Physical approach to the asymptotic behavior of stationary state for the centrally field

Dmitry L. Tsyganov¹

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Abstract

In this study, a simple approach describing the behaviour of the energy of the stationary state for the centrally symmetric field is proposed. The approach is based on the fundamental principle of quantum mechanics—the Heisenberg's uncertainty principle and allows estimating the energy of stationary states. The calculation accuracy increases with the increase of the orbital quantum number l . The received solutions represent the class of stationary states assuming Heisenberg equality, i.e. in the case of larger orbital quantum numbers l . The proposed approach can be also applied in solving other problems in quantum physics, as well as a teaching tool in higher education institutions.

Keywords Energy of stationary states · Heisenberg's uncertainty principle · The physical approach · Schrödinger equation · The ground state energy

1 Introduction

For a series of the physical phenomena occurring in the microcosm, it is important to find states in which the energy E_n has defined values, known as stationary states of the system. The stationary states E_n are described by wave functions Ψ_n of the operator Hamiltonian in the Schrödinger equation [1]: $i\hbar \frac{\partial \Psi_n}{\partial t} = \hat{H} \Psi_n = E_n \Psi_n$. Schrödinger equation has received lots of attention due to its application in various fields of science, such as electronic properties of semiconductors, superlattices band-structure, semiconductor heterostructures, heterojunctions, quantum dots, polarons, quantum liquids, solid-state electronics, etc. The Schrödinger equation analytical solution was described in the literature through different mathematical methodologies [1–18].

Generally, the exact solution of the Schrödinger equation can be found for a very limited number of problems and simple systems. For example, [1] for the centrally symmetric field solutions for four potentials only were found:

the spatial oscillator $U(r) = \frac{1}{2}m\omega^2 r^2$ with the frequency ω and weight m ; Colombian field $U(r) = -\frac{a}{r}$; the two centrally symmetric potentials $U(r) = \frac{A}{r^2} - \frac{B}{r}$ and $U(r) = \frac{A}{r^2} + Br^2$. Various approximate methods can be used to solve the equation [1–6]: the perturbation theory, different variation methods, the self-consistent field (SCF), etc.

The perturbation theory [1] is the most common and important method for solving the quantum mechanical tasks and is widely used in atomic physics, condensed matter and particle physics. Nevertheless, the perturbation theory of indignation demands “smallness” (i.e. small singular values) of the operator, which strongly limits the area of its utilization. Additionally, due to this limitation, it does not allow the indignation theory asymptotically to consider the transition from quantum to classical physics when Heisenberg inequality [7] transforms into equality (the case of larger orbital quantum numbers l) the perturbation theory cannot be used.

The other approach to the solution of the Schrodinger equation is based on the method of finding the approximate

✉ Dmitry L. Tsyganov, tdl-tdl@rambler.ru | ¹Instituto de Plasmas e Fusão Nuclear, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisbon, Portugal.

solution by the vibrational methods [1, 8, 9], such as the Ritz's method for instance [5]. The Ritz's method is based on the choice of a trial function, which has to minimize the particular function in the form of a superposition of the known functions and meet boundary conditions. This way the task is limited to searching for unknown superposition coefficients.

Another effective way to solve the Schrodinger equation is based on the self-consistent field (SCF) theory [1]. SCF theory studies the behaviour of large and complex stochastic systems by examining simple models [4], by transforming multiparticle problems to one-particle tasks, assuming that each particle moves in some average SCF created by all other particles in the system. In quantum mechanics, the methods using this approach are known as Hartree–Fock ‘...’ methods [1, 6, 10].

Despite the fact that Ritz's and Hartree–Fock ‘...’ are powerful and universal methods, the probability of errors exists. For example, ‘unsuccessful’ choice of the trial function, which has to minimize the particular function, in the case of the Ritz method can lead to inevitable mistakes [1, 6]. Also, it is well known, that with Hartree–Fock ‘...’ methods the solutions lose accuracy with the increase of the coefficient of interaction created by all other particles of the system [1].

