Analytical solution for free vibration of stiffened functionally graded cylindrical shell structure resting on elastic foundation

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Abstract

In this work, the analytical solution for free vibration of functionally graded cylindrical shell with stiffeners resting on Winkler–Pasternak foundation with several boundary conditions is reported. Material properties of the cylindrical shell are assumed to be varying smoothly and continuously across the thickness according to the power law distribution of the volume fraction of constituents. The governing equations of the stiffened functionally graded cylindrical shell resting on elastic foundation are obtained by using Hamilton's principle, the first-order shear deformation theory and the Lekhnitsky smeared stiffeners technique. The purpose of the present study is to illustrate a simple approach to get the natural frequency of the stiffened functionally graded cylindrical shell with several boundary conditions by using the Galerkin method with beam functions of the axial displacement fields. A good agreement is obtained as the present results in comparison with those of publications in the existing literature. In numerical investigations, some influences of volume fraction index, cylindrical shell's geometric ratios, boundary conditions and elastic foundation parameters on the fundamental frequency of the stiffened cylindrical shell resting on elastic foundation are given.

Keywords Free vibration · Stiffened functionally graded circular cylindrical shells · The first-order shear deformation theory · Different boundary conditions · Winkler–Pasternak foundation

1 Introduction

Shell and plate structures have a wide range of applications in engineering fields such as civil engineering, aircraft engineering, shipbuilding industry, and nuclear reactor, etc. Static and dynamic behaviors on these kinds of structures usually have gotten much more attentions by scientists throughout the world. In particular, there are rather many studies in the existing literature on behaviors of continuous systems such as beams, plates, and shells [1–10]. For isotropic and laminated composite materials, studies on free vibration of un-stiffened and stiffened cylindrical shell can be found in publications [11–26].

In the next stage, together with advanced technology, a kind of new material is proposed called functionally graded material (FGM). This material has two constituent

materials—ceramic and metal, and its material properties are assumed to be varying continuously and smoothly throughout the thickness of the structure. Thanks to this, the structures have many advantages of both ceramic and metal constituents such as high strength, toughness, and good capacity of corrosion and thermal resistance, etc. Interestingly, static and dynamic behaviors of the FGM structures have been still attracted many the scientists in the past years. In terms of the plate and shell theories, several models have been introduced. For instance, the classical plate theory, this theory was proposed first, based on the Kirchhoff's assumptions and ignoring the transverse shear deformation effects, hence this theory is only appropriate for the thin plate [27, 28]. To overcome that limitation, there are some models of the shear deformation theories are given such as the first-order shear

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deformation theory (FSDT) [29, 30], the higher-order shear deformation theories [31–33], and the simple first-order shear deformation theory [34–37]. In terms of the stiffened shell structures, studies about stiffened cylindrical shell were presented in many publications [38–41]. Many scientists also reported some researches on free vibration of functionally graded cylindrical shell resting on and not resting on elastic foundation are reinforced by rings or orthogonal stiffeners in the recent years [42–49].

On the other hand, modern numerical methods such as the finite element method (FEM) have been widely used in static and dynamic behaviors of the plate and shell structures [50–53]. Indeed, analytical approaches are generally rather limited, only the simply supported boundary conditions at ends are considered in the majority of the analytical studies on the stiffened plate and shell structures. There are rather few studies on the free vibration of the stiffened functionally graded cylindrical shell with different boundary conditions [15, 47, 54–56]. For example, Loy et al. [15] published a study on free vibration of isotropic cylindrical shell with various boundary conditions using generalized differential quadrature method. Wang and Wu recently published a study on free vibration of functionally graded porous (FGP) cylindrical shell structures subjected to different boundary conditions at ends [57], the natural frequency of the shell was obtained by utilizing the sinusoidal shear deformation theory (SSDT) and Rayleigh–Ritz method.

From the above reviews, this study focuses on the free vibration analysis of the stiffened functionally graded cylindrical shell with several boundary conditions resting on elastic foundation. Herein, the Lekhnitsky smeared stiffeners technique is used to smear the stiffeners into an equivalence layer of the cylindrical shell. Although this approach has several limitations, it is still a simple way and having a low computational cost. To obtain the natural frequency of the stiffened functionally graded cylindrical shell with several boundary conditions, the Galerkin method is used

with the beam functions of axial displacement fields. Due to the limitation of the proposed method, only three types of boundary conditions of the shell are considered as: simply supported–simply supported, clamped–simply supported, and clamped–clamped boundary conditions. In the investigation sections, the influences of volume fraction index, boundary conditions, geometrical ratios, and elastic foundation coefficients on the fundamental frequency of the stiffened FG cylindrical shell are given in details.

2 Material properties and theoretical formulations

In this section, the formulations and governing equations of a stiffened functionally graded (FG) cylindrical shell resting on elastic foundation will be presented. The geometric dimensions of the cylindrical shell structure as follows: R is radius, h is thickness, and L is length of the FG cylindrical shell structure in the coordinate system (x, θ, z) . The functionally graded cylindrical shell structure is reinforced by rings (circumferential stiffeners) and stringers (longitudinal stiffeners), as shown in Fig. 1.

