Effects of in-phase and anti-phase large amplitude nonlinear models for double-layer nanostructures

Ehsan Allahyari¹ · Masoud Asgari¹

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Abstract

Double layer nano plates have magnificent applications in nanostructures such as, nano sensors, nano actuators and nano transistors. So, the main aim of the presented study is to propose an analytical model based on Reddy's third-order shear deformation theory of plates which has a difficult procedure due to the great number of partial differential equations and nonlocal strain gradient theory. Due to the fact that numerical approaches aren't good enough accurate and need more time to calculate, the multiple scales method which is a useful method in systems characterized by disparate time scales and in a variety of situations for extracting the slow time dependence of patterns or other systems is adopted. Based on von Kármán equations as well as the theory of nonlocal strain gradient, the coupled nonlinear equations are derive. Compared to the other related published literature, acceptable agreement can be seen under different boundary conditions. Effect of size depended parameter on in-phase and anti-phase modes considering the SSSS and CCCC boundary condition is indescribable which means by increasing the nonlocal parameter nonlinear frequency and also the natural frequency of nanostructure reduces.

Keywords Double layer nano plates · Nonlinear vibration · Nonlocal strain gradient theory · Third-order shear deformation theory · IPM and APM modes

1 Introduction

There are various shell and plate theories which have been used for different structures and conditions. The simplest theory to analyzing the plates behavior is the classical plate theory (CPT). Based on this theory which is according to the Kirchhoff's assumptions the effect of shear and vertical deformation along the thickness does not take into account. In this theory, it is assumed that each cross section remains perpendicular to the middle layer of the plate after the deformation. Although the CPT is an appropriate choice for thin plates analysis with a thickness ratio of about 0.05, but due to the neglect of shear and vertical strains along the thickness, the use of this theory for thick plates are always accompanied by errors. In the first-order shear deformation plate

theory (FSDT), known as the Mindling–Reisner theory, it is assumed that the plane perpendicular to the middle layer of the plate remains not necessarily perpendicular after the deformation. The most commonly used theory of higher-order shear deformation theory is the third-order shear deformation plate theory (TSDT), which is proposed by Reddy. In this theory, unlike the first order theory, there is no need for a shear correction factor, hence the results [1]. Both FSDT and modified couple stress theory were used by Malikan [2] to derive the nonlinear equations based on Von-Karman hypothesis and Hamilton's principle. Free vibration of circular structures considering the TSDT with an simple and useful method were considered by Sahmani et al. [3]. By employing the nonlocal theory, Jomehzadeh et al. [4] developed the nonlinear large amplitude vibration

✉ Masoud Asgari, asgari@kntu.ac.ir; Ehsan Allahyari, allahyari@email.kntu.ac.ir | ¹ Faculty of Mechanical Engineering, K. N. Toosi University of Technology, P. O. Box 19395-1999, Tehran, Iran.

of a bilayer graphene. To solve the governing equations Hosseini-Hashemi et al. [5] analyzed the complex plate equations with an analytical approach based on nonlocal TSDT. Sahmani et al. [6] employed a TSDT to examine the nonlinear large-amplitude vibration of FGM plates based on an analytical solving methodology. Phung-Van et al. [7] studied the nonlinear transient dynamic response of functionally graded nanoplates in thermal environments based on TSDT. Jena and Chakraverty [8–11] studied the free vibration of Euler–Bernoulli nanobeam using the classic nonlocal Euler–Bernoulli beam theory based on differential transform method to get the mode frequency. They also analyzed the buckling behavior of Euler–Bernoulli nanobeam using the Differential Quadrature Method and MATLAB programmer [12].

Double-layered structures have outstanding applications in various systems like aerospace and devices such as Electrochemical equipment and medical diagnostics [13–20]. Therefore, in recent years, several investigations have been carried out to study the mechanical properties of double-layered structures. Circular double-layer graphene sheets were analyzed by Allahyari and Fadaee [21] based on an analytical investigation and surface effect theory. Hashemi et al. [22] studied free vibration of a two layer plate with orthotropic material. Based on the Eringen's theory Wang et al. [23] developed the free vibration of a nanoplate with two layer under thermal loading. A double-orthotropic microplate system was investigated by Jamalpoor and Hosseini [24] to obtain the complex motion equations for different cases. Free vibration of the graded double-refined-nanoplate systems in Hygro-thermal environments was presented by Barati and Shahverdi [25] based on strain gradient theory.