The quasi-classical approximation [1, 11, 12] is another effective and widely used approach. All the same, it is applicable only in the case when the characteristic action of the system is much higher than the Dirac constant $\hbar$ [1].

Up to date, in the literature possible ways to improve the different theories, such as the perturbation theory [13, 14], SCF [15], the fractional path integrals approach [16, 17] and Ritz's method [15, 18]. However, they are always present interest in finding faster and easier evaluation approach of the energy of the stationary state in the case of larger orbital quantum numbers l .

Having this in mind, the development of various approximation methods to solve quantum-mechanical problems are still an ongoing task. In this work, a new and simple approach is proposed, which allows faster and easier evaluation of the energy of the stationary state in the case of larger orbital quantum numbers l .

2 Physical approach

All existing approaches can be classified as rigorous mathematical approaches. The method proposed in this study is based on a physical approach with elements of mathematical analysis which does not include the use of strict mathematical justification. In spite of the fact that the physical approach is insufficiently rigorous in terms of mathematical logic, it simplifies the understanding and gives a physical interpretation of complex quantum-mechanical problems. The fundamental to quantum mechanics is HUP [1, 7]:

$$\Delta x \Delta p \geq \hbar \quad (1)$$

HUP introduces the concept of indeterminacy in atomic theory as well. It is considered a fundamental intrinsic limitation in the explanation of the atomic world [19]. In applied sciences and technologies, the HUP plays a leading role in quantum optics and nanophotonics [20, 21], in orbit descriptions of molecules [22], to certain phenomena found in software engineering [23], in quantum information science [24], and in microscopy [25], among others.

HUP is widely used in general physics lectures to explain why the electrons orbiting the nucleus without falling; to assess the size of a hydrogenous atom (Bohr radius) $r = \frac{4\pi\epsilon_0\hbar^2}{me^2}$; to assess the value of minimum energy (the ground state energy) of an electron in a hydrogenous atom $E_0 = -\left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{m_e e^4}{2\hbar^2}$ [1].

It is also possible to use the Generalized Uncertainty Principle (GUP) [26, 27] $\Delta x \Delta p \geq \frac{\hbar}{2}(1 + O(f(\Delta p)))$, which can lead to many important applications in various fields of theoretical physics, e.g., physics of the early universe, Hawking black hole thermodynamics, astrophysics and the entropic force law. It should be noted, that modification to GUP due to the existence of the minimal length and the maximal momentum studied here becomes important at or above the Planck energy [28–31]. Some evidence of these predictions may provide a direct test of the underlying quantum gravity scenario [26, 27]. These modifications are too small to be measured at the present time.

Here, HUP was used to estimate the energy of the stationary state for centrally symmetric fields $U(r)$. For this purpose, the use of the indeterminacy relation for an explanation of the minimum energy existence, other than zero, will be briefly considered [1].

The total energy E of one-dimensional motion is the sum of kinetic $\frac{p^2}{2m}$ and potential energies $U(x)$. For assessment of the minimum possible energy, it is necessary to assume the dispersion of the distances $\Delta x \approx x$ and the impulse $\Delta p \approx p$ [1]. Using the HUP (1), as an assessment of the impulse value is $p \approx \frac{\hbar}{x}$. Thus, the total energy E can be presented as the sum of kinetic $\frac{\hbar^2}{2mx^2}$ and potential energies $U(x)$:

$$E = \frac{\hbar^2}{2mx^2} + U(x) \quad (2)$$

If using the GUP, as an assessment of the impulse value is $p \approx \frac{\hbar}{x}(1 + O(f(x)))$ and the total energy is

$$E = \frac{\hbar^2((1+O(f(x)))^2)}{2mx^2} + U(x).$$

The behaviour of the total energy (2) depends on the behaviour of potential energy $U(x)$. Here cases, when the total energy (2) has a maximum or minimum, will be considered. If the total energy (2) has a minimum, the ground state energy E_0 , can be estimated from the function minimum $E_{\min}$ as:

