Research Article

Nonlinear vibrations of single- and double-walled carbon nanotubes resting on two-parameter foundation in a magneto-thermal environment

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Abstract
The excellent mechanical, electrical, structural and thermal properties coupled with high strength to weight ratio of carbon nanotubes have tremendously expanded their applications in various industrial, engineering, physical and natural sciences processes. In this work, nonlocal elasticity theory is used to analyze nonlinear vibrations of single and double-walled carbon nanotubes resting on two-parameter foundation in a thermal and magnetic environment. With the aid of Galerkin decomposition method, the systems of nonlinear partial differential equations are transformed into systems of nonlinear ordinary differential equations which are solved using homotopy perturbation method. The developed analytical solutions are used to investigate the influences of elastic foundations, magnetic field, temperature rise, interlayer forces, small scale parameter and boundary conditions on the frequency ratio. From the results, it is observed that the frequency ratio for all boundary conditions decreases as the number of walls increases from single to double. Also, it is established that the frequency ratio is highest for clamped–simple supported and lowest for clamped–clamped supported. Additionally, the results revealed that the frequency ratio decreases with increase in the value of spring constant (k_1) temperature and magnetic field strength. This work will enhance the applications of carbon nanotubes in structural, electrical, mechanical and biological applications especially in a thermal and magnetic environment.

Keywords Single and double-walled carbon nanotubes · Magneto-thermal environment · Nonlocal elastic theory · Small-scale effects · Elastic foundations

1 Introduction

The novel nanostructure materials discovered by Iijima [1] have led to considerable number of studies on carbon nanotubes due to their promising applications in nanodevices, nanoelectronics, and nanocomposites. Also, the excellent mechanical, electrical, structural and thermal properties coupled with high strength to weight ratio property of carbon nanotubes have continuously and tremendously expanded their applications in various industrial, engineering, physical and natural sciences processes. In fact, the nanostructures have merits when applied to the functionability of transistors and diodes. However, carbon nanotubes (CNTs) are capable of undergoing large deformations within the elastic limit and vibrate at frequency in the order of GHz and THz. Consequently, logical investigations and analysis of carbon nanotube have been a subject of interest such as the vibrations of a micro-resonator that is excited by electrostatic and piezoelectric actuations. Various studies have been carried out on beams, carbon nanotube, nano-wires, nano-rods and nano-beam so as to specifically understand and achieve their area of best fit [2–13]. In achieving this, the well know beam models were employed and dynamic

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ranges were obtained in the scope of the structures. In such studies, Liew et al. [5], Pantano et al. [6, 7], Qian et al. [8] and Salvetat et al. [9] examined the mechanics of single and multiwalled carbon nanotubes. Sears and Batra [10] analyzed carbon nanotubes buckling under the influence of axial compression. Yoon et al. [11] and Wang and Cai [12] investigated the impacts of initial stress on multiwall carbon nanotube with a focus on non-coaxial resonance. Wang et al. [13] explored the dynamic response of multi-walled carbon nanotubes using Timoshenko beam model. Zhang et al. [14] scrutinized the influence of compressive axial load on the transverse dynamic behaviour of double-walled carbon nanotubes (DWCNT). Another work on the vibration of double-walled carbon nanotubes was presented by Elishakoff and Pentaras [15]. Also, studies on nonlinear vibration of nanomechanical resonator, nanotube and nanowire-based electromechanical systems have been carried out by Buks and Yurke [16] and Postma et al. [17] while Fu et al. [18] examined nonlinear vibration analysis of embedded carbon nanotubes. In the same year, Xu et al. [19] considered the dynamic response of a double-walled carbon nanotube under the influence of nonlinear intertube van der Waals forces. The vibration of carbon nanotube-based switches with focus on static and dynamic responses was analyzed by Dequesnes et al. [20]. Few years later, Ouakad and Younis [21] investigated the nonlinear vibration of electrically actuated carbon nanotube resonators. In an earlier work, Zamanian et al. [22] presented the non-linear vibrations analysis of a micro-resonator subjected to piezoelectric and electrostatic actuations. As a continuation of the tremendous work, Abdel-Rahman, Hawwa, Hajnayeb, and Belhadj [23–26] performed a vibration and instability studies of DWCNT using a nonlinear model and considering an electrostatic actuation as an external excitation agent. In their work, a DWCNT was situated and conditioned to a direct and alternating voltage and different behaviors of the nanotubes were recorded as the exciting agent is varied. They went further to determine the bifurcation point of the DWCNT and concluded that both walls have the same frequency of vibration under the two resonant conditions considered. Belhadj et al. [26] carried out the vibration analysis of a pinned–pinned supported SWCNT employing nonlocal theory of elasticity and obtained natural frequency up to third mode. The authors also put forward an explanation on the advantages of the high frequency obtained in their work to optical applications. Lei et al. [27] studied the dynamic behaviour of DWCNT by employing the well-known Timoshenko theory of beam. The nonlinear governing equations generated by Sharabiani and Yazdi [28] derived relations in the application to nanobeams that are graded and have surface roughness. Wang [29] generated a close form model for the aforementioned surface roughness effect for an unforced fluid conveying nanotubes and beams based on nonlocal theory of elasticity and ascertained the significance of the study for reasonably small thickness of the tube considered. Interesting foundation studies have been considered after modelling of CNTs as structures resting or embedded on elastic foundations such as Winkler, Pasternak and Visco-Pasternak medium [30–35]. Other interesting works through modelling and experiment have also been presented to justify the widespread application of SWCNTs [36–41].

The dynamic behaviour of SWCNTs and DWCNTs have been characterized and their dynamic behaviour have been investigated with the aids of experimental measurements, density functional theory, molecular dynamics simulations, and continuum mechanics. However, there are difficulties in performing experiment at the nanoscale level. Consequently, over the years, the classical continuum models (which do not consider the small-scale effects) have been widely applied to the small-scale structures as reviewed in the preceding section. The demerit of such classical continuum theories is witnessed in their scale-free models as they cannot incorporate the small-scale effects in their formulations. For the purpose of correcting the inadequacy in the classical continuum models, Eringen [42–45] developed nonlocal continuum mechanics based on nonlocal elasticity theory. The nonlocal elasticity theory considers the stress state at a given point to be a function of the strain field at all points in the body. Therefore, in this work, nonlocal elasticity theory is used to analyzed nonlinear vibrations of single- and double-walled carbon nanotubes resting on two-parameter foundation in a thermal and magnetic environment. With the aid of van der Waals interlayer interaction, the nested slender double-walled nanotubes are coupled with each other. Such study on the simultaneous influences of thermal and magnetic field, two-parameter foundation on the vibration of single- and double-walled carbon nanotubes using nonlocal elasticity theory has not been presented in literature. Additionally, the development of analytical expressions for the frequencies, frequency ratio and deflections of the double-walled carbon nanotubes is shown to be another novel idea of the present study. The analytical solutions are used to investigate the influences of elastic foundations, magnetic field, temperature rise, interlayer forces, small scale parameter and boundary conditions on the frequency ratio.

2 Problem description and the governing equations

In order to develop the governing equations of motion for the SWCNTs and DWCNTs, we first consider a SWCNT under the influence of stretching effects and resting on

Winkler and Pasternak foundations in a thermal and magnetic environment as depicted in Fig. 1. With the aid of the Eringen's nonlocal elasticity theory, Euler–Bernoulli beam theory and Hamilton's principle, the governing equation of motion for the SWCNT is given by

$$\begin{aligned} &EI\frac{\partial^4 w}{\partial x^4} + m_c\frac{\partial^2 w}{\partial t^2} + k_1 w + k_3 w^3 - \left(\frac{EA}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx\right)\frac{\partial^2 w}{\partial x^2} \\ &- EA\alpha_x T\frac{\partial^2 w}{\partial x^2} - \eta A H_x^2\frac{\partial^2 w}{\partial x^2} + (e_o a)^2\left(m_c\frac{\partial^4 w}{\partial x^2\partial t^2} + k_1\frac{\partial^2 w}{\partial x^2}\right. \\ &+ 6k_3 w\left(\frac{\partial w}{\partial x}\right)^2 + 3k_3 w^2\left(\frac{\partial w}{\partial x}\right) - \left(\frac{EA}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx\right)\frac{\partial^4 w}{\partial x^4} \\ &\left. - EA\alpha_x T\frac{\partial^4 w}{\partial x^4} - \eta A H_x^2\frac{\partial^4 w}{\partial x^4}\right) = 0 \end{aligned} \tag{1}$$

where $w(x, t)$ is the bending deflection of the tube, t is the time coordinate, EI is the bending rigidity, m_c is the mass of tube per unit length. The term $EA\alpha_x T$ denotes the constant axial force due to thermal effects and the term $\eta A H_x^2$ is the magnetic force per unit length due to Lorentz force exerted on the tube in *z*-direction. Also, A is the cross-sectional area of the tube, α_x is the coefficient of thermal expansion and T is the change in temperature. Also, the term η is the magnetic field permeability and H_x is the magnetic field strength (Fig. 2).

For the purpose of incorporating the interlayer interactions for the DWCNTs with two layers, it is established that the pressure at any point between any two adjacent tubes depends on the difference in their deflections at that point. Therefore, one can express the linearized form of the van der Waals forces as

$$F_i = c_i(w_i - w_{i-1}). \tag{2}$$

where F_i is the van der Waals force between the ith tube and the $i-1$th tube, c_i is the coefficient of the van der Waals force between the *i*th tube and the $(i-1)$th tube. Assuming that the nested individual tubes of the DWCNT vibrate in the same plane, using the van der Waals forces in Eq. (2), the developed nonlinear governing equations of vibration for the embedded DWCNT in a thermal and magnetic environment with two layers are given as

$$\begin{aligned} &EI_1\frac{\partial^4 w_1}{\partial x^4} + m_{c1}1\frac{\partial^2 w_1}{\partial t^2} - \left(\frac{EA_1}{2L}\int_0^L \left(\frac{\partial w_1}{\partial x}\right)^2 dx\right)\frac{\partial^2 w_1}{\partial x^2} \\ &- EA_1\alpha_x T\frac{\partial^2 w_1}{\partial x^2} - \eta A_1 H_x^2\frac{\partial^2 w_1}{\partial x^2} - (e_o a)^2\left(m_{c1}\frac{\partial^4 w_1}{\partial x^2\partial t^2}\right. \\ &- \left(\frac{EA_1}{2L}\int_0^L \left(\frac{\partial w_1}{\partial x}\right)^2 dx\right)\frac{\partial^4 w_1}{\partial x^4} - E_1 A\alpha_x T\frac{\partial^4 w_1}{\partial x^4} \\ &\left. - \eta_1 A H_x^2\frac{\partial^4 w_1}{\partial x^4} + c_1\left(\frac{\partial^2 w_2}{\partial x^2} - \frac{\partial^2 w_1}{\partial x^2}\right)\right) = c_1(w_1 - w_2) \end{aligned} \tag{3}$$

