On the solution of time-fractional coupled system of partial differential equations

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Abstract

In this research, a newly proposed Laplace Padé reduced differential transform method (LPRDTM) has been applied for solving a coupled system of partial differential equations with time-fractional derivatives. Here the fractional derivatives are defined in the Caputo sense. This is a hybrid technique which is the coupling of laplace transform, Padé approximant, and the reduced differential transform method. Two examples are solved using the present method, and the obtained results are compared with the analytical solutions in terms of the tables, which recommend that LPRDTM provides exact solutions with a few iterations. The main benefit of applying this method is that it does not require any assumption, perturbation, and discretization for solving the governing fractional PDEs. Also, the computation time of this method is less compared to other techniques.

Keywords Padé approximant · Reduced differential transform method · Partial differential equations · Caputo fractional derivative

1 Introduction

Fractional differential equations (FDEs) have accomplished extraordinary consideration among scientists because of its extensive variety of utilization in engineering and science [1–6]. Recently a lot of powerful strategies have been developed for getting approximate and exact solution of FDEs such as shifted Chebyshev polynomials based method [7], variation iteration method [8, 9], generalized differential transform method (DTM) [10], local fractional variation iteration method [11], Laplace decomposition method [12], reproducing kernel Hilbert space method [13], Homotopy analysis method [14–16], Adomian decomposition method [17], Homotopy perturbation method (HPM) [18], Homotopy perturbation transform method [19], Residual power series method (RPSM) [20, 21] and reduced differential transform method (RDTM) [22–24] etc. Among all the above-noted techniques, DTM

is featured by its straightforwardness to explain the nonlinear differential equation. This method does not rely upon a perturbation parameter like other perturbation methods. The DTM was familiarized to the engineering science as a technique for obtaining the solutions of electrical circuits. DTM gives us series solutions which are based on the iterative procedure from Taylor's series expansion. This method is advantageous and efficient for solving different types of differential equations arising in science and engineering. Later RDTM was introduced in order to overcome the complex calculation of DTM. One may see the references [25–28] for different methods and their application on fractional calculus.

This paper deals with the implementation of a method which is the amalgamation of reduced differential transform method, Laplace transform method, and PA [29] to obtain the exact solutions of PDEs [30–34]. First, solutions of PDEs are obtained using RDTM. To obtain the better

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approximate solutions of the problems, LT is then applied to it, and further transformed series is then transformed to a holomorphic function by making its PA. Then the inverse LT of the PA is taken to find the exact solution. The main usefulness of implementing this method is that it increases the domain of convergence and leads to the exact solution. This asset of RDTM reduces the calculations and enhances the effectiveness of the technique in contrast with the other methods. Also, LPRDTM does not require any perturbation parameters like HPM.

The rest part of this paper has been sorted out as follows: necessary basic definitions of fractional calculus are included in Sect. 2. A brief idea and definitions of fractional RDTM are mentioned in Sect. 3. The primary idea behind the PA is presented in Sect. 4. In Sect. 5, the basic concept of the LPRDTM is incorporated. In Sect. 6, two examples are addressed to show the effectiveness and accuracy of the proposed method. In Sect. 7, numerical results and discussion are given. Lastly, a conclusion is included in Sect. 8.

2 Preliminaries

Definition 2.1 [5, 6] Let J_t^α be a Riemann–Liouville fractional integral operator, then

$$J_t^\alpha \psi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \psi(\tau) d\tau, \quad \alpha > 0, \quad (1)$$

$$J_t^0 \psi(t) = \psi(t).$$

where $t > 0$, $\alpha \in R^+$, $\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt$.

Moreover, if $\alpha, \beta \geq 0$ and $\gamma \geq -1$ then

$$\begin{aligned} J_t^\alpha J_t^\beta \psi(t) &= J_t^\beta J_t^\alpha \psi(t) = J_t^{\alpha+\beta} \psi(t), \\ J_t^\alpha t^\gamma &= \frac{\Gamma(1+\gamma)}{\Gamma(1+\alpha+\gamma)} t^{\alpha+\gamma}, \quad \gamma > -1, \quad \alpha > -\gamma - 1. \end{aligned} \quad (2)$$

Definition 2.2 The derivative of $\psi(t)$ in the Caputo sense is well-defined as [6]

$$D_t^\alpha \psi(t) = J_t^{m-\alpha} D_t^m \psi(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \psi^{(m)}(\tau) d\tau, & m-1 < \alpha < m, m \in N \\ \frac{d^m}{dt^m} \psi(t), & \alpha = m, m \in N. \end{cases} \quad (3)$$

Definition 2.3 [6]

$$\begin{aligned} D_t^\alpha J_t^\alpha \psi(t) &= \psi(t), \\ J_t^\alpha D_t^\alpha \psi(t) &= \psi(t) - \sum_{k=0}^p \psi^{(k)}(0^+) \frac{t^k}{k!}, \quad \text{for } p-1 < \alpha \leq p, p \in N. \end{aligned} \quad (4)$$

Table 1 Some operations of RDTM [35, 38, 39]

