Numerical study of non-linear thermal radiative heat transfer in a non-Darcy chemically reactive Casson fluid flow

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Abstract

Study of thermal radiative heat transport is relevant in several industrial applications. Analysis of radiative heat transfer in Newtonian fluids under different conditions has been studied extensively. However, not much attention has been paid to analyze the non-linear thermal radiative heat transfer in non-Darcy Casson fluid flows over stretching surfaces. Hence, in this paper an attempt is made to study the combined effect of non-linear thermal radiation and Navier slip in a magnetohydrodynamically and chemically reactive Casson fluid flow on a stretching surface immersed in a non-Darcy porous medium in the presence of thermo-diffusion. The physics of the problem is delineated by a set of partial differential equations. To solve these equations numerically scaling analysis and Runge–Kutta–Fehlberg method are sequentially applied. Computational results of the investigation are analyzed through graphs and tables. It is noted that the non-Newtonian rheology parameter, inertial and Lorentz forces are seen to suppress the flow while thermal and solutal buoyancy forces have an opposite effect. Parameters of temperature ratio and thermal radiation enhanced the temperature.

Keywords Non-Darcy porous medium · Non-Newtonian fluid · Soret effect · Non-linear thermal radiation · Joule heating

1 Introduction

Recently there has been a growing interest among scientists and technologists to study non-Newtonian fluids due to a diverse range of applications of these fluids in industry. Polymer solutions, jellies, blood, lubricants etc., represent some of such fluids. The rheological property of the non-Newtonian fluids leads the relation between the shear stress and shear rate to non-linear and thus, the apparent viscosity of the fluid is not a constant. Various constitutive equations have been proposed to model the appropriate non-Newtonian fluid. Casson fluid is one that belongs to this category. Casson proposed the rheological model during his study to describe the flow equation for suspension of pigments in Lithographic varnishes that is used to prepare printing inks. This model is reported to

be the best to describe suspensions of bentonite in water [1], suspension of Silicon [2] including certain polymers [3] used in industry. It is reported that [4–6] blood flow can be described as a Casson fluid very satisfactorily as blood shows a Newtonian nature when it flows through larger diameter vessels at higher rates of shear and shows a highly non-Newtonian behavior at smaller shear rates or while flowing through the smaller diameter vessels. Literature indicates that studies on heat and mass transfer in Casson fluid flows are very limited. Some of the researchers focused their study on these aspects in Casson fluids under different conditions [7–9].

Fluid flows over stretching surfaces find copious applications in industry including extrusion of polymer or rubber sheets, glass blowing and continuous casting. In the melt-spinning procedure the extrudate from the die is

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stretched into a filament or sheet followed by its solidification through gradual cooling. Heat transfer at the sheet has an important role on the finish of end product. Extensive research has been under taken for this type of stretching flows under different conditions [10–12].

The effect of magnetic field finds significant applications in science and engineering. Metallurgical applications include cooling of continuous strips or filaments in the process of drawing. In this case the finish of the end product depends on the rate of cooling to a major extent. Drawing such strips in an electrically conducting fluid in the presence of a magnetic field can affect the rate of cooling and stretching thereby fine tunes the quality of the finished product. Several researchers examined effect of magnetic field on fluid flows in different conditions [13–17].

At higher rates of flow when fluid travels through a porous medium, effect of inertia is dominant and this leads to non-Darcy nature. This non-Darcy behavior is accounted for by the inclusion of the square of velocity term in momentum equation. Singh and Tewari [18] examined the non-Darcy free convection flow from vertical surfaces in thermally stratified porous media. Recently, Tarakaramu et al. [19] discussed the non-linear thermal radiative heat transfer in a MHD nanofluid through a Darcy-Forchheimer porous medium. Dulal Pal and Mondal [20] examined the effects of thermal-diffusion and diffusion-thermo on non-Darcian mixed convection heat and mass transfer in an incompressible, electrically conducting fluid over a stretching sheet embedded in a porous medium in the presence of external magnetic and electrical fields and non-uniform heat source/sink.

