Scalar H_∞ autopilot synthesis for control systems and evaluation via HIL simulation

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Abstract

The improvement in calculation capabilities conciliate the design and performing of advanced robust control. Among the critical applications are the anti-tank missiles systems. The present work is concerned with enhancing the performance of an antitank guided missile system via robust synthesis of autopilot and guidance systems. This paper is concerned to the derivation of the missile airframe transfer functions. The obtained transfer functions are expanded with formulation for the system uncertainty to be considered during the robust design. The next objective for this paper is to design an autopilot using robust technique with complete justification against related work putting into consideration the overall requirements of flight tactics characteristics. The proposed design is implemented within the simulation model. The obtained results explain the capability of the designed controller to maintain the stability of the system in the presence of un-modeled dynamics. These autopilots proved its robustness to thrust uncertainties within 30% degradation, and about $\pm 30\%$ of nominal aerodynamic coefficients. It is limited to wind speed of about 20 m/s in both directions. It proved its capability of less sensitivity to measurement noise and reject disturbance of 50% within 0.09 s and 95% within 0.22 s. Then, the actuator hardware is implemented within the simulation via interfacing cards using the designed robust autopilot in presence of the prescribed uncertainty. The results show the accuracy via using the hardware-in-the loop simulation. The obtained results revealing that the developed autopilot has the capability to stabilize the system in presence of noise and disturbance.

Keywords Command guidance systems · CLOS · Robust control · H_∞ · Uncertainties

List of symbols

Symbols

$X_1, Y_1, \text{ and } Z_1$	Vectors components along the board reference axes	$F_{AX}, F_{AY} \text{ and } F_{AZ}$	Drag, lateral, and lift forces along the velocity axes
$X_g, Y_g, \text{ and } Z_g$	Vectors components along the ground reference axes	S	Characteristic area
$X, Y, \text{ and } Z$	Vectors components along the velocity reference axes	q	Dynamic pressure given by $q = 0.5\rho(V_m)^2$ (kg/m/s ²)
$\delta_{jp} \text{ and } \delta_{jy}$	Thrust jetivator angles in pitch and yaw planes	ρ	Air density (kg/m ³)
$F_{TX1}, F_{TY1} \text{ and } F_{TZ1}$	Thrust forces along the board reference axes	V_m	Missile velocity
		$C_x, C_y, \text{ and } C_z$	Dimension-less aerodynamic coefficients
		$\bar{g}$	Vector of gravity acceleration
		M	Mach number and given by $M = V_m/V_a$
		V_a	Sound velocity at missile position

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I_T	Perpendicular distance between the missile C.G. and the point of lateral thrust forces action
I_{TX}	Perpendicular distance between longitudinal axis and thrust force line
I_x, I_y, I_z	Characteristic linear dimensions of missile
m_{x_1}, m_{y_1} and m_{z_1}	Dimensionless aerodynamic coefficients
$\omega_{x_1}, \omega_{y_1}$ and ω_{z_1}	Airframe-turn rates along board coordinate axes
$\bar{J}$	Acceleration of missile
Ω	Angular velocity of VCS w.r.t GCS
I_{xx}, I_{yy} and I_{zz}	Moments of inertia components along the BCS
α	Angle of attack [angle of incidence] ($^\circ$)
β	Sideslip angle [angle of drift] ($^\circ$)
U, V, and W	Velocities Along board coordinate axis
U_d, V_d and W_d	Derivative of velocities along board coordinate axis
g_x, g_y and g_z	Gravity acceleration along board coordinate axis

Abbreviations

CLOS	Commanded to line of sight
AP	Autopilot
ATGM	Anti-tank guided missile
6-DOF	Six degrees of freedom
BCS	Board coordinate system
LFT	Linear fractional transformation
BTT	Bank to turn
STT	Skid to turn
C.G.	Centre of gravity
VCS	Velocity coordinate system
BCS	Body coordinate system

1 Introduction

The ever-increasing development of tanks capabilities necessitates the design of accurate control and guidance system for an antitank missile in presence of un-modelled dynamics, fin disturbances and circumstances noise. To achieve this objective, (Ouda, shady, Eslam, El Banna and El gabri) extracted a nonlinear mathematical model representing the dynamical behavior of the underlying missile for different flight phases with uncertainty quantification [1–5]. The system uncertainties included thrust variation due to different causes, variation in aerodynamic coefficients and parameters, wind velocity in different directions and different trim conditions. To overcome different sources of uncertainty, robust control is used to design the autopilot such that the system is

stable with the ability to overcome un-modelled dynamics, to reject the disturbances and minimize the effects of measurement noises overall the missile flight envelope as discussed by (Ouda, hassan, Islam and Abdallah) [6–9]. The performance specifications include overshoot, speed of response, steady state error, and system stability in addition to flight paths with different engagement scenarios. To overcome the effects due to uncertainties and achieve the performance requirements this paper is devoted to design a robust guidance and control for the underlying missile system using the H_∞ with evaluation. Amr and El Sheikh verified that this control system is said to be robust when it maintains a satisfactory level of stability and performance over a range of plant parameters, suddenly disturbances and circumstances noises [10, 11]. Thus, the objective is to investigate the robustness of the designed autopilot against uncertainties due to different sources. On the other hand, the recent advances in robust control theory discussed by (Zhu, Gabrel and Rastegar) offer all requirements of next generation missiles [12–14]. There is different goals obtained from using robust control approach such as better flexibility in the choice of airframe geometry, more tolerance to uncertainty in the underlying system autopilot design. The designed controller is implemented within the missile control system and should be insensitive to model uncertainties and be able to suppress disturbances and noise over the whole envelope of operation to prove its robustness. This paper is devoted to the autopilot design including the jetivator control using the H_∞ in state space form and its implementation for guidance and control performance analysis.

Several researches for missile autopilot design and guidance computer implementation was reviewed in this section. Yang and Li developed a control method using decoupling technique based on robust state feedback control scheme and also, developed a disturbance observer for the (BTT) missile system [15]. The (BTT) missile dynamic systems categorized into three sub-channels (roll, yaw, and pitch). Then disturbance observer was developed for each channel in order to estimate both the external disturbances and the nonlinear couplings. The robust state feedback controller was introduced to stabilize linear parts of each channel. Mattei investigated a nonlinear robust controller for a very highly maneuverable missile in the presence of a wide range of uncertainties in the circumstances [16]. The designed back-stepping robust controller was applied to the MIMO model in order to achieve both (BTT) and (STT) maneuvers. Liu emphasized the problem related to composite anti-disturbance autopilot design for missile systems with multiple disturbances [17]. The first part is developing an observer in order to estimate the disturbances. The finite time integral sliding mode control method was investigated to develop a controller in a feedback path. Lee and Singh developed an adaptive longitudinal autopilot to control the angle of attack of the missile in the presence external disturbance [18].

A control system must satisfy definite specifications and it must sustain uncertainties in the model. Dolye [19] illustrated that any feedback control system has three components: the process, sensors to measure the plant outputs, and a controller to generate the control signal as shown in Fig. 1. Generally, this system has three inputs that contribute to three outputs [the actual output $y(t)$, the tracking error $e(t)$, and the controller/actuator signal, $u(t)$] as described by

$$\begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} T/F & S & -T \\ R & -RF & -RF \\ S & -FS & -FS \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix} \quad (1a)$$

Where the sensitivity function (S), the complementary sensitivity function (T) and the control sensitivity function (R) are defined by El Sheikh [20] as:

$$S = \frac{1}{1+L}, \quad T = \frac{L}{1+L}, \quad R = \frac{C}{1+L} \quad (1b)$$

where L denotes the loop transfer function ($L = FPC$) and ($T = 1 - S$).

The main contribution of this work is the derivation of the missile airframe transfer functions representing the dynamics of the underlying system in pitch plane. The obtained transfer functions are expanded with a mathematical formulation for the system uncertainty to be considered during the robust (H_∞) design. Furthermore, the autopilot design using H_∞ technique putting into consideration the requirements of flight path characteristics. The designed autopilot is implemented within the simulation from which the obtained results revealing that the designed autopilot capable to stabilize the system with the validation via HIL experiment and in presence of noise and disturbance.

This paper is organized in seven sections ended by section seven that presents a conclusion of the paper. Section 1 presents the introduction of underlying missile and the objective for using robust (H_∞) controller. Section 2 is devoted to the types of uncertainties, controller performance evaluation and used model order reduction algorithms. Section 3 presents the problem of underlying missile and airframe dynamics with uncertainty modelling. Section 4 is devoted to nominal pitch channel modeling,

autopilot design using H_∞ loop shaping and autopilot robustness evaluation. Section 5 is devoted to flight performance evaluation concerning thrust variation with measurement noise, effect of yaw separation angle (ψ_s) and effect of target motion. Section 6 is devoted to flight performance evaluation concerning HIL experiment.

2 Robust control

One way to describe the performance of a control system can be considered as the size of interested signals. For example, the performance of a tracking system can be considered as the size of the error signal. There are several ways of defining a signal's size (i.e. several norms for signals), among these norms is the ∞ -Norm. The ∞ -Norm of a signal $u(t)$ is the least upper bound of its absolute value $\|u\|_\infty := \sup_t |u(t)|$. There are several robust techniques, among them is the H_∞ where a quantitative measure for the size of the system uncertainty is considered. Da, Ouda, Bahaa and Selim illustrated that the infinity norm of the transfer function relating the input to the output is the worst-case gain between the two, where both the input and output are measured either by their energy or peak value [21–24]. Other measures of gain can also characterize worst-case amplifications, but in ways which seem to be less useful in practice.

The set of all stable transfer functions whose infinity norms are finite forms a Hardy space as illustrated by Bryant and Nandakumar and denoted by $\|H\|_\infty$ [25, 26]. Moreover, it is the approach which gives much of recent robust control theory its name. The theory is of great interest because it gives solutions to realistic robust control problems known as $\|H\|_\infty$ optimization problems. One would expect it to be harder than LQG theory, because min-max optimization problems are usually harder than quadratic ones, but in fact recent developments have shown the theory to have remarkable similarities with the LQG theory, and LQG problems can even be seen as special cases of $\|H\|_\infty$ problems. In addition to the theoretical advances, one should add that a major reason why this theory is of practical interest is the availability of low-cost interactive software, like MATLAB, which makes it possible to perform all the necessary computations quickly and easily.

2.1 Types of uncertainties

Any real physical system cannot be precisely modeled by the mathematical system; there exist some level of uncertainty and consequently we cannot predict exactly what the output of the system will be even if the input is known [26]. The main challenge of designing a robust control

Fig. 1 Feedback control

system is to develop the control law which able to achieves system response and maintain the error signals within pre-specified limits however there exist an effects of uncertainties. Uncertainty may take many forms among them are the noise/disturbance signals and transfer function modeling errors in addition to un-modeled nonlinear distortion. Consequently, it had adopted a standard quantitative measure for the size of the uncertainty using H_∞ norm, as shown in Fig. 2 by Astrom [27]. The model error Δ can be represented by an unknown transfer function that indicates the difference between the model and the actual process. This general setup allows a designer to catch all these uncertainties, both structured and unstructured, and formulate them into the design. Kwakernaak illustrated that there are two types of uncertainty which are structured and unstructured [28].