This study investigates the nonlinear vibrational behavior of a nano double-layered plate (DLP) incorporating third-order shear deformation theory and von Karman type nonlinear kinematics. To the authors' best knowledge, rarely analytical approaches have been reported in the literature for the study of large amplitude nonlinear vibration of a DLP based on nonlocal strain gradient theory and TSDT. Therefore, the multiple scales method is utilized to observe the effects of different parameters on nonlinear behavior of DLP. By utilizing the Galerkin procedure and the Hamilton's principle the time dependent nonlinear ordinary differential equation of Duffing's type is derived. To show the accuracy of the presented results, a comparison with a related literature is done. Also, the van der Waals interaction between the upper and lower layers is modeled with the Lennard-Jones pair potential equation. A uniform temperature rise throughout the thickness direction is considered as a thermal loading.

2 Geometrical configuration

In Fig. 1 a nanostructure with two layer and dimensions a , b and uniform thickness h of each DLP resting on elastic medium with two different parameters K_W , K_G is plotted. Also, the mass density ρ , elasticity modulus E and Poisson's ratio ν are considered.

3 Formulation

3.1 The theory of nonlocal strain gradient

The theory Eringen's nonlocal strain gradient theory could be considered as a mixture of simple strain gradient theory and the size dependent theory of Eringen [26]. Therefore, in the first step the stress may be presented as follow

$$\sigma_{ij} = \sigma_{ij}^{(0)} - \nabla \cdot \sigma_{ij}^{(1)} \quad (1)$$

in which σ_{ij} is the total stress tensor, $\sigma_{ij}^{(0)}$ is the stress tensor and $\sigma_{ij}^{(1)}$ is a higher-order stress tensor. It also should be noted that the gradient symbol is shown by ∇ . In the theory of nonlocal strain gradient stress tensor and higher-order stress tensor are proposed as

$$\sigma_{ij}^{(0)} = \int_V \phi(x, x', \chi_0) C_{ijkl} e'_{kl}(x') dV \quad (2a)$$

$$\sigma_{ij}^{(1)} = \chi^2 \int_V \xi(x, x', \chi_1) C_{ijkl} e'_{kl,x}(x') dV \quad (2b)$$

In the above equations, C_{ijkl} , e'_{kl} , $\phi(x, x', \chi_0)$, $\xi(x, x', \chi_1)$ are called respectively, the fourth-order elastic modulus tensor, second order strain tensor, and the attenuation kernel functions. Moreover, Eringen nonlocal parameters are shown by χ_0 and χ_1 which χ in the mentioned equation is internal length parameter. Also, in the domain V , two various point are shown by x , x' . Equations (2a) and (2b) are the basic of nonlocal strain gradient theory, so these equations could be developed as a practical differential

Fig. 1 A rectangular isotropic DLP

form to easily used in different situations and problems. Therefore, the differential form of Eringen strain gradient theory considering the thermal loading is derived as [26]

$$(1 - \mu \nabla^2) \sigma_{ij} = C_{ijkl} (1 - \chi \nabla^2) [e_{kl} - \mathfrak{F} \Delta T] \quad (3)$$

that $\mathfrak{F}$, ΔT , μ states the coefficient of thermal expansion, changes of the temperature and Eringen nonlocal size-dependent parameter, respectively. The Laplacian operator is proposed by $\nabla^2 = (\partial/\partial x^2 + \partial/\partial y^2)$.

3.2 Reddy's third-order theory

The displacement field of the TSDT can be expressed as

$$\begin{aligned} u(x, y, z, t) &= u_0(x, y, t) + z\psi_x(x, y, t) + z^2\eta_x(x, y, t) + z^3\zeta_x(x, y, t) \\ v(x, y, z, t) &= v_0(x, y, t) + z\psi_y(x, y, t) + z^2\eta_y(x, y, t) + z^3\zeta_y(x, y, t) \\ w(x, y, z, t) &= w_0(x, y, t) \end{aligned} \quad (4)$$

where (u_0, v_0, w_0) are called the midplane displacement, (u, v, w) are the displacement along x , y and z directions and (ψ_x, ψ_y) are the rotations of a transverse normal about the y and x axes, respectively [27]. The transverse shear

$$\begin{aligned} \gamma_{xz} &= \psi_x(x, y, t) + 2z\eta_x(x, y, t) + 3z^2\zeta_x(x, y, t) + \frac{\partial w}{\partial x} \\ \gamma_{yz} &= \psi_y(x, y, t) + 2z\eta_y(x, y, t) + 3z^2\zeta_y(x, y, t) + \frac{\partial w}{\partial y} \end{aligned} \quad (5)$$