Soret effect is referred to the mass transfer induced by temperature gradients. The principle of Soret effect has been used for separation of isotopes and in mixture of gases with very light to moderate molecular weight. Soret effect is ignored in several studies taking its effect much lower than that described by Fick's law. Nevertheless, this effect cannot be overlooked as it is found that the flow field in a mixed convective boundary layer over a vertical surface through a porous medium is influenced effectively by Soret effect [21–24].

The key tenets of the Navier–Stokes theory are the no slip boundary conditions. However, there are several instances, especially in some non-Newtonian fluids such as certain polymer suspensions and emulsions wherein such condition at the surface does not hold as they show a macroscopic wall slip. It is essential to replace the no slip conditions by partial slip boundary condition. Navier [25] suggested a slip boundary condition where in the slip velocity is proportional to the local shear stress. Due to

the wall slip the behavior of the fluid flow differs from that corresponding to no slip condition. Some of the applications of such flows involve polishing of artificial valves and inertial cavities.

Heat transfer in fluid flows over a stretching sheet due to thermal radiation has significant industrial applications in solar power technology, furnace design, solar ponds, heat exchangers, satellites and space vehicles. Several engineering processes take place at very high temperatures and knowledge of radiation heat transfer is essential in the design and development of the pertinent equipment. Owing to these applications, several researchers examined the effect of thermal radiation on heat transfer of both Newtonian and non-Newtonian fluid flows over stretching surfaces [26–28]. In all these studies Rosse-land's approximation has been used which is valid when the temperature differences in the fluid region are small. In some situations the temperature difference may not be small. In such cases, while using the non-linear Rosse-land diffusion approximation, a new diffusion term arises in the energy transport equation. In view of these recently non-linear thermal radiation has been introduced to describe accurately such instances in industry [29–33]. It can be observed that due to the presence of non-linear thermal radiation, the non-linearity of the energy equation becomes stronger and contains an additional temperature parameter which is the ratio of wall and ambient temperatures.

Despite numerous investigations on Casson fluid flows, it seems that not much has been done in non-Darcy Casson fluid flows. Recently Seth et al. [34] investigated the Casson fluid flow near a vertical oscillating plate through a non-Darcy porous medium considering the effects of Ohmic and viscous dissipation, thermo-diffusion and linear thermal radiation. Qawasmeh et al. [35] studied the forced convection for non-Newtonian Casson fluid flow in non-Darcy porous media past a horizontal isothermal surface. To the best of the knowledge of the authors the effect of non-linear thermal radiation and wall velocity slip associated with stretching surfaces in a Casson fluid flow in the presence of electrical and magnetic fields is yet to be studied. Since thermal radiation has a significant effect on the thermal energy of a system, knowledge of radiative heat transfer in fluid flows enables us to predict rate of thermal energy transport. Further it is expected that the rate of heat transfer will be augmented due to the presence of Newtonian and Joule heating. This analysis concentrates on the impact of non-linear thermal radiation coupled with diffusion and wall slip in a chemically reactive non-Newtonian fluid on a surface of stretching through a non-Darcy porous medium.

2 Mathematical formulation

Consider the steady flow of an incompressible and electrically conducting Casson fluid induced over a vertical stretching sheet immersed in a porous medium. A Cartesian coordinate system is followed with x -axis along the sheet and y -axis normal to the sheet as shown in Fig. 1.

The flow region is subjected to a uniform transverse magnetic field $\vec{B} = (0, B_0, 0)$ and uniform electric field $\vec{E} = (0, 0, -E_0)$. The flow is caused by the stretching of an elastic boundary sheet by imposing two equal and opposite forces in such a way that velocity of the sheet is of linear order of the flow direction. The Maxwell's equations are $\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{E} = 0$. From the Ohm's law $\vec{J} = (\vec{E} + \vec{q} \times \vec{B})$, where $\vec{J}$ is the Joule current, σ is the magnetic permeability and $\vec{q}$ is the fluid velocity. The induced magnetic field due to the motion of electrically conducting fluid is negligible. The mathematical analysis is carried out in the presence of viscous dissipation, Joule heating, diffusion and wall slip effects. Characteristics of heat and mass transfer are analyzed with non-linear thermal radiation and first order chemical reaction. The constitutive equation of the fluid under study is taken following [12]

$$\tau_{ij} = \begin{cases} 2\left(\mu_B + \frac{P_y}{\sqrt{2\pi}}\right)e_{ij}, & \pi > \pi_c \\ 2\left(\mu_B + \frac{P_y}{\sqrt{2\pi_c}}\right)e_{ij}, & \pi < \pi_c \end{cases} \quad (1)$$

where τ_{ij} is the (i, j) th component of the stress tensor, μ_B is the plastic dynamic viscosity of the non-Newtonian fluid, P_y is the yield stress of the fluid, π is the product of the component of deformation rate with itself, namely, $\pi = e_{ij}e_{ij}$, and e_{ij} is the (i, j) th component of deformation

rate, and π_c is critical value of π based on non-Newtonian model.