Unstructured uncertainties assume less knowledge of the system and it is only assumed that the frequency response of the system lies between two bounds. This type of uncertainty comprises two types: additive and multiplicative uncertainty. Suppose that the actual system (P) is modelled by $\tilde{P}$, then the additive perturbation represents the absolute error in the model, whereas the multiplicative perturbation represents the relative error. The multiplicative perturbation is more frequently used because it is easier to manipulate.

- Additive errors

The most common form to represent uncertainty in a model is the additive model error structure shown in Fig. 3, where Δ_a is additive to the nominal model P so that the actual process is described by $\tilde{P} = P + \Delta_a$.

The additive uncertainties for the missile trajectory system can be considered as the unmodelled dynamics which is the uncertainties in determining some values such as the weight of missile which is vary during flight because of fuel consumption.

Fig. 2 Canonical robust control

Fig. 3 Additive model error

- Multiplicative errors

Another way to represent the model error is to use the multiplicative perturbation as shown in Fig. 4, where the perturbation is illustrated as being multiplicative on the process input. That is, the perturbed plant transfer function has the form: $\tilde{P} = (1 + \Delta_m W_m)P$.

Where W_m is the weight defined as a fixed stable transfer function and Δ_m is a variable stable transfer function satisfying $\|\Delta_m\|_\infty < 1$.

The multiplicative uncertainties for the missile trajectory system can be considered as the applied noise to measuring devices (sensors) such as gyros and accelerometers or the applied disturbances to mechanical actuators (fins).

Considering the realistic frequency range to analyze the system, the frequency response of the extracted pitch airframe transfer function (θ/δ_{jp}) at different operating conditions is clarifies that the max gain variation is 52.8 dB at low frequency and 26.1 dB at high frequency. In addition, the maximum phase variation is 12.7° at low frequency and 80° at high frequency. Also, the frequency response of the yaw airframe jetivator is examined for the ten operating points, where the maximum gain variation is 52.2 dB and the maximum phase variation is 359.8°.

2.2 H_∞ control theory

The methods of H_∞ synthesis are powerful tools for designing robust feedback control systems to achieve singular value loop shaping specifications. The standard H_∞ control problem is sometimes also called the H_∞ small gain

Fig. 4 Input multiplicative model error

Fig. 5 Small gain problem

problem. Michael illustrated that the small-gain theorem states that if a feedback loop consists of stable systems, and the product of all their gains is smaller than one, then the overall feedback loop is stable [29]. That is, assuming that the blocks P and C in Fig. 5 are stable, then the closed loop system remains stable if $\|T_{y_1 u_1}\| < 1$, where $\|T_{y_1 u_1}\|$ is the feedback closed loop transfer function. The small gain problem gives a general setup, and the problem of making $\|T_{y_1 u_1}\|_\infty \leq 1$ is considered as the main idea in designing a robust controller.

The generalized plant P , which is defined from the inputs $[u_1 u_2]^T$ to the outputs $[y_1 y_2]^T$, can be expressed in terms of its state space realization as follows:

$$P(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \quad (2)$$

Find a stabilizing feedback control law (u_2) such that the norm of the closed-loop transfer function matrix ($T_{y_1 u_1}$) is small; where

$$u_2(s) = C(s)y_2(s) \quad (3)$$

$$T_{y_1 u_1} = P_{11}(s) + P_{12}(s)[I - C(s)P_{22}(s)]^{-1}C(s)P_{21}(s) \quad (4)$$

The state-space model of an augmented plant $P(s)$ with weighting functions $W_p(s)$, $W_u(s)$ and $W_t(s)$ which penalize the error signal, control signal and output signal, respectively, as shown in Fig. 6, is considered so that the closed-loop transfer function matrix is the weighted mixed sensitivity;

$$T_{y_1 u_1} = \begin{bmatrix} W_p S \\ W_u R \\ W_t T \end{bmatrix} \quad (5)$$

where S , R and T are given by;

$$\begin{aligned} S &= (I + GC)^{-1} \\ R &= C(I + GC)^{-1} \\ T &= GC(I + GC)^{-1} \end{aligned} \quad (6)$$

In practice, it is usually not necessary to obtain a true optimal controller, but it is often simpler to find a sub-optimal controller. Suppose that $\gamma_{\min}$ is the minimum value of $\|F_\ell(P, C)\|_\infty$ over all possible stabilizing controllers C . Then,

Fig. 6 Pitch control system frequency response

the H_∞ sub-optimal control problem is to find all stabilizing controllers such that

$$\|F_\ell(P, C)\|_\infty < \gamma \quad (7)$$

where $F_\ell(P, C)$ is the lower linear fractional transformation of P and C .

Given that $\gamma > \gamma_{\min}$. This problem can be solved efficiently using the algorithm of Doyle et al. (1989), by reducing γ iteratively to yield the optimal solution [19].

2.3 Controller performance evaluation

Considering the feedback control system shown in Fig. 1, the stability margins and performance of such systems can be quantified using the singular values of the closed-loop transfer function matrices from r to each of the three outputs e , u , and y as defined in Eq. 1 and 5.

The singular value bode plot of each of the three transfer functions represent sensitivity, control sensitivity, and complementary sensitivity has an important influence in the design.

Considering the augmented plant shown in Fig. 6, it can be found that $S(s)$ is the closed-loop transfer function from disturbance (d) to the plant output (y). In order that, the disturbance attenuation can be determined by the singular values of $S(j\omega)$, thus disturbance attenuation performance specification can be written as:

$$\bar{\sigma}\{S(j\omega)\} \leq |W_p^{-1}(j\omega)| \quad (8)$$

Where $|W_p^{-1}(j\omega)|$ is the desired disturbance attenuation coefficient. Allowing $|W_p(j\omega)|$ to depend on frequency ω give us an opportunity to specify a different attenuation factor for each frequency ω .

The singular value Bode plot of $R(s)$ and of $T(s)$ are used to quantify the stability margins of feedback designs in the face of additive plant perturbations Δ_A and multiplicative plant perturbations Δ_M , respectively.

Multiplicative robustness: taking $\bar{\sigma}\{\Delta_M(j\omega)\}$ to be the 'size' of $\Delta_M(j\omega)$, the 'multiplicative' stability robustness is characterized by the size of the smallest destabilizing multiplicative uncertainty $\Delta_M(s)$ as

$$\bar{\sigma}\{\Delta_M(j\omega)\} \leq \left| \frac{1}{\bar{\sigma}\{T(j\omega)\}} \right| \quad (9)$$

The smaller is $\bar{\sigma}\{T(j\omega)\}$, the larger will be the size of the smallest destabilizing multiplicative perturbation and consequently the larger will be the stability margin.

Consider $\bar{\sigma}\{\Delta_A(j\omega)\}$ is the definition of the 'size' of $\Delta_A(j\omega)$ at certain frequency ω , the size of the smallest destabilizing additive uncertainty $\Delta_A(s)$ can be defined as

$$\bar{\sigma}\{\Delta_A(j\omega)\} \leq \left| \frac{1}{\bar{\sigma}\{R(j\omega)\}} \right| \quad (10)$$

As a result of Eqs. 9 and 10, the stability margin of the control system can be determined using singular value inequalities such as:

$$\begin{aligned} \bar{\sigma}\{R(j\omega)\} &\leq |W_u^{-1}(j\omega)| \\ \bar{\sigma}\{T(j\omega)\} &\leq |W_t^{-1}(j\omega)| \end{aligned} \quad (11)$$

where $|W_u(j\omega)|$ and $|W_t(j\omega)|$ are the sizes of the additive and multiplicative perturbations respectively. The effects of all uncertainty can be considered as a single multiplicative perturbation Δ_M , so that the control design requirements can be considered as:

$$\frac{1}{\bar{\sigma}_i\{S(j\omega)\}} \geq |W_p(j\omega)| \quad \text{and} \quad \bar{\sigma}_i\{T(j\omega)\} \leq |W_t^{-1}(j\omega)|.$$

The design requirements with the above inequalities are depicted in Fig. 7, where the upper half of the figure (above the 0 dB line) represents $\bar{\sigma}\{L(j\omega)\} \approx \frac{1}{\bar{\sigma}\{S(j\omega)\}}$, while the lower half (below the 0 dB line) represents $\bar{\sigma}\{L(j\omega)\} \approx \bar{\sigma}\{T(j\omega)\}$. This conclusion results from the fact that

$$\begin{aligned} S(s) &= \{I + L(s)\}^{-1} \approx L(s)^{-1}, \text{ if } \bar{\sigma}\{L(s)\} \gg 1 \\ T(s) &= L(s)\{I + L(s)\}^{-1} \approx L(s), \text{ if } \bar{\sigma}\{L(s)\} \ll 1 \end{aligned} \quad (12)$$

Where, L denotes the loop transfer function ($L = PC$).

The following trade-offs should be exercised:

- Good command following $\Leftarrow L$ large.
- Good disturbance rejection $\Leftarrow L$ large
- Good noise attenuation $\Leftarrow L$ small.
- Small input $u \Leftarrow C, L$ small.
- Nominal stability (stable plant) $\Leftarrow L$ small
- Robust stability (stable plant) $\Leftarrow L$ small

Fig. 7 Singular value specification on L, S, and T

Fortunately, these objectives are in different frequency ranges and consequently most of them can be met with a large gain ($|L| > 1$) for low frequencies (below crossover), and a small gain ($|L| < 1$) at high frequencies (above crossover). An alternative approach to the loop shaping is the H_∞ mixed-sensitivity loop shaping, where the performance and stability specifications, Eqs. (8, 9) are combined into a single infinity norm specification of the form $\|T_{y1u1}\| \leq 1$ in

$$\text{which } T_{y1u1} = \begin{bmatrix} W_p S \\ W_t T \end{bmatrix}.$$

The term $\|T_{y1u1}\|_\infty$ is called a mixed-sensitivity cost function because it penalizes both sensitivity $S(s)$ and the complementary sensitivity $T(s)$. Loop shaping is achieved via choosing W_p to have the target loop shape for frequencies $\omega < \omega_c$ and choosing $1/W_t$ to be the target for $\omega > \omega_c$. In choosing design specifications W_p and W_t for a controller design, it is required to ensure that the 0 dB crossover frequency for the Bode plot of W_p is below the 0 dB crossover frequency of $1/W_t$ as shown in Fig. 7. This is to allow a gap for the desired loop shape G_d to pass between the performance bound W_p and the robustness bound $1/W_t$; otherwise the performance and robustness requirements will not be achievable.

