Localized and complex soliton solutions to the integrable (4+1)-dimensional Fokas equation

Hitender Khatri¹ · Manjeet Singh Gautam² · Anand Malik³

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Abstract

We explore a family of exact traveling and localized soliton solutions of the nonlinear integrable (4+1)-dimensional Fokas equation by different approaches, namely, the Jacobian-function method, the Padé type transformation, He's semi-inverse variational technique and sine-cosine or triangle function approach. Some new exact traveling wave solutions of physical interest involving some constraint conditions are reported. The reported solutions are Jacobi doubly periodic wave solutions, fractional soliton, localized soliton when parameters are taken to be special values of doubly periodic functions and complex Bloch solitons. These obtained solutions may facilitate us in understanding the dynamical behavior of physical phenomenon governed by nonlinear integrable Fokas equation and show the applicability and efficacy of the used approaches that can be applied for higher dimensional nonlinear integrable equations as well as linear ones in mathematical physics. The differences and similarities between the used distinct approaches are discussed. To exhibits, the dynamical behavior of reported solutions, the two and three-dimensional structures are numerically simulated.

Keywords Localized and complex soliton · Fokas equation

1 Introduction

The localized and traveling wave solutions of nonlinear evolution equations (NLEEs) are requisite for understanding the dynamics of the nonlinear physical phenomenon which are observed in several fields such as mathematics, chemistry, biology, physics, fluid mechanics, oceanography, fiber communication engineering, and several others. Various powerful, effective and concise techniques are developed in the literature to find the exact and approximated solutions of these NLEEs [1–35]. The symbolic computation programs like Mathematica and Maple, enable us to perform the complex and tedious calculations on computers.

In this work, we extend our study to the following (4+1)-dimensional nonlinear Fokas equation which recast in the form

$$4\Psi_{TX} - \Psi_{XXX} + \Psi_{YYY} + 12\Psi_X\Psi_Y + 12\Psi\Psi_{XY} - 6\Psi_{ZW} = 0. \quad (1)$$

Fokas [36] derived this equation from integrable generalizations of the Kadomtsev–Petviashvili (KP) and Davey–Stewartson (DS) equations. The KP and DS equations have been vastly used as nonlinear wave models to pronounce the internal and surface waves in canals [1] and three-dimensional evolution of wave-packets on the water of finite depth [37], respectively. Consequently, the dynamics of the Fokas equation (1) trails essentially from the physical environment of the KP and DS equations. The significance of the higher dimensional integrable Fokas equation (1) recommends that the understanding of complexifying time can be examined in the framework of modern field theories through the occurrence of integrable nonlinear equations in four spatial dimensions comprising

✉ Hitender Khatri, hkkhatri24@gmail.com | ¹Department of Physics, Pt. Neki Ram Sharma Government College, Rohtak, Haryana 124001, India. ²Department of Physics, Government College, Alewa, Jind, Haryana 126111, India. ³Department of Physics, Chaudhary Bansi Lal University, Bhiwani, Haryana 127021, India.

complex time [36]. Very recently Fokas et al. extended their study to the complexification of the independent variables of nonlinear integrable evolution partial differential equations in multi-dimensions.

In recent times, the (4+1)-dimensional Fokas equation has been studied by several authors. The authors in [39] investigated Lie point symmetries and the potential symmetries of Eq. (1). Lee et al. [40] obtain the exact traveling wave solutions of Eq. (1) employing the three different approaches namely the modified tanh-coth method, the extended Jacobi elliptic function method and the Exp-function method. The authors in [41] used the (G'/G) -expansion method, the extended F-expansion method [42], the WTC method of Painlevé analysis [43], the modified and extended version of simplest equation method [18]. In ref. [44] authors obtained two homoclinic solutions and a rogue wave solution of Eq. (1) by utilizing the parameter limit method and examine the fusion and fission phenomena of the interaction between solitons, and investigate the parameter reasons of fusion and fission of solitons. Other significant results can be found by using the Hirota's bilinear method, the N-soliton solution and lump-type solutions of Eq. (1) in [7, 8]. Nevertheless, fractional and the complex Bloch solitons and other formal localized solitons of (4 + 1)- dimensional Fokas equation has not been reported in the previous works.

The paper is outlined as follows. In Sect. 2, we find the traveling and localized solutions of physical interest to (4+1) dimensional nonlinear Fokas equation with different types of methods. The final discussions and conclusions are presented in Sect. 3.