Substituting the $\gamma_{xz}(x, y, \pm h/2)$ and $\gamma_{yz}(x, y, \pm h/2)$ to zero, it can be concluded that

$$\begin{aligned} \zeta_x &= -\frac{4z^3}{3h^2} \left(\psi_x + \frac{\partial w}{\partial x} \right), \quad \eta_x = 0 \\ \zeta_y &= -\frac{4z^3}{3h^2} \left(\psi_y + \frac{\partial w}{\partial y} \right), \quad \eta_y = 0 \end{aligned} \quad (6)$$

Finally, the displacement field in Eq. (4) becomes

$$\begin{aligned} u(x, y, z, t) &= u_0(x, y, t) + z \left[\psi_x(x, y, t) - \frac{4}{3} \left(\frac{z}{h} \right)^2 \left(\psi_x(x, y, t) + \frac{\partial w}{\partial x} \right) \right] \\ v(x, y, z, t) &= v_0(x, y, t) + z \left[\psi_y(x, y, t) - \frac{4}{3} \left(\frac{z}{h} \right)^2 \left(\psi_y(x, y, t) + \frac{\partial w}{\partial y} \right) \right] \\ w(x, y, z, t) &= w_0(x, y, t) \end{aligned} \quad (7)$$

The nonzero strains of the general TSDT with the von Kármán nonlinearity are

$$\begin{aligned} e_{xx} &= \frac{\partial u_0}{\partial x} + z \frac{\partial \psi_x}{\partial x} - \frac{4z^3}{3h^2} \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \\ e_{yy} &= \frac{\partial v_0}{\partial y} + z \frac{\partial \psi_y}{\partial y} - \frac{4z^3}{3h^2} \left(\frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \\ e_{xy} &= \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) + z \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) - \frac{4z^3}{3h^2} \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \\ e_{xz} &= \frac{\partial w}{\partial x} + \psi_x - \frac{4z^2}{h^2} \left(\psi_x + \frac{\partial w}{\partial x} \right), \quad e_{yz} = \frac{\partial w}{\partial y} + \psi_y - \frac{4z^2}{h^2} \left(\psi_y + \frac{\partial w}{\partial y} \right), \quad e_{zz} = 0 \end{aligned} \quad (8)$$

Using the generalized Hook's law, the inplane stresses (S_{xx}, S_{yy}, S_{xy}) and transverse shear stresses (S_{xz}, S_{yz}) can be obtained using the theory of nonlocal strain gradient as

$$(1 - \mu \nabla^2) \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix} = (1 - \chi \nabla^2) \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{11} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{yz} \\ e_{xz} \\ e_{xy} \end{Bmatrix} - \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \mathfrak{F} \Delta T \quad (9)$$

where

$$Q_{11} = \frac{E}{1 - \nu^2}, \quad Q_{12} = \frac{\nu E}{1 - \nu^2}, \quad Q_{44} = Q_{55} = Q_{66} = \frac{E}{2(1 + \nu)} \quad (10)$$

stresses eliminate on the top and the bottom of the plate's surfaces $(+h/2, -h/2)$. Therefore, both strains on top and bottom of the plate are considered zero.

In the following equations, (N_{xx}, N_{yy}, N_{xy}) , (M_{xx}, M_{yy}, M_{xy}) , (P_{xx}, P_{yy}, P_{xy}) , (R_{xz}, R_{yz}) are defined the in-plane force resultants, moment resultants, higher order moment resultants, higher order force resultants and out of plane force resultants, respectively.

where the flexural rigidity of the nanoplate is developed as $D = Eh^3/12(1 - \nu^2)$.