Following Dulal Pal and Mondal [20], the equations governing the flow are expressed as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (2)$$

$$\frac{1}{\epsilon^2} \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \frac{\nu}{\epsilon} \left(1 + \frac{1}{\beta} \right) \frac{\partial^2 u}{\partial y^2} + \frac{\sigma}{\rho} (E_0 B_0 - B_0^2 u) - \frac{\nu}{k_1} u - \frac{C_b}{\sqrt{k_1}} u^2 + g\beta_T (T - T_\infty) + g\beta_C (C - C_\infty), \quad (3)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial y^2} + \frac{16\sigma^*}{3k^* \rho C_p} \frac{\partial}{\partial y} \left(T^3 \frac{\partial T}{\partial y} \right) + \frac{\mu}{\rho C_p} \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\sigma}{\rho C_p} (u B_0 - E_0)^2 \quad (4)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2} + \frac{Dk_T}{T_m} \frac{\partial^2 T}{\partial y^2} - k_0 (C - C_\infty), \quad (5)$$

where u and v are the fluid velocity components along x and y axes, respectively, ϵ is the porosity of the medium, ν is the kinematic viscosity, ρ is the density of the fluid, E_0 is the electric field strength in the transverse direction, B_0 is the transverse magnetic field strength, k_1 is permeability of the porous medium, $\beta = \mu_B \sqrt{2\pi_c} / P_y$ is the Casson parameter, C_b is the form of drag coefficient which is independent of viscosity and other physical properties of the fluid which depend on the geometry of the medium, g is the acceleration due to gravity, β_T is the coefficient of thermal expansion, β_C is the coefficient of concentration expansion, T is the temperature of the fluid inside the thermal boundary layer, T_w is the uniform temperature and T_∞ is the fluid temperature in the free stream with $T_w > T_\infty$, C is the fluid concentration, C_w is the uniform concentration and C_∞ is the fluid concentration in the free stream with $C_w > C_\infty$, c_p is the specific heat at constant pressure, μ is the dynamic viscosity of the fluid, k is the thermal conductivity of the medium, σ^* is the Stefan-Boltzman constant and k^* is the absorption coefficient, D is the mass diffusivity, T_m is the mean fluid temperature, k_T is the thermal-diffusion ratio and k_0 is the chemical reaction.

Boundary conditions are:

$$u = u_w(x) + N \left(1 + \frac{1}{\beta} \right) v \frac{\partial u}{\partial y}, \quad v = v_w,$$

$$T = T_w = T_\infty + A_0 \left(\frac{x}{l} \right)^2, \quad C = C_w = C_\infty + A_1 \left(\frac{x}{l} \right)^2 \quad \text{at } y = 0,$$

$$u \rightarrow 0, T \rightarrow T_\infty, C \rightarrow C_\infty, \quad \text{as } y \rightarrow \infty, \quad (6)$$

Fig. 1 Physical model and coordinate system

where $u_w(x) = bx$ is the velocity of the stretching surface, b is the proportionality constant of the stretching velocity, N is the velocity slip factor which changes with time, $v_w = -(\nu u_w/x)^{1/2}$ represent the mass transfer velocity at the surface of the sheet with $v_w > 0$ for injection and $v_w < 0$ for suction, and $v_w = 0$ corresponds to an impermeable sheet, A_0 is the parameter of temperature distribution on the stretching surface, A_1 is the parameter of concentration distribution on the stretching surface.

The stream function $\psi(x, y, t)$ is defined as

$$u = \frac{\partial \psi}{\partial y} = bxf'(\eta),$$

$$v = -\frac{\partial \psi}{\partial x} = -(\nu b)^{1/2}f(\eta),$$

which automatically satisfy the continuity Eq. (2).