In addition, the following trade-offs should be exercised: good command following and disturbance rejection necessitate L to be large while good noise attenuation and robust stability necessitates L to be small. That is, compromise between conflicting requirements should be experienced.

2.4 Model order reduction

The robust controller's design for complex systems necessitates model reduction to simplify the obtained controller to a reasonable order. The order reduction can be carried to the original system, to the obtained controller or to the system as a whole.

In this work a model reduction algorithm was applied to the control designed controller to reduce its complexity.

Glover proofed that eigenvalues define system stability whereas Hankel singular values define the energy of each state in the system [30].

In the design of controllers for complicated systems, model reduction arises in several places:

1. It is desirable to simplify the available model for the purpose of reduced order controller design.
2. In using certain design methods (including the H_∞ method), fictitious unobservable/uncontrollable states are generated by the algorithms which must be stripped away by a reliable model reduction algorithm.
3. Finally, if a modern control method such as H_∞ is employed for which the complexity of the control law is not explicitly constrained, the order of the resultant control law is likely to be considerably greater than is truly needed. Consequently, a good model reduction algorithm is applied to the control law to reduce its complexity with little change in control system performance.

Hankel singular values

In control theory, eigenvalues define the system stability, whereas Hankel singular values define the energy of each state in the system. Keeping larger energy states of a system preserves most of its characteristics in terms of stability, frequency, and time responses. Model reduction techniques presented in this paper are based on the Hankel singular values of a system. They can achieve a reduced-order model that preserves the majority of the system characteristics. Mathematically, given a stable state-space system (A, B, C, D) , its Hankel singular values are defined as:

$$\sigma_H = \sqrt{\lambda_i(P_g Q_g)} \quad (13)$$

where P_g and Q_g are controllability and observability grammians satisfying the following equations:

$$\begin{aligned} AP + PA^T &= -BB^T \\ A^T Q + QA &= -C^T C \end{aligned} \quad (14)$$

Robust control offers several algorithms for model approximation and order reduction which are used to control the absolute or relative approximation error based on the Hankel singular values of the system. Model reduction approaches can be put into two categories:

- Additive error method in which the reduced-order model G_{red} has an additive error bounded by an error criterion $\|G - G_{red}\|_\infty$.

- Multiplicative error method where the reduced-order model has a multiplicative or relative error bounded by an error criterion $\|G^{-1}(G - G_{red})\|_\infty$.

The error is measured in terms of peak gain across frequency (H_∞ norm), where the error bounds are functions of the neglected Hankel singular values.

Additive model reduction

Three methods are available to do the additive error model reduction:

- Square-root balanced model truncation
- Schur balanced model truncation
- Hankel minimum degree approximation

Each of the above methods possess the same infinity-norm error bound for a k th order reduced order model $\tilde{G}(S)$ of an m th order system $G(s)$:

$$\bar{\sigma}(G(j\omega) - \tilde{G}(j\omega)) \leq 2 \sum_{i=k+1}^m \sigma_i \forall \omega$$

Multiplicative Model Reduction

Combining the balanced stochastic truncation (BST) with the relative error bound (REM) can achieve the optimal solution for robust model reduction. Reschmr implements the Schur version of the BST-REM theory and yields the following “relative-error” and “multiplicative-error” bounds:

$$\begin{aligned} \|G^{-1}(G - \tilde{G})\|_\infty &\leq \Delta_{err} \\ \|\tilde{G}^{-1}(G - \tilde{G})\|_\infty &\leq \Delta_{err} \\ \Delta_{err} &= \sum_{i=k+1}^n \frac{2\sigma_i}{1 - \sigma_i} \end{aligned} \quad (15)$$

One method is available to do the multiplicative error model reduction known by balanced stochastic truncation. In this method, given a state space (A, B, C, D) of a system and the desired reduced order (k), the reduced order state space model is obtained via the following steps:

1. Find the controllability grammian P and observability grammian Q of the left spectral factor $\Phi = \Gamma(\sigma)\Gamma^* (-\sigma) = \Omega^* (-\sigma)\Omega(\sigma)$ by solving the following Lyapunov and Riccati equations;

$$AP + PA^T + BB^T = 0$$

$$B_W = PC^T + BD^T$$

$$QA + A^T Q + (QB_W - C^T)(-DD^T)(QB_W - C^T)^T = 0$$

- Find the Schur decomposition for PQ in both ascending and descending order, respectively;

$$V_A^T P Q V_A = \begin{bmatrix} \lambda_1 & \dots & \dots \\ 0 & \dots & \dots \\ 0 & 0 & \lambda_n \end{bmatrix} \quad V_D^T P Q V_D = \begin{bmatrix} \lambda_n & \dots & \dots \\ 0 & \dots & \dots \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

(a)

(b)

(c)

Fig. 8 **a** Ground and board axes, **b** ground and velocity axes, **c** velocity and body axes

- Find the left/right orthonormal Eigen-bases of PQ associated with the k th big Hankel singular values of the all-pass phase matrix $\{W^*(S)\}^{-1}G(S)$

$$V_A = [V_{R,SMALL}, \overbrace{V_{L,BIG}}^K]$$

$$V_D = [\overbrace{V_{R,BIG}}^K, V_{L,SMALL}]$$

- Find the SVD of $(V_{L,BIG}^T V_{R,BIG}) = U \Sigma \zeta^T$
- Form the left/right transformation for the final k th order reduced model;

$$S_{L,BIG} = V_{L,BIG} U \sum (1 : K, 1 : K)^{-1/2}$$

$$S_{R,BIG} = V_{R,BIG} V \sum (1 : K, 1 : K)^{-1/2}$$

- Finally, the reduced order model is obtained as:

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \begin{bmatrix} S_{L,BIG}^T A S_{R,BIG} & S_{L,BIG}^T B \\ C S_{R,BIG} & D \end{bmatrix}$$

3 Problem formulation

This section briefly describes the basic subsystems that form a guidance system which steers the missile towards its target according to a certain guidance law.

Missile equations of motion

When formulating and solving problems of flying vehicles, coordinate systems and reference frames have to be considered for the description of the various dynamical parameters including position, velocity, acceleration, forces, and moments. The forces acting on the missile are weight, thrust, and aerodynamic forces. These forces have different mother frames of reference and consequently coordinates transformation from a frame to another is indispensable. This transformation is carried out using Euler's angles transformation method. The mutual relation between ground, board and velocity coordinates is shown in Fig. 8a–c. The coordinate's transformation from body into the ground coordinate system using Euler's angles can be carried out using the following transformation matrix

$$\begin{bmatrix} x_g \\ y_g \\ z_g \end{bmatrix} = T_{bg} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} c\psi c\theta & -s\psi c\theta + c\psi s\theta s\phi & s\psi s\theta s\phi + c\psi s\theta c\phi \\ s\psi c\theta & c\psi c\theta + s\psi s\theta s\phi & -s\psi s\theta c\phi + c\psi s\theta c\phi \\ -s\theta & s\phi c\theta & c\theta c\phi \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad (16)$$

where X_1 , Y_1 , and Z_1 (X_g , Y_g and Z_g) are the vectors components along the board (ground) reference axes. The relative attitude between ground and velocity coordinates is represented by the following transformation matrix:

$$\begin{bmatrix} x_g \\ y_g \\ z_g \end{bmatrix} = T_{vg} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c\gamma c\vartheta & c\gamma s\vartheta s\chi - s\gamma c\chi & s\gamma s\chi + c\gamma s\vartheta c\chi \\ s\gamma c\vartheta & c\gamma c\chi + s\gamma s\vartheta s\chi & s\gamma s\vartheta c\chi - s\chi c\gamma \\ -s\vartheta & s\chi c\vartheta & c\vartheta c\chi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (17)$$

where X , Y , and Z (X_g , Y_g , and Z_g) are the vectors' components along the velocity (ground) reference-axes. The coordinates' transformation from velocity into body coordinates axes can be carried using the following matrix:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T_{vg} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} c\beta c\alpha & -s\beta & c\beta s\alpha \\ s\beta c\alpha & c\beta & s\beta s\alpha \\ -s\alpha & 0 & c\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad (18)$$

The thrust forces that act on the missile are inclined by angles δ_{jp} and δ_{jy} in the pitch and yaw planes. The thrust forces and moments are shown in Fig. 9

The equations of missile motion include three translational equations and three rotational equations (around missile c.g.; yawing, pitching and rolling). The simulation uses these nonlinear coupled differential equations that describe the behavior of a rigid missile can be summarized as follows:

$$\begin{aligned} \sum F_x &= mg_x + F_{Tx} + F_{Ax} = m(\dot{v}_x) = m\dot{v}_M \\ \sum F_y &= mg_y + F_{Ty} + F_{Ay} = m(v_x \Omega_z) = m v_M \Omega_z \\ \sum F_z &= mg_z + F_{Tz} + F_{Az} = -m v_M \Omega_y \end{aligned} \quad (19)$$

$$\begin{aligned} J_x &= g_x + (F_{Tx_1} c\beta c\alpha - F_{Ty_1} s\beta + F_{Tz_1} c\beta s\alpha - C_x s\varphi)/m \\ J_y &= g_y + (F_{Tx_1} s\beta c\alpha + F_{Ty_1} c\beta + F_{Tz_1} s\beta s\alpha - C_y s\varphi)/m \\ J_z &= g_z + (-F_{Tx_1} s\alpha + F_{Tz_1} c\alpha - C_z s\varphi)/m \end{aligned} \quad (20)$$

Fig. 9 Thrust forces and moments

$$\begin{aligned} M_{x_1} &= I_{xx} \dot{\omega}_{x_1} + (I_{zz} - I_{yy}) \omega_{z_1} \omega_{y_1} \\ M_{y_1} &= I_{yy} \dot{\omega}_{y_1} + (I_{xx} - I_{zz}) \omega_{z_1} \omega_{x_1} \\ M_{z_1} &= I_{zz} \dot{\omega}_{z_1} + (I_{yy} - I_{xx}) \omega_{x_1} \omega_{y_1} \end{aligned} \quad (21)$$

$$\begin{aligned} \dot{\theta} &= \omega_{y_1} \cos \phi - \omega_{z_1} \sin \phi \\ \dot{\psi} &= (\omega_{z_1} \cos \phi + \omega_{y_1} \sin \phi) / \cos \theta \\ \dot{\phi} &= \omega_{x_1} + (\omega_{z_1} \cos \phi + \omega_{y_1} \sin \phi) \tan \theta \end{aligned} \quad (22)$$

where Eq. (19) represents force components in the velocity reference frame, Eq. (20) represents the acceleration components in the velocity reference frame, Eq. (21) represents moment components in body reference frame, and Eq. (22) represents missile rotation around its center of gravity (c.g.) as shown in Fig. 10 that represent the Geometrical relation.