Original function	Transformed form
$\xi(x, t)$	$\xi_k(x) = \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} \xi(x, t) \right]$
$\xi(x, t) = \psi(x, t) + \zeta(x, t)$	$\xi_k(x) = \psi_k(x) + \zeta_k(x)$
$\xi(x, t) = x^m t^n \psi(x, t)$	$\xi_k(x) = \begin{cases} x^m \psi_{k-n}, & \text{if } k \geq n \\ 0, & \text{else} \end{cases}$
$\xi(x, t) = \psi(x, t) \cdot \zeta(x, t)$	$\xi_k(x) = \sum_{r=0}^k \psi_r(x) \zeta_{k-r}(x)$
$\xi(x, t) = \frac{\partial^r}{\partial t^r} \psi(x, t)$	$\xi_k(x) = \frac{(k+r)!}{k!} \psi_{k+r}(x)$
$\xi(x, t) = \frac{\partial^{na}}{\partial x^{na}} \psi(x, t)$	$\xi_k(x) = \frac{\partial^{na}}{\partial x^{na}} \psi_k(x)$
$\xi(x, t) = e^{\lambda t}$	$\xi_k(x) = \frac{\lambda^k}{k!}$
$\xi(x, t) = \frac{\partial^{na}}{\partial t^{na}} \psi(x, t)$	$\xi_k(x) = \frac{\Gamma(1+(k+n)\alpha)}{(1+k\alpha)} \psi_{k+n}(x)$

3 Fractional reduced differential transform method

Fractional reduced differential transform (FRDT) of an analytic and continuously differentiable function $\xi(x, t)$ is given by

$$\xi_k(x) = \frac{1}{\Gamma(\alpha k + 1)} [D_t^{\alpha k} \xi(x, t)]_{t=t_0}. \quad (5)$$

Applying inverse FRDT on $\xi_k(x)$, we obtain

$$\xi(x, t) = \sum_{k=0}^{\infty} \xi_k(x) (t - t_0)^{\alpha k}. \quad (6)$$

from Eqs. (5) and (6), we have

$$\xi(x, t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} [D_t^{\alpha k} \xi(x, t)]_{t=t_0} (t - t_0)^{\alpha k}. \quad (7)$$

In particular, at $t_0 = 0$, we get

$$\xi(x, t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} [D_t^{\alpha k} \xi(x, t)]_{t=0} t^{\alpha k}. \quad (8)$$

In order to demonstrate the elementary idea of RDTM, let us take the nonlinear PDE of the form as follows

$$L\psi(x, y, t) + R\psi(x, y, t) + N\psi(x, y, t) = h(x, y, t), \quad (9)$$

with initial condition (IC)

$$\psi(x, y, 0) = u(x), \quad (10)$$

where $L = \frac{\partial^\alpha}{\partial t^\alpha}$, R , N are the linear and nonlinear operator, $h(x, y, t)$ is an inhomogeneous source term. Using Table 1, Eq. (1) and applying RDTM to both sides of Eq. (9), we have

$$\frac{\Gamma(1 + \alpha k + \alpha)}{\Gamma(1 + \alpha k)} \psi_{k+1}(x, y) = H_k(x, y) - R\psi_k(x, y) - N\psi_k(x, y), \quad (11)$$

where $\psi_k(x, y)$ and $H_k(x, y)$ are the transformed form of $\psi(x, y, t)$ and $h(x, y, t)$ respectively. Applying RDTM to Eq. (10), we obtain

$$\psi_0(x, y) = u(x). \quad (12)$$

Substituting Eq. (12) into Eq. (11) and by the iterative scheme, we have $\psi_k(x, y)$ value. Then by using inverse FRDT of $\{\psi_k(x, y)\}_{k=0}^n$ gives n th order approximate solution as,

$$\psi_n(x, y, t) = \sum_{k=0}^n \psi_k(x, y) t^{\alpha k}. \quad (13)$$

So, the solution to Eq. (9) is given by

$$\psi(x, y, t) = \lim_{n \rightarrow \infty} \psi_n(x, y, t). \quad (14)$$

4 Padé approximant (PA) [36]

Considering the analytic function $\xi(x, y, t)$ with series form as follows

$$\xi(x, y, t) = \sum_{k=0}^n \xi_k(x, y) t^k. \quad (15)$$

Then PA to $\xi(x, y, t)$ of order $[K, L]$ which is denoted by $\left[\frac{K}{L}\right]_{\xi}(x, y, t)$ and is defined by [36] as

$$\left[\frac{K}{L}\right]_{\xi}(x, y, t) = \frac{p_0 + p_1 t + p_2 t^2 + \dots + p_K t^K}{q_0 + q_1 t + q_2 t^2 + \dots + q_L t^L}, \quad (16)$$

where $q_0 = 1$ and there has no common factor in the numerator and denominator polynomials. The numerator and denominator in Eq. (16) are formulated in such a way that $\xi(x, y, t)$ and $\left[\frac{K}{L}\right]_{\xi}(x, y, t)$ agree at $t = 0$ to $K + L$ that is

$$\xi(x, y, t) - \left[\frac{K}{L}\right]_{\xi}(x, y, t) = O(t^{K+L+1}). \quad (17)$$

From Eq. (17), we have

$$\xi(x, y, t) \sum_{n=0}^L q_n t^n - \sum_{n=0}^K p_n t^n = O(t^{K+L+1}). \quad (18)$$