To solve the governing boundary layer Eqs. (3)–(5), we employ the scaling analysis with the introduction of the following similarity transformations:

$$\psi(x, y, t) = (\nu b)^{1/2}xf(\eta), \eta = \sqrt{\frac{b}{\nu}}y,$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}. \quad (7)$$

Making use of the relations in (7), the governing Eqs. (3)–(5) reduce to the following non-linear ordinary differential equations

$$\left(1 + \frac{1}{\beta}\right) \frac{f'''}{\varepsilon} + \frac{ff''}{\varepsilon^2} - \frac{f'^2}{\varepsilon^2} + M^2(E_1 - f') - F^*f'^2 - k_p f' + \lambda(\theta + N\phi) = 0, \quad (8)$$

$$\theta'' + Nr \left[(1 + (\theta_w - 1)\theta)^3 \right] \theta'' + 3Nr \left[(\theta_w - 1)(1 + (\theta_w - 1)\theta)^2 \right] \theta'^2 + Pr \left[f\theta' - 2f'\theta + M^2 Ec(E_1 - f')^2 + Ec \left(1 + \frac{1}{\beta}\right) (f'')^2 \right] = 0, \quad (9)$$

$$\phi'' + Sc(f\phi' - 2f'\phi - \gamma\phi + Sr\theta'') = 0, \quad (10)$$

The boundary conditions in (6) take the form:

$$f(0) = fw, f'(0) = 1 + h \left(1 + \frac{1}{\beta}\right) f''(0), \theta(0) = 1, \phi(0) = 1 \text{ at } \eta = 0, \quad (11)$$

$$f'(\infty) = 0, \theta(\infty) = 0, \phi(\infty) = 0 \text{ as } \eta \rightarrow \infty, \quad (12)$$

where primes denote differentiation with respect to η , $M^2 = \sigma B_0^2 / \rho b$ is magnetic parameter, $k_p = \nu / k_1 b$ is porous parameter, $\lambda = Gr_x / Re_x^2$ is thermal buoyancy parameter, $N = Gr_x / Gr_x^*$ is solutal buoyancy parameter,

$Gr_x = \frac{g\beta_T(T - T_\infty)x^3}{\nu^2}$ is Grashof number due to temperature, $Gr_x^* = \frac{g\beta_C(\Omega^2 C_\infty)x^3}{\nu^2}$ is Grashof number due to concentration, $E_1 = E_0 / B_0 bx$ is electric field parameter, $F^* = \frac{C_b x}{\sqrt{k_1}}$ is inertial parameter, $Pr = \rho c_p \nu / k$ is Prandtl number, $Nr = 16\sigma^* T_\infty^3 / 3kk^*$ is thermal radiation parameter, $\theta_w = T_w / T_\infty$ is temperature ratio parameter, $Ec = b^2 l^2 / c_p A_0$ is Eckert number, $Sc = \nu / D$ is Schmidt number, $\gamma = k_0 / b$ is chemical reaction parameter, $Sr = \frac{Dk_T(T_w - T_\infty)}{\nu T_m(C_w - C_\infty)}$ is Soret number and $h = N\sqrt{bv}$ is velocity slip parameter.

Skin friction coefficient C_f , a measure of the drag force, Nusselt number Nu_x and Sherwood number Sh_x which reflect the rates of heat and mass transfer respectively, are defined by

$$C_f = \frac{2\tau_w}{\rho u_w^2}, Nu_x = \frac{xq_w}{K(T_w - T_\infty)}, Sh_x = \frac{xq_m}{D_m(C_w - C_\infty)}, \quad (13)$$

where τ_w is the wall shear stress, q_w is the heat transfer from the sheet, q_m is the mass transfer which are given by

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0}, q_w = -K \left(\frac{\partial T}{\partial y} \right)_{y=0}, q_m = -D_m \left(\frac{\partial C}{\partial y} \right)_{y=0}.$$

$$q_m = -D_m \left(\frac{\partial C}{\partial y} \right)_{y=0}. \quad (14)$$

Making use of Eq. (14) in Eq. (13), we obtain

$$\frac{1}{2} C_f \sqrt{Re_x} = \left(1 + \frac{1}{\beta}\right) f''(0),$$

$$Nu_x / Re_x^{1/2} = -(1 + Nr\theta_w^3) \theta'(0),$$

$$Sh_x / Re_x^{1/2} = -\phi'(0), \quad (15)$$

where $Re_x = \frac{u_w x}{\nu}$ is the local Reynolds number.