The material properties of the cylindrical shell vary continuously and smoothly throughout the thickness direction according to simple power-law distribution of the volume fraction of constituents. It can be assumed that the modulus of elasticity, mass of density and Poisson's ratio of the shell material as [58]:

$$\begin{aligned} E(z) &= (E_1 - E_2) \left(\frac{z}{h} + \frac{1}{2} \right)^p + E_2; \\ \rho(z) &= (\rho_1 - \rho_2) \left(\frac{z}{h} + \frac{1}{2} \right)^p + \rho_2 \\ \nu(z) &= (\nu_1 - \nu_2) \left(\frac{z}{h} + \frac{1}{2} \right)^p + \nu_2 \end{aligned} \quad (1)$$

where p is volume fraction index; and E_1 , ρ_1 , ν_1 are modulus of elasticity, mass of density and Poisson's ratio of the

Fig. 1 Geometry and coordinate system of the functionally graded cylindrical shell with stiffeners

material at the outer surface of the cylindrical shell, respectively. By contrast, E_2 , ρ_2 , ν_2 are modulus of elasticity, mass of density and Poisson's ratio of the material at the inner surface of the FG cylindrical shell.

According to the first-order shear deformation theory (FSDT), the displacement field of a point located in the middle surface of the functionally graded cylindrical shell structure is defined as [59, 60]:

$$\begin{aligned} u(x, \theta, z, t) &= u_0(x, \theta, t) + z\phi_x(x, \theta, t) \\ v(x, \theta, z, t) &= v_0(x, \theta, t) + z\phi_\theta(x, \theta, t) \\ w(x, \theta, z, t) &= w_0(x, \theta, t) \end{aligned} \quad (2)$$

herein u_0 , v_0 , w_0 are the components of displacement field of a point located in the middle surface of the cylindrical shell structure. Symbols $\phi_x(x, \theta, t)$, $\phi_\theta(x, \theta, t)$ are the rotation angle of the normal of middle surface to the axes θ, x respectively.

The relationships between strains and displacement components of the shell are in the form:

$$\begin{aligned} \varepsilon_x &= \varepsilon_x^0 + z\kappa_x \\ \varepsilon_\theta &= \varepsilon_\theta^0 + z\kappa_\theta \\ \gamma_{x\theta} &= \gamma_{x\theta}^0 + z\kappa_{x\theta} \\ \gamma_{\theta z} &= \gamma_{\theta z}^0 \\ \gamma_{xz} &= \gamma_{xz}^0 \end{aligned} \quad (3)$$

$$\begin{aligned} \text{in which: } \varepsilon_x^0 &= \frac{\partial u_0}{\partial x}; \quad \kappa_x = \frac{\partial \phi_x}{\partial x}; \quad \varepsilon_\theta^0 = \frac{\partial v_0}{R\partial\theta} + \frac{w_0}{R}; \quad \kappa_\theta = \frac{\partial \phi_\theta}{R\partial\theta}; \\ \gamma_{x\theta}^0 &= \frac{\partial u_0}{R\partial\theta} + \frac{\partial v_0}{\partial x}; \quad \kappa_{x\theta} = \frac{\partial \phi_x}{R\partial\theta} + \frac{\partial \phi_\theta}{\partial x}; \quad \gamma_{\theta z}^0 = \phi_\theta + \frac{\partial w_0}{R\partial\theta} - \frac{v_0}{R}; \\ \gamma_{xz}^0 &= \phi_x + \frac{\partial w_0}{\partial x}; \end{aligned}$$

In present study, it is assumed that the stiffeners' dimensions are small in comparison with the shell's geometric

dimensions; hence, the wrist of stiffeners can be neglected. On the other hand, the isotropic stiffeners are made from the same material with the shell surfaces that they attached, so the connection between the stiffeners and the shell are continuous ideally. Furthermore, it is assumed that the stiffeners are in the uniaxial stress state, omitting the wrist of stiffeners, using the Lekhnitsky smeared stiffeners technique, the force and moment resultants of the stiffened FG cylindrical shell can be obtained in the form as [46, 61–63]:

$$\begin{aligned} N_x &= \left(A_{11} + \frac{E_s A_s}{s_s} \right) \varepsilon_x^0 + A_{12} \varepsilon_\theta^0 + \left(B_{11} + \frac{E_s A_s z_s}{s_s} \right) \kappa_x + B_{12} \kappa_\theta \\ N_\theta &= A_{12} \varepsilon_x^0 + \left(A_{22} + \frac{E_r A_r}{s_r} \right) \varepsilon_\theta^0 + B_{12} \kappa_x + \left(B_{22} + \frac{E_r A_r z_r}{s_r} \right) \kappa_\theta \\ N_{x\theta} &= A_{66} \gamma_{x\theta}^0 + B_{66} \kappa_{x\theta} \end{aligned} \quad (4)$$