The governing equations of the TSDT can be derived using Hamilton's principle as follow

$$\begin{aligned}
 (1 - \mu \nabla^2) N_{xx} &= (1 - \chi \nabla^2) \left(\left[\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \left[\frac{Eh}{1 - \nu^2} \right] \right. \\
 &\quad \left. + \left[\frac{\partial v_0}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] \left[\frac{\nu Eh}{1 - \nu^2} \right] - \frac{E\mathfrak{F}}{1 - \nu} \int_{-h/2}^{h/2} \Delta T dz \right) \\
 (1 - \mu \nabla^2) N_{yy} &= (1 - \chi \nabla^2) \left(\left[\frac{\partial v_0}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] \left[\frac{Eh}{1 - \nu^2} \right] \right. \\
 &\quad \left. + \left[\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \left[\frac{\nu Eh}{1 - \nu^2} \right] - \frac{E\mathfrak{F}}{1 - \nu} \int_{-h/2}^{h/2} \Delta T dz \right) \\
 (1 - \mu \nabla^2) N_{xy} &= (1 - \chi \nabla^2) \left(\left[\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \left[\frac{Eh}{2(1 + \nu)} \right] - \frac{E\mathfrak{F}}{1 - \nu} \int_{-h/2}^{h/2} \Delta T dz \right) \\
 (1 - \mu \nabla^2) M_{xx} &= (1 - \chi \nabla^2) \left([D] \left(\nu \frac{\partial \psi_y}{\partial y} + \frac{\partial \psi_x}{\partial x} \right) - \frac{E\mathfrak{F}}{1 - \nu} \int_{-h/2}^{h/2} \Delta T z dz \right) \\
 (1 - \mu \nabla^2) M_{yy} &= (1 - \chi \nabla^2) \left(D \left(\frac{\partial \psi_y}{\partial y} + \nu \frac{\partial \psi_x}{\partial x} \right) - \frac{E\mathfrak{F}}{1 - \nu} \int_{-h/2}^{h/2} \Delta T z dz \right) \\
 (1 - \mu \nabla^2) M_{xy} &= (1 - \chi \nabla^2) \frac{D}{2} (1 - \nu) \left(\frac{\partial \psi_x}{\partial y} + \nu \frac{\partial \psi_y}{\partial x} \right) \\
 (1 - \mu \nabla^2) P_{xx} &= (1 - \chi \nabla^2) \left(\frac{Dh^2}{140} \left(16\nu \frac{\partial \psi_y}{\partial x} - 5\nu \frac{\partial^2 w}{\partial y^2} + 16 \frac{\partial \psi_x}{\partial x} - 5 \frac{\partial^2 w}{\partial y^2} \right) - \frac{E\mathfrak{F}}{1 - \nu} \int_{-h/2}^{h/2} \Delta T z^3 dz \right) \\
 (1 - \mu \nabla^2) P_{xy} &= (1 - \chi \nabla^2) \left(\frac{Dh^2}{140} \left(16\nu \frac{\partial \psi_y}{\partial y} - 5\nu \frac{\partial^2 w}{\partial y^2} + 16 \frac{\partial \psi_x}{\partial x} - 5 \frac{\partial^2 w}{\partial x^2} \right) - \frac{E\mathfrak{F}}{1 - \nu} \int_{-h/2}^{h/2} \Delta T z^3 dz \right) \\
 (1 - \mu \nabla^2) P_{yx} &= (1 - \chi \nabla^2) \frac{h^2}{140} D (1 - \nu) \left(8 \frac{\partial \psi_x}{\partial y} + 8 \frac{\partial \psi_y}{\partial x} - 5 \frac{\partial^2 w}{\partial x \partial y} \right) \\
 (1 - \mu \nabla^2) R_{xz} &= (1 - \chi \nabla^2) \frac{1}{5} D (1 - \nu) \left(\psi_x + \frac{\partial w}{\partial x} \right) \\
 (1 - \mu \nabla^2) R_{yz} &= (1 - \chi \nabla^2) \frac{1}{5} D (1 - \nu) \left(\psi_y + \frac{\partial w}{\partial y} \right) \\
 (1 - \mu \nabla^2) T_{xz} &= (1 - \chi \nabla^2) \frac{Eh}{3(1 + \nu)} \left(\psi_x + \frac{\partial w}{\partial x} \right) \\
 (1 - \mu \nabla^2) T_{yz} &= (1 - \chi \nabla^2) \frac{Eh}{3(1 + \nu)} \left(\psi_y + \frac{\partial w}{\partial y} \right)
 \end{aligned} \tag{11}$$

$$\int_0^T (\delta U + \delta V - \delta K) dt = 0 \quad (12)$$

The U , V , K may be called the strain energy, the work done by external applied forces and the kinetic energy, respectively. Employing the variational operator can be resulted as