3 Method of solution

The solution of the mathematical model presented in Eqs. (8)–(12) is obtained by enforcing the Runge–Kutta–Fehlberg method along with shooting technique. An iteration process is followed and continued until the required results are obtained within the following convergence criterion

$$\left| \frac{g_{i+1} - g_i}{g_{i+1}} \right| \leq 10^{-6}$$

Table 1 Comparison values of $-f''(0)$ for various values of M when $E_1 = F^* = \lambda = N = k_p = fw = h = 0$, $\beta \rightarrow \infty$ and $\varepsilon = 1.0$

M	Hayat et al. [36]	Mabood and Das [37]	Mabood et al. [38] (1)	Present results (2)	% Absolute error = $\left \frac{(1)-(2)}{(1)} \right \times 100$
0	-1.00000	-1.000008	-1.0000084	-1.000008428	2.79998×10^{-6}
1	1.4142	1.4142135	1.41421356	1.4142135642	2.9698×10^{-7}
5	2.4494	2.4494897	2.44948974	2.4494897427	1.10227×10^{-7}
10	3.31662	3.31662	3.316624	3.316624790	2.38194×10^{-5}
50	7.14142	7.1414284	7.14142843	7.1414284285	2.10042×10^{-8}
100	10.04987	10.049875	10.0498756	10.049875621	2.08958×10^{-7}
500	22.38302	22.383029	22.3830293	22.383029285	6.70151×10^{-8}
1000	31.63858	31.638584	31.6385840	31.638584046	1.45392×10^{-7}

Table 2 Comparison values of $-\theta'(0)$ for $M = E_1 = F^* = \lambda = N = k_p = fw = h = Nr = Ec = Sc = \gamma = Sr = 0$, $\theta_w = 1.0$ and $\varepsilon = 1.0$.

Pr	Chen [39]	Grubka and Bobba [40]	Dulal Pal and Mondal [20]	Present results
1.0	1.33334	1.3333	1.33333	1.33333
3.0	2.50972	2.5097	2.50971	2.50972
7.0	3.97150	–	3.97151	3.97151
10	4.79686	4.7969	4.79687	4.79686
100	15.7118	15.712	–	15.7119

where g stands for u, θ, ϕ and i denotes the iteration level. To validate the numerical code, collation of results of the present analysis i.e., $-f''(0)$ with the values evaluated by Hayat et al. [36], Mabood and Das [37], Mabood et al. [38] and $-\theta'(0)$ with those calculated by Chen [39], Grubka and Bobba [40], Dulal Pal and Mondal [20] is done and are respectively tabulated in Tables 1 and 2 as a limiting case. These two tables show that the results of the current

investigation are excellently agreed with them. Further, the percentage of absolute error of our results $[-f''(0)]$ comparing with Mabood et al. [38] is calculated for different values of M .

Fig. 2 Variation of β and fw on $f'(\eta)$

Fig. 3 Variation of M and E_1 on $f'(\eta)$

Fig. 4 Variation of λ and $f'(\eta)$ on $\theta(\eta)$

4 Results and discussion

In this analysis numerical solutions are obtained to investigate the MHD non-Darcy convective Casson fluid flow over a stretching surface to understand the influence of non-linear thermal radiation, chemical reaction and Navier slip. The computational results are shown through graphs and deliberated.

