),$$

$$(r_3)_{j-\frac{1}{2}} = -\theta_j + \theta_{j-1} + \frac{1}{2} h_j (s_j + s_{j-1}),$$

$$(r_4)_{j-\frac{1}{2}} = -(q_j - q_{j-1}) - \frac{1}{8} h_j (1 - \phi)^{5/2} \left(1 - \phi + \phi \frac{\rho_s}{\rho_f} \right) \\ \times (f_j + f_{j-1}) (q_j + q_{j-1}) \\ + \frac{1}{2} h_j \frac{(\sigma_e)_{nf}}{(\sigma_e)_f} (1 - \phi)^{5/2} M (p_j + p_{j-1} - 1)$$

and

$$(r_5)_{j-\frac{1}{2}} = -\frac{\kappa_{nf}}{\kappa_f} (s_j - s_{j-1}) \\ - \frac{1}{8} h_j \left[1 - \phi + \phi \frac{(\rho C_p)_s}{(\rho C_p)_f} \right] \\ \times Pr (f_j + f_{j-1}) (s_j + s_{j-1}) \\ - \frac{1}{4} h_j Br \left[\frac{1}{(1 - \phi)^{5/2}} (q_j + q_{j-1})^2 \right. \\ \left. + \frac{(\sigma_e)_{nf}}{(\sigma_e)_f} M (p_j + p_{j-1} - 1)^2 \right]$$

For all iterations, it is taken as

$$\delta f_0 = 0, \delta p_0 = 0, \delta s_0 = 0, \delta p_j = 0, \delta \theta_j = 0 \quad (35)$$

5.3 Method of block elimination

Block tridiagonal matrix form of Eqs. (30) to (34) is

$$\begin{bmatrix} [A_1] & [C_1] \\ [B_2] & [A_2] & [C_2] \\ & \ddots & \dots \\ & & [B_{J-1}] & [A_{J-1}] & [C_{J-1}] \\ & & & [B_J] & [A_J] \end{bmatrix} \begin{bmatrix} [\delta_1] \\ [\delta_2] \\ \vdots \\ [\delta_{J-1}] \\ [\delta_J] \end{bmatrix} = \begin{bmatrix} [r_1] \\ [r_2] \\ \vdots \\ [r_{J-1}] \\ [r_J] \end{bmatrix}$$

that is

$$[A][\delta] = [r] \quad (36)$$

where

$$[A_1] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{-h_j}{2} & 0 & 0 & \frac{-h_j}{2} & 0 \\ 0 & -1 & 0 & 0 & \frac{-h_j}{2} \\ a_6 & 0 & a_1 & a_5 & 0 \\ b_6 & 0 & b_1 & b_5 & b_7 \end{bmatrix};$$

$$[A_j] = \begin{bmatrix} \frac{-h_j}{2} & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & \frac{-h_j}{2} & 0 \\ 0 & -1 & 0 & 0 & \frac{-h_j}{2} \\ a_4 & 0 & a_1 & a_5 & 0 \\ b_4 & 0 & b_1 & b_5 & b_7 \end{bmatrix}, \quad 2 \leq j \leq J;$$

$$[B_j] = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \frac{-h_j}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{-h_j}{2} \\ 0 & 0 & a_4 & a_2 & 0 \\ 0 & 0 & b_4 & b_8 & b_2 \end{bmatrix}, \quad 2 \leq j \leq J;$$

$$[C_j] = \begin{bmatrix} \frac{-h_j}{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad 1 \leq j \leq J-1;$$

$$[\delta_1] = \begin{bmatrix} \delta q_0 \\ \delta \theta_0 \\ \delta f_1 \\ \delta q_1 \\ \delta s_1 \end{bmatrix}; \quad [\delta_j] = \begin{bmatrix} \delta p_{j-1} \\ \delta \theta_{j-1} \\ \delta f_j \\ \delta q_j \\ \delta s_j \end{bmatrix}, \quad 2 \leq j \leq J \quad \text{and}$$

$$[r_j] = \begin{bmatrix} (r_1)_j \\ (r_2)_j \\ (r_3)_j \\ (r_4)_j \\ (r_5)_j \end{bmatrix}, \quad 1 \leq j \leq J.$$