The mass moments of inertia are defined as [27]

$$I_i = \int_A \rho z^i dA, \quad i = 0, 1, 2, \dots, 6 \quad (15)$$

The elastic foundation and thermal changes can be considered as external loads which mean

$$\delta V = \int_A \left[(K_w w - K_G \nabla^2 w) \delta w + \frac{E \mathfrak{F}}{1 - \nu} \int_{-h/2}^{h/2} \Delta T dz \left(\frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} \right) \right] dA \quad (16)$$

Inserting Eqs. (13), (14), (16) into Eq. (12) and using integration by parts and assuming that all variations and their

$$\begin{aligned} \delta U = & \int_A \frac{1}{2} \left[N_{xx} \left[\frac{\partial \delta u_0}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \right] + M_{xx} \left[\frac{\partial \delta \psi_x}{\partial x} - \frac{\partial^2 \delta w}{\partial x^2} \right] \right. \\ & + N_{yy} \left[\frac{\partial \delta v_0}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} \right] + M_{yy} \left[\frac{\partial \delta \psi_y}{\partial y} - \frac{\partial^2 \delta w}{\partial y^2} \right] \\ & + N_{xy} \left[\frac{\partial \delta v_0}{\partial x} + \frac{\partial \delta u_0}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial y} \right] + M_{xy} \left[\frac{\partial \delta \psi_x}{\partial y} + \frac{\partial \delta \psi_y}{\partial x} \right] \\ & + \left(T_{yz} - \frac{4}{h^2} R_{yz} \right) \left[\delta \psi_y + \frac{\partial \delta w}{\partial y} \right] + \left(T_{xz} - \frac{4}{h^2} R_{xz} \right) \left[\delta \psi_x + \frac{\partial \delta w}{\partial x} \right] \\ & - \frac{4}{3h^2} P_{xx} \left[\frac{\partial \delta \psi_x}{\partial x} + \frac{\partial^2 \delta w}{\partial x^2} \right] - \frac{4}{3h^2} P_{yy} \left[\frac{\partial \delta \psi_y}{\partial y} + \frac{\partial^2 \delta w}{\partial y^2} \right] \\ & \left. - \frac{4}{3h^2} P_{xy} \left[\frac{\partial \delta \psi_x}{\partial y} + \frac{\partial \delta \psi_y}{\partial x} + 2 \frac{\partial^2 \delta w}{\partial x \partial y} \right] \right] \end{aligned} \quad (13)$$

