Isogeometric FE analysis of CNT-reinforced composite plates: free vibration

Ashish K. Singh¹ · A. Bhar¹

© Springer Nature Switzerland AG 2019

Abstract

Free vibration behavior of carbon nanotube-reinforced composite plates is studied here, using isogeometric analysis (IGA) based on a higher-order shear deformation theory (HSDT). Carbon nanotubes are considered dispersed into matrix in four different types of volume fraction distributions varying across the thickness. The formulation developed is approximated according to the HSDT model and is incorporated into a computer program developed in-house. IGA uses the same basis function for exact geometric modeling and for analysis, i.e., non-uniform rational B-spline basis function which can easily model the complex geometries exactly, and it can also achieve any desired degree of continuity through the choice of interpolation order so that this method can easily fulfill the C^1 continuity for the HSDT model. The effect of variation of width-to-thickness ratio, plate aspect ratio and other parameters, on natural frequencies, are evaluated from parametric studies after due validation with the given literature which is done in FORTRAN.

Keywords Isogeometric analysis (IGA) · Carbon nanotube-reinforced composite (CNTRC) · Higher-order shear deformation theory (HSDT) · Non-uniform rational B-spline (NURBS) · FORTRAN

1 Introduction

After the advent of carbon nanotubes (CNTs), relevant research was initially focused on finding their thermal, mechanical, electrical and other properties through experimental and computational means. Once their basic properties became evident, including properties like lightweight, high elastic modulus, strength and resilience, their potential for use as a reinforcing element on polymer/metal matrix-based composites was recognized. Subsequently, evaluations of effective material properties of such CNTRCs caught the attention of some researchers, e.g., Liu and Chen [1]. Since then, CNTs are being considered as important as conventional carbon fibers, but are potentially superior to the latter due to the possibility of tailoring the properties of fiber–matrix composites at the nanoscale. Once the excellent mechanical properties of such CNTRCs became evident, they started to be

researched for application as primary load-bearing structural components like plates. In this direction, Zhu et al. [2] investigated the static and vibration behavior of square CNTRC plates using the finite element method (FEM) based on Reissner–Mindlin theory. Some investigations on natural frequencies of skew CNTRC plates were presented very recently by Kiani [3] using FEM, by Ardestani et al. [4] using isogeometric analysis method and by Zhang [5] using mesh-free method. However, mostly square/rhombus plates having unit plate aspect ratio were considered therein. Using isogeometric analysis, static, dynamic and buckling behaviors of functionally graded rectangular and circular plate have been studied based on Reddy's theory by Tran et al. [6].

Basically, the motivation behind the isogeometric analysis was the bottleneck that has existed since a long time in the creation of suitable model. The creation of suitable model involves many steps that take a lot of

✉ Ashish K. Singh, aashish010389@gmail.com; A. Bhar, anindyab@mnnit.ac.in | ¹Department of Applied Mechanics, MNNIT Allahabad, Prayagraj, U.P. 211004, India.

time, so it is necessary to introduce the concept that bridges the gap between the computer-aided design (CAD) and the IGA studied by Hughes et al. [7]. On this subject, a lot of research has been published. IGA has attained much attention in the numerical field, especially in structures. IGA uses the same basis function that was used by CAD for geometry exactness as well as the approximation of the solution for the analysis. One of the most important features of IGA is that they achieve a desired degree of smoothness by using the choice of interpolation order in comparison with FEM. That is why IGA easily fulfills the C^1 continuity requirements of plate element using HSDT which is also used in the present study.

In this work, natural vibration behavior of CNTRC plates is investigated further using IGA based on the semi-refined HSDT (SRHSDT7) [8]. Considering plates of more general type of plan form become essential for use sometimes, non-rectangular plates of non-unit aspect ratio are considered herein. Four different types of volume fraction distributions of CNTs vary across the thickness, as per Van et al. [9]. The formulation developed is incorporated into a computer program developed in-house.