2 Exact traveling, localized and complex wave solutions

In the present work, we aimed to extract traveling and localized solitary wave solutions to Eq. (1). Therefore, to seek the traveling and localized wave solutions of Eq. (1), we utilize the following wave variable transformation

$$\begin{aligned}\Psi(X, Y, Z, W, T) &= \Psi(\xi), \\ \xi &= \alpha X + \beta Y + \chi Z + \rho W + \nu T + \xi_0,\end{aligned}\quad (2)$$

where ξ_0 is an arbitrary constant and $\alpha, \beta, \chi, \rho, \nu$ are all nonzero constants to transform Eq. (1) into the subsequent ordinary differential equation (ODE):

$$(\alpha\beta^3 - \alpha^3\beta)\Psi'''' + (4\alpha\nu - 6\chi\rho)\Psi'' + 12\alpha\beta(\Psi\Psi')' = 0. \quad (3)$$

Here ν signifies the velocity of propagation and a prime in Eq. (3) indicate the derivatives with respect to wave variable ξ . On integrating ODE (3) two times w.r.t. ξ and allowing the constants of integration to zero, we have a second order nonlinear differential equation of the form

$$a\Psi'' + b\Psi + c\Psi^2 = 0, \quad (4)$$

where $a = (\alpha\beta^3 - \alpha^3\beta)$, $b = (4\alpha\nu - 6\chi\rho)$ and $c = 6\alpha\beta$. This Eq. (4) will be further analyzed to construct the amplitude of traveling wave solutions in the following subsections via different approaches, namely, the Jacobian-function approach, the Pade's type transformation, sine-cosine or triangle function approach and He's semi-inverse variational technique, respectively.

2.1 Solution via Jacobi elliptic function approach

It is forthright to construct the following solution $\Psi(\xi)$ as

$$\Psi(\xi) = \tilde{A} + \tilde{B} \operatorname{cn}^2(\xi, k), \quad (5)$$

for solving the Eq. (4), where $\operatorname{cn}(\xi, k)$ is the Jacobi doubly periodic function with k being the modulus parameter ($0 \leq k \leq 1$) which signifies the degree of the energy localization of cnoidal waves and $\tilde{A}, \tilde{B}$ are the constant parameters to be calculated. It is observe that $\operatorname{cn}(\xi, 1) = \operatorname{sech}(\xi)$ and $\operatorname{cn}(\xi, 0) = \cos(\xi)$. With the increase in localization parameter k , the contribution from the non-linear terms increases, wave period goes to infinity and wave structure profiles are well controlled by hyperbolic functions. Substituting Eq. (5) in (4), we reach at the set of consistency conditions as

$$-2\tilde{B}a(k-1)\tilde{A} + c\tilde{A}^2 + b\tilde{A} = 0, \quad (6)$$

$$-4a(1-2k) + 2c\tilde{A} + b = 0, \quad (7)$$

$$-6ka + c\tilde{B} = 0, \quad (8)$$

which are readily solved to give $\tilde{A} = \frac{4a(1-2k)-b}{2c}$ and $\tilde{B} = \frac{6a}{c}k$ along with a constraint relation among the parameters as

$$b^2 = 16a^2(k^2 - k + 1). \quad (9)$$

Therefore, the general doubly periodic wave solution of Eq. (4) can then be specified as

$$\Psi(\xi) = \frac{2a}{c}[(1-2k) \mp \sqrt{k^2 - k + 1} + 3k \operatorname{cn}^2(\xi, k)], \quad (10)$$

representing a train of solitary wave and known as Jacobi elliptic function solution with double period. By selecting the special parameter case $k = 1$ results in localized wave solutions which exhibits that soliton solution can be found from the limits of Jacobian periodic solutions. From Eq. (9), two particular values of $b = \pm 4a$ are obtained. The positive root needs the occurrence of the constant parameter $\tilde{A}$, and the soliton profile assumes the form: $\Psi(\xi) = -\frac{b}{c}\left(1 - \frac{3}{2} \operatorname{sech}^2(\xi)\right)$ and substituting the value of parameter b and c , we have

$$\Psi(\xi) = -\frac{4\alpha\nu - 6\chi\rho}{6\alpha\beta}\left[1 - \frac{3}{2} \operatorname{sech}^2(\xi)\right], \quad (11)$$

Fig. 1 Two-dimensional density profile of double well solution (11) for $v = 0.1$ (red curve), $v = 0.5$ (green curve) and $v = 1.0$ (blue curve)

Fig. 2 Three dimensional plot of density profile of solution (11) for $v = 0.1$

representing a localized solitary wave train. The solitary wave speed is determined from the relation $v = (6\chi\rho + b)/4\alpha$. The intensity profiles of the localized W-type soliton is illustrated in Fig. 1 for parameter values $\alpha = \beta = \chi = \rho = 1.0$ and $v = 0.1, 0.5, 1.0$. It is very clearly depicted from Fig. 1 that the amplitude of the solitary wave increases with increase in soliton speed while other parameters remain unchanged. The three dimensional density profile of solution is shown in Fig. 2 for same values of $\alpha = \beta = \chi = \rho = 1.0$ and $v = 0.1$. With $k = 1$, the background must disappears for the negative root of Eq. (9), and we have the following solitary wave solution

$$\Psi(\xi) = -\frac{4\alpha v - 6\chi\rho}{4\alpha\beta} \operatorname{sech}^2(\xi). \quad (12)$$