$$E_0 \geq E_{\min} \quad (3)$$

If the total energy (2) has a maximum, the ground state energy E_0 cannot be easily estimated, except for the unique case: if it is initially known that the quantum system consists of one quantum state E_0 . It is the case, the ground state energy E_0 can be estimated from the function maximum $E_{\max}$ as:

$$E_0 \leq E_{\max} \quad (4)$$

The effective potential $U_{\text{eff}}^l(r)$ [1] of a particle moving in a centrally symmetric field $U(r)$ is:

$$U_{\text{eff}}^l(r) = U(r) + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2}, \quad l = 0, 1, 2, \dots \quad (5)$$

Therefore, the effective potential coincides with the total energy for one-dimensional motion (2), except for $l = 0$. Therefore, the task related to particle motion in the centrally symmetric field for each of orbital numbers l is possible to consider as the one-quantum task concerning one-dimensional motion with the kinetic $\frac{\hbar^2}{2m} \frac{l(l+1)}{r^2}$ ($l = 1, 2, \dots$) and potential $U(r)$ energies.

Considering that for the one-quantum system the ground state can be estimated according to (3) or (4), the following algorithm for assessment of stationary states energy E_l in case particle motion in the centrally symmetric field can be formulated:

1. to write down the efficient potential $U_{\text{eff}}^l(r)$ (5);
2. to identify a minimum and maximum of the effective potential $U_{\text{eff}}^l(r)$ function;
3. to find the energy $E_{\max}/E_{\min}$ of the effective potential $U_{\text{eff}}^l(r)$ in the minimum and maximum;
4. assuming that $E_{\max}/E_{\min}$ is the energies of the stationary state: $E_l \simeq E_{\min}(E_{\max})$.

Of course, in terms of rigorous mathematical logic, the suggestion of transition from the task of particle motion in the centrally symmetric field $U(r)$ to the task concerning one-dimensional motion with kinetic $E_{\text{kinetic}} = \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2}$ ($l = 1, 2, \dots$) and potential $U(r)$ is unacceptable. However, adopting here the physical approach certain discrepancy from the mathematical accuracy is reasonable.

The general issue concerning any approximation approach is the question of the accuracy of the calculation. In general physics, the correspondence Bohr principle [1]: the behaviour of systems described by the theory of quantum mechanics reproduces classical physics in the limit of large quantum numbers. Therefore, at large values of quantum numbers, l of the energy of the stationary state E_l will tend to $E_{\max}/E_{\min}$.

In order to evaluate the practical use of the proposed approach, it was tested for the calculation of stationary states E_n for four central potentials: spatial oscillator $U(r) = \frac{1}{2}m\omega^2 r^2$, Colombian field $U(r) = -\frac{\alpha}{r}$, the centrally

symmetric potential $U(r) = \frac{A}{r^2} - \frac{B}{r}$ and $U(r) = \frac{A}{r^2} + Br^2$. The exact and approximate solutions to these tasks can be found in [1].

3 Examples

3.1 Spatial oscillator: $U(r) = \frac{1}{2}m\omega^2 r^2$

The function of effective potential is found to be: (5) $U_{\text{eff}}^l(r) = \frac{1}{2}m\omega^2 r^2 + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2}$ for $r \in [0, +\infty]$ has a minimum at the point $r = \frac{\sqrt{(l^2+l)\hbar^2}}{\sqrt{m\omega}}$ and the corresponding value of

minimum energy is $E_{\min} = \sqrt{l(l+1)}\omega\hbar$. The exact solution for the spatial oscillator is well known: $E_{\text{ExactValue}} = E_{n_r} = (2n_r + l + \frac{3}{2})\omega\hbar$, where $n_r = 0, 1, 2, \dots$

The function of the error can be defined as:

$$\text{error} = \frac{E_{\min} - E_{\text{ExactValue}}}{E_{\text{ExactValue}}} \quad (6)$$

It can be seen, that the calculated error function (6) completely corresponds to the correspondence Bohr principle at $l \rightarrow +\infty$, i.e. the calculation error decreases with the increase of the quantum number l : $\text{error} = \frac{2n_r+1}{l} + O\left(\left(\frac{1}{l}\right)^2\right)$.