2 Methodology and problem description

As already mentioned, the present analysis is performed using IGA formulation based on HSDT. The volume fraction of CNT, considered for the four types of volume fraction distributions, varies with thickness direction z , such as uniformly distributed (UD) CNTRC, functionally graded V (FG-V)-type distribution, functionally graded X (FG-X)-type distribution and functionally graded O (FG-O)-type distribution of CNTRC plates as shown in Fig. 1 [9].

The distributions of CNTs along the thickness direction of the CNTRC plate are expressed by Eq. 1 [9]:

$$\begin{cases} V_{\text{CNT}} = V_{\text{CNT}}^* & (\text{UD CNTRC}) \\ V_{\text{CNT}}(z) = \left(1 + \frac{2z}{h}\right) V_{\text{CNT}}^* & (\text{FG-V CNTRC}) \\ V_{\text{CNT}}(z) = 2 \left(1 - \frac{2|z|}{h}\right) V_{\text{CNT}}^* & (\text{FG-O CNTRC}) \\ V_{\text{CNT}}(z) = 2 \left(\frac{2|z|}{h}\right) V_{\text{CNT}}^* & (\text{FG-X CNTRC}) \end{cases} \quad (1)$$

2.1 Isogeometric analysis

A set of coordinates in one dimension which is non-decreasing is knot vector:

Fig. 1 Configuration of CNTRC plates: **a** UD CNTRC plate, **b** FG-V CNTRC plate, **c** FG-O CNTRC plate and **d** FG-X CNTRC plate

$$\xi = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\} \quad (2)$$

where n and p are the basis function number and the order of polynomial which is used to construct B-splines. Basis functions $N_{i,p}(\xi)$ can be defined as:

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1} \quad (3)$$

When $P=0$,

$$N_{i,p} = \begin{cases} 1 & \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

To understand the process, consider the open knot vectors in both directions as $\{1, 2, \dots, 8\}$ and $\{1, 2, \dots, 8\} = \{0 \ 0 \ 1 \ 2 \ 3 \ 3 \ 3\}$.

Bivariate Basis Functions These are found by the tensor product of univariate basis functions:

$$N_{ij}^b(\xi, \eta) = N_{i,p}(\xi) M_{j,q}(\eta) \quad (5)$$

Considering the weight functions corresponding to every control point, NURBS basis function is written as:

$$N_i(\xi, \eta) = \frac{N_i^b w_i}{\sum_A^{m \times n} N_A^b(\xi, \eta) w_A} \quad (6)$$

2.2 Displacement field

The spatial displacements of the plate in terms of reference plane displacements using the higher-order shear deformation theory are (Bhar et al. [8])

$$\left. \begin{aligned} U(x, y, z, t) &= u(x, y, t) + z\theta_x(x, y, t) + z^3\theta_x^*(x, y, t) \\ V(x, y, z, t) &= v(x, y, t) + z\theta_y(x, y, t) + z^3\theta_y^*(x, y, t) \\ W(x, y, z, t) &= w(x, y, t) \end{aligned} \right\} \quad (7)$$

u, v and w are the membrane displacements, θ_x and θ_y are the rotation terms, and θ_x^* and θ_y^* are the higher-order terms.

The laminate strain vector $\{\bar{\epsilon}\}$ consists of the components:

$$\begin{aligned} \{\bar{\epsilon}\}^T &= \left\{ \left| \{\bar{\epsilon}\}_p \right| \left| \{\bar{\epsilon}\}_t \right| \right\} \\ &= \left\{ \left| \{\bar{\epsilon}_m\}^T \right| \left| \{\bar{\epsilon}_k\}^T \right| \left| \{\bar{\epsilon}_k^*\}^T \right| \left| \{\bar{\epsilon}_\varphi\}^T \right| \left| \bar{\epsilon}_\varphi^* \right| \right\} \end{aligned} \quad (8)$$

where the subscripts m, k and φ represent collectively the membrane bending/curvature and transverse shear laminate strains, respectively. Now, the spatial constitutive for the k th orthotropic lamina is given by

$$\{\sigma\}_k = \left\{ \begin{Bmatrix} \sigma_p \\ \sigma_t \end{Bmatrix}_k \right\} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{61} & \bar{Q}_{62} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{55} & \bar{Q}_{54} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{44} \end{bmatrix} \begin{Bmatrix} \epsilon_{XX} \\ \epsilon_{YY} \\ \gamma_{XY} \\ \gamma_{XZ} \\ \gamma_{YZ} \end{Bmatrix} \quad (9)$$