For localization parameter $k = 0$, the solution profiles vanishes, however for $(0 \leq k \leq 1)$, there is no issues with the periodic solutions; one can use any of the 12 Jacobi doubly periodic elliptic functions. Now we consider upfront the following periodic solution

$$\Psi(\xi) = \tilde{A} + \tilde{B} \operatorname{sn}^2(\xi, k), \quad (13)$$

for solving the Eq. (4), where $\operatorname{sn}(\xi, k)$ is the snoidal function, with k being the modulus parameter specifies in the interval $0 \leq k \leq 1$ and $\tilde{A}, \tilde{B}$ are the constant parameters to be calculated. It is observe that $\operatorname{sn}(\xi, 0) = \sin(\xi)$ and $\operatorname{sn}(\xi, 1) = \tanh(\xi)$. Replacement of Eq. (13) into Eq. (4), provide the set of consistency conditions as

$$6a\tilde{B}k^2 + c\tilde{B}^2 = 0, \quad (14)$$

$$\tilde{B}(-4ak^2 + 2c\tilde{A} + b - 4a) = 0, \quad (15)$$

$$2a\tilde{B} + c\tilde{A}^2 + b\tilde{A} = 0. \quad (16)$$

The above set of relations leads to $\tilde{A} = \frac{4ak^2 - b + 4a}{2c}$, and $\tilde{B} = -\frac{6a}{c}k^2$ along with a constraint condition between the parameters a and b specified as

$$b^2 = 16a^2(k^4 - k^2 + 1). \quad (17)$$

Therefore, the general doubly periodic wave solution can be specified as

$$\Psi(\xi) = \frac{2a}{c} \left[(k^2 + 1) \mp \sqrt{k^4 - k^2 + 1} - 3k^2 \operatorname{sn}^2(\xi, k) \right], \quad (18)$$

which is a solitary wave train and is a Jacobian periodic wave solution. For the limiting case $k = 0$ and 1 , respectively, the Jacobi elliptic functions interpolate between the trigonometric and hyperbolic functions. The special case $k = 1$ results in strongly localized wave solutions and it relates to two specific values of $b = \pm 4a$. The positive root needs the occurrence of the free parameter $\tilde{A}$, and the soliton profile appears in the form

$$\Psi(\xi) = \frac{\beta^2 - \alpha^2}{3} [1 - 3 \tanh^2(\xi)], \quad (19)$$

shows a localized solitary wave solution. The amplitude structure of the localized soliton is illustrated in Fig. 3 for different choice of α and β . When $\beta > \alpha$ the soliton takes the form of bell shape (in optical fibre known as bright soliton) and when $\beta < \alpha$ the soliton profile exhibits a dip in its center (in optical fibre called as dark soliton) and the amplitude profile of the solution matches with the structure profile shown in Fig. 3b of Ref. [18]. The graph is characterized by infinite tails for $\Psi \rightarrow 0$ as $\xi \rightarrow \pm\infty$. The soliton profiles for the negative root of Eq. (17), with parameter value $k = 1$ takes the form

$$\Psi(\xi) = (\beta^2 - \alpha^2)[1 - \tanh^2(\xi)]. \quad (20)$$

Fig. 3 Amplitude profile of solution (19) for $\beta = 0.2, \alpha = 0.1$ (blue curve) and $\beta = 0.1, \alpha = 0.2$ (red curve)

These obtained soliton solutions satisfy the existence condition $\beta^2 - \alpha^2 \neq 0$. Therefore, a wide variety of solutions emerge in appropriate parameters regime which include soliton trains and localized soliton. It is naturally appears from solutions (19) and (20) that the numerical values of parameters α and β controls the shape of the localized solitary wave solution and is clearly depicted in Fig. 3. It is worth to mention here that when localization parameter $k \rightarrow 0$ the amplitude profile of the solution (18) vanishes.