3.2 Colombian field: $U(r) = -\frac{\alpha}{r}, \alpha > 0$

The function of the effective potential (5) $U_{\text{eff}}^l(r) = -\frac{\alpha}{r} + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2}$ on $r \in [0, +\infty]$ has a minimum at the point $r = \frac{l^2+l}{\alpha m} \hbar^2$ and corresponding value of minimum energy is $E_{\min} = -\frac{m\alpha^2}{2(l^2+l)\hbar^2}$. The exact solution is $E_{\text{ExactValue}} = E_n = -\frac{1}{2n^2} \frac{m\alpha^2}{\hbar^2}$, where n —integral number and $n \geq l+1$.

The error function (6) also completely corresponds to the correspondence Bohr principle at $l \rightarrow +\infty$. Here also the error decreases with the increase of the orbital quantum number l : $\text{error} = \frac{1}{l} + O\left(\left(\frac{1}{l}\right)^2\right)$. Thus, for orbital quantum numbers $l > 10$, the error of calculation is less than 1%.

3.3 The centrally symmetric potential of Krater:

$$U(r) = \frac{A}{r^2} - \frac{B}{r}, \quad A \text{ and } B > 0$$

The function of the effective potential (5) $U_{\text{eff}}^l(r) = \frac{A}{r^2} - \frac{B}{r} + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2}$ at $r \in [0, +\infty]$ has a minimum at the point $r = \frac{2mA+l^2\hbar^2+l\hbar^2}{mB}$ and the value of the minimum energy is $E_{\min} = -\frac{mB^2}{4mA+2l(l+1)\hbar^2}$.

The exact solution is $E_{\text{ExactValue}} = E_p = -\frac{2mB^2}{\hbar^2} \left[2p+1 + \sqrt{(2l+1)^2 + \frac{8mA}{\hbar^2}} \right]^{-2}$, where $p=0, 1, 2, \dots$. In this case, also, the calculation error (6) decreases with the increase of the orbital quantum number l : $\text{error} = \frac{-2p-1}{l} + O\left(\left(\frac{1}{l}\right)^2\right)$.

3.4 Centrally symmetric potential $U(r) = \frac{A}{r^2} + Br^2$, A and $B > 0$

The function of effective potential (5) is $U_{\text{eff}}^l(r) = \frac{A}{r^2} + Br^2 + \frac{\hbar^2 l(l+1)}{2m r^2}$ at $r \in [0, +\infty]$ has a minimum at the point $r = \frac{\sqrt[4]{2Am + l^2 \hbar^2 + l \hbar^2}}{\sqrt[4]{2Bm}}$ and value of minimum energy $E_{\text{min}} = \sqrt[4]{4AB + \frac{2Bl(l+1)\hbar^2}{m}}$.

The exact solution is $E_{\text{ExactValue}} = E_n = \hbar \sqrt{\frac{B}{2m} \left[\sqrt{\frac{8Am}{\hbar^2} + (2l+1)^2} + 4n + 2 \right]}$, where $n=0, 1, 2, \dots$

Here also the error in the calculation (6) decreases with the increase of the orbital quantum number l : $\text{error} = \frac{2n+1}{l} + O\left(\left(\frac{1}{l}\right)^2\right)$.

3.5 Multi-electron atoms: helium atom

An atom with more than one electron comprises a complex system, where the electrons interact with each other. The forces of electron-electron interaction have been compared with the force of interaction of an electron with the nuclear charge of the atom. Accounting for the influence of the electron-electron and electron-nucleus interactions means that the total force is no longer central. Besides, the electron may orbit in different planes. Therefore, for accurate calculations, it is necessary to solve the problem in 3D space. Nevertheless, it is possible to use the described approach and to assess the energy of the atomic levels.