2.3 Governing equation of motion

Hamilton's variational principle is used to derive the governing equation of motion:

$$\delta \int_{t_1}^{t_2} (T - U - W) dt = 0 \quad (10)$$

After solving the above equation, we get the discretized system of equations:

$$(K - \omega^2 M) d = 0 \quad (11)$$

where

$$K = \int_{\Omega} \left(\begin{Bmatrix} B^m \\ B^{b_1} \\ B^{b_2} \end{Bmatrix}^T \begin{bmatrix} A & B & E \\ B & D & F \\ E & F & H \end{bmatrix} \begin{Bmatrix} B^m \\ B^{b_1} \\ B^{b_2} \end{Bmatrix} + \begin{Bmatrix} B^{s_0} \\ B^{s_1} \end{Bmatrix}^T \begin{bmatrix} A_s & B_s \\ B_s & D_s \end{bmatrix} \begin{Bmatrix} B^{s_0} \\ B^{s_1} \end{Bmatrix} \right) d\Omega \quad (12)$$

ω is the natural frequency and M is the global mass matrix that can be calculated by

$$M = \int_{\Omega} N^T m N d\Omega \quad (13)$$

where

$$N = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} \quad (14)$$

$$N_1 = \begin{bmatrix} N_i & 0 & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 & 0 \\ 0 & 0 & N_i & 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & 0 & N_i \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (15)$$

$$N_3 = \begin{bmatrix} 0 & 0 & N_{i,x} & N_i & 0 \\ 0 & 0 & N_{i,y} & 0 & N_i \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (16)$$

Table 1 Non-dimensional frequency parameters of CNTRC plate with simply supported boundary condition