Geometrical relations

It's clear that the relative attitude between reference frames can be described as follows:

- VCS w.r.t. GCS by angles $(\gamma, \vartheta, \chi)$
- VCS w.r.t. BCS by angles (α, β)
- BCS w.r.t. GCS by angles (ψ, θ, ϕ)

For complex or combined rotations, the transformation from, say, BCS into the GCS can be carried out using two approaches either directly or via the VCS. These approaches are described as follows:

Fig. 10 Geometrical relation

$$\begin{aligned}
 x &= T_{bv}x_1 \\
 x_1 &= T_{gb}x_g \\
 x &= T_{gv}x_g
 \end{aligned}
 \quad (23)$$

Manipulating these equations yields $T_{gb}=T_{vb}T_{gv}$, from which only three elements are selected from the above equations to yield the geometrical relations as follows:

$$\begin{aligned}
 \theta &= \vartheta + \beta \sin \chi + \alpha \cos \chi \\
 \phi &= \psi + \alpha \tan \vartheta \sin \chi - \beta \tan \vartheta \cos \chi \\
 \psi &= \gamma + ((\alpha \sin \chi - \beta \cos \chi) / \cos \vartheta)
 \end{aligned}
 \quad (24)$$

3.1 Pitch jetivator and airframe dynamics

Towards the autopilot design, the guidance equations derived in last section have to be linearized for extracting the necessary transfer function or state space models [1]. That is, consider Eqs. (19–24) which describe equations of motion for the intended guided missile, with the following assumptions:

- pitch and/or yaw motion
- piecewise constant velocity
- Neglect g and consider small firing angles (α, β) and small thrust jetivator angles δ_{jp} and δ_{jy} such that $F_{Tx_1} = F_{T_1}, F_{Ty_1} = -F_{T_1} \delta_{jy}$, and $F_{Tz_1} = -F_{T_1} \delta_{jp}$ [31–33].

The equations describing dynamics of guided missile c.g. motion, rotation around its c.g. and geometrical relations can be summarized as follows:

$$\begin{aligned}
 \ddot{\theta} &= M_{y1\alpha} \alpha + M_{y1\dot{\theta}} \dot{\theta} + M_{\delta_{jp}} \delta_{jp} \\
 \dot{\theta} &= n_{\alpha} \alpha + n_{\delta_{jp}} \delta_{jp} \\
 \vartheta &= \theta - \alpha
 \end{aligned}
 \quad (25)$$

$$M_{y1\alpha} = m_{y1}^{\alpha} sq l_y / I_{yy}$$

$$M_{y1\dot{\theta}} = m_{y1}^{\omega y_1} sq l_y / I_{yy}$$

$$M_{\delta_{jp}} = -F_{T_1} \times l_t / I_{yy}$$

$$n_{\delta_{jp}} = \frac{F_{T_1}}{mv_M}$$

$$n_{\alpha} = \frac{1}{mv_M} (C_z^{\alpha} sq + F_{T_1})$$

where S represents the characteristic area, (q) represents the dynamic pressure given by $q = 0.5 \rho (v_b)^2$ ($\text{kg/m}^2/\text{s}^2$), ρ the air density (kg/m^3), and v_b is the missile velocity in BCS. The Laplace transform of the above equations results in;

$$s \dot{\theta}(s) - M_{y1\dot{\theta}} \dot{\theta}(s) - M_{y1\alpha} \alpha(s) = M_{\delta_{jp}} \delta_{jp}(s) \quad (26)$$

$$s \vartheta(s) - n_{\alpha} \alpha(s) = n_{\delta_{jp}} \delta_{jp}(s) \quad (27)$$

$$\vartheta(s) - \theta(s) + \alpha(s) = 0 \quad (28)$$

Body rate transfer function

Manipulating Eq. (27) yields:

$$\begin{aligned}
 n_{\alpha} \alpha(s) &= s \vartheta(s) - n_{\delta_{jp}} \delta_{jp}(s) \\
 &= s[\theta(s) - \alpha(s)] - n_{\delta_{jp}} \delta_{jp}(s) \\
 \alpha(s)[n_{\alpha} + s] &= s \theta(s) - n_{\delta_{jp}} \delta_{jp}(s)
 \end{aligned}$$

Substituting into Eq. (26) yields:

$$\begin{aligned}
 s \dot{\theta}(s) - M_{y1\dot{\theta}} \dot{\theta}(s) - M_{y1\alpha} \left[\frac{s \theta(s) - n_{\delta_{jp}} \delta_{jp}(s)}{n_{\alpha} + s} \right] &= M_{\delta_{jp}} \delta_{jp}(s) \\
 s \dot{\theta}(s) - M_{y1\dot{\theta}} \dot{\theta}(s) - \frac{M_{y1\alpha}}{n_{\alpha} + s} \dot{\theta}(s) + \left[\frac{M_{y1\alpha} n_{\delta_{jp}}}{n_{\alpha} + s} \right] \delta_{jp}(s) &= M_{\delta_{jp}} \delta_{jp}(s) \\
 \dot{\theta}(s) \left[s - M_{y1\dot{\theta}} - \frac{M_{y1\alpha}}{n_{\alpha} + s} \right] &= \left[M_{\delta_{jp}} - \frac{M_{y1\alpha} n_{\delta_{jp}}}{n_{\alpha} + s} \right] \delta_{jp}(s) \\
 \dot{\theta}(s) \left[\frac{s(n_{\alpha} + s) - M_{y1\dot{\theta}}(n_{\alpha} + s) - M_{y1\alpha}}{n_{\alpha} + s} \right] &= \left[\frac{M_{\delta_{jp}}(n_{\alpha} + s) - (M_{y1\alpha} n_{\delta_{jp}})}{n_{\alpha} + s} \right] \delta_{jp}(s) \\
 \frac{\dot{\theta}(s)}{\delta_{jp}(s)} &= \frac{M_{\delta_{jp}}(n_{\alpha} + s) - (M_{y1\alpha} n_{\delta_{jp}})}{s^2 + n_{\alpha} s - M_{y1\dot{\theta}} s - M_{y1\alpha}} \\
 \frac{\dot{\theta}(s)}{\delta_{jp}(s)} &= \frac{M_{\delta_{jp}} s + (M_{\delta_{jp}} n_{\alpha} - M_{y1\alpha} n_{\delta_{jp}})}{s^2 + (n_{\alpha} - M_{y1\dot{\theta}}) s - (M_{y1\dot{\theta}} n_{\alpha} + M_{y1\alpha})} \quad (29)
 \end{aligned}$$

Substituting this equation into Eq. (26) results in;

$$\begin{aligned}
 s^2 \theta(s) - M_{y1\dot{\theta}} s \theta(s) - M_{y1\alpha} \left[\frac{s \theta(s) - n_{\delta_{jp}} \delta_{jp}(s)}{n_{\alpha} + s} \right] &= M_{\delta_{jp}} \delta_{jp}(s) \\
 s^2 \theta(s) - M_{y1\dot{\theta}} s \theta(s) - \frac{M_{y1\alpha}}{n_{\alpha} + s} s \theta(s) + \left[\frac{M_{y1\alpha} n_{\delta_{jp}}}{n_{\alpha} + s} \right] \delta_{jp}(s) &= M_{\delta_{jp}} \delta_{jp}(s) \\
 \theta(s) \left[s^2 - M_{y1\dot{\theta}} s - \frac{M_{y1\alpha}}{n_{\alpha} + s} s \right] &= \left[M_{\delta_{jp}} - \frac{M_{y1\alpha} n_{\delta_{jp}}}{n_{\alpha} + s} \right] \delta_{jp}(s) \\
 \theta(s) \left[\frac{s^2(n_{\alpha} + s) - s(M_{y1\dot{\theta}}(n_{\alpha} + s) + M_{y1\alpha})}{n_{\alpha} + s} \right] &= \left[\frac{M_{\delta_{jp}}(n_{\alpha} + s) - (M_{y1\alpha} n_{\delta_{jp}})}{n_{\alpha} + s} \right] \delta_{jp}(s) \\
 \frac{\theta(s)}{\delta_{jp}(s)} &= \frac{M_{\delta_{jp}} n_{\alpha} + M_{\delta_{jp}} s - (M_{y1\alpha} n_{\delta_{jp}})}{s^3 + n_{\alpha} s^2 - M_{y1\dot{\theta}} n_{\alpha} s - s^2 M_{y1\dot{\theta}} - M_{y1\alpha} s} \\
 \frac{\theta(s)}{\delta_{jp}(s)} &= \frac{M_{\delta_{jp}} s + (M_{\delta_{jp}} n_{\alpha} - M_{y1\alpha} n_{\delta_{jp}})}{s^3 + (n_{\alpha} - M_{y1\dot{\theta}}) s^2 - s(M_{y1\dot{\theta}} n_{\alpha} + M_{y1\alpha})} \quad (30)
 \end{aligned}$$

Angle of attack transfer function

Equation (27) clarifies that:

$$\begin{aligned}n_{\alpha}\alpha(s) &= s\vartheta(s) - n_{\delta_{jp}}\delta_{jp}(s) \\s\vartheta(s) &= n_{\alpha}\alpha(s) + n_{\delta_{jp}}\delta_{jp}(s) \\s[\vartheta(s) - \alpha(s)] &= n_{\alpha}\alpha(s) + n_{\delta_{jp}}\delta_{jp}(s) \\-s\alpha(s) + s\vartheta(s) &= n_{\alpha}\alpha(s) + n_{\delta_{jp}}\delta_{jp}(s) \\s\vartheta(s) &= \alpha(s)[n_{\alpha} + s] + n_{\delta_{jp}}\delta_{jp}(s)\end{aligned}$$

Manipulating Eq. (26) yields

$$\begin{aligned}s^2\vartheta(s) &= M_{y1\alpha}\alpha(s) + M_{y1\vartheta}s\vartheta(s) + M_{\delta_{jp}}\delta_{jp}(s) \\ \vartheta(s)[s^2 - M_{y1\vartheta}s] &= M_{y1\alpha}\alpha(s) + M_{\delta_{jp}}\delta_{jp}(s) \\ s\vartheta(s)[s - M_{y1\vartheta}] &= M_{y1\alpha}\alpha(s) + M_{\delta_{jp}}\delta_{jp}(s) \\ [\alpha(s)(n_{\alpha} + s) + n_{\delta_{jp}}\delta_{jp}(s)][s - M_{y1\vartheta}] &= M_{y1\alpha}\alpha(s) + M_{\delta_{jp}}\delta_{jp}(s) \\ \alpha(s)(n_{\alpha} + s)s + sn_{\delta_{jp}}\delta_{jp}(s) - M_{y1\vartheta}\alpha(s)(n_{\alpha} + s) - M_{y1\vartheta}n_{\delta_{jp}}\delta_{jp}(s) &= M_{y1\alpha}\alpha(s) + M_{\delta_{jp}}\delta_{jp}(s) \\ \alpha(s)[(n_{\alpha} + s)(s - M_{y1\vartheta}) - M_{y1\alpha}] &= [M_{\delta_{jp}} + M_{y1\vartheta}n_{\delta_{jp}} - sn_{\delta_{jp}}]\delta_{jp}(s) \\ \frac{\alpha(s)}{\delta_{jp}(s)} &= \frac{-sn_{\delta_{jp}} + M_{\delta_{jp}} + M_{y1\vartheta}n_{\delta_{jp}}}{(s^2 - s(M_{y1\vartheta} - n_{\alpha}) - M_{y1\vartheta}n_{\alpha} - M_{y1\alpha})} \\ \frac{\alpha(s)}{\delta_{jp}(s)} &= \frac{-sn_{\delta_{jp}} + M_{\delta_{jp}} + M_{y1\vartheta}n_{\delta_{jp}}}{(s^2 - s(M_{y1\vartheta} - n_{\alpha}) - (M_{y1\vartheta}n_{\alpha} + M_{y1\alpha}))}\end{aligned}\quad (31)$$