2.2 Solution via complex envelope and Padé type ansatz approach

The Bloch-type solitons of Eq. (4) can be specified by assuming a complex envelope ansatz in the following form

$$\Psi(\xi) = (\tilde{A} + i\tilde{B}) + (\tilde{C} + i\tilde{D})cn^2(\xi, k), \quad (21)$$

where $\tilde{A}, \tilde{B}, \tilde{C}$, and $\tilde{D}$ are all real constants to be evaluated. Replacing Eq. (21) in Eq. (4) leads to the equivalent constancy conditions:

$$c(\tilde{A} + i\tilde{B})\sqrt{\tilde{A}^2 + \tilde{B}^2} - b(\tilde{A} + i\tilde{B}) + 2a(\tilde{C} + i\tilde{D})(k - 1) = 0, \quad (22)$$

$$\begin{aligned} & [c^2(\tilde{A}^2 + \tilde{B}^2) - b^2](\tilde{A} + i\tilde{B})(\tilde{C} + i\tilde{D}) \\ & + c^2(\tilde{A}\tilde{C} + \tilde{B}\tilde{D})(\tilde{A} + i\tilde{B})^2 \\ & - 8a^2(\tilde{C} + i\tilde{D})^2(k - 1)(1 - 2k) \\ & - 2ba[(\tilde{C} + i\tilde{D})^2(k - 1) \\ & + 2(\tilde{A} + i\tilde{B})(1 - 2k)(\tilde{C} + i\tilde{D})] = 0, \end{aligned} \quad (23)$$

$$\begin{aligned} & c^2(\tilde{C}^2 + \tilde{D}^2)(\tilde{A} + i\tilde{B})^2 + (\tilde{A}^2 + \tilde{B}^2)(\tilde{C} + i\tilde{D})^2 \\ & + 4(\tilde{A}\tilde{C} + \tilde{B}\tilde{D})(\tilde{A} + i\tilde{B})(\tilde{C} + i\tilde{D}) - b^2(\tilde{C} + i\tilde{D})^2 \\ & - 4ba(\tilde{C} + i\tilde{D})[3(\tilde{A} + i\tilde{B})k^2(\tilde{C} + i\tilde{D})(1 - 2k) \\ & - 8a^2(\tilde{C} + i\tilde{D})^2[2(1 - 2k)^2 + 3k(k - 1)]] = 0, \end{aligned} \quad (24)$$

$$\begin{aligned} & c^2[(\tilde{A}\tilde{C} + \tilde{B}\tilde{D}) + (\tilde{C} - i\tilde{D})(\tilde{A} + i\tilde{B})] \\ & - 6bak - 24a^2k(1 - 2k) = 0, \end{aligned} \quad (25)$$

$$c^2(\tilde{C}^2 + \tilde{D}^2) - 36a^2k^2 = 0. \quad (26)$$

Since there in total are five simultaneous relations to fix three independent parameters, we can see that these are constraint solutions. With the help of symbolic computation we found the solutions of these constancy conditions through a tedious, but straightforward algebraic calculations as:

$$\begin{aligned} \tilde{A} &= \sqrt{(\mu - \tilde{B}^2)}, \\ \tilde{B} &= \sqrt{\frac{[\mu(c\sqrt{\mu} + b)^2 - 4\sigma a^2(k - 1)^2]}{[(c\sqrt{\mu} + b)^2 - 4a^2(k - 1)^2(\sigma/\mu)^2]}}, \\ \tilde{C} &= \sqrt{[v^2 - (\sigma/\mu)^2\tilde{B}^2]}, \tilde{D} = (\sigma/\mu)\tilde{B}, \end{aligned} \quad (27)$$

where

$$\begin{aligned} \mu &= \frac{1}{3c^2}[b^2 + 6(1 - 2k)ba + 4a^2(11k^2 - 11k + 2)], \\ v &= \frac{36a^2k^2}{c^2}, \sigma = \frac{3ak}{c^2}[4a(1 - 2k) + b], \end{aligned} \quad (28)$$

and $b^2 = 16a^2$. It is interesting to note that for $k = 1$, the condition $\tilde{A} = \tilde{C} = 0$, leading to the formerly found localized W-type soliton. Nevertheless, the periodic cnoidal wave profiles can obtained with complex background parameters.

Now we search for more general localized solutions, via using a Padé type ansatz of the following form

$$\Psi(\xi) = \frac{\tilde{A} + \tilde{B}f(\xi, k)}{1 + \tilde{C}f(\xi, k)}, \quad (29)$$

where $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are physical parameters to be evaluated from the constancy conditions, with $\tilde{A}\tilde{C} - \tilde{B} \neq 0$. The following constancy conditions can be inferred with the function choice $cn(\xi, k)$ as:

$$\tilde{A}^2b + 2\tilde{B}\tilde{C}(k - 1)a + \tilde{A}(c - 2\tilde{C}^2(k - 1)) = 0, \quad (30)$$

$$\begin{aligned} & \tilde{A}^2\tilde{C}b + \tilde{B}(c + (2k - 1)a) + \tilde{A}(2\tilde{B}b + \tilde{C}(a - 2ka + 2c)) \\ & = 0, \end{aligned} \quad (31)$$

$$\begin{aligned} & \tilde{B}^2b + \tilde{A}\tilde{C}^2(c + 2ka - a) + \tilde{B}\tilde{C}(2\tilde{A}b + a - 2ka + 2c) \\ & = 0, \end{aligned} \quad (32)$$

$$\tilde{B}^2\tilde{C}b - 2\tilde{B}ka + 2\tilde{A}\tilde{C}ka + \tilde{B}\tilde{C}^2c = 0. \quad (33)$$