$$\begin{aligned} M_x &= \left(B_{11} + \frac{E_s A_s z_s}{s_s} \right) \varepsilon_x^0 + B_{12} \varepsilon_\theta^0 + \left(D_{11} + \frac{E_s I_s}{s_s} \right) \kappa_x + D_{12} \kappa_\theta \\ M_\theta &= B_{12} \varepsilon_x^0 + \left(B_{22} + \frac{E_r A_r z_r}{s_r} \right) \varepsilon_\theta^0 + D_{12} \kappa_x + \left(D_{22} + \frac{E_r I_r}{s_r} \right) \kappa_\theta \\ M_{x\theta} &= B_{66} \gamma_{x\theta}^0 + D_{66} \kappa_{x\theta} \end{aligned} \quad (5)$$

$$\begin{cases} Q_\theta = K_s \left(A_{44} + \frac{G_r A_r}{s_r} \right) \gamma_{\theta z} \\ Q_x = K_s \left(A_{55} + \frac{G_s A_s}{s_s} \right) \gamma_{xz} \end{cases} \quad (6)$$

$K_s = 5/6$ is shear correction factor. In which:

$$\begin{aligned} A_{11} &= A_{22} = \int_{-h/2}^{h/2} \frac{E(z)}{1 - \nu^2(z)} dz; \quad B_{11} = B_{22} = \int_{-h/2}^{h/2} \frac{E(z)}{1 - \nu^2(z)} z dz; \quad D_{11} = D_{22} = \int_{-h/2}^{h/2} \frac{E(z)}{1 - \nu^2(z)} z^2 dz \\ A_{12} &= \int_{-h/2}^{h/2} \frac{\nu(z) E(z)}{1 - \nu^2(z)} dz; \quad B_{12} = \int_{-h/2}^{h/2} \frac{\nu(z) E(z)}{1 - \nu^2(z)} z dz; \quad D_{12} = \int_{-h/2}^{h/2} \frac{\nu(z) E(z)}{1 - \nu^2(z)} z^2 dz \\ A_{66} &= \int_{-h/2}^{h/2} \frac{E(z)}{2[1 + \nu(z)]} dz; \quad B_{66} = \int_{-h/2}^{h/2} \frac{E(z)}{2[1 + \nu(z)]} z dz; \quad D_{66} = \int_{-h/2}^{h/2} \frac{E(z)}{2[1 + \nu(z)]} z^2 dz \\ A_{44} &= A_{55} = \int_{-h/2}^{h/2} \frac{E(z)}{2[1 + \nu(z)]} dz \\ I_s &= \frac{b_s h_s^3}{12} + A_s z_s^2; \quad I_r = \frac{b_r h_r^3}{12} + A_r z_r^2; \quad z_s = \pm \frac{h_s + h}{2}; \quad z_r = \pm \frac{h_r + h}{2} \end{aligned}$$

Herein, let E_s, G_s and E_r, G_r denote the elastic modulus, the shear modulus of the stringer and ring materials, respectively. The height and width of stringer (ring) are h_s and b_s (h_r and b_r) respectively. Cross-sectional areas of stringer and ring are A_s and A_r , respectively. Let s_s and s_r denote distances between stringers and between rings, whereas z_s and z_r represent distances from the centroid of the stringer and ring to the middle surface of the cylindrical shell, respectively.

By applying Hamilton's principle, the governing equations of the stiffened functionally graded cylindrical shell resting on Winkler–Pasternak foundation are obtained as follows [43, 59, 60]:

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{\theta x}}{R \partial \theta} &= J_0 \ddot{u}_0 + J_1 \ddot{\phi}_x; \quad \frac{\partial N_{x\theta}}{\partial x} + \frac{\partial N_\theta}{R \partial \theta} + \frac{Q_\theta}{R} = J_0 \ddot{v}_0 + J_1 \ddot{\phi}_\theta \\ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_\theta}{R \partial \theta} - \frac{N_\theta}{R} - K_w w_0 + K_p \nabla^2 w_0 &= J_0 \ddot{w}_0 \\ \frac{\partial M_x}{\partial x} + \frac{\partial M_{\theta x}}{R \partial \theta} - Q_x &= J_1 \ddot{u}_0 + J_2 \ddot{\phi}_x; \quad \frac{\partial M_{x\theta}}{\partial x} + \frac{\partial M_\theta}{R \partial \theta} - Q_\theta = J_1 \ddot{v}_0 + J_2 \ddot{\phi}_\theta \end{aligned} \quad (7)$$

in which: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2}$ is Laplace operator.

Factors: K_w and K_p are the stiffness coefficients of the elastic foundation; The moments of inertia $J_i = \int_{-h/2}^{h/2} \rho_{eq}(z) z^i dz$ ($i = 0, 1, 2$) with $\rho_{eq} = \rho(z) + \frac{\rho_s A_s}{s_s h} + \frac{\rho_r A_r}{s_r h}$.