V_{CNT}^*	h/a	Mode	UD		FG-V		FG-O		FG-X	
			Van et al. [9]	Present (IGA)	Van et al. [9]	Present (IGA)	Van et al. [9]	Present (IGA)	Van et al. [9]	Present (IGA)
0.11	0.1	1	14.024	13.735	12.755	12.606	11.773	11.675	15.254	14.873
		2	17.943	17.914	17.128	17.237	16.469	16.418	18.825	18.904
		3	19.428	19.461	19.427	19.511	19.428	19.501	19.428	19.517
		4	19.428	19.461	19.427	19.511	19.428	19.501	19.428	19.517
		5	27.527	27.691	27.158	27.441	26.827	26.590	28.004	28.675
		6	34.769	33.343	33.227	32.094	31.858	30.869	35.980	34.466
	0.05	1	17.503	17.458	15.127	15.179	13.500	13.582	20.241	20.093
		2	21.343	21.646	19.606	20.007	18.371	18.580	23.573	23.958
		3	31.736	32.390	30.790	31.525	30.031	30.119	33.193	34.418
		4	38.855	38.923	38.855	39.023	38.855	39.003	38.855	39.034
		5	38.855	38.923	38.855	39.023	38.855	39.003	38.855	39.034
		6	48.714	49.610	48.298	48.222	44.759	44.302	49.581	51.700
	0.02	1	19.093	19.238	16.093	16.267	14.153	14.326	22.880	23.000
		2	22.968	23.475	20.683	21.212	19.154	19.453	26.183	26.848
		3	34.017	34.622	32.700	33.303	31.711	31.579	36.238	37.539
		4	53.664	53.456	53.040	52.846	52.422	50.824	55.066	56.323
		5	70.808	70.818	59.872	60.123	52.616	52.969	83.604	83.295
		6	72.569	73.168	62.118	62.948	55.123	55.978	83.703	83.722
0.14	0.1	1	14.925	14.553	13.653	13.451	12.662	12.501	16.104	15.669
		2	18.732	18.620	17.883	17.951	17.204	17.037	19.590	19.686
		3	19.775	19.812	19.774	19.899	19.775	19.885	19.775	19.907
		4	19.775	19.812	19.774	19.899	19.775	19.885	19.775	19.907
		5	28.288	28.389	27.878	28.156	27.539	27.109	28.750	29.593
		6	36.061	34.496	34.664	33.435	33.417	32.302	37.121	35.559
	0.05	1	19.196	19.082	16.606	16.625	14.838	14.882	22.084	21.867
		2	22.837	23.064	20.866	21.241	19.486	19.600	25.242	25.615
		3	33.024	33.621	31.882	32.632	31.017	30.904	34.638	36.038
		4	39.549	39.624	39.549	39.801	39.549	39.771	39.549	39.815
		5	39.549	39.624	39.549	39.801	39.549	39.771	39.549	39.815
		6	50.023	50.885	49.465	50.514	48.482	47.848	50.996	53.552
	0.02	1	21.290	21.417	17.879	18.048	15.701	15.859	25.528	25.631
		2	24.933	25.414	22.222	22.756	20.455	20.683	28.616	29.318
		3	35.678	36.278	34.013	34.663	32.840	32.520	38.313	39.845
		4	55.280	55.075	54.412	54.297	53.668	51.690	56.981	58.762
		5	78.467	78.324	66.438	66.640	58.490	58.809	85.793	86.646
		6	80.087	80.518	68.506	69.285	60.805	61.580	92.220	91.713
0.17	0.1	1	17.409	16.832	15.788	15.409	14.563	14.198	18.969	18.377
		2	22.352	22.172	21.329	21.381	20.518	20.144	23.458	23.702
		3	24.308	24.357	24.306	24.529	24.308	24.505	24.308	24.541
		4	24.308	24.357	24.306	24.529	24.308	24.505	24.308	24.541
		5	34.390	34.515	33.943	34.331	33.562	32.802	34.958	36.343
		6	43.353	41.335	41.341	39.764	39.599	38.095	44.908	43.018

Table 1 (continued)

V_{CNT}^*	h/a	Mode	UD		FG-V		FG-O		FG-X	
			Van et al. [9]	Present (IGA)	Van et al. [9]	Present (IGA)	Van et al. [9]	Present (IGA)	Van et al. [9]	Present (IGA)
0.05	1	1	21.624	21.102	18.632	18.303	16.625	16.326	25.049	24.422
		2	26.485	26.501	24.329	24.571	22.814	22.611	29.261	29.603
		3	39.567	40.155	38.427	39.268	37.509	36.967	41.362	43.337
		4	48.615	48.715	48.615	49.062	48.615	49.010	48.615	49.082
		5	48.615	48.715	48.6145	49.062	48.615	49.010	48.615	49.082
		6	60.862	61.847	60.404	58.716	55.256	53.726	61.906	65.683
0.02	1	1	23.528	23.077	19.777	19.488	17.398	17.136	28.228	27.653
		2	28.440	28.555	25.620	25.919	23.754	23.586	32.412	32.83
		3	42.362	42.764	40.781	41.376	39.579	38.668	45.090	46.977
		4	67.018	66.527	66.318	66.111	64.620	62.477	68.729	71.403
		5	87.328	85.139	73.560	72.007	65.571	63.343	103.301	100.566
		6	89.569	88.186	76.443	75.713	67.836	67.158	104.608	103.380

Table 2 Non-dimensional natural frequency $\bar{\omega} = \omega(a^2/h)\sqrt{\rho^m/E^m}$ for square plate with simply supported boundary condition

b/a	V_{CNT}^*	UD	FG-V	FG-O	FG-X
10	0.11	1.946	1.951	1.950	1.951
	0.14	1.981	1.990	1.988	1.990
	0.17	2.435	2.453	2.450	2.454
5	0.11	3.892	3.902	3.900	3.903
	0.14	3.962	3.980	3.977	3.981
	0.17	4.871	4.906	4.901	4.908
3.333	0.11	5.897	5.912	5.909	5.914
	0.14	6.003	6.030	6.026	6.032
	0.17	7.381	7.433	7.425	7.436
2.5	0.11	7.784	7.804	7.800	7.806
	0.14	7.924	7.960	7.954	7.963
	0.17	9.743	9.812	9.802	9.816