It has been observed that these equations are four in number and resulting the the general solution with constrained nature. Here, we consider localized solutions due to their physical interest. Therefore, by solving these equations self consistently and by selecting special parameter values ($k \rightarrow 1$), we can obtain distinct wave structure of solution: $\Psi(\xi) = -\frac{c}{b} \left(\frac{1 - 2 \operatorname{sech}(2\xi)}{1 + \operatorname{sech}(2\xi)} \right)$. On replacement of the parameter values b and c , we have the final form of localized solitary wave solution

$$\Psi(\xi) = -\frac{6\alpha\beta}{4\alpha\nu - 6\chi\rho} \left(\frac{1 - 2 \operatorname{sech}(2\xi)}{1 + \operatorname{sech}(2\xi)} \right) \quad (34)$$

Like the previously found case, the amplitude and width of the soliton solution are coupled and the soliton profile can be easily modified as

$$\Psi(\xi) = -\frac{6\alpha\beta}{4\alpha\nu - 6\chi\rho} \left(1 - \frac{3}{2} \operatorname{sech}^2(\xi) \right), \quad (35)$$

which is alike to the W-type soliton. The solution (35) can be recognized as the characteristic separatrix in the phase space of the solutions of Eq. (4), untying the periodic solutions with closed orbits from the open orbits i.e. unbounded ones. We now demonstrate the existence of bell-shaped soliton when all the parameters in Eq. (4) are nonzero i.e. $a \neq 0, b \neq 0, c \neq 0$. Eq. (4) can be solved by assuming the ansatz of bell type soliton solution of the form as

$$\Psi(\xi) = \frac{\lambda}{1 + \eta \cosh(q\xi)}, \quad (36)$$

where λ, η and q are parameters that will be determined as $\lambda = -\frac{5b}{3c}, \eta = 1$ and $q = \pm\sqrt{\frac{b}{3a}}$. Substituting parameter values yields an interesting soliton solution of the form

$$\Psi(\xi) = \frac{-5(4\alpha\nu - 6\chi\rho)}{18\alpha\beta(1 + \cosh(\pm\sqrt{\frac{(4\alpha\nu - 6\chi\rho)}{3(\alpha\beta^3 - \alpha^3\beta)}}\xi))} \quad (37)$$

for $\frac{4\alpha\nu - 6\chi\rho}{(\alpha\beta^3 - \alpha^3\beta)} > 0$ and $\alpha \neq \beta$. The two and three dimensional density profile of solution (37) is shown in Figs. 4 and 5, respectively, for parameter values $\alpha = 0.1, \beta = 1.0, \nu = 0.2, \chi = 0.1, \rho = 0.1$. The parameters values were satisfies the constraint relations.

2.3 Solutions via He's semi-inverse variational approach

Using the He's semi-inverse approach, we start with the following trial- functional $J(\Psi) = \int L d\xi$, where L is an

Fig. 4 The two-dimensional density profile of solution (37)

Fig. 5 The three-dimensional density profile of solution (37)

unknown function of Ψ and its derivatives. Therefore, we established the following variational principle:

$$J = \int_0^\infty \left[\frac{1}{2}(\Psi')^2 - \frac{b}{2a}\Psi^2 - \frac{c}{3a}\Psi^3 \right] d\xi. \quad (38)$$

Now we look for a solitary wave localized solution in the form $\Psi = A \operatorname{sech}(B\xi)$, where A and B represents the amplitude and inverse width of soliton solution respectively, which are to be determined in the course of the derivation of soliton solutions. Substituting the form of Ψ into Eq. (38) provides

$$\begin{aligned} J &= \int_0^\infty \left[\frac{1}{2}A^2B^2 \operatorname{sech}^2(B\xi) \cdot \tanh^2(B\xi) \right. \\ &\quad \left. - \frac{b}{2a}(A \operatorname{sech}(B\xi))^2 - \frac{c}{3a}(A \operatorname{sech}(B\xi))^3 \right] d\xi, \\ &= \frac{1}{2}A^2B^2 \int_0^\infty \operatorname{sech}^2(B\xi) \cdot \tanh^2(B\xi) d\xi \\ &\quad - \frac{b}{2a}A^2 \int_0^\infty \operatorname{sech}^2(B\xi) d\xi - \frac{c}{3a}A^3 \int_0^\infty \operatorname{sech}^3(B\xi) d\xi, \\ &= \frac{A^2B}{6} - \frac{bA^2}{2aB} - \frac{11cA^3}{42aB}. \end{aligned} \quad (39)$$