Equation (7) can be expressed in terms of the displacement fields ($u_0, v_0, w_0, \phi_x, \phi_\theta$) by using the expressions of Eqs. (3)–(6) as follows:

$$\begin{aligned} L_{11} u_0 + L_{12} v_0 + L_{13} w_0 + L_{14} \phi_x + L_{15} \phi_\theta &= 0 \\ L_{21} u_0 + L_{22} v_0 + L_{23} w_0 + L_{24} \phi_x + L_{25} \phi_\theta &= 0 \\ L_{31} u_0 + L_{32} v_0 + L_{33} w_0 + L_{34} \phi_x + L_{35} \phi_\theta &= 0 \\ L_{41} u_0 + L_{42} v_0 + L_{43} w_0 + L_{44} \phi_x + L_{45} \phi_\theta &= 0 \\ L_{51} u_0 + L_{52} v_0 + L_{53} w_0 + L_{54} \phi_x + L_{55} \phi_\theta &= 0 \end{aligned} \quad (8)$$

where L_{ij} ($i, j = 1, 2, \dots, 5$) are differential operators given in "Appendix".

3 Analytical solution

In this section, an analytical solution for the stiffened functionally graded cylindrical shell resting on elastic foundation with several boundary conditions will be presented.

First step, to solve the problem on free vibration of the cylindrical shell structure, the displacement functions are chosen as [57, 64, 65]:

$$\begin{aligned} u_0(x, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \phi_m^u(x) \phi_n^u(\theta) \cos(\omega t) \\ v_0(x, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \phi_m^v(x) \phi_n^v(\theta) \cos(\omega t) \\ w_0(x, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \phi_m^w(x) \phi_n^w(\theta) \cos(\omega t) \\ \phi_x(x, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} \phi_m^{\phi_x}(x) \phi_n^{\phi_x}(\theta) \cos(\omega t) \\ \phi_\theta(x, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_{mn} \phi_m^{\phi_\theta}(x) \phi_n^{\phi_\theta}(\theta) \cos(\omega t) \end{aligned} \quad (9)$$

in which:

$$\begin{aligned} \phi_n^u(\theta) &= \phi_n^w(\theta) = \phi_n^{\phi_x}(\theta, t) = \cos(n\theta) \\ \phi_n^v(\theta) &= \phi_n^{\phi_\theta}(\theta) = \sin(n\theta) \end{aligned} \quad (10)$$

where $A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn}$ are unknowns, m and n are natural numbers and expressing the number of half waves in the longitudinal and circumferential directions of the cylindrical shell, respectively. It can be noted that $\phi_m^k(x)$ ($k = v, w, \phi_\theta$) is the function in the longitudinal direction, satisfying the geometric boundary conditions, and can be written in the form:

$$\begin{aligned} \phi_m^k(x) &= T_1 \cosh\left(\frac{\lambda_m x}{L}\right) + T_2 \cos\left(\frac{\lambda_m x}{L}\right) \\ &\quad - \mu_m \left[T_3 \sinh\left(\frac{\lambda_m x}{L}\right) + T_4 \sin\left(\frac{\lambda_m x}{L}\right) \right] \end{aligned} \quad (11)$$

The remaining functions $\phi_m^i(x)$ ($i = u, \phi_x$) can be given by expressions: $\phi_m^i(x) = \partial \phi_m^k(x) / \partial x$. Coefficients $T_1, T_2, T_3, T_4, \lambda_m$ and μ_m in Eq. (11) with several boundary conditions at ends of the stiffened FG cylindrical shell are given as [57]:

Clamped–clamped (C–C):

$$T_1 = 1, T_2 = -1, T_3 = 1, T_4 = -1,$$

$$\mu_m = \frac{\cosh \lambda_m - \cos \lambda_m}{\sinh \lambda_m - \sin \lambda_m}, \quad \lambda_m = (2m + 1)\pi/2$$

Clamped–simply supported (C–S):

$$T_1 = 1, T_2 = -1, T_3 = 1, T_4 = -1,$$

$$\mu_m = \frac{\cosh \lambda_m - \cos \lambda_m}{\sinh \lambda_m - \sin \lambda_m}, \quad \lambda_m = (4m + 1)\pi/4$$