Matrix A is taken as a nonsingular matrix to solve the Eq. (36) then the matrix A can be written in the product of the lower triangular matrix and the upper triangular matrix as

$$[A] = [L][U] \quad (37)$$

where

$$[L] = \begin{bmatrix} [\alpha_1] & & & & \\ [B_2] & [\alpha_2] & & & \\ & \ddots & \ddots & & \\ & & [\alpha_{j-1}] & & \\ & & [B_j] & [\alpha_j] & \end{bmatrix} \text{ and}$$

$$[U] = \begin{bmatrix} I & [\beta_1] & & & \\ & I & [\beta_2] & & \\ & & \ddots & \ddots & \\ & & & I & [\beta_{j-1}] \\ & & & & I \end{bmatrix}$$

with I is the identity matrix of order 5×5 , and $[\alpha_j]$ and $[\beta_j]$ are the matrices of order 5×5 , whose elements are found through the following equations

$$[\alpha_1] = [A_1] \quad (38)$$

$$[A_1][\beta_1] = [C_1] \quad (39)$$

$$[\alpha_j] = [A_j] - [B_j][\beta_{j-1}], \quad j = 2, 3, \dots, J \quad (40)$$

In view of

$$[L][U][\delta_j] = [r_j] \quad (41)$$

with the assumption as

$$[U][\delta_j] = [W_j] \quad (42)$$

$$[L][W_j] = [r_j] \quad (43)$$

where $[W_j]$ is the column matrix of order 5×1 and elements of $[W_j]$ can be discovered by Eq. (43) as

$$[\alpha_1][W_1] = [r_1] \quad (44)$$

$$[\alpha_j][W_j] = [r_j] - [B_j][W_{j-1}], \quad 2 \leq j \leq J \quad (45)$$

the Eq. (42) disposes the solution, whose elements can be carried from the given relations

$$[\delta_j] = [W_j] - [\beta_j][\delta_{j+1}], \quad 1 \leq j \leq J - 1 \quad (46)$$

$$[\delta_j] = [W_j] \quad (47)$$

These iterative processes are repeated until some convergence regulation is satisfied with maintaining accuracy of 10^{-7} , while process is stopped when $|\delta q_0^{(i)}| \leq \xi$, where ξ is a small prescribed value.

6 Verification of the numerical method

Table 2 depicts the comparison made with earlier values by Makinde [39], to validate the present computational results of wall shear stress $f''(0)$ for several values of the solid volume fraction ϕ along with the nanofluids included base fluid water and different nanoparticles namely Cu, TiO_2 and Al_2O_3 likewise in the absence of magnetic field. This table shows that numerical results are in good agreement with the previously published data, which validate the proposed method.

Table 2 Comparison of $f''(0)$ for various values of ϕ corresponding to different types of nanofluids with $M = 0$

ϕ	Makinde [39]			Present results		
	Cu-water	TiO_2 -water	Al_2O_3 -water	Cu-water	TiO_2 -water	Al_2O_3 -water
0.000	0.3321	0.3321	0.3321	0.33258	0.33258	0.33258
0.008	0.3459	0.3398	0.3394	0.33945	0.33350	0.33315
0.014	0.3563	0.3456	0.3449	0.34429	0.33410	0.33350
0.016	0.3597	0.3476	0.3468	0.34585	0.33430	0.33360
0.020	0.3667	0.3515	0.3506	0.34888	0.33466	0.33379
0.100	0.5076	0.4362	0.4316	0.39015	0.33569	0.33418

7 Discussion of physical outcomes

To discuss the numerical values of the considered problem, the effects of the different types of nanofluids and the effects of the pertinent parameters namely the solid volume fraction ϕ , the magnetic parameter M , the Brinkmann number Br and the local Biot number Bi along with the Cu-water nanofluid on the velocity $f'(\eta)$ and the temperature $\theta(\eta)$ distributions are plotted graphically. Subsequently, the computational values of the wall shear stress

$f''(0)$ and the wall heat flux $\theta'(0)$ for the impacts of the different nanofluids as well as the specified parameters are given via table. It is also noted that due to find the impact of anyone specified parameter, all remaining controlling parameters are taken constant.