From Eq. (39) making J stationary with respect to A and B results in two coupled equations: $\frac{\partial J}{\partial A} = \frac{1}{3}AB - \frac{bA}{aB} - \frac{11}{14} \frac{cA^2}{aB} = 0$, $\frac{\partial J}{\partial B} = \frac{1}{6}A^2 + \frac{1}{2} \frac{bA^2}{aB^2} + \frac{11}{42} \frac{cA^3}{aB^2} = 0$. Solving these equations simultaneously, we obtain values of A and B and the solitary wave solution is well determined. From these equations, one can easily deduce the following relations: $A = -\frac{84b}{55c}$, $B = \left(-\frac{3b}{5a}\right)^{\frac{1}{2}}$. Hence, the solitary wave solution of nonlinear (4+1)-dimensional Fokas equation is approximated as

$$\Psi(\xi) = -\frac{14(4\alpha v - 6\chi\rho)}{55\alpha\beta} \operatorname{sech} \left(\sqrt{-\frac{3(4\alpha v - 6\chi\rho)}{5(\alpha\beta^3 - \alpha^3\beta)}} \xi \right), \quad (40)$$

$$(4\alpha v - 6\chi\rho)/(\alpha\beta^3 - \alpha^3\beta) < 0,$$

which is the solitary wave train solution of nonlinear (4+1)-dimensional Fokas equation of physical interest. Now we look for a localized solitary wave solution in the form $\Psi = A \operatorname{sech}^2(B\xi)$, where A and B needs to be determined. Replacing the form of Ψ into Eq. (38) and making J stationary for A and B results in two coupled equations. From solving these coupled equations simultaneously, the values of A and B are determined as $A = -\frac{3b}{2c}$ and $B = \pm \sqrt{\frac{-b}{4a}}$. Therefore, the exact solitary wave solution of the Fokas equation can be written as

$$\Psi(\xi) = -\frac{(4\alpha v - 6\chi\rho)}{4\alpha\beta} \operatorname{sech}^2 \left(\pm \sqrt{-\frac{(4\alpha v - 6\chi\rho)}{4(\alpha\beta^3 - \alpha^3\beta)}} \xi \right), \quad (41)$$

$$(4\alpha v - 6\chi\rho)/(\alpha\beta^3 - \alpha^3\beta) < 0,$$

which is characterized as a bell type soliton solution in the literature and the structure profile is similar to solution (37) as shown in Fig. 4. The structure profile of this solution matches with the solitary wave profile of Fig. 1 in Ref. [18].

2.4 Solution via Sine-cosine function approach

Now using the sine-cosine function method, we express directly the periodic wave solution of the reduced ODE (4) in the form

$$\Psi(\xi) = \lambda \cos^r(\mu\xi), \quad |\xi| \leq \frac{\pi}{2\mu}, \quad (42)$$

or in the form

$$\Psi(\xi) = \lambda \sin^r(\mu\xi), \quad |\xi| \leq \frac{\pi}{2\mu}, \quad (43)$$

where λ , μ , and $r \neq 0$ are parameters that will be determined later. μ and v are known as the wave number

and wave speed of the associated solution respectively. From the assumptions (42) and (43), we have $(\Psi^n)' = -n^2\mu^2r^2\lambda^n \cos^{nr}(\mu\xi) + n\mu^2\lambda^n r(nr-1) \cos^{nr-2}(\mu\xi)$ and $(\Psi^n)'' = -n^2\mu^2r^2\lambda^n \sin^{nr}(\mu\xi) + n\mu^2\lambda^n r(nr-1) \sin^{nr-2}(\mu\xi)$ and so on for other derivatives. Using these relations in the reduced ODE (4) provides a trigonometric equation in $\cos^N(\xi)$ or $\sin^N(\xi)$ terms. The parameters are evaluated by first balancing the exponents of each pair of cosines or sines to determine N . In the next step, we collect all the coefficients of the same power in $\cos^N(\xi)$ or $\sin^N(\xi)$, where these coefficients have to fixed to zero to acquire a system of algebraic equations among the unknowns r , λ , v and μ that will be found out.