Flight path transfer function

$$\begin{aligned}\dot{\vartheta}(s) &= n_{\alpha}\alpha(s) + n_{\delta_{jp}}\delta_{jp}(s) \\ \dot{\vartheta}(s) &= \left[n_{\alpha} \left(\frac{-sn_{\delta_{jp}} + (M_{\delta_{jp}} + M_{y1\vartheta}n_{\delta_{jp}})}{s^2 - s(M_{y1\vartheta} - n_{\alpha}) - (M_{y1\vartheta}n_{\alpha} + M_{y1\alpha})} \right) + n_{\delta_{jp}} \right] \delta_{jp}(s) \\ \dot{\vartheta}(s) &= \frac{-sn_{\delta_{jp}}n_{\alpha} + n_{\alpha}(M_{\delta_{jp}} + M_{y1\vartheta}n_{\delta_{jp}}) + n_{\delta_{jp}}(s^2 - s(M_{y1\vartheta} - n_{\alpha}) - (M_{y1\vartheta}n_{\alpha} + M_{y1\alpha}))}{s^2 - s(M_{y1\vartheta} - n_{\alpha}) - (M_{y1\vartheta}n_{\alpha} + M_{y1\alpha})} \delta_{jp}(s)\end{aligned}$$

$$\frac{\dot{\vartheta}(s)}{\delta_{jp}(s)} = \frac{n_{\delta_{jp}}s^2 - M_{y1\vartheta}n_{\delta_{jp}}s - n_{\delta_{jp}}M_{y1\alpha} + n_{\alpha}M_{\delta_{jp}}}{s^2 - s(M_{y1\vartheta} + n_{\alpha}) + M_{y1\vartheta}n_{\alpha} + M_{y1\alpha}}\quad (32)$$

Normal acceleration transfer function

The normal acceleration in pitch plane is given by $J_z = -V_M\dot{\vartheta}$ and using the angular rate Eq. (32) yield;

$$\frac{J_z(s)}{\delta_{jp}(s)} = -V_M \left(\frac{n_{\delta_{jp}}s^2 - M_{y1\vartheta}n_{\delta_{jp}}s - n_{\delta_{jp}}M_{y1\alpha} + n_{\alpha}M_{\delta_{jp}}}{s^2 - s(M_{y1\vartheta} + n_{\alpha}) + M_{y1\vartheta}n_{\alpha} + M_{y1\alpha}} \right)\quad (33)$$

The perturbed variables are n_{α} , $n_{\delta_{jp}}$, $M_{y1\alpha}$, $M_{\delta_{jp}}$ and $M_{y1\vartheta}$ in pitch plane and n_{β} , $n_{\delta_{jy}}$, $M_{y1\beta}$, $M_{\delta_{jy}}$ and $M_{y1\psi}$ in yaw plane.

The pitch control system structure is shown in Fig. 11, where the gyro is a free gyro used to measure the body angle in the elevation plane. The airframe transfer function (ϑ/δ_{jp}) expressed in Eq. (30) is obtained via conducting the 6-DOF simulation with target at distance 500 m and considering ten operating points during the flight envelope. The extracted airframe transfer function has the form $\frac{\vartheta(s)}{\delta_{jp}(s)} = \frac{a_1s^2 + a_2s + a_3}{s^3 + b_1s^2 + b_2s}$.

The frequency response of the airframe at different operating conditions is shown in Fig. 12 which clarifies that the max gain variation is 53.9 dB at low frequency and 26.2 dB at high frequency. In addition, the maximum phase variation is 22.5° at low frequency and 80° at high frequency.

3.2 Pitch airframe-jetivator with uncertainty modelling

The main variation of coefficients for perturbed motion happens in the aerodynamics coefficients C_x , C_y^{α} , m_z^{α} , $m_z^{\omega_z}$ which are usually determined experimentally as functions of the Mach number that may vary in sufficiently wide intervals. Thus, the aerodynamic coefficients are supposed to exercise about 25% variations in the perturbed motion and this level of uncertainty will be useful for justifying the designed controller from the robustness point of view. In deriving the uncertain model of the system dynamics the angle ϑ is eliminated by using the relationship $\vartheta = \theta - \alpha$ (Eq. 25) which allows avoiding the use of the angle θ and

works only with the derivatives $\dot{\theta}$ and $\ddot{\theta}$. This in turn makes it possible to avoid the usage of an additional integrator in the plant dynamics that violates the conditions for

Fig. 11 Pitch control system schematic

Fig. 12 Pitch control system frequency response

controller existence in the H_∞ design. The five uncertain coefficients of the perturbed motion equations are $M_{y1\alpha}$, $M_{y1\dot{\theta}}$, $M_{\delta_{jp}}$, n_α and $n_{\delta_{jp}}$ with about 25% for each. Each uncertain coefficient (c) may be represented in the form $c = \bar{c}(1 + P_c \delta_c)$ where c is the nominal value of the coefficient $\bar{c}$ (at a given time instant), p_c is the relative uncertainty ($p_c = 0.3$ for uncertainty 25%) and $-1 \leq \delta_c \leq 1$. The uncertain coefficient can be represented as an upper linear fractional transformation (LFT) in δ_c : $c = F_u(M_c, \delta_c)$ where $M_c = \begin{bmatrix} 0 & \bar{c} \\ p_c & \bar{c} \end{bmatrix}$ is shown in Fig. 13.

The uncertainty model corresponding to the system of Eq. (25) is difficult to be obtained directly. That is why the uncertain model is derived corresponding to its individual

Fig. 13 LFT uncertain plant

equations and combining them in a common model. Consider the equation $(\dot{\alpha} = -n_\alpha \alpha - n_{\delta_{jp}} \delta_{jp} + \dot{\theta})$ for which the uncertain model is shown in Fig. 14, where $n_\alpha = \begin{bmatrix} 0 & \bar{n}_\alpha \\ p_\alpha & \bar{n}_\alpha \end{bmatrix}$,

$$n_{\delta_{jp}} = \begin{bmatrix} 0 & \bar{n}_{\delta_{jp}} \\ p_{jp} & \bar{n}_{\delta_{jp}} \end{bmatrix}, p_\alpha = 0.3, p_{jp} = 0.25, |\delta_{n_\alpha}| \leq 1, |\delta_{n_{\delta_{jp}}}| \leq 1 \text{ and}$$

the nominal values of the coefficients are denoted by the bar. The uncertain model represented by equation $(\ddot{\theta} = M_{y1\alpha} \alpha + M_{y1\dot{\theta}} \dot{\theta} + M_{\delta_{jp}} \delta_{jp})$ is shown in

Fig. 15, where $M_{y1\alpha} = \begin{bmatrix} 0 & \bar{M}_{y1\alpha} \\ p_{y1\alpha} & \bar{M}_{y1\alpha} \end{bmatrix}$, $M_{y1\dot{\theta}} = \begin{bmatrix} 0 & \bar{M}_{y1\dot{\theta}} \\ p_{y1\dot{\theta}} & \bar{M}_{y1\dot{\theta}} \end{bmatrix}$,

$$M_{\delta_{jp}} = \begin{bmatrix} 0 & \bar{M}_{\delta_{jp}} \\ p_{\delta_{jp}} & \bar{M}_{\delta_{jp}} \end{bmatrix}, p_{y1\alpha} = 0.3, p_{y1\dot{\theta}} = 0.3, p_{\delta_{jp}} = 0.3,$$

$|\delta_{M_{y1\alpha}}| \leq 1, |\delta_{M_{y1\dot{\theta}}}| \leq 1, |\delta_{M_{\delta_{jp}}}| \leq 1$ and the nominal values of coefficients are denoted by (bar).

Pulling out the uncertain parameters from the known part of the model yields uncertain model in the form of upper LFT as illustrated by (looye, Michael and Paulo) as shown in Fig. 16 with a (5×5) matrix (Δ) of uncertain parameters, i.e. $\Delta = \text{diag}(\delta_{n_\alpha}, \delta_{n_{\delta_{jp}}}, \delta_{M_{y1\alpha}}, \delta_{M_{\delta_{jp}}}, \delta_{M_{y1\dot{\theta}}})$ [34, 35].

Fig. 14 Uncertain model for α

Fig. 15 Uncertain model for θ

Due to the complexity of the plant, the easiest way in simulation and design is to define the uncertainty model

$$\frac{\theta}{\Delta_{p_{command}}} = \frac{1.024 \times 10^{-8}s^2 - 1.362 \times 10^5s - 5.385 \times 10^4}{s^5 + 392.6s^4 + 4.934 \times 10^4s^3 + 4.246 \times 10^6s^2 + 1.959 \times 10^6s} \quad (34)$$

Fig. 16 Upper LFT plant model

and implement the interconnection system, where the plant input is considered as the reference signal $u(t)$ to the fins servo-actuator, and the plant output is the body angle θ . The equations describing dynamics of guided missile c.g motion, rotation around its c.g and geometrical relations are shown in Eq. (25). Consequently, the pitch stabilization system can be depicted as shown in Fig. 17.

The extracted transfer function has the form $\frac{\theta}{\Delta_{p_{command}}} = \frac{a_1s^4 + a_2s^3 + a_3s^2 + a_4s + a_5}{s^5 + b_1s^4 + b_2s^3 + b_3s^2 + b_4s + b_5}$, the coefficients of which are obtained by conducting the 6-DOF simulation with a target at range 500 m and the flight envelope is divided into 10 operating points. The frequency response of the pitch airframe jetivator is shown in Fig. 18 for the ten operating points. This figure clarifies maximum gain variation of about 53.3 dB and phase variation of about 387.5°.