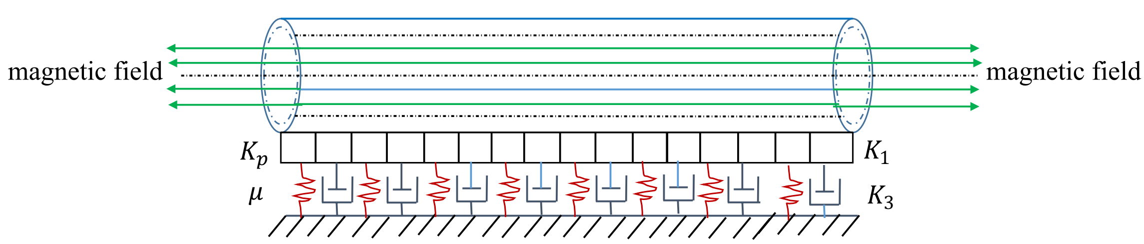


Fig. 1 The SWCNT on two-parameter elastic foundation in a thermal and magnetic field influnce

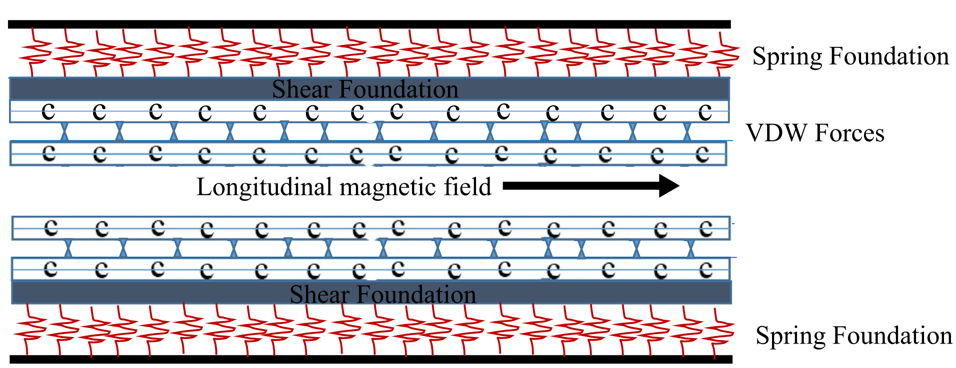


Fig. 2 The embedded DWCNT in a thermal and magnetic environment

$$EI_2\frac{\partial^4 w_N}{\partial x^4}+m_{c2}\frac{\partial^2 w_2}{\partial t^2}+k_1w_2+k_3w_2^3 -\left(\frac{EA_2}{2L}\int_0^L\left(\frac{\partial w_2}{\partial x}\right)^2dx\right)\frac{\partial^2 w_2}{\partial x^2}-EA_2\alpha_xT\frac{\partial^2 w_2}{\partial x^2} -\eta A_2H_x^2\frac{\partial^2 w_2}{\partial x^2}-(e_oa)^2\left(m_{c2}\frac{\partial^4 w_2}{\partial x^2\partial t^2} +k_1\frac{\partial^2 w_2}{\partial x^2}+6k_3w_2\left(\frac{\partial w_2}{\partial x}\right)^2+3k_3w_2^2\left(\frac{\partial w_2}{\partial x}\right) -\left(\frac{EA_2}{2L}\int_0^L\left(\frac{\partial w_2}{\partial x}\right)^2dx\right)\frac{\partial^4 w_2}{\partial x^4}-E_2A\alpha_xT\frac{\partial^4 w_2}{\partial x^4} -\eta A_2H_x^2\frac{\partial^4 w_2}{\partial x^4}+c_1\left(\frac{\partial^2 w_2}{\partial x^2}-\frac{\partial^2 w_1}{\partial x^2}\right)\right)=c_1(w_2-w_1) \tag{4}$$

It should be noted that the k_1 and k_3 will not enter into the equations of the inner tubes since only the outer tube interacts with the elastic medium.

The displacements of the nanotubes are subjected to the following boundary conditions:

For simply supported (S–S) nanotube,

$$w_i(0,t)=0,\quad \frac{\partial^2 w_i(0,t)}{\partial^2 x}=0,\quad w_i(L,t)=0,\quad \frac{\partial^2 w_i(L,t))}{\partial^2 x}=0. \tag{5}$$

For clamped–clamped supported (C–C) nanotube,

$$w_i(0,t)=0,\quad \frac{\partial w_i(0,t)}{\partial x}=0,\quad w_i((L,t)=0,\quad \frac{\partial w_i(L,t)}{\partial x}=0. \tag{6}$$

For a clamped–simply supported (C–S) nanotube,

$$w_i(0,t)=0,\quad \frac{\partial w_i(0,t)}{\partial x}=0,\quad w_i(L,t)=0,\quad \frac{\partial^2 w_i((L,t)}{\partial^2 x}=0. \tag{7}$$

3 Solution methodology

Using the Galerkin's decomposition procedure to separate the spatial and temporal parts of the lateral displacement functions,

$$w_i(x,t)=\phi(x)W_i(t)\quad i=1,2. \tag{8}$$

where $w_i(x,t)$ is the lateral displacement functions, $W_i(t)$ is the time-dependent parameter or time-dependent maximum amplitude of oscillation of the i-th layer of the nanotube and $\phi(x)$ is a trial/comparison function that will satisfy both the geometric and natural boundary conditions.

Applying one-parameter Galerkin to a generalized form of the Eqs. (3) and (4), we have

$$\int_0^L R(x,t)\phi(x)dx=0 \tag{9}$$

where $R_N(x,t)$ is the equation of motion for each wall. For the outer wall of double-walled carbon nanotubes,

$$R(x,t)=EI_2\frac{\partial^4 w_2}{\partial x^4}+m_{c2}\frac{\partial^2 w_2}{\partial t^2}+k_1w_2 +k_3w_2^3-\left(\frac{EA_2}{2L}\int_0^L\left(\frac{\partial w_2}{\partial x}\right)^2dx\right)\frac{\partial^2 w_2}{\partial x^2} -EA_2\alpha_xT\frac{\partial^2 w_2}{\partial x^2}-\eta A_2H_x^2\frac{\partial^2 w_2}{\partial x^2} -(e_oa)^2\left(m_{c2}\frac{\partial^4 w_2}{\partial x^2\partial t^2}+k_1\frac{\partial^2 w_2}{\partial x^2} +6k_3w_2\left(\frac{\partial w_2}{\partial x}\right)^2+3k_3w_2^2\left(\frac{\partial w_2}{\partial x}\right) -\left(\frac{EA_2}{2L}\int_0^L\left(\frac{\partial w_2}{\partial x}\right)^2dx\right)\frac{\partial^4 w_2}{\partial x^4} -E_2A\alpha_xT\frac{\partial^4 w_2}{\partial x^4}-\eta A_2H_x^2\frac{\partial^4 w_2}{\partial x^4} +c_1\left(\frac{\partial^2 w_2}{\partial x^2}-\frac{\partial^2 w_1}{\partial x^2}\right)\right)=c_1(w_2-w_1) \tag{10}$$

One arrives at

$$\alpha_1EI_2W_2+\alpha_2m_{C2}\frac{d^2W_2}{dt^2}+\alpha_2k_1W_2+\alpha_3k_3W_2^3 -\alpha_4\frac{EA_2}{2L}W_N^3-(a_0a)^2\alpha_5m_{c2}\frac{d^2W_2}{dt^2} -(a_0a)^2\alpha_5k_1W_2-6\alpha_6(a_0a)^2k_3W_2^3 -3\alpha_7(a_0a)^2k_3W_2^3+\alpha_8(a_0a)^2\frac{EA_2}{2L}W_2^3 +\alpha_1(a_0a)^2(EA_2\alpha_xT_t+\eta A_2H_x^2)W_2 -\alpha_5(a_0a)^2c_1(W_2-W_1)-\alpha_5(EA_2\alpha_xT_t +\eta A_2H_x^2)W_2+\alpha_2(a_0a)^2c_1(W_2-W_1)=0 \tag{11}$$

After collecting like terms, we have

$$\frac{d^2W_2}{dt^2}+\left(\frac{\alpha_1EI_1+\alpha_2k_1-\alpha_5\mu k_1+(\alpha_1\mu-\alpha_5)\left(EA_2\alpha_xT_t+\eta A_2H_x^2\right)}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)W_2 +\left(\frac{(\alpha_2-\alpha_5\mu)c_1}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)(W_2-W_1) +\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)W_2^3=0 \tag{12}$$

where

$$\alpha_1=\int_0^L\phi(x)\frac{d^4\phi(x)}{dx^4}dx,\quad \alpha_2=\int_0^L\phi^2(x)dx,\quad \alpha_3=\int_0^L\phi^4(x)dx,$$
$$\alpha_4=\int_0^L\left(\phi(x)\left(\int_0^L\left(\frac{d\phi(x)}{\partial x}\right)^2dx\right)\frac{d^2\phi(x)}{dx^2}dx\right),$$
$$\alpha_5=\int_0^L\phi(x)\frac{d^2\phi(x)}{dx^2}dx,\quad \alpha_6=\int_0^L\phi^2(x)\left(\frac{d\phi(x)}{dx}\right)^2dx \tag{13}$$
$$\alpha_7=\int_0^L\phi^3(x)\frac{d\phi(x)}{dx}dx,\quad \alpha_8=\int_0^L\left(\phi(x)\left(\int_0^L\left(\frac{d\phi(x)}{\partial x}\right)^2dx\right)\frac{d^4\phi(x)}{dx^4}dx\right),$$
$$\mu=(e_0a)^2,\quad m_{cN}=\rho A_N,$$

Similarly, the same procedure is applied to other inner walls appropriately.

Therefore, the governing equations of motion for nonlinear vibrations of embedded DWCNTs in a thermal and magnetic environment in ODE form is obtained as,

$$\frac{d^2W_1}{dt^2}+\left(\frac{\alpha_1EI_1+(\alpha_1\mu-\alpha_5)\left(EA_1\alpha_xT_t+\eta A_1H_x^2\right)}{(\alpha_2-\alpha_5\mu)\rho A_1}\right)W_1 -\frac{c_1}{\rho A_1}(W_2-W_1)+\left(\frac{(\alpha_8\mu-\alpha_4)\frac{EA_1}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_1}\right)W_1^3=0 \tag{14}$$

$$\frac{d^2W_2}{dt^2}+\left(\frac{\alpha_1EI_2+(\alpha_1\mu-\alpha_5)\left(EA_2\alpha_xT_t+\eta A_2H_x^2\right)}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)W_2 +\frac{c_1}{\rho A_2}(W_2-W_1)+\left(\frac{(\alpha_8\mu-\alpha_4)\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)W_2^3=0 \tag{15}$$

and the initial conditions are

$$W_1(0)=X\quad\text{and}\quad\frac{dW_1(0)}{dt}=0$$
$$W_2(0)=X\quad\text{and}\quad\frac{dW_2(0)}{dt}=0 \tag{16}$$

3.1 Homotopy perturbation method

The nonlinear terms in Eqs. (14) and (15) make the development of exact analytical solution. Therefore, for the purpose of generating a symbolic solution for the nonlinear equations, we made a recourse homotopy perturbation method. The principle and the procedures of the method can be found in our previous works [46, 47].