In Fig. 2 the effect of Casson parameter (β) and mass flux parameter (fw) on velocity is plotted. β represents the non-Newtonian nature of the fluid and higher values of β indicate increase in plastic dynamic coefficient viscosity and

$\frac{1}{\beta} = 0$ corresponds to a Newtonian case. $fw < (>) 0$ corresponds to injection (suction) and $fw = 0$ refers to an impermeable wall. It is concluded that the velocity decreases throughout the boundary layer region with increasing values of β leading to thinning of hydrodynamic boundary layers. At higher values of β the plastic dynamic coefficient viscosity is dominant which restricts the fluid motion. The velocity in the presence of injection is seen to be higher than that of an impermeable wall case. Effect of suction is seen to be more than injection in reducing velocity across the fluid layers. Figure 3 illustrates the variation of

Fig. 5 Variation of N and $f'(\eta)$ on $\phi(\eta)$

magnetic field (M) and electric field (E_1) parameters on velocity. It is revealed that the magnetic parameter suppresses the flow across the boundary layer leading to thinning of the boundary layers. This is attributed to the body force namely, Lorentz force, developed due to the application of transverse magnetic field, which resists the fluid motion. The velocity in the presence of electric field ($E_1 \neq 0$) is seen to be higher than in the absence of electric field ($E_1 = 0$). It is evident that the presence of electric parameter (E_1) increases the velocity distribution throughout the hydrodynamic boundary layer near the boundary. However, a little away from the stretching sheet, the

enhancement in velocity is more pronounced. This shows that the streamlines are drifted away from the surface. We may conclude that the electric force serves as an accelerating force to overcome the frictional resistance.

Figure 4 is the plots of velocity and temperature distributions across the fluid for different values of thermal buoyancy parameter (λ). Positive values of λ assist the flow and negative values of λ oppose the flow. It is seen that velocity enhances with increase in thermal buoyancy parameter λ since thermal buoyancy force acts as a favorable pressure gradient in accelerating the fluid flow in the boundary layer whereas the temperature experiences a

Fig. 6 Variation of F^* and ε on $f'(\eta)$

Table 3 Skin friction coefficient, Nusselt number and Sherwood number for various values of pertinent parameters when $Pr = 2.5; Nr = 0.5; \theta_w = 1.1; Ec = 0.2; Sc = 1.2; Sr = 0.2; \gamma = 0.1$

β	M	E_1	F^*	λ	N	k_p	fw	h	$\left(1 + \frac{1}{\beta}\right)f''(0)$	$-(1 + Nr\theta_w^3)\theta'(0)$	$-\phi'(0)$
0.5	2.0	1.0	0.1	0.1	0.1	0.1	0.1	0.1	-0.483353	3.171539	1.529082
1.0									-0.416623	3.172060	1.528379
1.5									-0.388790	3.171358	1.527716
2.0									-0.373318	3.170712	1.527244
0.5	2.0	1.0	0.1	0.1	0.1	0.1	0.1	0.1	-0.483353	3.171539	1.529082
	3.0								-0.306358	3.234910	1.554425
	4.0								-0.211204	3.266369	1.567305
	5.0								-0.154507	3.284047	1.574664
0.5	2.0	0.01	0.1	0.1	0.1	0.1	0.1	0.1	-2.857979	1.656899	1.134517
		0.04							-2.782869	1.753690	1.156273
		0.07							-2.707850	1.841864	1.175609
		0.1							-2.632981	1.923681	1.193236
0.5	2.0	1.0	0.0	0.1	0.1	0.1	0.1	0.1	-0.436849	3.188803	1.534951
			1.0						-0.834979	3.025576	1.485285
			2.0						-1.133060	2.881033	1.448996
			3.0						-1.372620	2.751470	1.420424
			4.0						-1.573370	2.633937	1.396909
0.5	2.0	1.0	0.1	-0.2	0.1	0.1	0.1	0.1	-0.559939	3.154186	1.250575
				-0.1					-0.534089	3.161174	1.252634
				0.1					-0.482640	3.174882	1.256725
				0.2					-0.457037	3.181601	1.258758
0.5	2.0	1.0	0.1	0.1	0.1	0.1	0.1	0.1	-0.482640	3.174882	1.256725
					0.2				-0.479789	3.175685	1.256969
					0.3				-0.476939	3.176486	1.257213
					0.4				-0.474090	3.177286	1.257457
0.5	2.0	1.0	0.1	0.1	0.1	0.1	0.1	0.1	-0.483083	3.171644	1.256588
						0.5			-0.678406	3.091449	1.234410
						1.0			-0.900696	2.987426	1.209180
						1.5			-1.102258	2.880854	1.186342
0.5	2.0	1.0	0.1	0.1	0.1	0.1	-0.2	0.1	-0.471532	2.783772	1.392739
							-0.1		-0.475317	2.909286	1.437201
							0.0		-0.479111	3.039651	1.482710
							0.1		-0.482909	3.174778	1.529257
0.5	2.0	1.0	0.1	0.1	0.1	0.1	0.1	0.1	-0.482909	3.174778	1.529257
								0.2	-0.366508	3.145453	1.516497
								0.3	-0.295398	3.126669	1.508741
								0.4	-0.247425	3.113614	1.503526