Simply supported (S–S):

$$T_1 = 0, T_2 = 0, T_3 = 0, T_4 = -1, \quad \mu_m = 1, \quad \lambda_m = m\pi$$

Next, substituting expressions from Eq. (9) into Eq. (8), and then applying the Galerkin method for the obtained results, we have equations as follows [64, 65]:

$$\begin{aligned} \int_t \int_\theta \int_x (L_{11}u_0 + L_{12}v_0 + L_{13}w_0 + L_{14}\phi_x + L_{15}\phi_\theta)u_0 dx d\theta dt &= 0 \\ \int_t \int_\theta \int_x (L_{21}u_0 + L_{22}v_0 + L_{23}w_0 + L_{24}\phi_x + L_{25}\phi_\theta)v_0 dx d\theta dt &= 0 \\ \int_t \int_\theta \int_x (L_{31}u_0 + L_{32}v_0 + L_{33}w_0 + L_{34}\phi_x + L_{35}\phi_\theta)w_0 dx d\theta dt &= 0 \\ \int_t \int_\theta \int_x (L_{41}u_0 + L_{42}v_0 + L_{43}w_0 + L_{44}\phi_x + L_{45}\phi_\theta)\phi_x dx d\theta dt &= 0 \\ \int_t \int_\theta \int_x (L_{51}u_0 + L_{52}v_0 + L_{53}w_0 + L_{54}\phi_x + L_{55}\phi_\theta)\phi_\theta dx d\theta dt &= 0 \end{aligned} \quad (12)$$

Table 1 The geometric and material properties for the stiffened FG cylindrical shell structure

Characteristics	Geometrical data					
	Unstiffened	Stringers	Rings	Stringers	Orthogonal rings/stringers	Orthogonal rings/stringers
Model	R1	R2	R3	R4	R5	R6
Number of stiffeners		60	19	04	13/20	05/10
Radius (m)	1	0.242	0.49759	0.1945	0.203	0.2
Thickness (m)	0.05	6E−4	1.65E−3	4.64E−4	2.04E−3	0.002
Length (m)	20	0.6096	0.3945	0.9868	0.813	1
Height of stiffeners (m)		0.00702	0.005334	0.0101	0.006/0.006	0.006/0.006
Width of stiffeners (m)		0.002554	0.003175	0.00104	0.004/0.008	0.002/0.002
E_0 (N/m ²)		6.9E+10	6.9E+10	2.00E+11	2.07E+11	2.00E+11
ν_0		0.3	0.3	0.3	0.3	0.3
ρ_0 (kg/m ³)		2714	2762	7770	7430	5700
E_1 (N/m ²)	2.07788E+11					2.00E+11
ν_1	0.317756					0.3
ρ_1 (kg/m ³)	8166					5700
E_2 (N/m ²)	2.05098E+11					7.0E+10
ν_2	0.31					0.3
ρ_2 (kg/m ³)	8900					2702
Stiffening type		External	External	Internal	Internal	External

Table 2 Comparison the natural frequency of the stiffened isotropic cylindrical shell

Mode number		Natural frequencies (Hz)							
		R2 (stringers)		R3 (rings)		R4 (stringers)		R5 (orth. stiffened)	
m	n	Ref. [41]	Present	Ref. [41]	Present	Ref. [41]	Present	Ref. [41]	Present
1	1	1141	1143.1	1204	1225.6	778	778.1	942	936.3
	2	674	672.3	1587	1584.4	317	317.2	439	436.8
	3	427	425.3	4462	4391.5	159	159.3	337	330.8
	4	296	294.5	8559	8311.9	99.6	101.6	482	476.7
	5	225	223.2	13,780	13,188.3	91.5	91.4	740	737.6

Afterward, to perform the integration for Eq. (12), the obtained results can be written in the matrix form as:

$$([K]_{5 \times 5} - \omega^2[M]_{5 \times 5})\{\delta\}_{5 \times 1} = \{0\} \quad (13)$$

where $\{\delta\}_{5 \times 1} = \{A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn}\}^T$. Factors K_{ij} and M_{ij} can be obtained by using MATLAB software. Doing the solution of eigenvalue problem of Eq. (13), we can get natural frequencies of free vibration of the stiffened functionally graded cylindrical shell resting on elastic foundation, and the corresponding vibration mode.

4 Result and discussions

In the following examples, the geometric and material properties for the functionally graded cylindrical shell and stiffeners used are in Table 1.

4.1 Validation of present study

Firstly, in order to verify the natural frequency of the functionally graded cylindrical shell with stiffeners, a comparison between the present results and those reported by Mustafa and Ali [41], geometric and material properties used are R2, R3, R4 and R5 models in Table 1, the comparison results between them are presented in Table 2. It can be pointed out that the maximum discrepancy is only 4.29% with R3 model. It can be noted that the publication of Mustafa and Ali used the energy method and Flugge's shell theory with simply supported boundary conditions at ends [41].

Besides, the fundamental frequencies of the stiffened FG cylindrical shell from the present study are compared with those of Tran and Nguyen [46] using Navier-type solution. Material and geometric properties used is R6 model in Table 1, with the notice that the stiffeners' dimensions used are $b_s = b_r = 0.002$ m, $h_s = h_r = 0.012$ m. The results of the comparison are listed in Table 3, it can be seen that the discrepancies is relatively small, maximum error is 0.1%.