3.1.1 Analysis of single-walled carbon nanotube

For SWCNT, the governing equation is given by

$$\frac{d^2W}{dt^2}+\left(\frac{\alpha_1EI+\alpha_2k_1-\alpha_5\mu k_1+(\alpha_1\mu-\alpha_5)\left(EA\alpha_xT_t+\eta AH_x^2\right)}{(\alpha_2-\alpha_5\mu)\rho A_N}\right)W +\left(\frac{\alpha_3k_3-\alpha_4\frac{EA}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA}{2L}}{(\alpha_2-\alpha_5\mu)\rho A}\right)W^3=0 \tag{17}$$

Introducing the following dimensionless quantities,

$$r = \sqrt{\frac{I}{A}}, \quad \tau = \omega_0 t, \quad a = \frac{W}{r} \tag{18}$$

After applying the dimensionless parameters in Eq. (18), Eq. (17) is transformed to

$$\omega_0^2 \frac{d^2 a}{d\tau^2} + f_1 a + f_2 a^3 = 0 \tag{19}$$

where f_1 and f_2 are defined as

$$f_1 = \frac{\alpha_1 EI + \alpha_2 k_1 - \alpha_5 \mu k_1 + (\alpha_1 \mu - \alpha_5)\left(EA\alpha_x T_t + \eta AH_x^2\right)}{(\alpha_2 - \alpha_5 \mu)\rho A} = \omega^2 \tag{20}$$

$$f_2 = \left(\frac{\alpha_3 k_3 - \alpha_4 \frac{EA}{2L} - 6\alpha_6 \mu k_3 - 3\alpha_7 \mu k_3 + \alpha_8 \mu \frac{EA}{2L}}{(\alpha_2 - \alpha_5 \mu)\rho A} \right) \cdot \frac{I}{A} \tag{21}$$

and the initial conditions are

$$a(0) = X \quad \text{and} \quad \frac{da(0)}{d\tau} = 0 \tag{22}$$

It is shown from Eq. (19) that

$$\omega = \sqrt{f_1} = \sqrt{\frac{\alpha_1 EI + \alpha_2 k_1 - \alpha_5 \mu k_1 + (\alpha_1 \mu - \alpha_5)\left(EA\alpha_x T_t + \eta AH_x^2\right)}{(\alpha_2 - \alpha_5 \mu)\rho A}} \tag{23}$$

It should be noted that ω is the linear forced vibration frequency, and ω_0 is an unknown nonlinear angular frequency to be determined.

In order to solve Eq. (17), we construct the following homotopy with ω_0 as the initial approximation for the angular nonlinear frequency as

$$(1-p)\left\{ \omega_0^2 \left(\frac{d^2 a}{d\tau^2} + a \right) \right\} + p\left(\omega_0^2 \frac{d^2 a}{d\tau^2} + f_1 a + f_2 a^3 \right) = 0 \tag{24}$$

It should be noted that the solutions of $a = a(\tau, p)$ and $\omega = \omega(p)$ of the homotopy change from their initial approximations $a_0 = a(\tau)$ and ω_0 to the required solutions $a(\tau)$ and ω_o of (19) as the embedding parameter p travels from 0 to 1.

Assuming that the solution of Eq. (19) take the form of:

$$a(\tau) = a_0(\tau) + pa_1(\tau) + p^2 a_2(\tau) + p^3 a_3(\tau) + \cdots, \tag{25a}$$

$$\omega_0 = \omega_0 + p\omega_1 + p^2\omega_2 + p^3\omega_3 + \cdots, \tag{25b}$$

After substituting Eq. (25) into the homotopy Eq. (24) and rearranging the coefficients of the terms with identical powers of p, we have a series of linear differential equations of the form

$$p^0 : \omega_0^2 \left(\frac{d^2 a_0}{d\tau^2} + a_0 \right) = 0, \tag{26a}$$

with initial conditions

$$a_0(0) = X \quad \text{and} \quad \frac{da_0(0)}{d\tau} = 0,$$

$$p^1 : \omega_0^2 \left[\frac{d^2 a_1}{d\tau^2} + a_1 - \left(\frac{d^2 a_0}{d\tau^2} + a_0 \right) \right] + \omega_0^2 \frac{d^2 a_0}{d\tau^2} + f_1 a_0 + f_2 a_0^3 = 0 \tag{26b}$$

the corresponding initial conditions are

$$a_1(0) = 0 \quad \text{and} \quad \frac{da_1(0)}{d\tau} = 0,$$

$$p^2 : \omega_0^2 \left(\frac{d^2 a_2}{d\tau^2} + a_2 \right) - \omega_0^2 a_1 + 2\omega_0\omega_1 \frac{d^2 a_0}{d\tau^2} + f_1 a_1 - f_2 a_0^2 a_1 = 0 \tag{26c}$$

And the initial conditions are given as

$$a_2(0) = 0 \quad \text{and} \quad \frac{da_0(0)}{d\tau} = 0,$$

Since $\frac{d^2 a_0}{d\tau^2} + a_0 = 0$ and $\frac{d^2 a_0(0)}{d\tau^2} = -a_0$ from Eq. (26a), we can write Eq. (26b) as

$$\omega_0^2 \left(\frac{d^2 a_1}{d\tau^2} + a_1 \right) - \omega_0^2 a_0 + f_1 a_0 + f_2 a_0^3 = 0, \tag{27}$$

The solution of the initial zeroth approximation is given by

$$a_0 = X\cos\tau, \tag{28}$$

On substituting Eq. (28) into the first approximation equation in Eq. (27), one arrives at

$$\omega_0^2 \left(\frac{d^2 a_1}{d\tau^2} + a_1 \right) - \omega_0^2 X\cos\tau + f_1 X\cos\tau + f_2 (X\cos\tau)^3 = 0, \tag{29}$$

After the application of trigonometry identities to the fourth-term in the LHS of Eq. (29), we have

$$\omega_0^2 \left(\frac{d^2 a_1}{d\tau^2} + a_1 \right) - \omega_0^2 X\cos\tau + f_1 X\cos\tau + \frac{3}{4} f_2 X^3 \cos\tau + \frac{1}{4} f_2 X^3 \cos 3\tau = 0, \tag{30}$$

in order to eliminate the secular terms, we set the coefficient of $\cos\tau$ in Eq. (30) to zero

$$-\omega_0^2 X\cos\tau + f_1 X\cos\tau + \frac{3}{4} f_2 X^3 \cos\tau = 0, \tag{31}$$

It gives the nonlinear natural frequency as,

$$\omega_0 = \sqrt{f_1 + \frac{3}{4} f_2 X^2}, \tag{32}$$

It should be noted that the frequency ratio is given as $\psi = \frac{\omega_o}{\omega}$.

Therefore, from Eqs. (32) and (23), we have the frequency ratio as

$$\psi = \frac{\sqrt{f_1 + \frac{3}{4}f_2X^2}}{\sqrt{f_1}} = \sqrt{1 + \frac{3}{4}\frac{f_2}{f_1}X^2} \tag{33}$$

On substituting Eqs. (20) and (21) into Eq. (33), we have

$$\psi = \sqrt{1 + \frac{3}{4}\left\{\frac{\left(\alpha_3 k_3 - \alpha_4 \frac{EA}{2L} - 6\alpha_6 \mu k_3 - 3\alpha_7 \mu k_3 + \alpha_8 \mu \frac{EA}{2L}\right).\frac{I}{A}}{\alpha_1 EI + \alpha_2 k_1 - \alpha_5 \mu k_1 + (\alpha_1 \mu - \alpha_5)\left(EA\alpha_x T_t + \eta A H_x^2\right)}\right\}X^2} \tag{34}$$

The solution of Eq. (30) produces

$$a_1(\tau) = \frac{f_2X^3}{32f_1 + 24f_2X^2}(cos3\tau - \cos\tau) \tag{35}$$

Hence, the first approximate solution of Eq. (19) can be written as

$$a(\tau) = a_0(\tau) + a_1(\tau) = Xcos\tau + \frac{f_2X^3}{32f_1 + 24f_2X^2}(cos3\tau - \cos\tau) \tag{36}$$

From Eqs. (8) and (18), we have

$$w(x,t) = \phi(x)\mathrm{W}(t), \quad W = a(\tau)\sqrt{\frac{I}{A}} \tag{37}$$

Therefore, the displacement of the nanotube to be expressed as

$$w(x,t) = \phi(x)a(\tau)\sqrt{\frac{I}{A}}, \tag{38}$$

Substituting Eq. (36) and the shape functions in the Table 1 into Eq. (38), for simple simply support, we have

$$w(x,t) = \left[\left(Xcos\tau + \frac{f_2X^3}{32f_1 + 24f_2X^2}(cos3\tau - \cos\tau)\right)\right]\sqrt{\frac{I}{A}}sin\left(\frac{n\pi x}{l}\right) \tag{39}$$

while for clamped–clamped gives

$$w(x,t) = \left[Xcos\tau + \frac{f_2X^3}{32f_1 + 24f_2X^2}(cos3\tau - \cos\tau)\right] \sqrt{\frac{I}{A}}\left\{\left(\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right) - \left(\frac{\sinh\beta + \sin\beta}{\cosh\beta - \cos\beta}\right)\left(\sinh\left(\frac{\beta x}{L}\right) - sin\left(\frac{\beta x}{L}\right)\right)\right\} \tag{40}$$

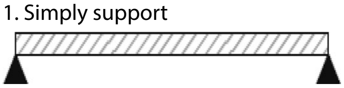
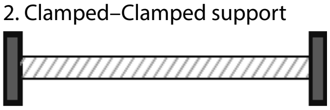
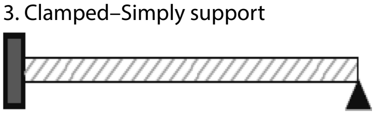
and for clamped–simply support, one obtains

$$w(x,t) = \left[Xcos\tau + \frac{f_2X^3}{32f_1 + 24f_2X^2}(cos3\tau - \cos\tau)\right] \sqrt{\frac{I}{A}}\left\{\left(\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right) - \left(\frac{\cosh\beta - \cos\beta}{\sinh\beta - \sin\beta}\right)\left(\sinh\left(\frac{\beta x}{L}\right) - sin\left(\frac{\beta x}{L}\right)\right)\right\} \tag{41}$$

3.1.2 Analysis of double-walled carbon nanotube

For a DWCNT, the governing equation is given by

Table 1 The basic functions corresponding to the above boundary conditions [37]

Cases	Mode shape, $\phi(x)$	Value of β
1. Simply support 	$sin\left(\frac{\beta x}{L}\right)$	π
2. Clamped–Clamped support 	$\left(\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right) - \left(\frac{\sinh\beta + \sin\beta}{\cosh\beta - \cos\beta}\right)\left(\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right)$	4.730041
3. Clamped–Simply support 	$\left(\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right) - \left(\frac{\cosh\beta - \cos\beta}{\sinh\beta - \sin\beta}\right)\left(\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right)$	3.926602

$$\frac{d^2W_1}{dt^2}+\left(\frac{\alpha_1EI_1+(\alpha_1\mu-\alpha_5)\left(EA_1\alpha_xT_t+\eta A_1H_x^2\right)}{(\alpha_2-\alpha_5\mu)\rho A_1}\right)W_1 - \frac{c_1}{\rho A_1}(W_2-W_1)+\left(\frac{(\alpha_8\mu-\alpha_4)\frac{EA_1}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_1}\right)W_1^3=0 \tag{42}$$

$$\frac{d^2W_2}{dt^2}+\left(\frac{\alpha_1EI_2+\alpha_2k_1-\alpha_5\mu k_1+(\alpha_1\mu-\alpha_5)\left(EA_2\alpha_xT_t+\eta A_2H_x^2\right)}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)W_2 + \frac{c_1}{\rho A_2}(W_2-W_1)+\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)W_2^3=0 \tag{43}$$