From Eq. (18) the following algebraic linear system of equations is obtained as

$$\left. \begin{aligned} \xi_K q_1 + \xi_{K+1} q_2 + \xi_{K+2} q_3 + \dots + \xi_{K-L+1} q_L &= -\xi_{K+1}, \\ \xi_{K+1} q_1 + \xi_{K+2} q_2 + \xi_{K+3} q_3 + \dots + \xi_{K-L+2} q_L &= -\xi_{K+2}, \\ \xi_{K+2} q_1 + \xi_{K+3} q_2 + \xi_{K+4} q_3 + \dots + \xi_{K-L+3} q_L &= -\xi_{K+3}, \\ &\vdots \\ \xi_{K+L-1} q_1 + \xi_{K+L-2} q_2 + \xi_{K+L-3} q_3 + \dots + \xi_K q_L &= -\xi_{K+L}. \end{aligned} \right\} \quad (19)$$

Comparing the equal power of t on both sides of Eq. (18), we have

$$\begin{aligned} p_0 &= \xi_0, \\ p_1 &= \xi_1 + \xi_0 q_1, \\ p_2 &= \xi_2 + \xi_1 q_1 + \xi_0 q_2, \\ &\vdots \\ p_K &= \xi_K + \xi_{K-1} q_1 + \xi_{K-2} q_2 + \dots + \xi_0 q_K. \end{aligned} \quad (20)$$

From Eq. (18), all the coefficients of q_n , $1 < n < L$ are calculated. The coefficients of p_n , $0 < n < K$ are determined by Eq. (20).

5 Laplace–Padé reduced differential transform method

There are various methods that give the series solutions. However, sometimes, these types of solutions lack a large domain of convergence. Hence the Laplace–Padé resummation method [11, 30, 31, 33] is applied in this article to increase the convergence of the solutions. Here, the LPRDTM which is the amalgamation of RDTM and Laplace–Padé resummation method is depicted as follows [36].

1. First, find the series solution of given equation by using RDTM
2. Secondly, the LT is then applied to the obtained result.
3. Next, s is substituted for $\frac{1}{t}$ in the obtained result (2).
4. After that, the transformed series obtained in the result (3) is converted into a meromorphic function with the aid of forming its PA of order $[K/L]$ where K and L are arbitrarily chosen. In this step, the domain of the obtained series solution is increased by the PA to acquire better convergence and accuracy.
5. Then, t is substituted for $\frac{1}{s}$ in the obtained result.
6. Finally, the exact solution is obtained by using the inverse LT.

6 Numerical examples

In this section, the present method has been implemented to solve titled problems. Solutions of these examples are also compared with exact solutions.

Example 1 Consider the system of nonlinear time-fractional PDE [37] as

$$\begin{aligned} D_t^\alpha \xi + \psi_x \zeta_y - \psi_y \zeta_x &= -\xi, \\ D_t^\alpha \psi + \zeta_x \xi_y + \xi_x \zeta_y &= \psi, \\ D_t^\alpha \zeta + \xi_x \psi_y + \xi_y \psi_x &= \zeta. \end{aligned} \quad (21)$$

with IC [33]

$$\xi(x, y, 0) = e^{x+y}, \quad \psi(x, y, 0) = e^{x-y}, \quad \zeta(x, y, 0) = e^{-x+y}, \quad 0 < \alpha < 1, \quad (22)$$

Using RDTM and Table 1, Eq. (21) leads to a recurrence relation as

$$\begin{aligned} \frac{\Gamma(1 + \alpha k + \alpha)}{\Gamma(1 + \alpha k)} \xi_{k+1}(x, y) + \sum_{r=0}^k \frac{\partial}{\partial x} \psi_{k-r}(x, y) \frac{\partial}{\partial y} \zeta_r(x, y) \\ - \sum_{r=0}^k \frac{\partial}{\partial y} \psi_{k-r}(x, y) \frac{\partial}{\partial x} \zeta_r(x, y) = -\xi_k(x, y). \end{aligned} \quad (23)$$

$$\begin{aligned} \frac{\Gamma(1 + \alpha k + \alpha)}{\Gamma(1 + \alpha k)} \psi_{k+1}(x, y) + \sum_{r=0}^k \frac{\partial}{\partial x} \zeta_{k-r}(x, y) \frac{\partial}{\partial y} \xi_r(x, y) \\ + \sum_{r=0}^k \frac{\partial}{\partial x} \xi_{k-r}(x, y) \frac{\partial}{\partial y} \zeta_r(x, y) = \psi_k(x, y). \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{\Gamma(1 + \alpha k + \alpha)}{\Gamma(1 + \alpha k)} \zeta_{k+1}(x, y) + \sum_{r=0}^k \frac{\partial}{\partial x} \xi_{k-r}(x, y) \frac{\partial}{\partial y} \psi_r(x, y) \\ + \sum_{r=0}^k \frac{\partial}{\partial y} \xi_{k-r}(x, y) \frac{\partial}{\partial x} \psi_r(x, y) = \zeta_k(x, y). \end{aligned} \quad (25)$$

Again applying RDTM to initial conditions, we may have

$$\xi_0(x, y) = e^{x+y}, \quad \psi_0(x, y) = e^{x-y}, \quad \zeta_0(x, y) = e^{-x+y}. \quad (26)$$