Finally, to validate the natural frequencies of cylindrical shell with several boundary conditions (simply supported–simply supported, clamped–simply supported and clamped–clamped). One comparison on the non-dimensional natural frequency of the isotropic cylindrical shell between the present results with those of Wang and Wu [57] using the sinusoidal shear deformation theory and Rayleigh–Ritz method are carried out. The comparison

results are shown in Table 4, the non-dimensional natural frequency used is $\varpi = R\omega\sqrt{(1-\nu^2)\rho/E}$. It can be shown that the maximum discrepancy is 0.61%.

From above verifications, it can be found that the present results have small differences in comparison with the works of Mustafa and Ali [41], Tran and Nguyen [46], and Wang and Wu [57]. It can be concluded that the program codes and formulations of the present study on the free vibration of the stiffened functionally graded cylindrical shell resting on elastic foundation is reliable.

4.2 Numerical investigations

In investigations, unless otherwise specified, geometric and material properties used are in R6 model of Table 1. In this model, the functionally graded cylindrical shell are made of functional graded materials Al/ZrO₂.

Next, some investigations related to the influences of volume fraction index, geometric ratios, boundary conditions, elastic foundation parameters on the fundamental frequency of the stiffened FG circular cylindrical shell are presented in the following subsections.

4.2.1 Effects of volume fraction index

Figure 2 illustrates the variation of fundamental frequency of the stiffened FG cylindrical shell resting on Winkler and Pasternak foundation versus the volume fraction index with three boundary conditions: simply supported–simply supported (SS), clamped–simply supported (CS), clamped–clamped (CC).

As Fig. 2 shown, the fundamental frequencies have some slightly fluctuations as the volume fraction index increase. Besides, the influences of volume fraction index on the fundamental frequency of the cylindrical shell structure are more clearly as the elastic foundation parameters K_w (N/m³), K_p (N/m) rise up.

Table 3 Comparison of the fundamental frequency of stiffened FG cylindrical shell structure

p	Fundamental frequencies (Hz)		
	Ref. [46]	Present	Discrepancy (%)
0	369.13	368.78	0.10
3	356.04	356.11	0.02
20	344.93	344.90	0.01

Table 4 Comparison of non-dimensional natural frequency for unstiffened isotropic cylindrical shell

n	Boundary conditions								
	Simply supported–simply supported (m = 1, h/R = 1/100, L/R = 20)			Clamped–clamped (m = 1, h/R = 2/1000, L/R = 20)			Clamped–simply supported (m = 1, h/R = 2/1000, L/R = 20)		
	Ref. [57]	Present	Discrepancy (%)	Ref. [57]	Present	Discrepancy (%)	Ref. [57]	Present	Discrepancy (%)
1	0.0161	0.0160	0.61	0.0340	0.0340	0.05	0.024830	0.024721	0.44
2	0.0094	0.0094	0.42	0.0119	0.0119	0.10	0.008411	0.008379	0.38
3	0.0221	0.0221	0.12	0.0072	0.0072	0.38	0.005897	0.005884	0.23
4	0.0421	0.0420	0.17	0.0090	0.0090	0.28	0.008717	0.008704	0.15
5	0.0680	0.0679	0.17	0.0137	0.0138	0.37	0.013682	0.013663	0.14

Fig. 2 The variation of fundamental frequency with respect to volume fraction index

Fig. 3 Influence of L/R ratio on the natural fundamental frequency

For boundary conditions, it can be seen that the clamped–clamped boundary condition shell produces fundamental frequencies higher than those of other boundary conditions. This phenomenon is because of the clamped–clamped boundary condition enhances the cylindrical shell's stiffness.

4.2.2 Effects of geometric ratios and boundary conditions

This section, the influences of geometric ratios (L/R , h/R) and boundary conditions (CC, CS, SS) on the fundamental frequency of the stiffened functionally graded cylindrical shell resting on elastic foundation will be presented.

Figure 3 shows the variation of fundamental frequency with respect to L/R ratio of the stiffened

functionally graded cylindrical shell resting on elastic foundations, three types of boundary conditions namely clamped–clamped (CC), clamped–simply supported (CS), and simply supported–simply supported (SS) are illustrated. In particular, one example is presented the results of the stiffened FG cylindrical shell resting on Winkler foundation in Fig. 3a, whereas the results for the Pasternak foundation are plotted in Fig. 3b.

As the graphs in Fig. 3, it can be seen that the fundamental frequencies of the stiffened FG cylindrical shell generally fall down as the ratio L/R increase at all cases. The smaller the ratios L/R are, the faster the fundamental frequency of stiffened FG cylindrical shell decrease. Besides, the shell's small L/R ratio is sensitive on fundamental frequency of the stiffened FG cylindrical

Fig. 4 Influence of h/R ratio on the fundamental frequency

shell. Moreover, Fig. 3b demonstrates that when the geometric ratio $L/R > 18$ and Pasternak foundation $K_w = 3.10^7 \text{ N/m}^3$, $K_p = 3.10^7 \text{ N/m}$, the fundamental frequencies are almost identical at all boundary conditions, the effects of the boundary conditions on the fundamental frequency of the shell are small.