Geometric characteristics of the plate are $a/h = 10$

Table 3 Non-dimensional natural frequency with clamped boundary condition under uniform load

b/h	V_{CNT}^*	UD	FG-V	FG-O	FG-X
10	0.11	18.120	17.676	17.145	18.604
	0.14	18.649	18.287	17.778	19.138
	0.17	22.526	21.995	21.225	23.316
20	0.11	29.157	27.026	25.201	31.245
	0.14	30.734	28.703	26.880	32.753
	0.17	35.823	33.121	30.738	38.702
30	0.11	35.466	31.798	29.020	39.372
	0.14	38.076	34.322	31.404	42.014
	0.17	43.211	38.637	35.143	48.312
40	0.11	39.182	34.536	31.228	44.458
	0.14	42.523	37.573	33.997	48.036
	0.17	47.520	41.802	37.715	54.223
50	0.11	41.635	36.408	32.810	47.825
	0.14	45.450	39.747	35.793	52.079
	0.17	50.380	44.001	39.595	58.121

Assuming square plate ($a/b = 1$)

3 Numerical results

In this section, analysis is being done by NURBS-based isogeometric analysis on bare plates to ensure the validity of the developed formulation into a computer program using FORTRAN. The material properties of the matrix are assumed as [9] Poisson's ratio (ν^m) = 0.34 and elastic modulus (E^m) = 2.1 GPa. The values of elastic modulus, shear modulus and Poisson's ratio of CNT are: $E_{11}^{\text{CNT}} = 5.6466 \times 10^3$ GPa, $E_{22}^{\text{CNT}} = 7.08 \times 10^3$ GPa, $G_{12}^{\text{CNT}} = 1.9445 \times 10^3$ GPa and $\nu_{12}^{\text{CNT}} = 0.175$. The CNT efficiency parameter of $V_{\text{CNT}}^* = 0.11$, $\eta_1 = 0.149$ and $\eta_2 = 0.934$, for $V_{\text{CNT}}^* = 0.14$, $\eta_1 = 0.150$ and $\eta_2 = 0.941$ and for $V_{\text{CNT}}^* = 0.17$, $\eta_1 = 0.149$ and $\eta_2 = 1.381$, where η_1 and η_2 are the CNTs' efficiency parameters. In this paper, simply supported

(SSSS) and clamped (CCCC) boundary conditions are used. The results are obtained in terms of non-dimensional frequency parameter $\tilde{\omega} = (\omega a^2/h)\sqrt{\rho^m/E^m}$ using NURBS-based isogeometric analysis for obtaining eigensolution. As evident from Table 1, the present results in terms of non-dimensional fundamental natural frequency are found to be in close conformity with those from the published literature, for rectangular CNTRC plates with simply supported boundary condition.

First, we analyze the effect of aspect ratio and volume fraction of CNTs with simply supported (SSSS) boundary condition, and it shows that as the aspect ratio decreases,

Fig. 2 Convergence of rectangular plate having clamped boundary conditions

Fig. 3 Non-dimensional natural frequency of CNTRC plates with $h/a=0.1$ **a** UD, **b** FG-V, **c** FG-O and **d** FG-X

the non-dimensional natural frequency increases and it also increases with the increase of volume fraction of carbon nanotube as shown in Table 2.

Next, we analyze the effect of width-to-thickness ratio with clamped (CCCC) boundary condition, and it can be seen that the non-dimensional natural frequency increases with the increase of b/h ratio and it also increases with the increase in volume fraction of CNTs as shown in Table 3.

From Fig. 2, convergence takes place at 8×8 elements with clamped boundary condition.