Now the variation of kinetic energy for TSDT is concluded as

$$\begin{aligned} \delta K = & \frac{1}{2} \rho \int_V \delta [\dot{u}^2 + \dot{v}^2 + \dot{w}^2] dV \\ = & \rho \int_V [\dot{u} \delta \dot{u} + \dot{v} \delta \dot{v} + \dot{w} \delta \dot{w}] dV \\ = & \int_A \left[I_0 (\dot{u}_0 \delta \dot{u}_0 + \dot{v}_0 \delta \dot{v}_0 + \dot{w} \delta \dot{w}) + I_1 (\dot{u}_0 \delta \dot{\psi}_x + \dot{v}_0 \delta \dot{\psi}_y + \dot{\psi}_x \delta \dot{u}_0 + \dot{\psi}_y \delta \dot{v}_0) \right. \\ & + I_2 (\dot{\psi}_x \delta \dot{\psi}_x + \dot{\psi}_y \delta \dot{\psi}_y) - I_3 \frac{4}{3h^2} \left(\dot{u}_0 \delta \dot{\psi}_x + \dot{v}_0 \delta \dot{\psi}_y + \dot{u}_0 \frac{\partial \delta \dot{w}}{\partial x} + \dot{v}_0 \frac{\partial \delta \dot{w}}{\partial y} \right. \\ & + \dot{\psi}_x \delta \dot{u}_0 + \dot{\psi}_y \delta \dot{v}_0 + \delta \dot{u}_0 \frac{\partial \dot{w}}{\partial x} + \delta \dot{v}_0 \frac{\partial \dot{w}}{\partial y} \left. \right) - I_4 \frac{4}{3h^2} \left(\dot{\psi}_x \frac{\partial \delta \dot{w}}{\partial x} + \dot{\psi}_y \frac{\partial \delta \dot{w}}{\partial y} \right. \\ & + \delta \dot{\psi}_x \frac{\partial \dot{w}}{\partial x} + \delta \dot{\psi}_y \frac{\partial \dot{w}}{\partial y} + 2 \dot{\psi}_x \delta \dot{\psi}_x + 2 \dot{\psi}_y \delta \dot{\psi}_y \left. \right) + I_6 \frac{16}{9h^4} (\dot{\psi}_x \delta \dot{\psi}_x + \dot{\psi}_y \delta \dot{\psi}_y \\ & + \dot{\psi}_x \frac{\partial \delta \dot{w}}{\partial x} + \dot{\psi}_y \frac{\partial \delta \dot{w}}{\partial y} + \delta \dot{\psi}_x \frac{\partial \dot{w}}{\partial x} + \delta \dot{\psi}_y \frac{\partial \dot{w}}{\partial y} \frac{\partial \dot{w}}{\partial x} + \frac{\partial \dot{w}}{\partial y} \frac{\partial \delta \dot{w}}{\partial y} \left. \right) \left. \right] dA \end{aligned} \quad (14)$$

derivatives are zero at $t = 0$, $t = T$, the Euler–Lagrange equations can be presented as [28]

$$\delta u_0 : \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = l_0 \frac{\partial^2 u_0}{\partial t^2} + \hat{l}_1 \frac{\partial^2 \psi_x}{\partial t^2} - l_3 \frac{4}{3h^2} \frac{\partial^3 w}{\partial t^2 \partial x} \quad (17a)$$

$$\delta v_0 : \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = l_0 \frac{\partial^2 v_0}{\partial t^2} + \hat{l}_1 \frac{\partial^2 \psi_y}{\partial t^2} - l_3 \frac{4}{3h^2} \frac{\partial^3 w}{\partial t^2 \partial y} \quad (17b)$$

where

$$\hat{l}_1 = l_1 - \frac{4}{3h^2} l_3, \quad \hat{l}_2 = l_2 - \frac{8}{3h^2} l_4 + \frac{16}{9h^4} l_6, \quad \hat{l}_4 = l_4 - \frac{4}{3h^2} l_6 \quad (18)$$

The boundary conditions may be expressed as

$$\delta u_0 : N_{xx} n_x + N_{xy} n_y = 0 \quad (19a)$$

$$\delta v_0 : N_{xy} n_x + N_{yy} n_y = 0 \quad (19b)$$

$$\begin{aligned} \delta w : n_x \left[T_{xx} + \left(K_G + \frac{E\mathfrak{S}}{1-\nu} \int_{-h/2}^{h/2} \Delta T dz \right) \frac{\partial w}{\partial x} - \frac{4}{h^2} R_{xz} + \frac{4}{3h^2} \left(\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} \right) \right. \\ \left. - l_3 \frac{4}{3h^2} \frac{\partial^2 u_0}{\partial t^2} - \hat{l}_4 \frac{4}{3h^2} \frac{\partial^2 \psi_x}{\partial t^2} + l_6 \frac{19}{9h^4} \frac{\partial^3 w}{\partial t^2 \partial x} \right] + n_y \left[T_{yy} + \left(K_G + \frac{E\mathfrak{S}}{1-\nu} \int_{-h/2}^{h/2} \Delta T dz \right) \frac{\partial w}{\partial y} \right. \\ \left. - \frac{4}{h^2} R_{yz} + \frac{4}{3h^2} \left(\frac{\partial P_{yy}}{\partial y} + \frac{\partial P_{xy}}{\partial x} \right) - l_3 \frac{4}{3h^2} \frac{\partial^2 v_0}{\partial t^2} - \hat{l}_4 \frac{4}{3h^2} \frac{\partial^2 \psi_y}{\partial t^2} + l_6 \frac{19}{9h^4} \frac{\partial^3 w}{\partial t^2 \partial y} \right] = 0 \end{aligned} \quad (19c)$$