Using the following dimensionless parameters,

$$r=\sqrt{\frac{I_1}{A_1}},\quad a_1=\frac{W_1}{r},\quad a_2=\frac{W_2}{r}\quad \text{and}\quad \tau=\omega_0t \tag{44}$$

On substituting the dimensionless parameters in Eq. (44) into Eqs. (42) and (43), we have the following dimensionless nonlinear system of equations

$$\omega_0^2\frac{d^2a_1}{d\tau^2}+f_1a_1+f_2a_1^3-f_3a_2=0, \tag{45}$$

$$\omega_0^2\frac{d^2a_2}{d\tau^2}+g_1a_2+g_2a_2^3-g_3a_1=0. \tag{46}$$

where

$$\begin{aligned}
f_1&=\frac{\alpha_1EI_1+(\alpha_1\mu-\alpha_5)\left(EA_1\alpha_xT_t+\eta A_1H_x^2\right)}{(\alpha_2-\alpha_5\mu)\rho A_1}+\frac{c_1}{\rho A_1},\\
f_2&=\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1},\quad f_3=\frac{c_1}{\rho A_1}\\
g_1&=\left(\frac{\alpha_1EI_2+\alpha_2k_1-\alpha_5\mu k_1+(\alpha_1\mu-\alpha_5)\left(EA_2\alpha_xT_t+\eta A_2H_x^2\right)}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)+\frac{c_1}{\rho A_2}\\
g_2&=\frac{\left(\alpha_3k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}\right)}{(\alpha_2-\alpha_5\mu)\rho A_2}\left(\frac{I_1}{A_1}\right),\\
g_3&=\frac{c_1}{\rho A_2}
\end{aligned} \tag{47}$$

In a similar manner to SWCNT, we construct a homotopy on Eqs. (45) and (46) as follows

$$(1-p)\left\{\omega_0^2\left(\frac{d^2a_1}{d\tau^2}+a_1\right)\right\} + p\left\{\omega_0^2\frac{d^2a_1}{d\tau^2}+f_1a_1+f_2a_1^3-f_3a_2\right\}=0, \tag{48a}$$

$$(1-p)\left\{\omega_0^2\left(\frac{d^2a_2}{d\tau^2}+a_2\right)\right\} + p\left\{\omega_0^2\frac{d^2a_2}{d\tau^2}+g_1a_2+g_2a_2^3-g_3a_1\right\}=0. \tag{48b}$$

The solutions of $a_1=a_1(\tau,p)$, $a_2=a_2(\tau,p)$ and $\omega=\omega(p)$ of the homotopy change from their initial approximations $a_{10}=a_1(\tau)$,$a_{20}=a_2(\tau)$ and ω_0 to the required solutions $a_1(\tau),a_2(\tau)$ and ω_o of Eqs. (49) and (50) as the embedding parameter p travels from 0 to 1.

Assuming the solution of Eqs. (45) and (46) to be in the following form

$$a_1(\tau)=a_{10}(\tau)+pa_{11}(\tau)+p^2a_{12}(\tau)+p^3a_{13}(\tau)+\cdots, \tag{49a}$$

$$a_2(\tau)=a_{20}(\tau)+pa_{21}(\tau)+p^2a_{22}(\tau)+p^3a_{23}(\tau)+\cdots, \tag{49b}$$

$$\omega=\omega_0+p\omega_1+p^2\omega_2+p^3\omega_3+\cdots \tag{49c}$$

Substituting Eqs. (45a–c) into the homotopy in Eqs. (44a) and (44b), collecting and rearranging the coefficients of the terms with identical powers of p, we have a series of linear differential equations

$$p^0:\begin{cases}\frac{d^2a_{10}}{d\tau^2}+a_{10}=0,\ a_{10}(0)=X_1,\ \frac{da_{10}(0)}{d\tau}=0,\\ \frac{d^2a_{20}}{d\tau^2}+a_{20}=0,\ a_{20}(0)=X_1,\ \frac{da_{20}(0)}{d\tau}=0\end{cases} \tag{50a}$$

$$p^1:\begin{cases}\omega_0^2\left\{\frac{d^2a_{11}}{d\tau^2}+a_{11}\right\}-\omega_0^2a_{10}+f_1a_{10}+f_2a_{10}^3-f_3a_{20}=0,\quad a_{11}(0)=0\ \ \frac{da_{11}(0)}{d\tau}=0,\\ \omega_0^2\left\{\frac{d^2a_{21}}{d\tau^2}+a_{21}\right\}-\omega_0^2a_{20}+g_1a_{20}+g_2a_{20}^3-g_3a_{10}=0,\ a_{21}(0)=0,\ \frac{da_{21}(0)}{d\tau}=0\end{cases} \tag{50b}$$

$$p^2:\begin{cases}\omega_0^2\left\{\frac{d^2a_{12}}{d\tau^2}+a_{12}\right\}-\omega_0^2a_{11}+2\omega_0\omega_1\frac{d^2a_{10}}{d\tau^2}+f_1a_{11}+3f_2a_{10}^2a_{11}-f_3a_{21}=0,\quad a_{12}(0)=0\ \ \frac{da_{12}(0)}{d\tau}=0,\\ \omega_0^2\left\{\frac{d^2a_{22}}{d\tau^2}+a_{22}\right\}-\omega_0^2a_{21}+2\omega_0\omega_1\frac{d^2a_{20}}{d\tau^2}+g_1a_{21}+3g_2a_{20}^2a_{21}-g_3a_{11}=0,\ a_{22}(0)=0,\ \frac{da_{22}(0)}{d\tau}=0\end{cases} \tag{50c}$$

The solution of the initial zeroth approximation in Eq. (50a) is simply given by

$$a_{10}(0) = X_1 cos\tau, \tag{51}$$

$$a_{20}(0) = X_2 cos\tau, \tag{52}$$

Substituting Eqs. (51) and (52) into the first approximation in Eq. (50b), eliminating the coefficient of cos τ in the above system to avoid the secular terms, we have the following nonlinear system of equations:

$$-X_1\omega_0^2 + f_1X_1 + \frac{3}{4}f_2X_1^3 - f_3X_2 = 0 \tag{53}$$

$$-X_2\omega_0^2 + g_1X_2 + \frac{3}{4}g_2X_2^3 - g_3X_1 = 0 \tag{54}$$

From Eq. (53),

$$X_2 = \frac{-X_1\omega_0^2 + f_1X_1 + \frac{3}{4}f_2X_1^3}{f_3}. \tag{55}$$

After the substitution of Eq. (55) into Eq. (53), we have

$$-\omega_0^2\left(\frac{-X_1\omega_0^2 + f_1X_1 + \frac{3}{4}f_2X_1^3}{f_3}\right) + g_1\left(\frac{-X_1\omega_0^2 + f_1X_1 + \frac{3}{4}f_2X_1^3}{f_3}\right) + \frac{3}{4}g_2\left(\frac{-X_1\omega_0^2 + f_1X_1 + \frac{3}{4}f_2X_1^3}{f_3}\right)^3 - g_3X_1 = 0 \tag{56}$$

After expansion of Eq. (56) and collecting like terms, we arrived at

$$\begin{aligned}&\frac{3}{4}\frac{g_2}{f_3^3}X_1^3\omega_0^6 + \left(\frac{X_1}{f_3} - \frac{3}{4}\frac{g_2}{f_3^3}X_1^2\left(f_1X_1 + \frac{3}{2}f_2X_1^3\right)\right.\\ &\left.-\frac{9}{8}\frac{g_2}{f_3^3}f_1f_2X_1^8\right)\omega_0^4 + \left(\frac{f_1X_1 + \frac{3}{4}f_2X_1^3 + g_1X_1}{f_3}\right.\\ &+\frac{3}{4}\frac{g_2}{f_3^3}f_1^2X_1^3 + \frac{9}{16}f_2^2X_1^7 + 2X_1\left(f_1^2X_1^2 + \frac{9}{16}f_2^2X_1^6\right)\\ &\left.+\frac{9}{2}f_1f_2X_1^5\right)\omega_0^2 + g_3X_1 - g_1\left(\frac{f_1X_1 + \frac{3}{4}f_2X_1^3}{f_3}\right)\\ &-\frac{3}{4}\frac{g_2}{f_3^3}\left\{f_1^3X_1^3 + \frac{27}{64}f_2^3X_1^6 + \frac{3}{4}f_1^2f_2X_1^5\right.\\ &\left.+\frac{9}{16}f_1f_2^2X_1^7 + 2\left[\frac{3}{4}f_1^2f_2X_1^5 + \frac{9}{16}f_1f_2^2X_1^7\right]\right\} = 0\end{aligned} \tag{57}$$

Equation (57) can be written as

$$\lambda_1\omega_0^6 + \lambda_2\omega_0^4 + \lambda_3\omega_0^2 + \lambda_4 = 0 \tag{58}$$

where

$$\begin{aligned}\lambda_1 &= \frac{3}{4}\frac{g_2}{f_3^3}X_1^3\\ \lambda_2 &= \left(\frac{X_1}{f_3} - \frac{3}{4}\frac{g_2}{f_3^3}X_1^2\left(f_1X_1 + \frac{3}{2}f_2X_1^3\right) - \frac{9}{8}\frac{g_2}{f_3^3}f_1f_2X_1^8\right)\\ \lambda_3 &= \left(\frac{f_1X_1 + \frac{3}{4}f_2X_1^3 + g_1X_1}{f_3} + \frac{3}{4}\frac{g_2}{f_3^3}f_1^2X_1^3 + \frac{9}{16}f_2^2X_1^7\right.\\ &\left.\quad+2X_1\left(f_1^2X_1^2 + \frac{9}{16}f_2^2X_1^6\right) + \frac{9}{2}f_1f_2X_1^5\right)\\ \lambda_4 &= g_3X_1 - g_1\left(\frac{f_1X_1 + \frac{3}{4}f_2X_1^3}{f_3}\right) - \frac{3}{4}\frac{g_2}{f_3^3}\left\{f_1^3X_1^3\right.\\ &\quad+\frac{27}{64}f_2^3X_1^6 + \frac{3}{4}f_1^2f_2X_1^5 + \frac{9}{16}f_1f_2^2X_1^7\\ &\quad\left.+2\left[\frac{3}{4}f_1^2f_2X_1^5 + \frac{9}{16}f_1f_2^2X_1^7\right]\right\}\end{aligned} \tag{59}$$