From Eqs. (23)–(25), we have

$$\begin{aligned} \xi_{k+1}(x, y) = \frac{\Gamma(1 + \alpha k)}{\Gamma(1 + \alpha k + \alpha)} \\ \left[-\xi_k(x, y) - \sum_{r=0}^k \frac{\partial}{\partial x} \psi_{k-r}(x, y) \frac{\partial}{\partial y} \zeta_r(x, y) \right. \\ \left. 1 + \sum_{r=0}^k \frac{\partial}{\partial y} \psi_{k-r}(x, y) \frac{\partial}{\partial x} \zeta_r(x, y) \right]. \end{aligned} \quad (27)$$

$$\begin{aligned} \psi_{k+1}(x, y) = \frac{\Gamma(1 + \alpha k)}{\Gamma(1 + \alpha k + \alpha)} \\ \left[\psi_k(x, y) - \sum_{r=0}^k \frac{\partial}{\partial x} \zeta_{k-r}(x, y) \frac{\partial}{\partial y} \xi_r(x, y) \right. \\ \left. - \sum_{r=0}^k \frac{\partial}{\partial x} \xi_{k-r}(x, y) \frac{\partial}{\partial y} \zeta_r(x, y) \right]. \end{aligned} \quad (28)$$

$$\begin{aligned} \zeta_{k+1}(x, y) = \frac{\Gamma(1 + \alpha k)}{\Gamma(1 + \alpha k + \alpha)} \\ \left[\zeta_k(x, y) - \sum_{r=0}^k \frac{\partial}{\partial x} \xi_{k-r}(x, y) \frac{\partial}{\partial y} \psi_r(x, y) \right. \\ \left. - \sum_{r=0}^k \frac{\partial}{\partial y} \xi_{k-r}(x, y) \frac{\partial}{\partial x} \psi_r(x, y) \right]. \end{aligned} \quad (29)$$

Using Eq. (26) in Eqs. (27)–(29), the following recursive value of $\xi_k(x)$, $\psi_k(x)$ and $\zeta_k(x)$ for $k = 1, 2, 3, \dots$ are obtained

$$\begin{aligned} \xi_1(x, y) = -\frac{e^{x+y}}{\Gamma(1 + \alpha)}, \quad \xi_2(x, y) = \frac{e^{x+y}}{\Gamma(1 + 2\alpha)}, \\ \xi_3(x, y) = -\frac{e^{x+y}}{\Gamma(1 + 3\alpha)}, \quad \xi_4(x, y) = \frac{e^{x+y}}{\Gamma(1 + 4\alpha)}, \dots \end{aligned}$$

$$\begin{aligned} \psi_1(x, y) = \frac{e^{x-y}}{\Gamma(1 + \alpha)}, \quad \psi_2(x, y) = \frac{e^{x-y}}{\Gamma(1 + 2\alpha)}, \\ \psi_3(x, y) = \frac{e^{x-y}}{\Gamma(1 + 3\alpha)}, \quad \psi_4(x, y) = \frac{e^{x-y}}{\Gamma(1 + 4\alpha)}, \dots \end{aligned}$$

$$\begin{aligned} \zeta_1(x, y) = \frac{e^{-x+y}}{\Gamma(1 + \alpha)}, \quad \zeta_2(x, y) = \frac{e^{-x+y}}{\Gamma(1 + 2\alpha)}, \\ \zeta_3(x, y) = \frac{e^{-x+y}}{\Gamma(1 + 3\alpha)}, \quad \zeta_4(x, y) = \frac{e^{-x+y}}{\Gamma(1 + 4\alpha)}, \dots \end{aligned}$$

Now the solution of the Eq. (21) maybe written as

$$\begin{aligned} \xi(x, y, t) = \sum_{k=0}^{\infty} \xi_k(x, y) t^{\alpha k} \\ = e^{x+y} \left[1 - \frac{1}{\Gamma(1 + \alpha)} t^\alpha + \frac{1}{\Gamma(1 + 2\alpha)} t^{2\alpha} \right. \\ \left. - \frac{1}{\Gamma(1 + 3\alpha)} t^{3\alpha} + \frac{1}{\Gamma(1 + 4\alpha)} t^{4\alpha} - \dots \right]. \end{aligned} \quad (30)$$

Similarly

$$\begin{aligned} \psi(x, y, t) = e^{x-y} \left[1 + \frac{1}{\Gamma(1 + \alpha)} t^\alpha + \frac{1}{\Gamma(1 + 2\alpha)} t^{2\alpha} \right. \\ \left. + \frac{1}{\Gamma(1 + 3\alpha)} t^{3\alpha} + \frac{1}{\Gamma(1 + 4\alpha)} t^{4\alpha} + \dots \right]. \end{aligned} \quad (31)$$

$$\zeta(x, y, t) = e^{-x+y} \left[1 + \frac{1}{\Gamma(1+\alpha)} t^\alpha + \frac{1}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{1}{\Gamma(1+3\alpha)} t^{3\alpha} + \frac{1}{\Gamma(1+4\alpha)} t^{4\alpha} + \dots \right]. \quad (32)$$

So

$$\xi(x, y, t) = e^{x+y} \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(1+i\alpha)} t^{i\alpha} \right). \quad (33)$$

$$\psi(x, y, t) = e^{x-y} \left(\sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i\alpha)} t^{i\alpha} \right). \quad (34)$$

$$\zeta(x, y, t) = e^{-x+y} \left(\sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i\alpha)} t^{i\alpha} \right). \quad (35)$$

For $\alpha = 1$, the exact solutions are

$$\left. \begin{aligned} \xi(x, y, t) &= e^{x+y-t} \\ \psi(x, y, t) &= e^{x-y+t} \\ \zeta(x, y, t) &= e^{-x+y+t} \end{aligned} \right\} \quad (36)$$

For $\alpha = 0.5$, the exact solutions are

$$\left. \begin{aligned} \xi(x, y, t) &= e^{x+y} \left(e^t + e^t \left(\operatorname{erfc}(\sqrt{t}) - 1 \right) \right) \\ \psi(x, y, t) &= e^{x-y} \left(e^t + e^t \left(-\operatorname{erfc}(\sqrt{t}) + 1 \right) \right) \\ \zeta(x, y, t) &= e^{-x+y} \left(e^t + e^t \left(-\operatorname{erfc}(\sqrt{t}) + 1 \right) \right) \end{aligned} \right\} \quad (37)$$

where $\operatorname{erfc}(t) = 1 - \operatorname{erf}(t)$ and $\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t (e^{-z^2}) dz$.