$$\delta w : \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} - (K_w w - K_G \nabla^2 w) + \frac{E\mathfrak{S}}{1-\nu} \int_{-h/2}^{h/2} \Delta T dz \nabla^2 w$$

$$+ \frac{4}{3h^2} \left(\frac{\partial^2 P_{xx}}{\partial x^2} + 2 \frac{\partial^2 P_{xy}}{\partial x \partial y} + \frac{\partial^2 P_{yy}}{\partial y^2} \right) - \frac{4}{h^2} \left(\frac{\partial R_{xz}}{\partial x} + \frac{\partial R_{yz}}{\partial y} \right) \quad (19d)$$

$$\begin{aligned} &= l_0 \frac{\partial^2 w}{\partial t^2} + l_3 \frac{4}{3h^2} \left(\frac{\partial^3 u_0}{\partial t^2 \partial x} + \frac{\partial^3 v_0}{\partial t^2 \partial y} \right) \\ &- l_6 \frac{16}{9h^4} \left(\frac{\partial^4 w}{\partial t^2 \partial x^2} + \frac{\partial^4 w}{\partial t^2 \partial y^2} \right) + \hat{l}_4 \frac{4}{3h^2} \left(\frac{\partial^3 \psi_x}{\partial t^2 \partial x} + \frac{\partial^3 \psi_y}{\partial t^2 \partial y} \right) \end{aligned} \quad (19e)$$

The by van der Waals forces are considered between two layers and may be computed by $F = \kappa(\pm w^1 \mp w^2)$ that κ is the is the van der Waals interaction coefficient. Using the Lennard-Jones pair potential [29] the van der Waals interaction coefficient can be obtained as

$$\kappa = - \left(\frac{4\sqrt{3}}{9a} \right)^2 \frac{24\zeta}{\delta^2} \left(\frac{\delta}{a} \right)^8 \left[\frac{3003\pi}{256} \sum_{k=0}^5 \frac{(-1)^k}{2k+1} \binom{5}{k} \left(\frac{\delta}{a} \right)^6 \frac{1}{(\bar{z}_1 - \bar{z}_2)^{12}} - \frac{35\pi}{8} \sum_{k=0}^2 \frac{(-1)^k}{2k+1} \frac{1}{(\bar{z}_1 - \bar{z}_2)^6} \right] \quad (20)$$

$$\begin{aligned} \delta \psi_x : \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - T_{xz} - \frac{4}{3h^2} \left(\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} \right) + \frac{4}{h^2} R_{xz} \\ = \hat{l}_1 \frac{\partial^2 u_0}{\partial t^2} + \hat{l}_2 \frac{\partial^2 \psi_x}{\partial t^2} - \hat{l}_4 \frac{4}{3h^2} \frac{\partial^3 w}{\partial t^2 \partial x} \end{aligned} \quad (17d)$$

$$\begin{aligned} \delta \psi_y : \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - T_{yz} - \frac{4}{3h^2} \left(\frac{\partial P_{yy}}{\partial y} + \frac{\partial P_{xy}}{\partial x} \right) + \frac{4}{h^2} R_{yz} \\ = \hat{l}_1 \frac{\partial^2 v_0}{\partial t^2} + \hat{l}_2 \frac{\partial^2 \psi_y}{\partial t^2} - \hat{l}_4 \frac{4}{3h^2} \frac{\partial^3 w}{\partial t^2 \partial y} \end{aligned} \quad (17e)$$

where a is the characteristic internal length of a C–C bond. $\zeta = 2.968$ meV and $\delta = 3.407$ Å are parameters chosen to fit the physical properties of graphene sheets. $\bar{z}_j = z_j/a$ ($j = 1, 2$), where z_j is the coordinate of the j th layer in the direction of thickness with the origin at the midplane of the graphene sheets. Here the subscripts 1 and 2 denote the quantities associated with the upper

and lower layers of the DLGS, respectively. In the case of in-phase mode (IPM), two nanoplates vibrate synchronously so that there is no relative motion between them

($w^1 - w^2 = 0$). In the case of anti-phase modes (APM), two nanoplates vibrate asynchronously ($w^1 - w^2 \neq 0$).