The roots of the sexic equations are

$$(\omega_0)_1 = \sqrt{\begin{aligned}&\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) + \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}}\\ &+\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) - \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}} - \frac{\lambda_2}{3\lambda_1}\end{aligned}} \tag{60a}$$

$$(\omega_0)_2 = -\sqrt{\begin{aligned}&\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) + \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}}\\ &+\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) - \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}} - \frac{\lambda_2}{3\lambda_1}\end{aligned}} \tag{60b}$$

$$(\omega_0)_3=\sqrt{\frac{-1}{2\lambda_1}\left[\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)+\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}+\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)-\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}\right]+\frac{\sqrt{-3}}{2\lambda_1}\left[\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)+\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}-\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)-\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}\right]-\frac{\lambda_2}{3\lambda_1}} \tag{60c}$$

$$(\omega_0)_4=-\sqrt{\frac{-1}{2\lambda_1}\left[\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)+\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}+\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)-\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}\right]+\frac{\sqrt{-3}}{2\lambda_1}\left[\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)+\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}-\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)-\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}\right]-\frac{\lambda_2}{3\lambda_1}} \tag{60d}$$

$$(\omega_0)_5 = \sqrt{\begin{array}{l}\frac{-1}{2\lambda_1}\left[\begin{array}{l}\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) + \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}} \\ +\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) - \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}}\end{array}\right] \\ \frac{-\sqrt{-3}}{2\lambda_1}\left[\begin{array}{l}\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) + \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}} \\ -\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) - \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}}\end{array}\right] - \frac{\lambda_2}{3\lambda_1}\end{array}} \tag{60e}$$

$$(\omega_0)_6 = -\sqrt{\begin{array}{l}\frac{-1}{2\lambda_1}\left[\begin{array}{l}\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) + \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}} \\ +\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) - \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}}\end{array}\right] \\ \frac{-\sqrt{-3}}{2\lambda_1}\left[\begin{array}{l}\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) + \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}} \\ -\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) - \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}}\end{array}\right] - \frac{\lambda_2}{3\lambda_1}\end{array}} \tag{60f}$$

The nonlinear natural frequency (ω_0) for embedded DWCNTs is obtained from the above solution. The smallest real value of ω_0 is the nonlinear natural frequency for DWCNTs. From Eq. (60),

$$\omega_0 = \sqrt{\begin{array}{l}\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) + \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}} \\ +\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) - \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}} - \frac{\lambda_2}{3\lambda_1}\end{array}} \tag{61}$$

To calculate the linear natural frequencies for DWNT, substitute

$$a_{10}(\tau) = X_1 cos\omega t \tag{62a}$$

$$a_{20}(\tau) = X_2 cos\omega t \tag{62b}$$

into Eq. (50b) and neglecting the nonlinear terms give

$$-X_1\omega^2 + f_1X_1 - f_3X_2 = 0 \tag{63}$$

$$-X_2\omega^2 + g_1X_2 - g_3X_1 = 0 \tag{64}$$

which can be written in matrix form as

$$\begin{bmatrix} -\omega^2 + f_1 & -f_1 \\ -g_3 & -\omega^2 + g_1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{65}$$

Since $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ cannot be equal to zero, for nontrivial case to occur, then

$$\begin{bmatrix} -\omega^2 + f_1 & -f_1 \\ -g_3 & -\omega^2 + g_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{66}$$

By equating the determinant of the matrix in Eq. (66) to zero, the frequency characteristic equation is obtained as

$$\omega^4 - (f_1 + g_1)\omega^2 + f_1g_1 - f_3g_3 = 0 \tag{67}$$

where the roots of the quartic equation are

$$(\omega)_1 = \sqrt{\frac{(f_1 + g_1) + \sqrt{(f_1 + g_1)^2 - 4(f_1g_1 - f_3g_3)}}{2}} \tag{68a}$$

$$(\omega)_2 = -\sqrt{\frac{(f_1 + g_1) + \sqrt{(f_1 + g_1)^2 - 4(f_1g_1 - f_3g_3)}}{2}} \tag{68b}$$

$$(\omega)_3 = \sqrt{\frac{(f_1 + g_1) - \sqrt{(f_1 + g_1)^2 - 4(f_1g_1 - f_3g_3)}}{2}} \tag{68c}$$

$$(\omega)_4 = -\sqrt{\frac{(f_1 + g_1) - \sqrt{(f_1 + g_1)^2 - 4(f_1g_1 - f_3g_3)}}{2}} \tag{68d}$$

The linear natural frequency of DWNTs is the lowest root of the Eq. (67). From Eq. (68), it is

$$\omega = \sqrt{\frac{(f_1 + g_1) - \sqrt{(f_1 + g_1)^2 - 4(f_1g_1 - f_3g_3)}}{2}} \tag{69}$$

We should recall that frequency ratio is given by $\psi = \frac{\omega_o}{\omega}$. Therefore

$$\psi = \sqrt{\frac{\left(\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) + \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}} + \sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) - \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}} - \frac{\lambda_2}{3\lambda_1}\right)}{\left(\frac{(f_1+g_1) - \sqrt{(f_1+g_1)^2 - 4(f_1g_1 - f_3g_3)}}{2}\right)}} \tag{70}$$

On substituting Eqs. (51) and (52) into Eq. (50b), we have

$$\begin{cases} \omega_0^2\left\{\frac{d^2a_{11}}{d\tau^2} + a_{11}\right\} - \omega_0^2X_1cos\tau + f_1X_1cos\tau + f_2\left(X_1cos\tau\right)^3 - f_3X_2cos\tau = 0, \quad a_{11}(0) = 0 \quad \frac{da_{11}(0)}{d\tau} = 0, \\ \omega_0^2\left\{\frac{d^2a_{21}}{d\tau^2} + a_{21}\right\} - \omega_0^2X_2cos\tau + g_1X_2cos\tau + g_2\left(X_2cos\tau\right)^3 - g_3X_1cos\tau = 0, \ a_{21}(0) = 0, \ \frac{da_{21}(0)}{d\tau} = 0 \end{cases} \tag{71}$$

After the application of trigonometry identities to the fourth-term in the LHS

$$\begin{cases} \omega_0^2\left\{\frac{d^2a_{11}}{d\tau^2} + a_{11}\right\} - \omega_0^2X_1cos\tau + f_1X_1cos\tau + \frac{3}{4}f_2X_1^3cos\tau + \frac{1}{4}f_2X_1^3cos3\tau - f_3X_2cos\tau = 0, \quad a_{11}(0) = 0 \quad \frac{da_{11}(0)}{d\tau} = 0, \\ \omega_0^2\left\{\frac{d^2a_{21}}{d\tau^2} + a_{21}\right\} - \omega_0^2X_2cos\tau + g_1X_2cos\tau + \frac{3}{4}g_2X_2^3cos\tau + \frac{1}{4}g_2X_2^3cos3\tau - g_3X_1cos\tau = 0, \quad a_{21}(0) = 0, \quad \frac{da_{21}(0)}{d\tau} = 0 \end{cases} \tag{72}$$

The solutions of Eqs. (72) are

$$a_{11}(\tau)=\frac{f_2X^3(cos3\tau-\cos\tau)}{32\left(\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)+\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}+\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)-\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}-\frac{\lambda_2}{3\lambda_1}\right)} \tag{73}$$

$$a_{21}(\tau)=\frac{g_2X^3(cos3\tau-\cos\tau)}{32\left(\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)+\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}+\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)-\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}-\frac{\lambda_2}{3\lambda_1}\right)} \tag{74}$$

Therefore, the first approximate solution of Eqs. (45) and (46) can be written as follows:

$$a_1(x,t)=X_1cos\tau+\frac{f_2X^3(cos3\tau-\cos\tau)}{32\left(\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)+\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}+\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)-\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}-\frac{\lambda_2}{3\lambda_1}\right)} \tag{75}$$

$$a_2(x,t)=X_2cos\tau+\frac{g_2X^3(cos3\tau-\cos\tau)}{32\left(\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)+\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}+\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)-\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}-\frac{\lambda_2}{3\lambda_1}\right)} \tag{76}$$

The displacements of the nanotubes to be expressed as for simple simply support as

$$w_1(x,t)=\left(X_1cos\tau+\frac{f_2X^3(cos3\tau-\cos\tau)}{32\left(\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)+\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}+\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)-\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}-\frac{\lambda_2}{3\lambda_1}\right)}\right)\sqrt{\frac{I}{A}}sin\left(\frac{n\pi x}{l}\right) \tag{77}$$

$$w_2(x,t)=\left(X_2 cos\tau+\frac{g_2X^3(cos3\tau-\cos\tau)}{32\left(\begin{array}{c}\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)+\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}\\+\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)-\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}-\frac{\lambda_2}{3\lambda_1}\end{array}\right)}\right)\sqrt{\frac{I}{A}}sin\left(\frac{n\pi x}{l}\right) \quad (78)$$

while for clamped–clamped are given as

$$w_1(x,t)=\left(X_1 cos\tau+\frac{f_2X^3(cos3\tau-\cos\tau)}{32\left(\begin{array}{c}\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)+\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}\\+\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)-\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}-\frac{\lambda_2}{3\lambda_1}\end{array}\right)}\right)$$
$$\times\sqrt{\frac{I}{A}}\left\{\begin{array}{l}\left(\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right)\\-\left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left(\sinh\left(\frac{\beta x}{L}\right)-sin\left(\frac{\beta x}{L}\right)\right)\end{array}\right\} \quad (79)$$

$$w_2(x,t)=\left(X_2 cos\tau+\frac{g_2X^3(cos3\tau-\cos\tau)}{32\left(\begin{array}{c}\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)+\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}\\+\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)-\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}-\frac{\lambda_2}{3\lambda_1}\end{array}\right)}\right)$$
$$\times\sqrt{\frac{I}{A}}\left\{\begin{array}{l}\left(\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right)\\-\left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left(\sinh\left(\frac{\beta x}{L}\right)-sin\left(\frac{\beta x}{L}\right)\right)\end{array}\right\} \quad (80)$$

And for clamped–simply supports, we have

$$w_1(x,t)=\left(X_1 cos\tau+\frac{f_2X^3(cos3\tau-\cos\tau)}{32\left(\begin{array}{l}\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)+\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}\\+\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)-\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}-\frac{\lambda_2}{3\lambda_1}\end{array}\right)}\right)\times\sqrt{\frac{I}{A}}\left\{\begin{array}{l}\left(\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right)\\-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left(\sinh\left(\frac{\beta x}{L}\right)-sin\left(\frac{\beta x}{L}\right)\right)\end{array}\right\} \tag{81}$$

$$w_2(x,t)=\left(X_2 cos\tau+\frac{g_2X^3(cos3\tau-\cos\tau)}{32\left(\begin{array}{l}\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)+\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}\\+\sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)-\sqrt{\left(\frac{\lambda_3}{3\lambda_1}-\frac{\lambda_2^2}{9\lambda_1^2}\right)^3+\left(\frac{-\lambda_2^3}{27\lambda_1^3}+\frac{\lambda_2\lambda_3}{6\lambda_1^2}-\frac{\lambda_4}{2\lambda_1}\right)^2}}-\frac{\lambda_2}{3\lambda_1}\end{array}\right)}\right)\times\sqrt{\frac{I}{A}}\left\{\begin{array}{l}\left(\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right)\\-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left(\sinh\left(\frac{\beta x}{L}\right)-sin\left(\frac{\beta x}{L}\right)\right)\end{array}\right\} \tag{82}$$

4 Results and discussion

Using the material and geometric parameters of the carbon nanotubes, $E=1.1$ TPa, $\rho=1300$ kg/m^3, $l=45$ nm, the outer diameter $do=3$ nm, and the thickness of each layer, $h=0.68$ nm, the frequency ratio against non-dimensional maximum amplitude for the nonlinear vibrations of SWCNTs and DWCNTs in a thermal and magnetic environment are given in Figs. 3, 4, 5, 6, 7, 8, 9 and 10. The results of the simulation and the effects of various parameters on the frequency ratio of nonlinear vibrations of embedded single- and double-walled carbon nanotubes in a thermal and magnetic environment are presented and discussed.