4 Robust autopilot synthesis

4.1 Nominal pitch channel modeling

Let us consider either the 6th pitch airframe transfer function or the 4th one as a nominal airframe which has a moderate frequency response compared with the remainder trim points, then find the overall plant transfer function which is the jetivator ($\delta_{jp}/\Delta_{p_{command}}$) cascaded with the airframe (θ/δ_{jp}) as shown in Fig. 17. The jetivator transfer function has the form: $\frac{\delta_{jp}}{\Delta_{p_{command}}} = \frac{2.85}{6.33 \times 10^{-5}s^2 + 0.0079s + 1}$

- Case-A: The 6th pitch airframe transfer function is considered and thus the overall plant transfer function is

- Case-B: The 4th pitch airframe transfer function is considered and yields the overall plant transfer function is

$$\frac{\theta}{\Delta_{p_{command}}} = \frac{2.559 \times 10^{-9}s^2 - 1.387 \times 10^5s - 3.342 \times 10^4}{s^5 + 234.1s^4 + 2.947 \times 10^4s^3 + 1.731 \times 10^6s^2 + 5.179 \times 10^5s} \quad (35)$$

Fig. 17 Pitch stabilization system

Fig. 18 Bode Diagram of the extracted T.F. $\theta/\Delta_{command}$

4.2 Autopilot design using H_∞ loop shaping

This approach is utilized for the autopilot design without uncertainty modelling i.e. there is no certain structure for the uncertainty during the design.

4.2.1 Autopilot design trials

Let us consider the following weights:

$$W_p = \frac{0.1s + 1.5}{s + 0.015}, \quad W_t = \frac{10s + 15}{1.9s + 5} \quad (36)$$

$$W_p = \frac{0.1s + 1.5}{s + 0.015}, \quad W_t = \frac{2s + 3}{1.9s + 5} \quad (37)$$

$$W_p = \frac{0.1s + 1.5}{s + 0.015}, \quad W_t = \frac{3.5s + 2.9954}{1.9s + 5} \quad (38)$$

- **Trial-1:** Considering the nominal plant Case-A, the weights (36) and the optimal H_∞ robust control yield the controller CA1 as:

Fig. 19 **a** Sensitivity and output weighting inverse, **b** complementary and error weighting inverse, **c** sensitivity and complementary weighting, **d** closed loop step response

$$C_{A1} = \frac{2.867 \times 10^5 s^6 + 1.133 \times 10^8 s^5 + 1.444 \times 10^{10} s^4 + 1.255 \times 10^{12} s^3 + 3.765 \times 10^{12} s^2 + 1.478 \times 10^{12} s + 143.3}{s^7 + 612.4 s^6 + 1.553 \times 10^5 s^5 + 2.327 \times 10^7 s^4 + 1.994 \times 10^9 s^3 + 3.272 \times 10^{10} s^2 + 1.311 \times 10^{10} s + 1.892 \times 10^8}$$

(c)

(d)

Fig. 19 (continued)

The frequency response of sensitivities and weights are shown in Fig. 19a, b, c and the closed loop step response using the obtained controller is shown in Fig. 19d.

- **Trial-2:** Considering the nominal plant Case-A, the weights (37) and the optimal H_∞ robust control yield the controller CA2 as:

$$C_{A2} = \frac{8.628 \times 10^6 s^6 + 3.41 \times 10^9 s^5 + 4.346 \times 10^{11} s^4 + 3.776 \times 10^{13} s^3 + 1.133 \times 10^{14} s^2 + 4.448 \times 10^{13} s + 2189}{s^7 + 1063 s^6 + 5.178 \times 10^5 s^5 + 1.494 \times 10^8 s^4 + 2.544 \times 10^{10} s^3 + 6.925 \times 10^{11} s^2 + 2.801 \times 10^{11} s + 4.045 \times 10^9}$$

- **Trial-3:** Considering the nominal plant Case-A, the weights (38) and the optimal H_∞ robust control yield the controller CA3 as:

$$C_{A3} = \frac{1.237 \times 10^7 s^6 + 4.888 \times 10^9 s^5 + 6.23 \times 10^{11} s^4 + 5.412 \times 10^{13} s^3 + 1.624 \times 10^{14} s^2 + 6.376 \times 10^{13} s + 5103}{s^7 + 1200 s^6 + 6.681 \times 10^5 s^5 + 2.211 \times 10^8 s^4 + 4.314 \times 10^{10} s^3 + 1.932 \times 10^{12} s^2 + 7.86 \times 10^{11} s + 1.136 \times 10^{10}}$$

- **Trial-4:** Considering the nominal plant Case-B, the weights (36) and the optimal H_∞ robust control yield the controller CB1 as:

$$C_{B1} = \frac{2.403 \times 10^5 s^6 + 5.69 \times 10^7 s^5 + 7.231 \times 10^9 s^4 + 4.347 \times 10^{11} s^3 + 1.219 \times 10^{12} s^2 + 3.276 \times 10^{11} s + 31.43}{s^7 + 507.7 s^6 + 1.278 \times 10^5 s^5 + 1.94 \times 10^7 s^4 + 1.701 \times 10^9 s^3 + 2.767 \times 10^{10} s^2 + 6.978 \times 10^9 s + 9.842 \times 10^7}$$

- **Trial-5:** Considering the nominal plant Case-B, the weights (37) and the optimal H_∞ robust control yield the controller CB2 as:

$$C_{B2} = \frac{1.906 \times 10^6 s^6 + 4.513 \times 10^8 s^5 + 5.735 \times 10^{10} s^4 + 3.448 \times 10^{12} s^3 + 9.671 \times 10^{12} s^2 + 2.598 \times 10^{12} s + 187.7}{s^7 + 833.3 s^6 + 2.874 \times 10^5 s^5 + 5.84 \times 10^7 s^4 + 7.151 \times 10^9 s^3 + 1.568 \times 10^{11} s^2 + 3.969 \times 10^{10} s + 5.601 \times 10^8}$$

The results obtained from these design trials can be summarized as shown in Table 1.

The results clarify that the weights (37, 38) yield designs with $\|W_p S\|_\infty$ and $\|W_t T\|_\infty < 1$ and the closed loop step response has settling and rise times smaller than that obtained from another controller obtained using the weights (36) which yields that $\|W_p S\|_\infty < 1$ and $\|W_t T\|_\infty > 1$. In addition, the angle of departure between sensitivity and

complementary weighting functions in the first case with $\|W_p S\|_\infty$ and $\|W_t T\|_\infty < 1$ is greater than those obtained in the second case where $\|W_p S\|_\infty < 1$ and $\|W_t T\|_\infty > 1$.

4.2.2 Reduced order autopilot

The reduced order obtained from designed autopilots in the previous section can be illustrated as follows:

- **Trial-1**

The reduced order controller is obtained as:

$$C_{A1(red)} = \frac{2.867 \times 10^5 s^4 + 3.599 \times 10^7 s^3 + 4.331 \times 10^9 s^2 + 1.187 \times 10^{10} s - 3.066 \times 10^4}{s^5 + 343.6 s^4 + 6.068 \times 10^4 s^3 + 6.517 \times 10^6 s^2 + 1.133 \times 10^8 s + 1.383 \times 10^6}$$

Table 1 Obtained controllers characteristics

Trial number	Obtained controller	Rise time	Settling time	$\ W_p S\ _\infty$	$\ W_t T\ _\infty$
1	C_A1	0.99	2.21	0.4635	3.0008
2	C_A2	0.756	1.79	0.3297	0.6786
3	C_A3	1.36	2.91	0.6438	0.8713
4	C_B1	0.99	2.21	0.4635	3.0008
5	C_B2	0.752	1.79	0.3330	0.6810

The frequency response of controllers is shown in Fig. 20a while the closed loop step response is shown in Fig. 20b.

- **Trial-2,3,4,5**

The reduced controllers are obtained as;

Fig. 20 **a** Frequency response of original C_A1 and reduced controllers, **b** closed loop step response using C_A1 and C_A_red

$$C_{A2(red)} = \frac{8.628 \times 10^6 s^4 + 9.2 \times 10^8 s^3 + 1.25 \times 10^{11} s^2 + 3.682 \times 10^{11} s - 4.375 \times 10^6}{s^5 + 784.1 s^4 + 2.763 \times 10^5 s^3 + 7.075 \times 10^7 s^2 + 2.424 \times 10^9 s + 3.13 \times 10^7}$$

$$C_{A3(red)} = \frac{1.237 \times 10^7 s^4 + 1.31e \times 10^9 s^3 + 1.949 \times 10^{11} s^2 + 5.709 \times 10^{11} s - 2.779 \times 10^7}{s^5 + 920.7 s^4 + 3.86 \times 10^5 s^3 + 1.126 \times 10^8 s^2 + 7.358 \times 10^9 s + 9.459 \times 10^7}$$

Fig. 21 Step response (original and designed)

$$C_B1(red) = \frac{2.403 \times 10^5 s^4 + 2.885 \times 10^7 s^3 + 2.895 \times 10^9 s^2 + 8.373 \times 10^9 s - 5.297 \times 10^4}{s^5 + 394s^4 + 7.754 \times 10^4 s^3 + 9.757 \times 10^6 s^2 + 1.961 \times 10^8 s + 2.251 \times 10^6}$$

$$C_B2(red) = \frac{1.906 \times 10^6 s^4 + 1.833 \times 10^8 s^3 + 2.183 \times 10^{10} s^2 + 6.809 \times 10^{10} s - 4.736 \times 10^5}{s^5 + 700.5s^4 + 1.761 \times 10^5 s^3 + 3.323 \times 10^7 s^2 + 1.109 \times 10^9 s + 1.353 \times 10^7}$$

The obtained results clarify that the multiplicative reduction method yields a better result which is consistent with the non-reduced controller and consequently it will be used forward in this work.

4.2.3 Autopilot robustness evaluation

- Un-modelled dynamics

The five designed controllers are implemented at the remainder operating points. The results clarify that the controller obtained taking the 6th operating point as a nominal transfer function (Case-A) is not robust against all unmodelled dynamics with the 1st and 2nd controllers in the early operating points while it is robust against all unmodelled dynamics for the third controller (Trial-3) but with slower response. On the other hand the controller obtained taking the 4th operating point (Case-B) as a nominal transfer function is robust against all unmodelled dynamics but with slower response than that obtained with Case-A especially in the final operating points. These results are summarized in Fig. 21, which clarifies that the controller CA2 which has the smallest values of $\|\omega_p S\|_\infty$ and $\|\omega_t T\|_\infty$ has faster response than CA1 and CA3. However, the obtained

controller has a large control effort at the beginning of action with lowest values at steady state compared to the original controller as shown in Fig. 22.

- Noise attenuation

Applying noise on the gyro output the obtained control effort is shown in Fig. 23, which clarify that CA1 and CB1 which have $\|\omega_t T\|_\infty > 1$ is less sensitive to additive noise compared to other controllers.

- Disturbance rejection

Applying a disturbance on the jetivator output, the obtained closed loop step response is shown in Fig. 24 which clarifies that the controller CA2 is the best compared to other controllers as it rejects 50% within 0.1 s and 95% within 0.5 s. Also control effort is shown in Fig. 25 which reveals that this controller has the lowest control effort after applying the disturbance. These results clarify that the designed robust controllers CA1 and CA2 is less sensitive to noise and disturbance and consequently they have the lowest control effort, but they are not robust against un-modelled dynamics in the early operating conditions.