reverse effect. From Fig. 5 it is seen that the effect of solutal buoyancy parameter (N) on velocity is exactly similar to that of λ while concentration has a contrary behavior. Figure 6 depicts the influence of inertial parameter (F^*) and porosity parameter (ϵ) of porous medium on velocity. Higher values of F^* refer to increased Darcy resistance. It is clear that velocity gets diminished with increased F^* , which is corroborated by the values of drag coefficient presented in Table 3. The impact of porosity parameter ϵ is seen to enhance the velocity. Figure 7 indicates that velocity distribution shows a decreasing tendency with increasing values of porous permeability parameter (k_p) as increasing values of k_p correspond to decreasing permeability of the medium. The impact of velocity slip parameter

is observed to repress the velocity as illustrated in Fig. 8 and the values of shear stress are increased.

It is evident from Fig. 9 that temperature drops for larger values of Prandtl number (Pr) across the thermal boundary layer. Higher the Prandtl number lesser is the thermal conductivity of the fluid. At higher Prandtl numbers convection plays a dominant role in transferring energy from the boundary compared to conduction. Hence, in fluids with higher Prandtl numbers, reduction in temperature is more rapid. It is seen that larger values of Eckert number (Ec) enhance the temperature due to release of stored energy owing to frictional heating and hence augmentation of temperature takes place. Figure 10 indicates the variation of thermal radiation parameter (Nr) and temperature ratio parameter (θ_w) on temperature. Increase in Nr results in

Fig. 7 Variation of k_p on $f'(\eta)$ **Fig. 8** Variation of h on $f'(\eta)$

enhancement of thermal energy in the fluid leading to higher temperatures. The parameter θ_w arises solely due to non-linear thermal radiation. It is clear that variation of θ_w is qualitatively similar to that of Nr .

Figure 11 shows the influence of Schmidt's number (Sc) and chemical reaction (γ) parameter on concentration. The concentration gets diluted for increasing Sc and γ . This may be explained as larger values of Sc correspond to smaller values of mass diffusivity and hence the solutal boundary layers become thinner for larger values of Sc . Figure 12 portrays that increasing Soret number (Sr) results in stronger concentrations. This is due to the fact that Soret

effect yields a mass flux from lower to higher species concentration driven by temperature gradient.

Surface drag, heat and mass transfer coefficients for different variations of governing parameters are tabulated in Table 3. It is noted that the magnitude of the three transport coefficients diminish with increasing β while λ and N show a contrary effect. Electric field parameter is seen to enhance the surface drag coefficient, Nusselt number and Sherwood number. It is observed that when the magnetic field parameter assumes higher values, the absolute values of surface drag coefficient and the rates of heat and mass transfer decrease. This reduction is due to the dominance of electrical force. The inertial parameter is

Fig. 9 Variation of Pr and Ec on $\theta(\eta)$ **Fig. 10** Variation of Nr and θ_w on $\theta(\eta)$

observed to have a decreasing influence on surface drag coefficient and increases the rate of heat and mass transfer. The drag coefficient, Nusselt number and Sherwood number are seen to be decreasing functions of permeability parameter. Increased values of suction parameter decrease surface drag due to the reduced force exerted on the boundary and the Nusselt number and Sherwood number are enhanced while an opposite influence is felt due to velocity slip parameter.