Here Laplace Padé approximant is applied to the truncated series up to n th order of perturbation that is

$$\left. \begin{aligned} \xi_n(x, y, t) &= e^{x+y} \left(\sum_{i=0}^n \frac{(-1)^i}{\Gamma(1+i\alpha)} t^{i\alpha} \right) \\ \psi_n(x, y, t) &= e^{x-y} \left(\sum_{i=0}^n \frac{1}{\Gamma(1+i\alpha)} t^{i\alpha} \right) \\ \zeta_n(x, y, t) &= e^{-x+y} \left(\sum_{i=0}^n \frac{1}{\Gamma(1+i\alpha)} t^{i\alpha} \right) \end{aligned} \right\} \quad (38)$$

Then applying LT on $\xi_n(x, y, t)$, $\psi_n(x, y, t)$ and $\zeta_n(x, y, t)$ yields

$$L[\xi_n(x, y, t)] = e^{x+y} \left[\sum_{i=0}^n \frac{(-1)^i}{s^{\alpha i+1}} \right]. \quad (39)$$

$$L[\psi_n(x, y, t)] = e^{x-y} \left[\sum_{i=0}^n \frac{1}{s^{\alpha i+1}} \right]. \quad (40)$$

$$L[\zeta_n(x, y, t)] = e^{-x+y} \left[\sum_{i=0}^n \frac{1}{s^{\alpha i+1}} \right]. \quad (41)$$

For simplicity substitute $s = \frac{1}{t}$ in Eq. (39)–(41), we have

$$\left. \begin{aligned} L[\xi_n(x, y, t)] &= e^{x+y} \left[\sum_{i=0}^n (-1)^i t^{\alpha i+1} \right] \\ L[\psi_n(x, y, t)] &= e^{x-y} \left[\sum_{i=0}^n (t^{\alpha i+1}) \right] \\ L[\zeta_n(x, y, t)] &= e^{-x+y} \left[\sum_{i=0}^n (t^{\alpha i+1}) \right] \end{aligned} \right\} \quad (42)$$

All of the $\left[\frac{K}{L} \right]$ PA of Eq. (43) with $K \geq 1$, $L \geq 1$ and $K + L \leq n$ give

$$\left. \begin{aligned} \left[\frac{K}{L} \right]_{\xi_n}(x, y, t) &= \frac{e^{x+y} t}{1 + t^\alpha} \\ \left[\frac{K}{L} \right]_{\psi_n}(x, y, t) &= \frac{e^{x-y} t}{1 - t^\alpha} \\ \left[\frac{K}{L} \right]_{\zeta_n}(x, y, t) &= \frac{e^{-x+y} t}{1 - t^\alpha} \end{aligned} \right\} \quad (43)$$

Substituting $t = \frac{1}{s}$ we have $\left[\frac{K}{L} \right]_{\xi_n}$, $\left[\frac{K}{L} \right]_{\psi_n}$ and $\left[\frac{K}{L} \right]_{\zeta_n}$ in term of s as follows

$$\left. \begin{aligned} \left[\frac{K}{L} \right]_{\xi_n}(x, y, s) &= \frac{e^{x+y} s^{\alpha-1}}{s^\alpha + 1} \\ \left[\frac{K}{L} \right]_{\psi_n}(x, y, s) &= \frac{e^{x-y} s^{\alpha-1}}{s^\alpha - 1} \\ \left[\frac{K}{L} \right]_{\zeta_n}(x, y, s) &= \frac{e^{-x+y} s^{\alpha-1}}{s^\alpha - 1} \end{aligned} \right\} \quad (44)$$

Finally, applying the inverse LT on both sides of Eq. (44), we have the exact solution as follows

$$\left. \begin{aligned} \xi_n(x, y, t) &= e^{x+y} L^{-1} \left\{ \frac{s^{\alpha-1}}{s^\alpha + 1} \right\} \\ \psi_n(x, y, t) &= e^{x-y} L^{-1} \left\{ \frac{s^{\alpha-1}}{s^\alpha - 1} \right\} \\ \zeta_n(x, y, t) &= e^{-x+y} L^{-1} \left\{ \frac{s^{\alpha-1}}{s^\alpha - 1} \right\} \end{aligned} \right\} \quad (45)$$

Putting $\alpha = 1$ in Eq. (45) we have

$$\left. \begin{aligned} \xi_n(x, y, t) &= e^{x+y} L^{-1} \left\{ \frac{1}{s+1} \right\} = e^{x+y-t} \\ \psi_n(x, y, t) &= e^{x-y} L^{-1} \left\{ \frac{1}{s-1} \right\} = e^{x-y+t} \\ \zeta_n(x, y, t) &= e^{-x+y} L^{-1} \left\{ \frac{1}{s-1} \right\} = e^{-x+y+t} \end{aligned} \right\} \quad (46)$$

One may see that the exact solution of Eq. (21) solved by the HPM [37] is the same as the solution solved by the present method, and thus, the solutions are validated for $\alpha = 1$.