Figure 4 shows the variation of fundamental frequency of the stiffened FG cylindrical shell with respect to h/R ratio, and corresponding to three types of boundary conditions: clamped–clamped (CC), clamped–simply supported (CS) and simply supported–simply supported (SS). The fundamental frequencies of the stiffened FG cylindrical shell resting on Winkler foundation are demonstrated in Fig. 4a, c; and the results for Pasternak foundation are shown in Fig. 4b, d.

As we can observed, in Fig. 4a, c, it is clear that that fundamental frequency of the stiffened FG cylindrical shell has significant fluctuations, first decrease and then rise back with respect to h/R ratio in the case of the Winkler foundation. However, as can be seen, in Fig. 4b, d, for Pasternak foundation, the results fall down versus the h/R ratio.

In all cases of the h/R ratios, the influences of the boundary conditions on the fundamental frequencies are larger in case of the stiffened FG cylindrical shell resting on Winkler foundation. The clamped stiffened FG cylindrical shells have the highest fundamental frequencies, whereas the simply supported stiffened FG cylindrical shells produce the smallest values.

4.2.3 Influence of elastic foundation parameters

Herein, the influences of elastic foundation parameters K_w and K_p on the fundamental frequency of the stiffened FG

Fig. 5 The influence of elastic foundation parameter K_w on the fundamental frequency

Fig. 6 The influence of elastic foundation parameter K_p on the fundamental frequency

cylindrical shell will be presented. In particular, the effects of the elastic foundation parameter K_w on the fundamental frequency are shown in Fig. 5, while the effects of the foundation parameter K_p in case of $K_w = 3.10^7 \text{ N/m}^3$ on the results is plotted in Fig. 6, three types of boundary conditions are also considered.

As can be seen, these two figures have some points in common when the foundation parameters increase. Firstly, the fundamental frequencies of the stiffened FG cylindrical shell structure rise up at all cases. The increment of foundation moduli lead to the enhancing of the cylindrical shell's stiffness, the higher the elastic foundation parameters are, the harder the cylindrical shell structure's stiffness are. Besides, it can be seen that most of the fundamental frequencies of the stiffened FG cylindrical shell with the volume fraction index $p = 20$ increase more rapidly than those with the volume fraction index $p = 0$.

In contrast, it can be seen that these two figures show some different influences stemming from two foundation parameters. The fundamental frequency of the cylindrical shell have a gradual increase versus the foundation modulus K_w , whereas the results goes up significantly at the beginning and then goes slowdown in the case of the foundation modulus K_p . Furthermore, the clamped stiffened cylindrical shells usually produce results higher than those of other boundary conditions do.

5 Conclusions

Based on the first-order shear deformation theory (FSDT), Lekhnitsky smeared stiffeners technique, and the Galerkin method, this study presents an analytical solution for the

free vibration of stiffened functionally graded cylindrical shell resting on Winkler–Pasternak foundation with several boundary conditions. Some investigations relating to the influences of volume fraction index, geometric ratios, boundary conditions and elastic foundation parameters on the fundamental frequency of the stiffened FG cylindrical shell are reported. Some key points can be given:

- The fundamental frequencies of the stiffened FG cylindrical shell fall down as the ratios L/R increase. The small L/R ratios are sensitive on fundamental frequency of the stiffened FG cylindrical shell. The influence of the boundary conditions on the fundamental frequency of the stiffened FG cylindrical shell structure resting on Pasternak foundation is rather small as ratio $L/R > 18$.
- For Winkler foundation, the fundamental frequency of the stiffened FG cylindrical shell first decreases, and then rising back with respect to h/R ratio. However, for Pasternak foundation, the results fall down versus the h/R ratio.

- The clamped stiffened FG cylindrical shells always get highest fundamental frequencies, whereas the simply supported stiffened FG cylindrical shells produce smallest values in all cases investigated.
- The effects of volume fraction index on the fundamental frequency of the stiffened FG cylindrical shell are more clearly as the elastic foundation parameters K_w, K_p rise.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

Differential operators of Eq. (8):