Figures 3 and 4 display the influences of the four various types of nanofluids on the velocity $f'(\eta)$ and the temperature $\theta(\eta)$ fields respectively. It can be seen from these figures that the fluid flow reduces and the fluid temperature enhances as the nanofluid changes in the

Fig. 3 Effect of different types of nanofluids on dimensionless velocity for $\phi = 0.07$ and $M = 0.01$

Fig. 4 Effect of different types of nanofluids on dimensionless temperature for $\phi = 0.07$, $M = 0.01$, $Pr = 6.2$, $Br = 0.1$ and $Bi = 0.1$

serial as Ag-water, Cu-water, TiO₂-water and Al₂O₃-water respectively. This is true with the fact that the velocity and the temperature profiles modify according to the change of solid nanoparticles beside in base fluid water, because various types of nanoparticles have distinct mechanical and physical characteristics like as dynamic viscosity, density and thermal expansion coefficients.

Impacts on the dimensionless velocity $f'(\eta)$ and the dimensionless temperature $\theta(\eta)$ for the several values of the solid volume fraction ϕ are presented by Figs. 5 and

6 respectively. These figures indicate that the velocity profile as well as the temperature profile rises with increment of the solid volume fraction ϕ . Because, nanofluids have solid nanoparticles, which transport the more flow resistance. Further, enhancement of solid nanoparticles volume fraction leads to increment in the nanofluid thermal conductivity that in turn outcomes the enhancement in the temperature.

Figures 7 and 8 illustrate the influence of the magnetic parameter M on the velocity $f'(\eta)$ and the temperature $\theta(\eta)$

Fig. 5 Effect of ϕ with Cu-water nanofluid on dimensionless velocity for $M = 0.01$

distributions respectively. It can be noted from these figures that the fluid velocity and the temperature increase along with the increment of the magnetic parameter M , while reverse is happened in temperature for $\eta > 1$. Physically it occurs because ratio of electromagnetic force and the viscous force is equivalent to the magnetic parameter. So the Lorentz force rises with the increasing values of the magnetic parameter, which creates more resistance to the transport circumstance and opposes the fluid motion which composed heat.

Behavior of the Brinkmann number Br on the fluid temperature $\theta(\eta)$ is described in Fig. 9. This figure expresses that an enlargement in the Brinkmann number Br leads to rise in temperature. This is occurred due to the reason that viscous dissipation impact on the flow area is to raise the energy, which accommodating the higher fluid temperature and buoyancy force. Increment in the buoyancy force along the developing dissipation parameter increases the temperature field.

Fig. 6 Effect of ϕ with Cu-water nanofluid on dimensionless temperature for $M = 0.01$, $Pr = 6.2$, $Br = 0.1$ and $Bi = 0.1$

Figure 10 portrayed to indicate the temperature $\theta(\eta)$ profile for several values of the local Biot number Bi . As the value of the local Biot number Bi step-up, temperature develops. From the physical point of view, surface thermal resistance reduces with the increasing nature of the local Biot number and so convective heat transfer to the fluid enhances.

Reflexion of the four different types of the nanofluids and several values of the solid volume fraction ϕ , the magnetic parameter M , the Brinkmann number Br and the local Biot number Bi on the surface shear stress $f''(0)$ and the heat transfer rate $\theta'(0)$ are exhibit through Table 3. It can be observed by Eq. (15) that $f''(0)$ and $\theta'(0)$ are proportional to the local skin friction

Fig. 7 Effect of M with Cu-water nanofluid on dimensionless velocity for $\phi = 0.07$

coefficient C_f and the local Nusselt number Nu_x respectively. This table represents that the local skin friction coefficient C_f decline with the sequence of nanofluids such as Ag-water, Cu-water, TiO_2 -water and Al_2O_3 -water, although the local Nusselt number Nu_x evolves for the

accrual manner of nanofluids like Cu-water, Ag-water, TiO_2 -water and Al_2O_3 -water respectively. It is also clear from this table that the wall shear stress $f''(0)$ and the wall heat flux $\theta'(0)$ boost for the booming values of the solid volume fraction ϕ and the magnetic parameter M .