Figure 3 shows the four types of CNTRC plates with different boundary conditions, and it can be seen that greater the width-to-thickness ratio greater will be the non-dimensional natural frequency. From all the four types of graphs, it can also be seen that the value of frequency in UD and FG-X type is greater than the FG-V and FG-O because the distribution of volume fraction of CNTs can affect the stiffness of the plate. In UD and FG-X, the reinforcement distribution is more in the top and bottom than the mid-plane. The stiffness is mainly influenced by the top and the bottom surface, i.e., if the volume fraction of CNTs is more in the top and bottom, the stiffness is more. So, the non-dimensional natural frequency is more in UD and FG-X distribution type than that in the other two types of distributions.

4 Conclusions

This paper presents the plate formulation using NURBS-based isogeometric analysis with the semi-refined HSDT for free vibration of FG-CNTRC plate. The comparison and the convergence are carried out to assure the correctness of the given method. The following are the conclusions made with the present analysis: It can be seen that the non-dimensional frequency increases when the plate aspect ratio (b/a) decreases. As the volume fraction of CNTs increases, the non-dimensional natural frequency increases. As the b/h ratio increases, i.e., the thickness of the plate decreases, the non-dimensional frequency increases. It is concluded that the non-dimensional natural frequency in UD and FG-X type is more than that in the FG-V and FG-O type.

From the conclusion, it has been suggested that to reduce the non-dimensional natural frequency, the aspect

ratio and volume fraction of CNT should be small and the thickness of the plate should be more.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Liu YJ, Chen XL (2003) Evaluations of the effective material properties of carbon nanotube-based composites using a nanoscale representative volume element. *Mech Mater* 35(1–2):69–81. [https://doi.org/10.1016/S0167-6636\(02\)00200-4](https://doi.org/10.1016/S0167-6636(02)00200-4)
2. Zhu P, Lie ZX, Liew KM (2012) Static and free vibration analyses of carbon nanotube-reinforced composite plates using finite element method with first order shear deformation plate theory. *Compos Struct* 94:1450–1460. <https://doi.org/10.1016/j.composstruct.2011.11.010>
3. Kiani Y (2016) Free vibration of FG-CNT reinforced composite skew plates. *Aero Sci Technol* 58:178–188. <https://doi.org/10.1016/j.ast.2016.08.018>
4. Ardestani MM, Zhang LW, Liew KM (2017) Isogeometric analysis of the effect of CNT orientation on the static and vibration behaviors of CNT-reinforced skew composite plates. *Comput Methods Appl Mech Eng* 317:341–379. <https://doi.org/10.1016/j.cma.2016.12.009>
5. Zhang LW (2017) On the study of the effect of in-plane forces on the frequency parameters of CNT-reinforced composite skew plates. *Compos Struct* 160:824–837. <https://doi.org/10.1016/j.compstruct.2016.10.116>
6. Tran LV, Ferreira AJM, Xuan HN (2013) Isogeometric analysis of functionally graded plates using higher-order shear deformation theory. *Comp Part B Eng* 51:368–383. <https://doi.org/10.1016/j.compositesb.2013.02.045>
7. Hughes TJR, Cottrell JA, Bazilevs Y (2005) Isogeometric analysis: CAD, finite element, NURBS, exact geometry and mesh refinement. *Comput Methods Appl Mech Eng* 194(39–41):4135–4195. <https://doi.org/10.1016/j.cma.2004.10.008>
8. Bhar A, Phoenix SS, Satsangi SK (2010) Finite element analysis of laminated composite stiffened plates using FSDT and HSDT: a comparative perspective. *Compos Struct* 92(2):312–321. <https://doi.org/10.1016/j.compstruct.2009.08.002>
9. Van PP, Wahab MA, Liew KM, Bordas SPA, Xuan HN (2015) Isogeometric analysis of functionally graded carbon nanotube-reinforced composite plates using higher-order shear deformation theory. *Compos Struct* 123:137–149. <https://doi.org/10.1016/j.compstruct.2014.12.021>

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.