Putting $\alpha = 0.5$, Eq. (45) reduces

$$\left. \begin{aligned} \xi_n(x, y, t) &= e^{x+y} L^{-1} \left\{ \frac{s^{-0.5}}{s^{0.5}+1} \right\} = e^{x+y} \left(e^t \operatorname{erfc}(\sqrt{t}) \right) = e^{x+y+t} \operatorname{erfc}(\sqrt{t}) \\ \psi_n(x, y, t) &= e^{x-y} L^{-1} \left\{ \frac{s^{-0.5}}{s^{0.5}-1} \right\} = e^{x-y} \left(e^t \operatorname{erfc}(-\sqrt{t}) \right) = e^{x-y+t} \operatorname{erfc}(-\sqrt{t}) \\ \zeta_n(x, y, t) &= e^{-x+y} L^{-1} \left\{ \frac{s^{-0.5}}{s^{0.5}-1} \right\} = e^{-x+y} \left(e^t \operatorname{erfc}(-\sqrt{t}) \right) = e^{-x+y+t} \operatorname{erfc}(-\sqrt{t}) \end{aligned} \right\} \quad (47)$$

One may see that the exact solution of Eq. (21) solved by the RDTM is the same as the solution solved by the present method and the solutions are validated for $\alpha = 0.5$.

Example 2 Considering another fractional coupled non-linear system of Burger equation [32]

$$\left. \begin{aligned} \xi_t^\alpha - \xi_{xx} - 2\xi\xi_x + (\xi\psi)_x &= 0 \\ \psi_t^\alpha - \psi_{xx} - 2\psi\psi_x + (\xi\psi)_x &= 0 \end{aligned} \right\} \quad (48)$$

with IC [28]

$$\xi(x, 0) = \sin(x), \psi(x, 0) = \sin(x). \quad (49)$$

Applying RDTM to Eq. (48), we may obtain

$$\begin{aligned} \frac{\Gamma(1+\alpha k+\alpha)}{\Gamma(1+\alpha k)} \xi_{k+1}(x) - \frac{\partial^2}{\partial x^2} \xi_k(x) - 2 \\ \sum_{r=0}^k \xi_{k-r}(x) \frac{\partial}{\partial x} \xi_r(x) + \sum_{r=0}^k \frac{\partial}{\partial x} (\xi_{k-r}(x) \psi_r(x)) = 0. \end{aligned} \quad (50)$$

$$\begin{aligned} \frac{\Gamma(1+\alpha k+\alpha)}{\Gamma(1+\alpha k)} \psi_{k+1}(x) - \frac{\partial^2}{\partial x^2} \psi_k(x) - 2 \\ \sum_{r=0}^k \psi_{k-r}(x) \frac{\partial}{\partial x} \psi_r(x) + \sum_{r=0}^k \frac{\partial}{\partial x} (\xi_{k-r}(x) \psi_r(x)) = 0. \end{aligned} \quad (51)$$

Again applying RDTM to IC, we have

$$\xi_0(x) = \sin(x), \psi_0(x) = \sin(x). \quad (52)$$

From Eq. (50), (51), we have

$$\begin{aligned} \xi_{k+1}(x) &= \frac{\Gamma(1+\alpha k)}{\Gamma(1+\alpha k+\alpha)} \\ &\left[\frac{\partial^2}{\partial x^2} \xi_k(x) + 2 \sum_{r=0}^k \xi_{k-r}(x) \frac{\partial}{\partial x} \xi_r(x) - \sum_{r=0}^k \frac{\partial}{\partial x} (\xi_{k-r}(x) \psi_r(x)) \right]. \end{aligned} \quad (53)$$

$$\begin{aligned} \psi_{k+1}(x) &= \frac{\Gamma(1+\alpha k)}{\Gamma(1+\alpha k+\alpha)} \\ &\left[\frac{\partial^2}{\partial x^2} \psi_k(x) + 2 \sum_{r=0}^k \psi_{k-r}(x) \frac{\partial}{\partial x} \psi_r(x) - \sum_{r=0}^k \frac{\partial}{\partial x} (\xi_{k-r}(x) \psi_r(x)) \right]. \end{aligned} \quad (54)$$

Using Eq. (52) in Eqs. (53), (55), the following recursive

value of $\xi_k(x)$ and $\psi_k(x)$ are obtained

$$\begin{aligned} \xi_1(x) &= -\frac{\sin(x)}{\Gamma(1+\alpha)}, \xi_2(x) = \frac{\sin(x)}{\Gamma(1+\alpha)}, \\ \xi_3(x) &= -\frac{\sin(x)}{\Gamma(1+\alpha)}, \xi_4(x) = \frac{\sin(x)}{\Gamma(1+\alpha)}, \dots \end{aligned} \quad (55)$$

$$\begin{aligned} \psi_1(x) &= -\frac{\sin(x)}{\Gamma(1+\alpha)}, \psi_2(x) = \frac{\sin(x)}{\Gamma(1+\alpha)}, \\ \psi_3(x) &= -\frac{\sin(x)}{\Gamma(1+\alpha)}, \psi_4(x) = \frac{\sin(x)}{\Gamma(1+\alpha)}, \dots \end{aligned} \quad (56)$$