The coordinate system will be placed in the atomic nucleus, for simplicity and to understand the electron-electron interaction consequences. The distance from the origin of coordinates to an i -electron is designated as r_i , while the angles between i -electron and j -electron are designated as φ_{ij} . Here the next assumption is made: the influence of gravitational interaction of electrons with each other and with the nucleus of an atom is negligible in comparison with the electromagnetic interactions.

The potential part of the Lagrangian function consists of the potential energy of electron-nucleus interactions (charge of the atom Ze) and the potential energy of electron-electron interactions:

$$L_{\text{potential}} = \frac{1}{4\pi\epsilon\epsilon_0} \left(-\sum_i^N \frac{Ze^2}{|r_i|} + \sum_{i \neq j}^N \frac{e^2}{|r_{ij}|} \right) = \frac{1}{4\pi\epsilon\epsilon_0} \left(-\sum_i^N \frac{Ze^2}{r_i} + \sum_{i \neq j}^N \frac{e^2}{\sqrt{r_i^2 + r_j^2 - 2r_i r_j \cos(\varphi_{ij})}} \right) \quad (7)$$

where ϵ —dielectric permittivity of the medium; ϵ_0 —dielectric permittivity in vacuum, $8.8541 \times 10^{-12} \text{ Fm}$; e —elementary charge, $1.6021766208 \times 10^{-19} \text{ C}$, N —the total number of electrons.

For this case, the effective potential (5) can be written as:

$$U_{\text{eff}}^l(r_i, \varphi_{ij}) = \sum_i^N \frac{\hbar^2 l(l+1)}{2m_e r_i^2} + \frac{1}{4\pi\epsilon\epsilon_0} \left(-\sum_i^N \frac{Ze^2}{r_i} + \sum_{i \neq j}^N \frac{e^2}{\sqrt{r_i^2 + r_j^2 - 2r_i r_j \cos(\varphi_{ij})}} \right) \quad (8)$$

Further, the proposed algorithm was used: identifying the global minimum of the effective potential function at $r_i \in [0, +\infty]$:

$$\begin{cases} \frac{\partial U_{\text{eff}}^l(r_i, \varphi_{ij})}{\partial r_i} \equiv 0, \\ \frac{\partial U_{\text{eff}}^l(r_i, \varphi_{ij})}{\partial \varphi_{ij}} \equiv 0, \end{cases} \quad i = 1 \dots N, j = 1 \dots N, i \neq j \quad (9)$$

From the equation $\frac{\partial U_{\text{eff}}^l(r_i, \varphi_{ij})}{\partial r_i} \equiv 0$, the maximum number of the electrons which can be in one orbit can be calculated. This equation has two solutions: $\varphi_{ij} = 2\pi n$ and $\varphi_{ij} = \pi + 2\pi n$ (where n is integer number).

The solutions $\varphi_{ij} = 2\pi n$ are not physically possible since the electrons should keep a maximum distance from one another. The solutions $\varphi_{ij} = \pi + 2\pi n$ are the correct ones, according to which, one orbital can host a maximum of two electrons. Since the electron is a fermion, this solution completely corresponds to Pauli's exclusion principle [1]: two or more identical fermions cannot occupy the same quantum state within a quantum system simultaneously.

Generally, the analytical solution of the equations is difficult. Helium atom will be further considered. For the system with two electrons, the effective potential is:

$$U_{\text{eff}}^l(r_1, r_2, \varphi_{1,2}) = \sum_i^2 \frac{\hbar^2 l(l+1)^2}{2m_e r_i^2} - \frac{1}{4\pi\epsilon\epsilon_0} \sum_i^2 \frac{Ze^2}{r_i} + \frac{1}{4\pi\epsilon\epsilon_0} \frac{e^2}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\varphi_{1,2})}}.$$

By solving the system of Eq. (9), the obtained results are $r_1 = r_2 = 4\pi\epsilon\epsilon_0 \frac{4l(1+l)\hbar^2}{e^2 m_e (-1+4Z)}$ and $E_l = 4\pi\epsilon\epsilon_0 \frac{e^4 m_e (-1+4Z)^2}{8l(1+l)\hbar^2}$, ($l = 1, 2, \dots$).