4.1 Effects of boundary conditions on the frequency ratio of the carbon nanotubes

Figures 3 and 4 show the effects of boundary conditions on the frequency ratio for the nonlinear vibrations of SWCNTs and DWCNTs, respectively in thermal and magnetic environment ($k_1=10^7$ N/m^2, $k_3=10^8$ N/m^2, $T=40$ K, $H_x=10^7$A/m, $e_oa=1.5\times10^{-9}$ and $c_1=c_2=c_3=0.3\times10^{12}$ N/m^2). As it is depicted in the figures, the frequency ratio for all boundary conditions decreases as the number of wall increases. This is due to the fact that carbon nanotubes generally have weak shear interactions between adjacent tubes and become more predominant as the number of walls increases. It could therefore be inferred that in an application where linear vibration is preferred for system stability, DWCNTs will perform better than SWCNTs of the same geometry and size. Also, the figures show that for both the SWCNTs and DWCNTs, the frequency ratio is highest for clamped simple supported and least for clamped–clamped supported. This establishes that the clamped–clamped supported system provided the best grip (support) for the nanotubes and this can be used to control nonlinear vibration of the system.

4.2 Effects of spring stiffness (k_1) on the frequency ratio of the carbon nanotubes

The impacts of the spring stiffness (k_1) on the dimensionless frequency ratio of the single- and double-walled carbon nanotubes in thermal and magnetic environment are shown in Fig. 5 and 6. It is depicted that he frequency ratio decreases with increases in the value of spring constant (k_1) for CNTs. This is because, the linear frequency increases as the value spring constant increases. At large value of

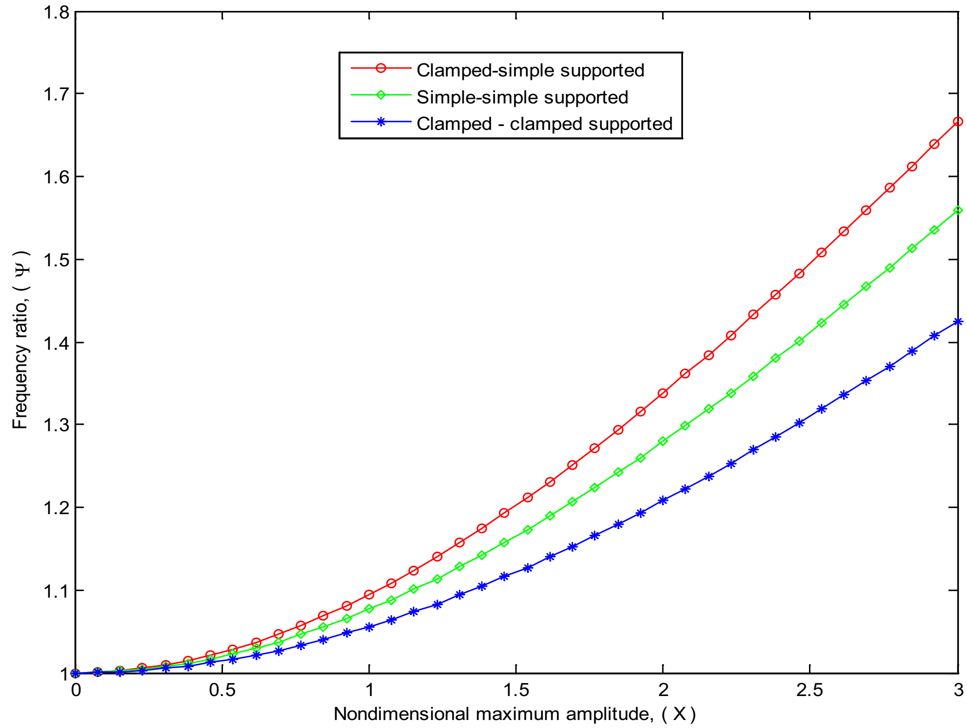


Fig. 3 Frequency ratio versus non-dimensional amplitude for SWCNT under various boundary conditions

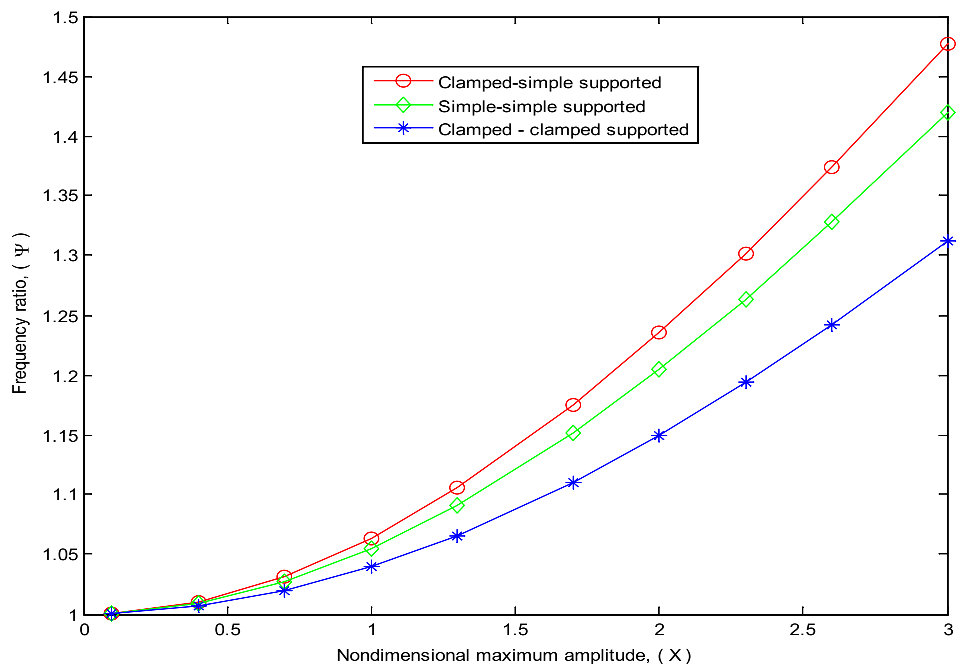


Fig. 4 Frequency ratio versus non-dimensional amplitude for DWCNT under various boundary conditions

k_1 ($k_1 = 10^{10}$ N/m^2), the vibration of the system becomes stable and this can be used as good measure in controlling nonlinear vibration of the system.

4.3 Effects of nonlinear spring stiffness (k_3) on the frequency ratio of the carbon nanotubes

Figure 7 shows the effect of nonlinear spring stiffness (k_3) on the frequency ratio of outer walled of embedded DWCNTs in a thermal and magnetic environment. It could be

Fig. 5 Effect of Winkler constant (k1) on amplitude–frequency ratio curve for SWCNT

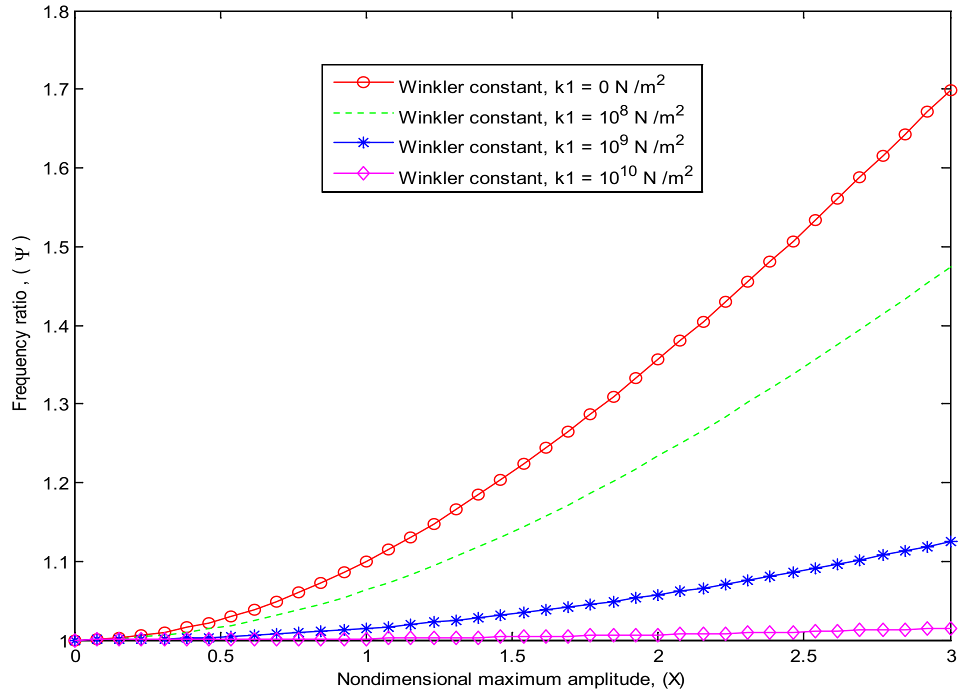
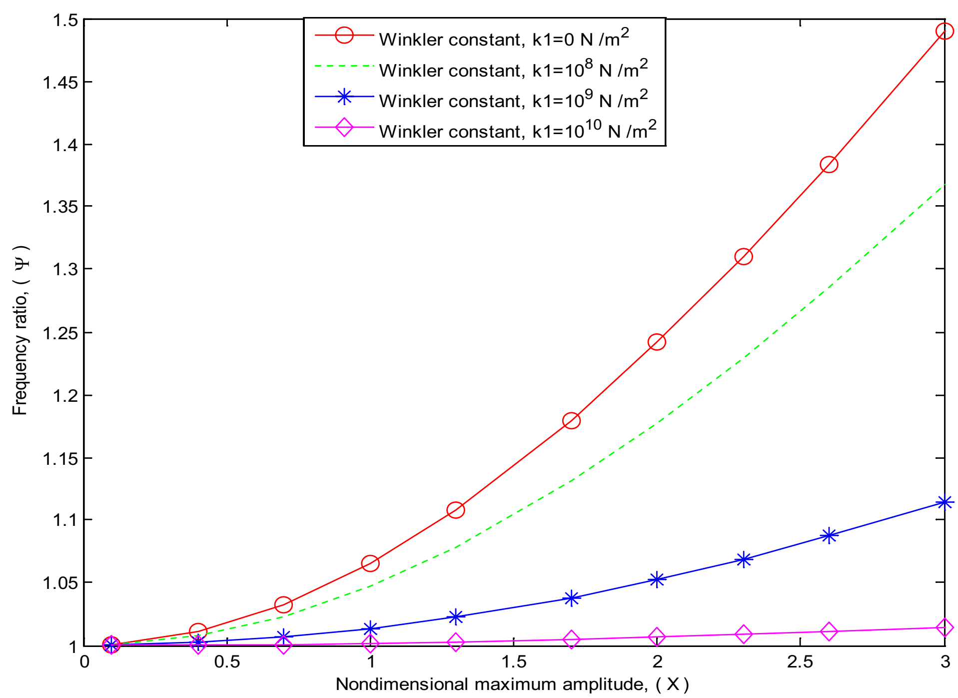


Fig. 6 Effect of Winkler constant (k_1) on amplitude–frequency ratio curve for DWCNTs



seen that the frequency ratio increases with increases in the value of the nonlinear spring constant. This is because the nonlinear frequency increases as the value of the nonlinear spring constant increases without producing any effect on the linear frequency. The value of nonlinear spring constant should be kept as low as possible since it makes the vibration of the system to be nonlinear and this can lead to the instability of the system.