Fig. 22 **a** Control signal (original and designed) at the start of action, **b** control signal (original and designed) at the steady state

4.3 Autopilot synthesis with uncertainty modelling

This section is devoted to the design of a robust system for attitude/horizontal stabilization of a time-varying thrust vector control. The linearized equations of the longitudinal motion are derived with the consideration of variations in the aerodynamic coefficients as parametric uncertainties in the design such that the desired closed-loop performs in the presence of uncertainty, disturbances and noises.

4.3.1 Pitch plane performance requirements

The stabilization challenge is to achieve the desired body angle in the presence of circumstances uncertainties. A general diagram represents the overall system including uncertainty and weighting functions is shown in Fig. 26.

This system has a reference signal r and two weighted outputs e_p and e_u which characterize performance requirements. The transfer function W_g represent the free gyro dynamics that measures θ . The system M is the ideal model to be matched by the designed closed loop system. The dotted box represents the perturbed plant model

Fig. 23 Control signal (original and designed)

Fig. 24 Step response with applied disturbance

$G = F_u(G_{mis}, \Delta)$, where G_{mis} is the nominal model of the missile and Δ parameterizes the model uncertainty. The matrix Δ is unknown but has a diagonal structure and is norm bounded, i.e. $\|\Delta\|_\infty < 1$. For robust performance, it is required that the transfer function matrix from r to e_p and e_u should be small in the sense of $\|\cdot\|_\infty$, for all possible uncertain matrices Δ . The transfer function matrices W_p and W_u are employed to represent the relative significance

of performance requirements over different frequency ranges. The measured output feedback signal is $y = W_g \theta$ and the gyro transfer function is chosen as $W_g = \frac{0.044}{0.022s+1}$.

The ideal system model satisfies the requirements to the closed-loop dynamics is chosen as $M = \frac{100^2}{s^2+80s+100^2}$ and the performance weighting functions are $W_p(s) = \frac{0.1s+5}{3s+0.015}$ and $W_u(s) = \frac{0.4s+0.9}{5.9s+5}$. The performance weighting functions

Fig. 25 **a** Control signal (original and designed) at the start of action, **b** control signal at the steady state period

are chosen so as to ensure an acceptable tradeoff between the nominal performance and robust performance of the closed-loop system with control action which satisfies the constraint imposed. The frequency response of W_p^{-1} is shown in Fig. 27 which clarifies that over the low frequency range it is required to have a small difference between the system and model and small effect on the system output due to disturbances. This ensures good reference tracking and small error in the case of low-frequency disturbances.

The internal structure of the open loop interconnection for the missile stabilization system with 7 inputs and

8 outputs is shown in Fig. 28, where the open loop system is of 10th order. The reference signal is the variable (ref), the control action is the variable control (u) and the measured output is the variable (y). Figure 29 showing the specific input/output ordering for the system variables. Consider the 6th operating point as a nominal transfer function (Case-A) with the previous weights from which the designed controller obtained is of 10th order. Figure 30 shows the frequency response of the structured singular value (μ) for the case of robust stability analysis, where the maximum value of structured singular value $\mu_{\max} = 0.47578$, which means that the stability of the

Fig. 26 The closed loop system with perturbation

Fig. 27 Frequency response of the inverse weighting function

Fig. 28 Open loop system with performance requirements

closed-loop system is preserved under all perturbations that satisfy $\|\Delta\|_{\infty} < \frac{1}{0.47578}$.

The nominal performance of the closed loop system transfer matrix is tested via the frequency response as

Fig. 29 Open-loop system schematic

shown in Fig. 31. The obtained peak value of γ is 2.5863 which is not less than 1 and shows that the nominal performance has not been achieved. The robust performance of the closed-loop system with the uncertainty matrix Δ using the H_{∞} controller is also investigated by means of the μ analysis. The robust performance (in respect to the uncertainty and performance weighting functions) is achieved if and only if, over a range of frequency under

Fig. 30 Robust closed-loop stability **Fig. 31** Nominal closed-loop performance **Fig. 32** Robust closed-loop performance

Fig. 33 Closed loop structure

consideration, the structured singular value $\mu_{\Delta_p}(j\omega)$ at each ω is less than 1. The frequency response of μ for the case of robust performance analysis is given in Fig. 32,

Fig. 34 Closed loop transient response

Fig. 35 Singular values of closed loop poles

where the peak value of μ is 3.2273, which shows that the system does not preserve performance under all relative parameter changes.

To check if the designed controller achieves robust stability and robust performance of the closed-loop system at other time instants of flight, further analysis should be conducted with corresponding dynamics. The closed loop simulation is conducted via a program designed and corresponds to the structure shown in Fig. 33, in which the performance weighting functions W_p and W_u are absent. The simulation shows the transient responses of the closed loop system with the designed H_∞ controller for a step command is shown in Fig. 34, where the system is under damped with accepted characteristics.

The singular value plot of closed loop poles is shown in Fig. 35, which reveals that there exist some singular values > 1 and that the H_∞ norm of the closed loop system is

greater than 1, i.e. the condition $\|W_p(I + GC)^{-1}\|_\infty < 1$ is not satisfied in this case.

- Redesign the robust controller using the following performance weighting functions $W_p(s) = \frac{0.1s+5}{27.2522s+0.015}$ and $W_u(s) = \frac{0.4s+0.9}{5.9s+5}$ at the same operating point which results in a controller of 10th order. The frequency response of the structured singular value for the case of robust stability analysis is considered, where $\mu_{\max} = 0.47705$, from the frequency response of the nominal performance it is seen that the obtained peak value of γ is 1.001 and less than 1 in the high frequency range which shows that the nominal performance has achieved. In addition, the peak value of μ is 1.6, and less than 1 in the high frequency range which shows that the robust performance has achieved. That is, the system does preserve performance under all relative parameter changes. The transient response of the closed loop system with the designed H_∞ controller for a step command is obtained, where the system is under damped with accepted characteristics. The singular value plot of closed loop poles reveals that all singular values < 1 which show that the H_∞ norm of the closed loop system is less than 1, and the condition $\|W_p(I + GC)^{-1}\|_\infty < 1$ is satisfied in this case.

4.3.2 Yaw plane performance requirements

The performance weighting functions are $W_p(s) = \frac{0.1s+4.261}{18.5s+0.02}$ and $W_u(s) = \frac{0.4s+0.9}{5.9s+5}$ from which the designed controller obtained is of 10th order. For the case of robust stability analysis, the maximum value of structured singular value $\mu_{\max} = 0.40335$, which means that the stability of the closed-loop system is preserved under all perturbations that satisfy $\|\Delta\|_\infty < \frac{1}{0.40335}$. The nominal performance of the closed loop system transfer matrix is tested via the frequency response. The obtained peak value of γ is 1.0009 and less than 1 at high frequency range which shows that the nominal performance has been achieved. For the case of robust performance analysis, the peak value of μ is 1.5821, and less than 1 at high frequency which shows that the robust performance has been achieved. In other words, the system does preserve performance under all relative parameter changes with uncertainty range (25%). The simulation shows the transient responses of the closed loop system with the designed H_∞ controller for step command where the system is under-damped with accepted characteristics. The singular value plot of closed loop poles is considered, which reveals that all singular values < 1 and that the H_∞ norm of the closed loop system is less than 1. Note that, the model order reduction techniques are applied to the obtained controller using the multiplicative method and yields 4th order autopilots.

4.3.3 Autopilot robustness evaluation

- Unmodeled dynamics

The two designed controllers in pitch plane and the designed controller in yaw plane are implemented at the different operating points, two (early and final operating points) for each are illustrated in Figs. 36, 37 and 38, which clarify the robustness of these designed controllers against all unmodelled dynamics.

The results are summarized in Fig. 39 and clarified that the designed robust controllers with uncertainty modeling have a faster response than these designed without uncertainty modeling. In addition, the obtained robust controller with uncertainty modeling have the lowest control effort at the steady state compared to others as shown in Figs. 40 and 41 which illustrate the fast Fourier transform spectrum of the control signal and reveal that the control effort obtained via designed robust controllers with uncertainty modeling has the lowest band of operating frequencies.

- Noise attenuation

Applying noise to the gyro output the obtained control effort is shown in Figs. 42 and 43 which clarify that the designed robust controller with uncertainty modeling is less sensitive to additive noise compared to the designed controller without uncertainty modeling.

- Disturbance rejection

Applying disturbance on the jetivator output the obtained step response of closed loop system is shown in Fig. 44 which clarifies that the designed robust controllers with uncertainty modelling is the best compared to other controllers as it rejects 50% within 0.1 s and 95% within 0.25 s. Also the control effort shown in Figs. 45 and 46 reveal that this autopilot has the lowest steady state control effort.

The same evaluation of yaw plane clarifies that the designed robust controllers with uncertainty modeling have faster response a lowest steady state control effort than classical one and the obtained robust controller without modeling the uncertainty. In addition, applying noise clarify that designed robust controllers with uncertainty modeling is less sensitive to additive noise compared to others. Also the response results to disturbance on the jetivator output clarify its capability reject 50% within 0.09 s and 95% within 0.22 s.

Fig. 36 Closed loop step response pitch sub-optimal-1

- Flight path evaluation

The obtained controllers are evaluated with the flight path trajectory at the minimum and maximum tactical data (500 m), (2800 m), respectively, at different flight conditions. For simplicity a sample of obtained results is shown.

Thrust variation The designed autopilots are evaluated with the flight path against classical autopilot and using different thrust values as shown in Figs. 47 and 48.

The robust autopilot proved its robustness to thrust uncertainties to about 30% degradation with little oscillation at the gathering phase compared to nominal thrust case.

Fig. 37 Closed loop step response pitch sub-optimal-2

Variation in aerodynamic coefficients

The designed autopilots are evaluated against perturbations in the aerodynamic coefficients of about $\pm 30\%$ and the results are shown in Figs. 49 and 50.

The results obtained with varying aerodynamics clarify that the robustness of designed autopilots without uncertainty modeling is limited to about 5% to -20% of nominal values after which the miss distance will be large

or missile ground impact occurs. While the robustness of designed autopilot, with uncertainty modeling, is limited to about $\pm 30\%$ of nominal value.

Wind speed in X direction

The simulation is conducted with considering the wind speed along the X-axis from which the results are shown in Figs. 51, 52, 53 and 54.

Fig. 38 Closed loop step response yaw sub-optimal

The obtained results clarify the following points:

- Figures 51 and 52 clarify that all flight path trajectories have an acceptable trajectory and miss distance when the wind speed along the X1-axis reaches 30 m/s.
- Its clear from Fig. 53 that at maximum tactical range when the wind speed along the X1-axis reaches -5 m/s a ground impact occurred by the missile using the classical controller at 2580.7 m, C_A1 at 2704 m, C_A2 at 2619.2 m, C-A3 at 2401 m, C-B1 At 2494 m, C-B2 at 2545 m.
- Its clear from Fig. 54 that at maximum tactical range when the wind speed along the X1-axis reaches -10 m/s a ground impact occurred by the missile using the controller C_A1 at 2453 m, C-A2 at 2405 m, C-A3 at 2261 m, C-B1 at 2271 m, C-B2 at 2303 m. While, a ground impact occurred by the missile using the classical controller at 734.7 m.