Now the solution of Eq. (48) is written as

$$\begin{aligned} \xi(x, t) &= \sum_{k=0}^{\infty} (\xi_k(x) t^{\alpha k}) \\ &= \sin(x) \left[1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \dots \right]. \end{aligned} \quad (57)$$

$$\begin{aligned} \psi(x, t) &= \sum_{k=0}^{\infty} (\psi_k(x) t^{\alpha k}) \\ &= \sin(x) \left[1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \dots \right]. \end{aligned} \quad (58)$$

In compact form, Eqs. (57), (58) may be written as

$$\begin{aligned} \xi(x, t) &= \sin(x) \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(1+i\alpha)} t^{i\alpha} \right) \\ \psi(x, t) &= \sin(x) \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(1+i\alpha)} t^{i\alpha} \right) \end{aligned} \quad (59)$$

For $\alpha = 1$, the analytical solutions are

$$\left. \begin{aligned} \xi(x, t) &= e^{-t} \sin(x) \\ \psi(x, t) &= e^{-t} \sin(x) \end{aligned} \right\} \quad (60)$$

For $\alpha = 0.5$, the exact solutions are

$$\left. \begin{aligned} \xi(x, t) &= \sin(x) \left(e^t + e^t \left(\operatorname{erfc}(\sqrt{t}) - 1 \right) \right) \\ \psi(x, t) &= \sin(x) \left(e^t + e^t \left(\operatorname{erfc}(\sqrt{t}) - 1 \right) \right) \end{aligned} \right\} \quad (61)$$

Here LPAM is applied to the truncated series up to n th order of perturbation that is

$$\left. \begin{aligned} \xi_n(x, t) &= \sin(x) \left(\sum_{i=0}^n \frac{(-1)^i}{\Gamma(1+i\alpha)} t^{i\alpha} \right) \\ \psi_n(x, t) &= \sin(x) \left(\sum_{i=0}^n \frac{(-1)^i}{\Gamma(1+i\alpha)} t^{i\alpha} \right) \end{aligned} \right\} \quad (62)$$

Then applying LT on $\xi_n(x, t)$ and $\psi_n(x, t)$ yields

$$\left. \begin{aligned} L[\xi_n(x, t)] &= \sin(x) \left[\sum_{i=0}^n \frac{(-1)^i}{s^{\alpha i+1}} \right] \\ L[\psi_n(x, t)] &= \sin(x) \left[\sum_{i=0}^n \frac{(-1)^i}{s^{\alpha i+1}} \right] \end{aligned} \right\} \quad (63)$$

For simplicity substituting $s = \frac{1}{t}$ in Eq. (63), we get

$$\left. \begin{aligned} L[\xi_n(x, t)] &= \sin(x) \left[\sum_{i=0}^n (-1)^i t^{\alpha i+1} \right] \\ L[\psi_n(x, t)] &= \sin(x) \left[\sum_{i=0}^n (-1)^i t^{\alpha i+1} \right] \end{aligned} \right\} \quad (64)$$

All of the $\left[\frac{K}{L} \right]$ PA of Eq. (64) with $K \geq 1, L \geq 1$ and $K + L \leq n$ give

$$\left. \begin{aligned} \left[\frac{K}{L} \right]_{\xi_n}(x, t) &= \frac{\sin(x) \cdot t}{1 + t^\alpha} \\ \left[\frac{K}{L} \right]_{\psi_n}(x, t) &= \frac{\sin(x) \cdot t}{1 + t^\alpha} \end{aligned} \right\} \quad (65)$$

Substituting $t = \frac{1}{s}$ we have $\left[\frac{K}{L} \right]_{\xi_n}$ and $\left[\frac{K}{L} \right]_{\psi_n}$ in term of s as follows

$$\left. \begin{aligned} \left[\frac{K}{L} \right]_{\xi_n}(x, s) &= \frac{\sin(x) \cdot s^{\alpha-1}}{s^\alpha + 1} \\ \left[\frac{K}{L} \right]_{\psi_n}(x, s) &= \frac{\sin(x) \cdot s^{\alpha-1}}{s^\alpha - 1} \end{aligned} \right\} \quad (66)$$

Finally, applying the inverse LT on both sides of Eq. (67), we get the exact solution as follows

$$\left. \begin{aligned} \xi_n(x, t) &= \sin(x) L^{-1} \left\{ \frac{s^{\alpha-1}}{s^\alpha + 1} \right\} \\ \psi_n(x, t) &= \sin(x) L^{-1} \left\{ \frac{s^{\alpha-1}}{s^\alpha - 1} \right\} \end{aligned} \right\} \quad (67)$$

Putting $\alpha = 1$ in the above two equations we have

$$\left. \begin{aligned} \xi_n(x, t) &= \sin(x) L^{-1} \left\{ \frac{1}{s+1} \right\} = \sin(x) e^{-t} \\ \psi_n(x, t) &= \sin(x) L^{-1} \left\{ \frac{1}{s-1} \right\} = \sin(x) e^{-t} \end{aligned} \right\} \quad (68)$$

One may see that the exact solution of Eq. (48) solved by the HPM [32] is the same as the solution solved by the present method and the solutions are validated for $\alpha = 1$.