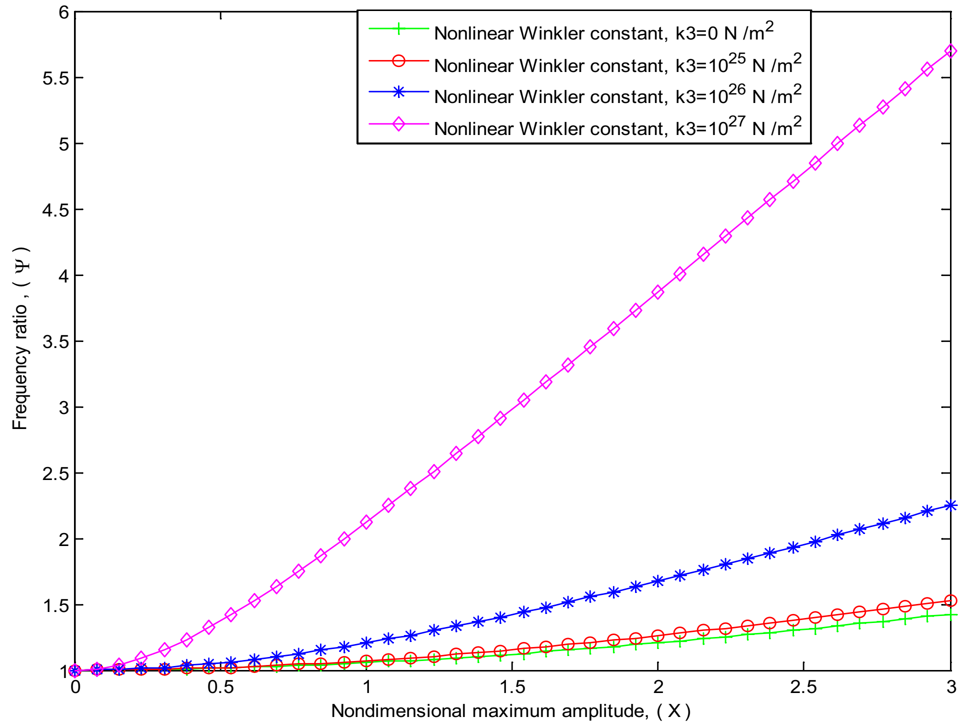


Fig. 7 Effect of nonlinear spring constant constant (k_3) on amplitude–frequency ratio curve for DWCNTs

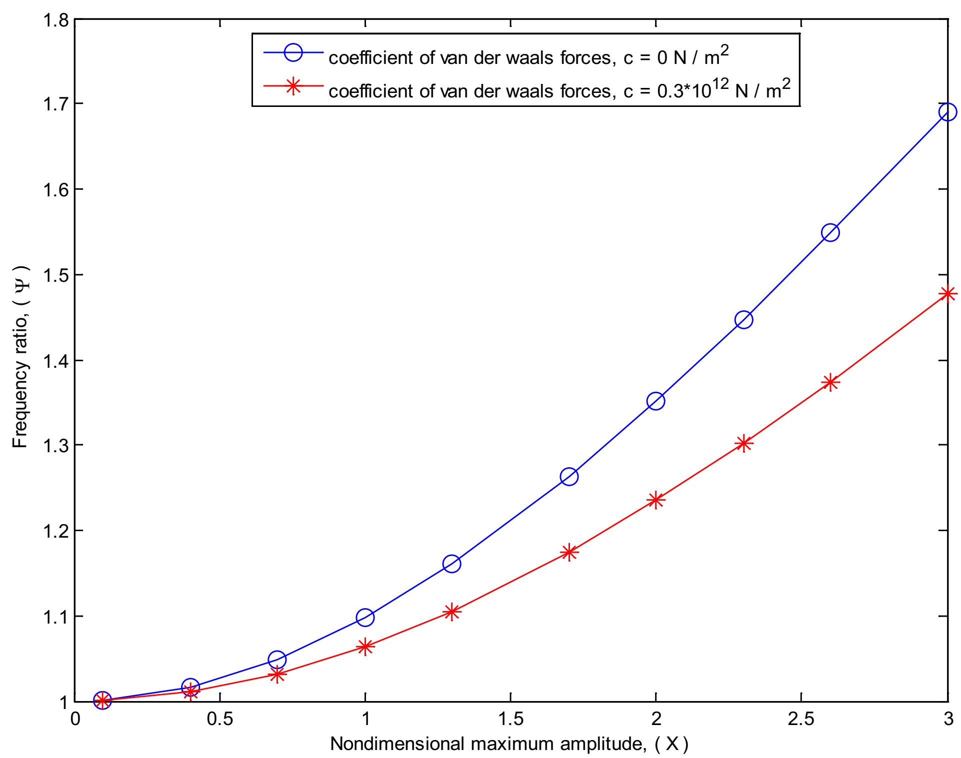


Fig. 8 Effect of Van der waals force on amplitude–frequency ratio curve for DWCNTs

4.4 Effects of Van der Waal force on the frequency ratio of the carbon nanotubes

Figure 8 presents the effects of Van der Waal force on the frequency ratio of the SWCNTs and DWCNTs, respectively, in a thermal and magnetic environment. It can be seen that when the coefficient of the van der Waals forces is zero i.e. c = 0 N/m^2, it means a single-walled carbon nanotube with the same dimension and geometry with double-walled carbon nanotubes. The results reveals that the frequency

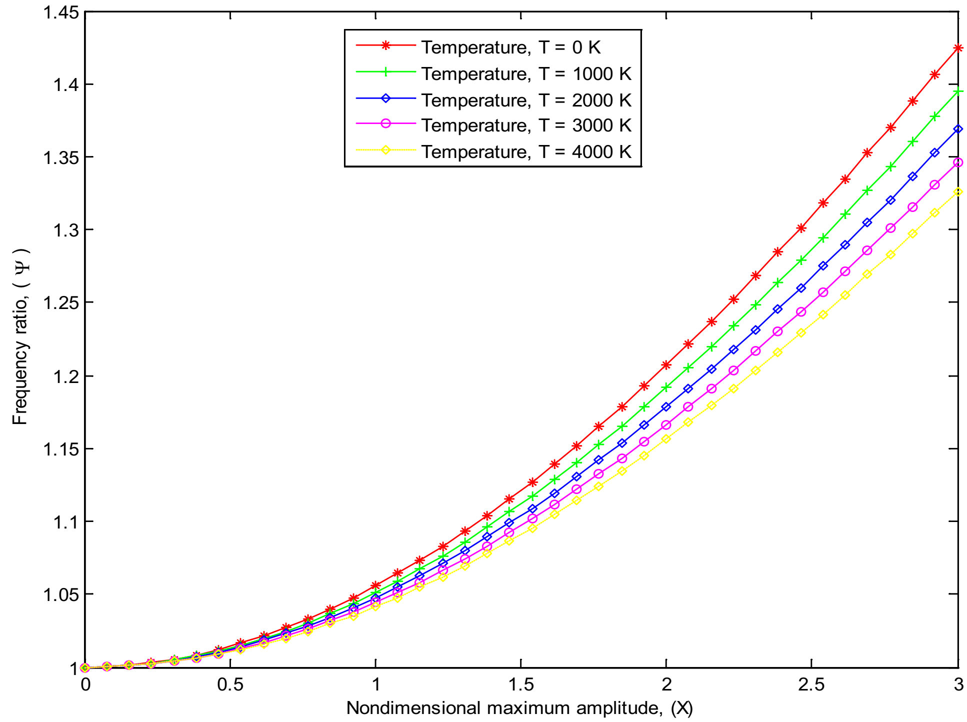


Fig. 9 Effect of temperature on amplitude–frequency ratio curve on outerwall of DWCNTs

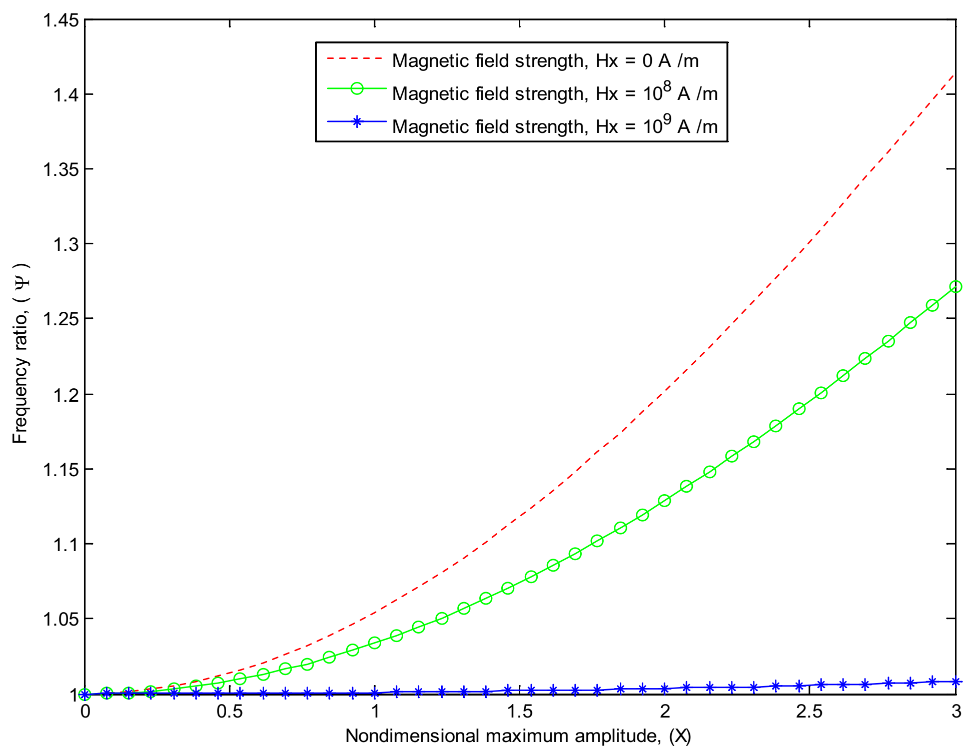


Fig. 10 Effect of magnetic force strength on amplitude–frequency ratio curve on outerwall of DWCNTs

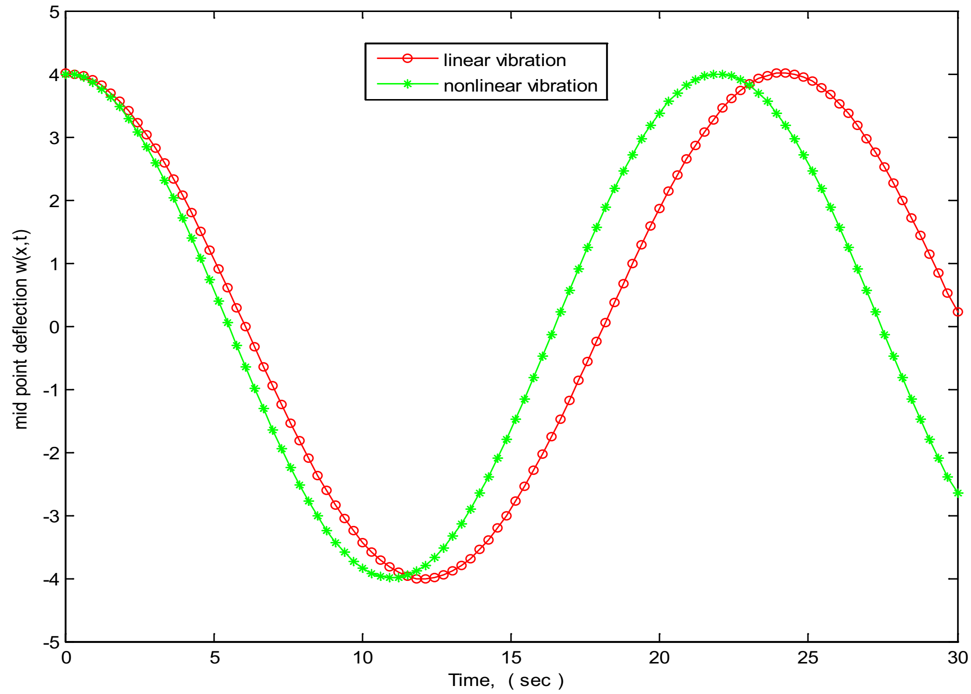


Fig. 11 The linear and nonlinear vibration deflection-time curve of outer wall of DWCNTs

ratio decreases as the number of walls increases. Increasing the number of walls can be used as a good parameter to control the nonlinear vibration of the system.