Fig. 39 Step response of the original and designed controller

Fig. 40 Control signal of the original and designed controller

- The results obtained with considering wind velocity along the X-axis clarify that the robustness of designed autopilots without uncertainty modeling is limited to about $-5:30$ m/s, while the robustness of designed autopilots with uncertainty modeling is limited to about $-19:30$ m/s.

Wind speed in Y direction

The simulation is conducted with considering the wind speed along the Y1-axis from which the results are shown in Figs. 55 and 56.

The obtained results clarify the following points:

Fig. 41 FFT **a** original, **b** CA2, **c** Optimal-1, **d** Optimal-2 controllers

Fig. 42 Control signal of the original and obtained controllers

- Its clear from Fig. 55 that at maximum tactical range when the wind speed along the Y1-axis reaches ± 20 m/s a ground impact occurred by the missile using the controller C-A1 at 2784.5 m, C-A2 at 2753.5 m, C-A3 at 2525.5 m, C-B1 at 2656.5 m, C-B2 at 2714.5 m.
- It's clear from Fig. 56 that at maximum tactical range when the wind speed along the Y1-axis reaches ± 40 m/s a ground impact occurred by the missile using the controller C-A1 at 2445 m, C-A2 at 2450 m, C-A3 at 2351 m, C-B1 at 2337.5 m, C-B2 at 2714.5 m.

Fig. 43 FFT **a** original, **b** CA2, **c** Optimal-1, **d** Optimal-2 controllers

Fig. 44 Step response of the original and obtained controller

2395 m. While, a ground impact occurred by the missile using the classical controller at 800 m.

- The results obtained with considering wind velocity along the Y-axis clarify that the robustness of designed autopilots with uncertainty modeling is limited to about ± 30 m/s, while the designed robust autopilots without uncertainty modeling is limited to about ± 15 m/s.

The results obtained reveal that the designed robust controller with uncertainty modelling which satisfy the equality $\|W_p(I + GC)^{-1}\|_{\infty} < 1$ (sub-optimal2) is the best design. This controller proves its robustness against unmodelled dynamics, stable flight path with the consideration of different sources of uncertainties (thrust degradation, aerodynamic coefficient variation and wind speed),

Fig. 45 Control signal of original and obtained controller

Fig. 46 FFT **a** original, **b** CA2, **c** Optimal-1, **d** Optimal-2 controllers

low miss-distance, low control effort, and less sensitivity to additive noise and disturbance. The remainder designed autopilots are sorted in descending order from the best performance to lowest one as: sub-optimal1, CA2, CA1, CB2, CB1, CA3 and finally the classical autopilot.

5 Flight performance evaluation

The obtained sub-optimal controllers in both pitch and yaw planes are evaluated via the flight path trajectory at different flight scenarios with existence of additive random noise applying on the measuring devices (gyros).

Fig. 47 Trajectory obtained with controllers at nominal thrust

Fig. 48 Trajectory obtained with controllers at 75% thrust

Fig. 49 Trajectory with –20% variations in aerodynamic coefficients

5.1 Thrust variation with measurement noise

The 6-DOF simulation is conducted with target at 2800 m (500 m) and separated from the line of sight in yaw plane with angle $\psi_s = 2.8^\circ$ at thrust (nominal—90–85%) values and additive noise on pitch and yaw gyros. The additive noise is shown in Fig. 57 and the flight path trajectories are shown in Fig. 58.

These results reveal that the optimal autopilot has the least sensitivity to measurement noise in addition that the conventional autopilot has less stable trajectory compared to the optimal one. The miss-distance m is summarized as shown in Table 2.

Fig. 50 Trajectory with 30% variations in aerodynamic coefficients

Fig. 51 Trajectory with $V_{W_x} = 10$ m/s

Fig. 52 Trajectory with $V_{W_x} = 30$ m/s

5.2 Effect of yaw separation angle (ψ_s)

The 6-DOF simulation is conducted using the target distance at 500 m and separated from the line of sight in yaw plane with an angle ($\psi_s = 1.5^\circ$, $\psi_s = 1^\circ$, $\psi_s = 0.5^\circ$ (9mils)) at 90% thrust values and additive noise on pitch and yaw gyros. The sample influence of the measurement noise on the flight path trajectories is shown in Fig. 59.

The results reveal that the conventional autopilot has less stable trajectory compared with the optimal one and the miss-distance m is summarized as shown in Table 3.

5.3 Effect of target motion

In this case outgoing/incoming target with average speed $VT = 65$ km/h and range 2800 m (500 m) is considered. In

Fig. 53 Trajectory with $V_{w_x} = -5$ m/s

Fig. 54 Trajectory with $V_{w_x} = -10$ m/s

Fig. 55 Trajectory with $V_{w_y} = \pm 20$ m/s

addition, the LOS separation angle in Yaw plane is $\psi_s = 2.8^\circ$ ($\psi_s = 0.5^\circ$) without measurement noise and at nominal thrust value. The 6-DOF simulation is conducted using the designed autopilots from which the flight path trajectories are shown in Fig. 60.

These results reveal that the sub-optimal autopilots have successful engagements against outgoing/incoming targets and the miss-distance m is summarized as shown in Table 4.

Fig. 56 Trajectory with $V_{wy} = \pm 40$ m/s

Fig. 57 Additive noise on pitch and yaw gyros

Fig. 58 Pitch Trajectory with controllers at 85% thrust

6 Flight path performance evaluation within HIL experiment

The disassembled hardware section consists of an actuator and jetivator assembly, Electronic Pack. The actuator and jetivator assembly and Electronic Pack are attached together via two potentiometers to feedback the nozzle

movement. The input signal to the hardware section and the potentiometer voltage are transmitted from the simulation program to the hardware section and vice versa through a multi-channel input–output data acquisition module. The hardware assembly has an external power supply and a source of pressed air necessary for the actuator servo. Figure 61 shows the hardware implementation of the control section within the simulation program.

Table 2 Obtained controllers miss-distances with thrust variation

Range (m)	Thrust (%)	Conventional-pitch Conventional-yaw	Optimal-pitch Conventional-yaw	Optimal-pitch Optimal-yaw
500	100	2.3174	3.329	2.2893
	90	2.1433	3.414	1.2436
	85	Ground impact	2.733	1.2086
2800	90	0.9317	1.596	0.8417
	85	Ground impact	1.78	0.4254

Table 3 Obtained controllers miss-distances with yaw separation angle variation

ψ_s (°)	Conventional-pitch Conventional-yaw	Optimal-pitch Conventional-yaw	Optimal-pitch Optimal-yaw
1.5°	2.0569	2.9406	1.1713
1°	1.9040	2.7252	1.1600
0.5°	1.9520	2.9014	1.7157

Fig. 59 Pitch Trajectory with autopilots at $\psi_s = 0.5^\circ$ **Fig. 60** **a** Pitch trajectory with outgoing target, **b** pitch trajectory with incoming target**Table 4** Obtained controllers miss-distances with moving targets

Range (m)	Target	Conventional-pitch Conventional-yaw	Optimal-pitch Conventional-yaw	Optimal-pitch Optimal-yaw
3000	Outgoing	ground impact	1.038	1.1782
	Incoming	0.4180	0.754	0.6466
500	Outgoing	1.6183	1.3756	1.4593
	Incoming	3.6435	2.04	2.2389

Fig. 61 Hardware in the loop simulation

The evaluation of the conventional autopilot and designed sub-optimal robust autopilot against different uncertainties in HIL environment is illustrated in Figs. 62, 63, 64 and 65.

6.1 Nominal thrust value—nominal aerodynamic coefficient—zero wind

Fig. 62 **a** Pitch trajectory obtained at nominal conditions, **b** commanded error at nominal conditions

6.2 (75%) thrust value—nominal aerodynamic coefficient—zero wind speed

Fig. 6.3 **a** Pitch trajectory obtained at 75% thrust value, **b** commanded error at 75% thrust value

6.3 Nominal thrust value—(+ 20%) aerodynamic coefficient variation—zero wind speed

Fig. 64 **a** Pitch trajectory obtained at (+ 20%) aerodynamic coefficient variation, **b** commanded error at (+ 20%) aerodynamic coefficient variation

6.4 Nominal thrust value—nominal aerodynamic coefficient— $V_{wx} = -30$ m/s

Fig. 65 **a** Pitch trajectory obtained at $V_{wx} = -30$ m/s, **b** commanded error at $V_{wx} = -30$ m/s

The obtained results with HIL experiments reveal the following observations:

- (1) At nominal flight and low uncertainties level the flight path trajectory shows stable and successful engagements for both classical and robust autopilots. In addition, the control effort obtained with the classical controller at this low level of uncertainties is lower than that obtained with a robust controller.
- (2) Increasing the margin of uncertainties (lower thrust values—change of aerodynamic coefficients—wind speed) the classical autopilot has unstable flight path trajectory and ground impact, while the robust autopilot yields successful engagements. The control effort obtained using robust autopilots has the same level as that obtained at low uncertainties level, while the classical controller has a higher control effort.
- (3) The miss-distance obtained via robust autopilot especially at low tactical target range is lower than that

obtained via classical autopilot and within an acceptable margin.

7 Conclusion

This paper presented the robust control theory in the form of two approaches; the H_∞ and sub-optimal H_∞ designs with different sensitivities and norms. In addition, it presented the model reduction techniques that can be utilized for reducing the controller order. Then, the underlying system is formulated in structures appropriate for utilizing these design techniques. The autopilots designed using the two techniques are evaluated against stability, un-modeled dynamics, disturbance rejection, noise attenuation and flight path. The obtained results clarify that the sub-optimal H_∞ controller, where the uncertainties are modeled during the design process, is more robust than the other H_∞ technique, where the design process

is carried out without modeling the uncertainties, and the classical one. These autopilots proved its robustness to thrust uncertainties within $\pm 30\%$ degradation, and about $\pm 30\%$ of nominal aerodynamic coefficients. In addition, it is limited to wind speed of about $V_{wx} = -19:30$ m/s, $V_{wy} = \pm 30$ m/s, and $V_{wz} = \pm 20$ m/s. It proved its capability of faster response with the lowest steady state control effort, less sensitivity to measurement noise and reject disturbance of 50% within 0.09 s and 95% within 0.22 s.

The HIL experiment is described in detail. Then, the system performance is evaluated using different types of models' structures and at different flight phases. The results reveal the robustness of the designed the optimal H_∞ controller via fewer excursions in the flight path which leads to less flight time, possible increased range and less possibility to ground hit.

Compliance with ethical standards

Conflict of interest The author(s) declare that they have no competing interests.

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