For the helium atom ($Z=2$), the energy E_0 of the ground state is $-\frac{49}{16} Da$ or -3.06 eV. The experimental value of zero-point energy is -2.9 eV. The value of zero-point energy is -2.75 eV [1] if we considered the interaction between electrons as perturbation and -2.85 eV [1] if the variation method to the wave function $\psi = \frac{Z^2}{\pi} \exp(-Z(r_1 + r_2))$ was used. Therefore, the calculation error is 5%, which is approximately the same if the interaction between electrons as the perturbation is considered. However, the error is higher if the variation method with the wave function $\psi = \frac{Z^2}{\pi} \exp(-Z(r_1 + r_2))$ is used.

3.6 The collision cross-section

The effective potential in classical mechanics is written as:

$$U_{\text{eff}}(r) = \frac{p_{\infty}^2}{2m} \frac{\sigma}{\pi r^2} + U(r) \quad (10)$$

where $\sigma = \pi \rho^2$ is the collision cross-section, m^2/ρ is partial distance, m/p_{∞} is momentum at infinity, m/s.

The effective potential (10) was used in [32] for calculation of the collision cross-section, known in the classical mechanics as an effective potential method (EPM). EPM is based on the same mathematical principles that are used in the proposed approach. Therefore, the use of EPM for assessment of the collision cross-section σ is possible in quantum mechanics.

The classical approach of the finding the collision cross-section [1] in quantum mechanics consists of the finding of the sum of partial cross-section $\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2(\delta_l) = \sum_{l=0}^{\infty} \sigma_l$ under the influence of potential $U(r)$. Using this approach, the exact solution can be found, however, the level of complexity of calculations is very high. For this reason, a simplified approximate physical approach is proposed.

By comparing the effective potential (5) with the effective potential (10), a concept of partial distance ρ_l as $\rho_l^2 = \frac{\hbar^2}{p^2} l(l+1) = \frac{l(l+1)}{k^2}$ (where $k = \frac{p_{\infty}}{\hbar}$ is the wavenumber, m^{-1}) is proposed and the partial collision cross-section is $\sigma_l = \pi \rho_l^2 = \pi \frac{l(l+1)}{k^2}$. The total collision cross-section is:

$$\sigma = \sum_{l=0}^L \sigma_l = \sum_{l=0}^L \pi \frac{l(l+1)}{k^2} = \frac{\pi}{3} \frac{L(1+L)(2+L)}{k^2} \quad (11)$$

where L is the number of partial phases.

The formula (11) is fully consistent with the conclusion made in [1], i.e. $\sigma \sim k^{-2}$. The number of partial phases L is defined from the condition of “falling” in the centre of the field [32, 33]: “Fall” in the centre of the field those particles whose energy at infinity $E_{\infty} = \frac{p_{\infty}^2}{2m}$ minus the total energy losses Q less than the sum maximal for effective potential (5): $\sum_{l=0}^L U_{\text{eff}}(r_0^l) \leq \frac{p_{\infty}^2}{2m} - Q$. The condition of the maximum

function $\frac{dU_{\text{eff}}(r)}{dr} = \frac{dU(r)}{dr} - \frac{2l(l+1)\hbar^2}{mr^3} \equiv 0$ ($l = 1, 2, \dots$) is used to find corresponding r_0^l .