Substituting Eqs. (42) or (43) with their derivatives into Eq. (4), yields

$$a\mu^2r^2\lambda \cos^r(\mu\xi) + a\mu^2\lambda r(r-1) \cos^{r-2}(\mu\xi) + b\lambda \cos^r(\mu\xi) + c\lambda^2 \cos^{2r}(\mu\xi) = 0. \quad (44)$$

We use the balance between the exponents of the cosine functions $\cos^N(\mu\xi)$ and this require that Eq. (44) is satisfied only if the subsequent system of algebraic equations holds:

$$r(r-1) \neq 0, \quad (45)$$

$$2r = r-2, \quad (46)$$

$$-a\mu^2r^2\lambda + b\lambda = 0, \quad (47)$$

$$a\mu^2\lambda r(r-1) + c\lambda^2 = 0. \quad (48)$$

Solving the above equations simultaneously leads to the solution set: $r = -2$, $a = \frac{b}{4\mu^2}$, $c = -\frac{3b}{2\lambda}$. The results can be easily obtained if we also use the sine function assumption (43). Therefore the solution of the integrable nonlinear (4+1)-dimensional Fokas equation is written in periodic form as $\Psi(\xi) = -\frac{3b}{2c} \cos^{-2}(\pm \sqrt{\frac{b}{4a}} \xi)$. Therefore, we specify the singular periodic solution as

$$\Psi(\xi) = -\frac{4\alpha v - 6\chi\rho}{4\alpha\beta} \sec^2 \left(\pm \sqrt{\frac{4\alpha v - 6\chi\rho}{4(\alpha\beta^3 - \alpha^3\beta)}} \xi \right), \quad (49)$$

$$(4\alpha v - 6\chi\rho)/(\alpha\beta^3 - \alpha^3\beta) > 0.$$

which is equivalently written as

$$\Psi(\xi) = -\frac{4\alpha v - 6\chi\rho}{4\alpha\beta} \operatorname{sech}^2 \left(\pm \sqrt{\frac{6\chi\rho - 4\alpha v}{4(\alpha\beta^3 - \alpha^3\beta)}} \xi \right), \quad (50)$$

$$(4\alpha v - 6\chi\rho)/(\alpha\beta^3 - \alpha^3\beta) < 0,$$

which is an analytical bell-shaped soliton solution characterize by sech function with an exponential decaying

wing. This solution is the same solution (41) which is obtained by the semi-inverse method. From this, we conclude that from the sine–cosine function method, one can recover the results of the semi-inverse method. The Eq. (50) also matches with the solitary wave solution (12) which is obtain as a limiting case of Jacobi cn function when parameter k approaches to specific value 1. It is interesting to note that if r is a positive fraction in (42) then the subsequent solution is known as compacton solution which is well-defined as a soliton with finite wavelength. For an integer value of r , the resultant solution is called traveling wave solution and if r is a positive fraction and the resulting solution comprises a hyperbolic function, the solution is named as a solitary wave solution.

To obtain the rational triangle functions (sine and cosine functions), we assume the solution $\Psi(x, t)$ of the form

$$\Psi(\xi) = \frac{\alpha_0}{\alpha_1 + \alpha_2 f(\xi)}, \quad (51)$$

where α_0 , α_1 and α_2 are the parameters to be determined and the rational triangle function $f(\xi) = \sin(\mu\xi)$ or $f(\xi) = \cos(\mu\xi)$, where $\xi = \alpha X + \beta Y + \chi Z + \rho W + \nu T + \xi_0$. The rational function approach can be used directly to Eq. (4) in direct manner. In the rational sine and cosine function approach, we replace the rational triangle function (51) into Eq. (4) with different form of $f(\xi)$. Collecting the resultant coefficient of triangle functions and subsequently solving the resultant set of equations, we get $\alpha_0 = -\frac{3b\alpha_2}{c}$, $\alpha_1 = \alpha_2$ and $\mu = \pm\sqrt{\frac{b}{a}}$. Therefore we obtain the following triangle wave function solution as

$$\Psi(\xi) = -\frac{3(4\alpha\nu - 6\chi\rho)}{\alpha\beta\left[1 + \sin\left(\pm\sqrt{\frac{4\alpha\nu - 6\chi\rho}{\alpha\beta^3 - \alpha^3\beta}}\xi\right)\right]}, \quad (52)$$

$$(4\alpha\nu - 6\chi\rho)/(\alpha\beta^3 - \alpha^3\beta) > 0,$$

$$\Psi(\xi) = -\frac{3(4\alpha\nu - 6\chi\rho)}{\alpha\beta\left[1 + \cos\left(\pm\sqrt{\frac{4\alpha\nu - 6\chi\rho}{\alpha\beta^3 - \alpha^3\beta}}\xi\right)\right]}, \quad (53)$$

$$(4\alpha\nu - 6\chi\rho)/(\alpha\beta^3 - \alpha^3\beta) > 0.$$

These are the periodic wave solutions of nonlinear integrable (4+1)-dimensional Fokas equation obtained with constraint conditions. In Figs. 6 and 7, we depict the two and three-dimensional profile of periodic solution (52), respectively, for parameter values $\alpha = 4.0$, $\nu = 0.1$, $\chi = 0.1$, $\rho = 0.1$, $\beta = 0.5$, $\xi_0 = 0$. The structure profile of the solution (52) matches with the periodic profile of solution (18) in Ref. [42] shown in Fig. 2a.