From Table 4 it is remarked that the coefficient of heat transfer enhances with increased Pr. This is due to the fact that the fluid with low Prandtl number has larger thermal

conductivity and thus absorbs and convects heat immediately and leads to raising of the fluid temperature. Consequently the temperature gradient at the surface reduces leading to a reduction in surface heat flux. Sherwood number is seen to diminish with increasing values of Prandtl number. Nusselt number is found to grow with higher thermal radiation parameter as well as temperature ratio parameter while Eckert number has an opposite influence. The presence of non-linear thermal radiation leads to a 3% elevation in Nusselt number value compared to linear thermal radiation. Mass transfer rate is seen to increase with Sc and γ while it decreases with Sr. The coupled effect

Fig. 11 Variation of Sc and γ on $\phi(\eta)$ **Fig. 12** Variation of Sr on $\phi(\eta)$

of thermal diffusion and chemical reaction is to cause a reduction of 12% in Sherwood number.

5 Conclusions

In this paper, we studied the effect of thermo-diffusion, electrical field, non-linear thermal radiation on the mixed convective MHD non-Darcy flow over a permeable stretching surface. Some of the significant findings of the investigations are that the non-Newtonian nature of the fluid and Lorentz force have a reducing influence on the fluid flow while electrical force has a contrary impact. Both the

thermal and solutal buoyancy forces accelerate the fluid flow. Viscous and Ohmic heating augment the rate of heat transfer. Effect of non-linear thermal radiation enhanced the thermal energy resulting in higher temperatures. Non-linear thermal radiation causes a 3% elevation in heat transfer compared to linear thermal radiation. Effect of mass diffusivity and chemical reaction is in thinning of solutal boundary layers with weaker concentration. The presence of thermo diffusion and chemical reaction leads to a 12% increase in rate of mass transfer.

Table 4 Skin friction coefficient, Nusselt number and Sherwood number for various values of pertinent parameters when $\beta = 0.5; M = 2.0; E_1 = 2.0; F^* = \lambda = N = k_p = fw = h = 0.1$

Pr	Nr	θ_w	Ec	Sc	γ	Sr	$\left(1 + \frac{1}{\beta}\right)f''(0)$	$-(1 + Nr\theta_w^3)\theta'(0)$	$-\phi'(0)$
0.7	0.5	1.1	0.2	1.2	0.1	0.2	-0.473359	1.630610	1.689489
1.0							-0.476029	1.963624	1.658434
2.0							-0.481222	2.823630	1.568907
3.0							-0.484149	3.497811	1.492122
2.5	0.0	1.0	0.0	1.2	0.1	0.2	-0.486819	2.566991	1.397930
			0.2				-0.486288	2.542545	1.400831
			0.5				-0.483999	3.105540	1.496968
			0.2				-0.483490	3.074623	1.499104
2.5	0.5	1.1	0.2	1.2	0.1	0.2	-0.482909	3.174778	1.529257
							-0.480514	3.693468	1.589902
							-0.478666	4.143531	1.625878
							-0.477169	4.546375	1.649915
2.5	0.5	1.1	0.2	1.2	0.1	0.2	-0.483353	3.171539	1.529082
							-0.481317	3.536320	1.613455
							-0.478965	3.992211	1.677398
							-0.476462	4.529263	1.720058
2.5	0.5	1.1	0.0	1.2	0.1	0.2	-0.483413	3.206169	1.527507
			0.1				-0.483161	3.190444	1.528383
			0.4				-0.482411	3.143617	1.531001
			0.7				-0.481672	3.097307	1.533605
			1.0				-0.480945	3.051501	1.536196
2.5	0.5	1.1	0.2	0.5	0.1	0.2	-0.482259	3.176426	1.001688
				1.0			-0.482729	3.176236	1.399994
				1.5			-0.482993	3.176136	1.704530
				2.0			-0.483171	3.176074	1.961682
2.5	0.5	1.1	0.2	1.2	0.0	0.0	-0.483080	3.174715	1.741218
2.5	0.5	1.1	0.2	1.2	0.0	0.2	-0.482882	3.174788	1.491252
					0.1		-0.482909	3.174778	1.529257
					0.2		-0.482936	3.174768	1.566471
					0.3		-0.482961	3.174759	1.602938
					0.4		-0.482985	3.174750	1.638699
2.5	0.5	1.1	0.2	1.2	0.1	0.0	-0.483103	3.174707	1.776306
					0.1	0.1	-0.483006	3.174742	1.652783
					0.2	0.2	-0.482909	3.174778	1.529257
					0.3	0.3	-0.482813	3.174813	1.405728
					0.4	0.4	-0.482716	3.174849	1.282198

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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