$$\begin{aligned}
 L_{11} &= \left(A_{11} + \frac{E_s A_s}{s_s} \right) \frac{\partial^2}{\partial x^2} + \frac{A_{66}}{R^2} \frac{\partial^2}{\partial \theta^2}; \quad L_{21} = \frac{(A_{12} + A_{66})}{R} \frac{\partial^2}{\partial x \partial \theta}; \quad L_{31} = -\frac{A_{12}}{R} \frac{\partial}{\partial x} \\
 L_{41} &= \left(B_{11} + \frac{E_s A_s z_s}{s_s} \right) \frac{\partial^2}{\partial x^2} + \frac{B_{66}}{R^2} \frac{\partial^2}{\partial \theta^2}; \quad L_{51} = \frac{(B_{12} + B_{66})}{R} \frac{\partial^2}{\partial x \partial \theta} \\
 L_{12} &= \frac{(A_{12} + A_{66})}{R} \frac{\partial^2}{\partial x \partial \theta}; \quad L_{22} = A_{66} \frac{\partial^2}{\partial x^2} + \frac{1}{R^2} \left(A_{22} + \frac{E_r A_r}{s_r} \right) \frac{\partial^2}{\partial \theta^2} - \frac{k_s}{R^2} \left(A_{44} + \frac{G_r A_r}{s_r} \right) \\
 L_{32} &= -\frac{1}{R^2} \left[\left(A_{22} + \frac{E_r A_r}{s_r} \right) + k_s \left(A_{44} + \frac{G_r A_r}{s_r} \right) \right] \frac{\partial}{\partial \theta}; \quad L_{42} = \frac{(B_{12} + B_{66})}{R} \frac{\partial^2}{\partial x \partial \theta} \\
 L_{52} &= B_{66} \frac{\partial^2}{\partial x^2} + \frac{1}{R^2} \left(B_{22} + \frac{E_r A_r z_r}{s_r} \right) \frac{\partial^2}{\partial \theta^2} + \frac{k_s}{R} \left(A_{44} + \frac{G_r A_r}{s_r} \right) \\
 L_{13} &= \frac{A_{12}}{R} \frac{\partial}{\partial x}; \quad L_{23} = \frac{1}{R^2} \left[\left(A_{22} + \frac{E_r A_r}{s_r} \right) + k_s \left(A_{44} + \frac{G_r A_r}{s_r} \right) \right] \frac{\partial^2}{\partial \theta} \\
 L_{33} &= \left[k_s \left(A_{55} + \frac{G_s A_s}{s_s} \right) + K_p \right] \frac{\partial^2}{\partial x^2} + \left[\frac{k_s}{R^2} \left(A_{44} + \frac{G_r A_r}{s_r} \right) + \frac{K_p}{R^2} \right] \frac{\partial^2}{\partial \theta^2} - \frac{1}{R^2} \left(A_{22} + \frac{E_r A_r}{s_r} \right) - K_w \\
 L_{43} &= \left[\frac{B_{12}}{R} - k_s \left(A_{55} + \frac{G_s A_s}{s_s} \right) \right] \frac{\partial}{\partial x}; \quad L_{53} = \left[\frac{1}{R^2} \left(B_{22} + \frac{E_r A_r z_r}{s_r} \right) - \frac{k_s}{R} \left(A_{44} + \frac{G_r A_r}{s_r} \right) \right] \frac{\partial}{\partial \theta} \\
 L_{14} &= \left(B_{11} + \frac{E_s A_s z_s}{s_s} \right) \frac{\partial^2}{\partial x^2} + \frac{B_{66}}{R^2} \frac{\partial^2}{\partial \theta^2}; \quad L_{24} = \frac{(B_{12} + B_{66})}{R} \frac{\partial^2}{\partial x \partial \theta}; \quad L_{34} = \left[k_s \left(A_{55} + \frac{G_s A_s}{s_s} \right) - \frac{B_{12}}{R} \right] \frac{\partial}{\partial x} \\
 L_{44} &= \left(D_{11} + \frac{E_s I_s}{s_s} \right) \frac{\partial^2}{\partial x^2} - k_s \left(A_{55} + \frac{G_s A_s}{s_s} \right) + \frac{D_{66}}{R^2} \frac{\partial^2}{\partial \theta^2}; \quad L_{54} = \frac{(D_{12} + D_{66})}{R} \frac{\partial^2}{\partial x \partial \theta} \\
 L_{15} &= \frac{(B_{12} + B_{66})}{R} \frac{\partial^2}{\partial x \partial \theta}; \quad L_{25} = B_{66} \frac{\partial^2}{\partial x^2} + \frac{1}{R^2} \left(B_{22} + \frac{E_r A_r z_r}{s_r} \right) \frac{\partial^2}{\partial \theta^2} + \frac{k_s}{R} \left(A_{44} + \frac{G_r A_r}{s_r} \right) \\
 L_{35} &= \left[\frac{k_s}{R} \left(A_{44} + \frac{G_r A_r}{s_r} \right) - \frac{1}{R^2} \left(B_{22} + \frac{E_r A_r z_r}{s_r} \right) \right] \frac{\partial}{\partial \theta}; \quad L_{45} = \frac{(D_{12} + D_{66})}{R} \frac{\partial^2}{\partial x \partial \theta} \\
 L_{55} &= D_{66} \frac{\partial^2}{\partial x^2} - k_s \left(A_{44} + \frac{G_r A_r}{s_r} \right) + \frac{1}{R^2} \left(D_{22} + \frac{E_r I_r}{s_r} \right) \frac{\partial^2}{\partial \theta^2}
 \end{aligned}$$

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