Fig. 6 The two-dimensional profile of periodic solution (52)

Fig. 7 The three-dimensional profile of periodic solution (52)

3 Discussions and conclusions

In this section, we will highlight the similarities and differences between the used distinct approaches to solving the nonlinear integrable (4+1)-dimensional Fokas equation and lastly, the achieved results are concluded. In the Jacobi function approach the functions $sn(\xi, k)$ and $cn(\xi, k)$ have oscillating nature with double period, unity amplitude and can take both positive and negative signs, whereas the function $dn(\xi, k)$ is always positive and has the form of oscillations superimposed over the uniform background. When localization parameter k approaches to zero, the doubly periodic functions $sn(\xi, k)$ and $cn(\xi, k)$ functions are well estimated by trigonometric functions $\sin(\xi)$ and $\cos(\xi)$, respectively with period of

2π , which correspond to the case of weak localization. The localization parameter $k \rightarrow 1$ drives to the case of strong localization and the sn -wave and cn -wave profiles are well specified by the hyperbolic functions i.e. $\tanh(\xi)$ and $\operatorname{sech}(\xi)$, respectively. Henceforth, the Jacobi functions method provides doubly periodic solutions, the trigonometric, as well as hyperbolic function solutions as the limiting case of doubly periodic solutions of the nonlinear integrable Fokas equation. The Padé type function approach is simply an extended version of the Jacobian function approach and provides the fractional periodic as well as localized solutions as a limiting case of nonlinear integrable Fokas equation. When the parameter $\tilde{C} = 0$ in ansatz (29), then Padé type ansatz render the solutions which can be obtained by the Jacobian function approach. We also used the He's semi-inverse variational procedure to obtain the solitary wave solutions of nonlinear integrable Fokas equation. It is shown that He's semi-inverse method is a robust mathematical tool to the search for variational formulation for nonlinear Fokas equation without employing a Lagrangian multiplier. In this approach a variational principle for the nonlinear equations is established and based upon this formulation, a solitary wave solution is easily obtained utilizing the Ritz method. In the He's semi-inverse method one can depend on the initial trial function for obtaining the solitary wave solution of nonlinear equations and all variables in the trial functional (38) is in first-order differentiations which aid in numerical simulation of the obtained solution. The next approach which we have used to obtain the traveling wave solution of the nonlinear Fokas equation is the sine–cosine function approach. The main power of the sine–cosine function approach is that it greatly reduces the size of computational work compared to the other existing techniques. From the sine–cosine function approach, one can recover the results of He's semi-inverse variational technique.

Hence from the above discussion, we infer that the Padé type ansatz is a more customary type of transformation that contains the solutions of the Jacobian function approach. The Jacobian function method usually recovers all the results of the sine–cosine method and He's semi-inverse variational method underneath some parametric constraint limits. However, no identified method can be utilized for all types of nonlinear evolution equations. These methods were proved to be robust, reliable and competent in handling a large number of nonlinear equations for obtaining single traveling wave solutions. The entire task will be transformed from solving a nonlinear differential equation to simply solving a system of algebraic equations which can be readily solved by symbolic computation.

In conclusion, using a traveling wave reduction technique, we analytically constructed various new types of exact traveling wave and localized solutions of the (4+1)-dimensional nonlinear Fokas equation by distinct approaches. These obtained solutions contain localized wave solutions, fractional soliton solutions, Jacobi elliptic doubly periodic traveling wave solutions and complex type soliton solutions. It is apprehended that some of these solutions transformed as solitary wave solutions and trigonometric function solutions at a certain limit condition. With the help of symbolic computation software Mathematica the exactness of the obtained results is confirmed by substituting them in the original equation. It is easy to see that the used approaches are more simplistic and straightforward. The form of the solutions acquired in our paper is also modest, concise and further enriched the list of solutions of integrable (4+1) dimensional nonlinear Fokas equation. We believe that these reported results are quite significant in understanding the dynamics of the underlying physical phenomenon described by the nonlinear Fokas equation. In future we can extend our study to other higher dimensional nonlinear evolution equations of physical interest and results may be reported soon.

Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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