4.5 Effects of temperature on the frequency ratio of outer wall of DWCNTs

Figure 9 illustrates the influence of temperature on the frequency ratio on the outer wall of DWCNTs in a thermal and magnetic environment. The result presents that as the temperature increases, the frequency ratio decreases. This shows that increase in temperature leads to increase in the value of linear natural frequency of the system.

4.6 Effects of magnetic force strength on the frequency ratio of outer wall of DWCNTs

Figure 10 presents the impact of magnetic force strength on the dimensionless frequency ratio. From the figure, it is established that as the magnetic field strength increases, the vibration of the system approaches linear vibration and become linear at higher value of magnetic force strength, $H = 10^9$ A/m.

4.7 The linear and nonlinear vibration deflection-time curve of outer wall of DWCNTs

Figure 11 shows the comparison of the linear vibration with nonlinear vibration of the DWCNT. It could be seen in the figure that the discrepancy between the linear and nonlinear amplitudes increases and the vibration time progresses.

5 Conclusion

In this work, nonlocal elasticity theory has been used to analyze the nonlinear vibrations of single and double-walled carbon nanotubes resting on two-parameter foundation in a thermal and magnetic environment. The effects of the various controlling parameters on the non-linear to linear frequency ratio have been analyzed and discussed. The results established that the frequency ratio for all boundary conditions decreases as the number of walls increases from single to double. Also, it is established that the frequency ratio is highest for clamped–simple supported and least for clamped–clamped supported. Additionally, the results revealed that the frequency ratio decreases with increase in the value of spring constant (k_1) temperature and magnetic field strength. This work will enhance the applications of carbon nanotubes in structural, electrical, mechanical and biological applications especially in a thermal and magnetic environment.

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Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interest regarding the publication of this research work.

References

1. Iijima S (1991) Helical micro tubes of graphitic carbon. Nature 354:56–58
2. Terrones M, Banhart F, Grobert N, Charlier J, Terrones C, Ajayan H (2002) Molecular junctions by joining single-walled carbon nanotubes. Phys Rev Lett 89:07550
3. Nagy P, Ehlich R, Biro LP, Gjyulai J (2000) Y-branching of single walled carbon nanotubes. Appl Phys A Mater 70:481–483
4. Chernozatonskii LA (1992) Carbon nanotubes connectors and planar jungle gyms. Appl Phys A 172:173–176
5. Liew KM, Wong CH, He XQ, Tan MJ, Meguid SA (2004) Nanomechanics of single and multiwalled carbon nanotubes. Phys Rev B 69:115429
6. Pantano A, Boyce MC, Parks DM (2004) Mechanics of axial compression of single and multi-wall carbon nanotubes. J Eng Mater Technol 126:279–284
7. Pantano A, Parks DM, Boyce MC (2004) Mechanics of deformation of single- and multi-wall carbon nanotubes. J Mech Phys Solids 52:789–821
8. Qian D, Wagner GJ, Liu WK, Yu MF, Ruoff RS (2002) Mechanics of carbon nanotubes. Appl Mech Rev 55:495–533
9. Salvetat JP, Bonard J-M, Thomson NH, Kulik AJ, Forro L, Benoit W, Zuppiroli L (1999) Mechanical properties of carbon nanotubes. Appl Phys A 69:255–260
10. Sears A, Batra RC (2006) Buckling of carbon nanotubes under axial compression. Phys Rev B 73:085410
11. Yoon J, Ru CQ, Mioduchowski A (2002) Noncoaxial resonance of an isolated multiwall carbon nanotube. Phys Rev B 66:233402
12. Wang X, Cai H (2006) Effects of initial stress on non-coaxial resonance of multi-wall carbon nanotubes. Acta Mater 54:2067–2074
13. Wang CM, Tan VBC, Zhang YY (2006) Timoshenko beam model for vibration analysis of multi-walled carbon nanotubes. J Sound Vib 294:1060–1072
14. Zhang Y, Liu G, Han X (2005) Transverse vibrations of double-walled carbon nanotubes under compressive axial load. Phys Lett A 340:258–266
15. Elishakoff I, Pentaras D (2009) Fundamental natural frequencies of double-walled carbon nanotubes. J Sound Vib 322:652–664
16. Buks E, Yurke B (2006) Mass detection with nonlinear nanomechanical resonator. Phys Rev E 74:046619
17. Postma HWC, Kozinsky I, Husain A, Roukes ML (2005) Dynamic range of nanotube- and nanowire-based electromechanical systems. Appl Phys Lett 86:223105
18. Fu YM, Hong JW, Wang XQ (2006) Analysis of nonlinear vibration for embedded carbon nanotubes. J Sound Vib 296:746–756
19. Xu KY, Guo XN, Ru CQ (2006) Vibration of a double-walled carbon nanotube aroused by nonlinear intertube van der Waals forces. J Appl Phys 99:064303
20. Dequesnes M, Tang Z, Aluru NR (2004) Static and dynamic analysis of carbon nanotube-based switches. Trans ASME 126:230–237
21. Ouakad HM, Younis MI (2010) Nonlinear dynamics of electrically actuated carbon nanotube resonators. J Comput Nonlinear Dyn 5:011009
22. Zamanian M, Khadem SE, Mahmoodi SN (2009) Analysis of non-linear vibrations of a microresonator under piezoelectric and electrostatic actuations. J Mech Eng Sci 223:329–344
23. Abdel-Rahman EM, Nayfeh AH (2003) Secondary resonances of electrically actuated resonant microsensors. J Micromech Microeng 13:491–501
24. Hawwa MA, Al-Qahtani HM (2010) Nonlinear oscillations of a double-walled carbon nanotube. Comput Mater Sci 48:140–143
25. Hajnayeb A, Khadem SE (2012) Nonlinear vibration and stability analysis of a double-walled carbon nanotube under electrostatic actuation. J Sound Vib 331:2443–2456
26. Belhadj A, Boukhalfa A, Belalia S (2016) Carbon nanotube structure vibration based on non local elasticity. J Mod Mater 3(1):9–13. https://doi.org/10.21467/jmm.3.1.9-13
27. Lei XW, Natsuki T, Shi JX, Ni QQ (2012) Surface effects on the vibrational frequency of double-walled carbon nanotubes using the nonlocal Timoshenko beam model. Compos B 43:64–69
28. Sharabiani PA, Yazdi MRH (2013) Nonlinear free vibrations of functionally graded nanobeams with surface effects. Compos B 45:581–586
29. Wang L (2010) Vibration analysis of fluid-conveying nanotubes with consideration of surface effects. Physica E 43:437–439
30. Biro LP, Horvat ZE, Mark GI, Osvath Z, Koos AA, Santucci S, Kenny JM (2004) Carbon nanotube Y junctions: growth and properties. Diam Relat Mater 13:241–249
31. Lin RM (2012) Nanoscale vibration characterization of multi-layered graphene sheets embedded in an elastic medium. Comput Mater Sci 53:44–52
32. Pradhan SC, Phadikar JK (2009) Small scale effect on vibration of embedded multilayered graphene sheets based on nonlocal continuum models. Phys Lett A 373:1062–1069
33. Ghorbanpour AA, Zarei MS, Amir S, Khoddami MZ (2013) Non-linear nonlocal vibration of embedded DWCNT conveying fluid using shell model. Phys B 410:188–196
34. Ghorbanpour AA, Amir S (2013) Electro-thermal vibration of visco-elastically coupled BNNT systems conveying fluid embedded on elastic foundation via strain gradient theory. Phys B 419:1–6
35. Eichler A, Moser J, Chaste J, Zdrojek M, Wilson-Rae RI, Bachtold A (2011) Nonlinear damping in mechanical resonators made from carbon nanotubes and grapheme. Nat Nanotechnol 6:339–342
36. Dodds HL, Runyan H (1965) Effects of high velocity fluid flow in the bending vibrations and static divergence of a simply supported pipe. National Aeronautical and Space Administration report, NASA TN, D-2870
37. Sobamowo MG (2017) Nonlinear thermal and flow induced vibration analysis of fluid-conveying carbon nanotube resting on Winkler and Pasternak foundations. Therm Sci Eng Prog 4:133–149
38. Liu F, Wagterveld RM, Gebben B, Otto MJ, Biesheuvel PM, Hamelers HV (2014) Carbon nanotube yarns as strong flexible conductive capacitive electrodes. Colloid Interface Sci Commun 3:9–12
39. Sobamowo MG, Yinusa AA (2018) Thermo-fluidic parameters effects on nonlinear vibration of fluid-conveying nanotube resting on elastic foundations using homotopy perturbation method. J Therm Eng 4(4):2211–2233
40. Coşkun SB, Atay MT, Öztürk B (2011) Transverse vibration analysis of Euler–Bernoulli beams using analytical approximate techniques. In: Ebrahimi F (ed) Advances in vibration analysis research. ISBN: 978-953-307-209-8, InTech
41. Sobamowo MG, Yinusa AA (2017) Power series-after treatment technique for nonlinear cubic duffing and double-well duffing oscillators. J Comput Appl Mech 48(2):297–306
42. Eringen AC (1983) On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. J Appl Phys 54(9):4703–4710

43. Eringen AC (1972) Linear theory of nonlocal elasticity and dispersion of plane waves. Int J Eng Sci 10(5):425–435
44. Eringen AC, Edelen DGB (1972) On nonlocal elasticity. Int J Eng Sci 10(3):233–248
45. Eringen AC (2002) Nonlocal continuum field theories. Springer, New York
46. Sobamowo MG, Akinshiloa AT (2018) Double diffusive magnetohydrodynamic squeezing flow of nanofluid between two parallel disks with slip and temperature jump boundary conditions. Appl Comput Mech 11(2):167–182
47. Sobamowo MG, Adeleye OA (2018) Homotopy perturbation method for kinetic analysis of thermal inactivation of jack bean urease. Karbala Int J Mod Sci 4:187–199