As an example, the elastic collision cross-section at the dipole-quadrupole interaction of nitrogen with an electron is calculated: $U(r) = -\frac{1}{4\pi\epsilon_0} \frac{\alpha e^2}{2r^4}$, where α is the polarizability of nitrogen, $\alpha = 2.15 \text{ \AA}^3$. The condition for the finding of the number of partial phases L is written as: $\sum_{l=0}^L \frac{\pi\epsilon_0\hbar^4 l^2(l+1)^2}{2e^2 m^2 \alpha} = \frac{\pi\epsilon_0\hbar^4}{30e^2 m^2 \alpha} L(L+1)(L+2)(3L^2+6L+1) \leq \frac{p_{\infty}^2}{2m}$. Unfortunately, the last equation cannot be solved analytically, however, the numerical solution can be easily found for L .

A comparison of various approaches of the elastic collision cross-section calculation of the dipole-quadrupole interaction of nitrogen with an electron is presented in Fig. 1. Calculation within classical mechanics predicts that at $v \rightarrow 0$ of the collision cross-section tends to infinity, while at $v \rightarrow \infty$ the collision cross-section tends to 0 as v^{-1} . As can be seen, the graph is smooth and monotonous.

Opposite to the classical mechanic's calculations, the calculation of the collision cross-section within the proposed approach, exhibit a resonant character, which is caused by the quantum behaviour of the particle. The range of amplitude of resonance decreases at higher energy. The collision cross-section begins to grow with some initial velocity v^* . It should be noted, that the existence of initial velocity v^* within the present assumptions is characteristic of any potential. This is due to the fact that the material point approximation was used [33]. It is well-known, that the material point has weight, but extremely small size. Therefore, according to the optical theorem, at the material point, the section of the collision tends to zero at $v \rightarrow 0$ m/s. The collision cross-section σ is $\sim k^{-2}$.

Fig. 1 The collision cross-section at the dipole-quadrupole interaction of nitrogen with an electron according to: 1—the classical approximation [32, 33]; 2—the quasi-classical approximation [1]; 3—the physical approach (11)

The calculation of the collision cross-section within the quasi-classical approximation [1] well correlates with calculation within our assumptions. The small discrepancy is most probably related to the neglect of phases with small l for the quasi-classical approximation.

4 Conclusion

The physical approach of calculation of an asymptotical behaviour of the energy of stationary states in the centrally symmetric field is proposed and described. The physical approach is based on the fundamental principle of quantum mechanics, the HUP and allows quick and easy evaluation of the energy of the stationary state for the centrally symmetric field as the function of an orbital quantum number l . The accuracy increases with the increase of an orbital quantum number l . The received solutions represent the class of stationary states when Heisenberg inequality turn over equality (the case of larger orbital quantum numbers l).

This approach was used to calculate the energy of stationary states E_n for four central potentials: spatial oscillator $U(r) = \frac{1}{2}m\omega^2 r^2$, Colombian field $U(r) = -\frac{\alpha}{r}$, the centrally symmetric potential $U(r) = \frac{A}{r^2} - \frac{B}{r}$ and $U(r) = \frac{A}{r^2} + Br^2$; the energy of the stationary state E_n of the helium atom; and to calculate the collision cross-sections at the dipole-quadrupole interaction of nitrogen atom with an electron. Based on these examples, it was shown that using this method took less effort than using the classic methods [1]; the calculation error decreases with the increase of the orbital quantum number l as $O\left(\left(\frac{1}{l}\right)^2\right)$; the offered approach completely corresponds to the Bohr principle, Pauli's exclusion principle and to others quantum principles.

In spite of the fact that the physical approach is insufficiently rigorous in terms of mathematical logic, it provides simplicity and imparts to the reader physical thinking describing very complicated phenomena of quantum nature. The proposed approach can be used as a teaching tool in higher education institutions and as an approximate method in applied problems of quantum physics. It can be used in applied science, such as electronic properties of semiconductors, superlattices band-structure, semiconductor heterostructures, heterojunctions, quantum dots, polarons, quantum liquids, solid-state electronics, etc.

The proposed approach can be used also in theoretical physics, i.e. physics of the early universe, Hawking black hole thermodynamics, astrophysics and the entropic force law, etc.

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Compliance with ethical standards

Conflict of interest I, on behalf of the author, wish to declare that there are no conflicts of interest of any type in this work.

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