Fig. 8 Effect of M with Cu-water nanofluid on dimensionless temperature for $\phi = 0.07$, $Pr = 6.2$, $Br = 0.1$ and $Bi = 0.1$

Further, increment in the Brinkmann number Br implies the enhancement in the local Nusselt number Nu_x , while reverse trend is true for the local Biot number Bi . Moreover, negative sign of the heat flux leads that there is a heat flow into the surface.

Finally, in the previous studies such as Chaudhary and Choudhary [8], and Chaudhary et al. [36], the magnetohydrodynamic flows of ordinary fluid have been examined

and Galerkin finite element method applied to find the solution of the governing equations. Whereas the magnetohydrodynamic flow of various types of nanofluids have been explored by using the Keller-Box scheme in the present analysis. From these illustrations, it can be observed that nanofluids flow gives the comparatively better performance of the flow system than the ordinary fluids.

Fig. 9 Effect of Br with Cu-water nanofluid on dimensionless temperature for $\phi = 0.07$, $M = 0.01$, $Pr = 6.2$ and $Bi = 0.1$

8 Conclusions

In the present analysis, a steady MHD nanofluids boundary layer flow past a flat surface with viscous dissipation and Joule heating influences along with the Newtonian

heating is analyzed. The presence of four various kinds of nanoparticles namely Ag, Cu, TiO_2 and Al_2O_3 with water based fluid is considered also. By using the similarity variables, the governing equations are converted into non-dimensional form and solved by the Keller-Box method.

Fig. 10 Effect of Bi with Cu -water nanofluid on dimensionless temperature for $\phi = 0.07$, $M = 0.01$, $Pr = 6.2$ and $Br = 0.1$

Moreover, the impacts of different nanofluids and pertinent parameters on the velocity, temperature, surface shear stress and the surface heat flux are examined and discussed. The main observations of present investigation are given as follows

- (1) The velocity profile and the surface gradient are higher for Ag -water nanofluid than remaining nanofluids like Cu -water, TiO_2 -water and Al_2O_3 -water

- respectively. Whereas, Al_2O_3 -water nanofluid has higher improvement on the temperature profile and the surface heat flux than other nanofluids.
- (2) The momentum boundary layer, the thermal boundary layer, the local skin friction coefficient and the local Nusselt number increase as the solid volume fraction and the magnetic parameter develop, while opposite phenomenon occurs in the thermal bound-

Table 3 Computed values of $f''(0)$ and $\theta'(0)$ for distinct types of nanofluids and several values of specified parameters when $Pr = 6.2$

Nanofluids	ϕ	M	Br	Bi	$f''(0)$	$-\theta'(0)$
Ag-water	0.07	0.01	0.1	0.1	0.40348	0.0835793
Cu-water					0.39077	0.0836163
TiO ₂ -water					0.35026	0.0834815
Al ₂ O ₃ -water					0.34770	0.0833913
Cu-water	0.01				0.35439	0.0843381
	0.04				0.37516	0.0840001
	0.10				0.40207	0.0831959
	0.07	1.00			1.06816	0.0801331
		2.00			1.46935	0.0775452
		3.00			1.78315	0.0754303
		0.01	1.0		0.39077	0.0636384
			2.0			0.0414407
			3.0			0.0192430
			0.1	0.5		0.2668798
				1.0		0.3675850
				10.0		0.5566177

ary layer for increment in the magnetic parameter when $\eta > 1$.

- (3) In case of enhancement in the Brinkmann number and the local Biot number, the fluid temperature as well as the heat transfer rate step-up, while reverse trend is happened in the heat transfer rate for the booming value of the local Biot number.

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Compliance with ethical standards

Conflict of interest Authors declare that they have no conflict of interest.

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