Putting $\alpha = 0.5$, the Eq. (15) becomes

$$\left. \begin{aligned} \xi_n(x, t) &= \sin(x) L^{-1} \left\{ \frac{s^{-0.5}}{s^{0.5} + 1} \right\} = \sin(x) \left(e^t \operatorname{erfc}(\sqrt{t}) \right) \\ \psi_n(x, t) &= \sin(x) L^{-1} \left\{ \frac{s^{-0.5}}{s^{0.5} - 1} \right\} = \sin(x) \left(e^t \operatorname{erfc}(\sqrt{t}) \right) \end{aligned} \right\} \quad (69)$$

One may see that the exact solution of Eq. (27) solved by the RDTM is the same as the solution solved by the present method and the solutions are validated for $\alpha = 0.5$.

7 Numerical results and discussions

In this research, the reduced differential transform method (RDTM) has been carried out as a tool to solve coupled fractional PDEs. The combination of RDTM and Laplace–Padé enables the authors to obtain the solution of the problems with the aid of a few iterations. It is worth mentioning that a higher index fractional PDEs are often challenging to treat numerically as well as analytically without reducing its index to one. In this regard, one may solve these equations by using index-reduction. However, this index reduction may not always give the solution of the original fractional PDEs. As I mentioned earlier that to expand the convergence region of the RDTM solution, Laplace–Padé is applied to the truncated series obtained by RDTM, which leads to the closed-form solution. The RDTM does not involve any lengthy computation, which is the main demerit for the perturbation methods like HPM. This primary aim of applying this RDTM is that it reduces the size of the computation. It is also noted that the high complexity of these problems may effectively handle by

Fig. 1 The solution plots of $\xi(x,y,t)$, $\psi(x,y,t)$ and $\zeta(x,y,t)$ (left) in comparison with $\xi_n(x,y,t)$, $\psi_n(x,y,t)$ and $\zeta_n(x,y,t)$ (right) at $t = 0$ and $\alpha = 1$ of Example 1

Fig. 2 The solution plots of $\xi(x, y, t)$, $\psi(x, y, t)$ and $\zeta(x, y, t)$ (left) in comparison with $\xi_n(x, y, t)$, $\psi_n(x, y, t)$ and $\zeta_n(x, y, t)$ (right) at $t = 1$ and $\alpha = 0.5$ of Example 2

the LPRDTM method due to the flexibility of RDTM and resummation capability of Laplace–Padé. It is remarked that even if the Laplace–Padé resummation method sometime may fail to obtain the exact solution of fractional PDEs, but it may still provide a good approximation in the enlarged domain of convergence. Figures 1, 2, 3, 4

show the comparison plots of the present solution with the exact solution of two test examples at $\alpha = 1$ and 0.5. A comparative study between the present solutions with other existing solutions are given in Tables 1, 2, 3. It is clear from the tables and plots that both the results are matching with each other.

Fig. 3 The solution plots of $\xi(x, t)$ and $\psi(x, t)$ (left) in comparison with $\xi_n(x, y, t)$ and $\psi_n(x, y, t)$ (right) at $\alpha = 1$ of Example 2

8 Conclusion

In this study, LPRDTM is presented to obtain the solution of coupled fractional differential equations. Firstly, the solutions of the fractional nonlinear system of equations are found in series form by using RDTM. It also increases the domain of convergence series. This technique substantially

improves the convergence rate and gives the exact solution. Moreover, RDTM is a powerful method as it does not consider any perturbation parameter to calculate the solutions like HPM. So LPRDTM is a useful technique to find exact solutions. From the obtained results, it is clear that LPRDTM gives exact solutions. Hence, LPRDTM is very useful, and it enables us to have higher accuracy, efficiency, a

Fig. 4 The solution plots of $\xi(x, t)$ and $\psi(x, t)$ (left) in comparison with $\xi_n(x, y, t)$ and $\psi_n(x, y, t)$ (right) at $\alpha = 0.5$ of Example 2

Table 2 Comparison of the present solution with Ganji et al. [37] at $\alpha = 1$

(x, y, t)	Present solution			Ref. [37]		
	$\xi_n(x, y, t)$	$\psi_n(x, y, t)$	$\zeta_n(x, y, t)$	$\xi(x, y, t)$	$\psi(x, y, t)$	$\zeta(x, y, t)$
(0.2,0.2,0.2)	1.22140	1.22140	1.22140	1.22140	1.22140	1.22140
(0.4,0.4,0.4)	1.49182	1.49182	1.49182	1.49182	1.49182	1.49182
(0.6,0.6,0.6)	1.82211	1.82211	1.82211	1.82211	1.82211	1.82211
(0.8,0.8,0.8)	2.22554	2.22554	2.22554	2.22554	2.22554	2.22554
(1,1,1)	2.71828	2.71828	2.71828	2.71828	2.71828	2.71828

Table 3 Comparison of the present solution with Sweilam and Khader [32] at $\alpha = 1$

(x, t)	Present solution		Ref. [32]	
	$\xi_n(x, y, t)$	$\psi_n(x, y, t)$	$\xi(x, y, t)$	$\psi(x, y, t)$
(0.2,0.2)	0.162656	0.162656	0.162656	0.162656
(0.4,0.4)	0.261034	0.261034	0.261034	0.261034
(0.6,0.6)	0.309882	0.309882	0.309882	0.309882
(0.8,0.8)	0.322328	0.322328	0.322328	0.322328
(1,1)	0.309559	0.309559	0.309559	0.309559

mathematical tool for PDEs. Finally, the ongoing appearance of nonlinear fractional differential equations as models in certain fields, it is important to investigate this approach of solutions for some interesting problems arising in every branch of science and engineering and we hope that this work is a step in this path.

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Compliance with ethical standards

Conflict of interest The authors state that they have no conflict of interest.

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