The generalized differential equation for the DLP will be derived as

Nanoplate1:

$$\begin{aligned} (1 - \chi \nabla^2) & \left[\frac{4Eh}{15(1 + \nu)} \left(\frac{\partial \psi_y^1}{\partial y} + \frac{\partial \psi_x^1}{\partial x} + \frac{\partial^2 w^1}{\partial y^2} + \frac{\partial^2 w^1}{\partial x^2} \right) + \frac{4Eh^3}{315(1 - \nu^2)} \right. \\ & \left. \left(\frac{\partial^3 \psi_y^1}{\partial y^3} + \frac{\partial^3 \psi_x^1}{\partial x^3} + \frac{\partial^3 \psi_x^1}{\partial x \partial y^2} + \frac{\partial^3 \psi_x^1}{\partial y \partial x^2} \right) - \frac{Eh^3}{252(1 - \nu^2)} \left(\frac{\partial^4 w^1}{\partial y^4} + \frac{\partial^4 w^1}{\partial x^4} + 2 \frac{\partial^4 w^1}{\partial x^2 \partial y^2} \right) \right] \\ & = (1 - \mu \nabla^2) \left[I_0 \frac{\partial^2 w^1}{\partial t^2} + (K_w w^1 - K_G \nabla^2 w^1) - \left(\frac{E \mathfrak{F}}{1 - \nu} \int_{-h/2}^{h/2} \Delta T dz \nabla^2 w^1 \right) + \kappa (w^1 - w^2) \right. \\ & \left. - I_6 \frac{16}{9h^4} \left(\frac{\partial^4 w^1}{\partial t^2 \partial x^2} + \frac{\partial^4 w^1}{\partial t^2 \partial y^2} \right) + \hat{I}_4 \frac{4}{3h^2} \left(\frac{\partial^3 \psi_x^1}{\partial t^2 \partial x} + \frac{\partial^3 \psi_y^1}{\partial t^2 \partial y} \right) \right] \end{aligned} \quad (21a)$$

$$\begin{aligned} (1 - \chi \nabla^2) & \left[-\frac{4Eh}{15(1 + \nu)} \left(\psi_x^1 + \frac{\partial w^1}{\partial x} \right) + \frac{17Eh^3}{630(1 + \nu)} \left(\frac{\partial^2 \psi_x^1}{\partial y^2} \right) + \frac{17Eh^3}{630(1 - \nu^2)} \left(\frac{\partial^2 \psi_y^1}{\partial y \partial x} \right) \right. \\ & \left. + \frac{17Eh^3}{315(1 - \nu^2)} \left(\frac{\partial^2 \psi_x^1}{\partial x^2} \right) - \frac{4Eh^3}{315(1 - \nu^2)} \left(\frac{\partial^3 w^1}{\partial x^3} + \frac{\partial^3 w^1}{\partial x \partial y^2} \right) \right] = (1 - \mu \nabla^2) \left(\hat{I}_2 \frac{\partial^2 \psi_x^1}{\partial t^2} - \hat{I}_4 \frac{4}{3h^2} \frac{\partial^3 w^1}{\partial t^2 \partial x} \right) \end{aligned} \quad (21b)$$

$$\begin{aligned} (1 - \chi \nabla^2) & \left[-\frac{4Eh}{15(1 + \nu)} \left(\psi_y^1 + \frac{\partial w^1}{\partial y} \right) + \frac{17Eh^3}{630(1 + \nu)} \left(\frac{\partial^2 \psi_y^1}{\partial x^2} \right) - \frac{4Eh^3}{315(1 - \nu^2)} \left(\frac{\partial^3 w^1}{\partial y \partial x^2} \right) \right. \\ & \left. + \frac{17Eh^3}{315(1 - \nu^2)} \left(\frac{\partial^2 \psi_y^1}{\partial y^2} + \frac{1}{2} \frac{\partial^2 \psi_x^1}{\partial y \partial x} \right) - \frac{4Eh^3}{315(1 - \nu^2)} \left(\frac{\partial^3 w^1}{\partial y^3} \right) \right] = (1 - \mu \nabla^2) \left(\hat{I}_2 \frac{\partial^2 \psi_y^1}{\partial t^2} - \hat{I}_4 \frac{4}{3h^2} \frac{\partial^3 w^1}{\partial t^2 \partial y} \right) \end{aligned} \quad (21c)$$

Nanoplate2:

$$\begin{aligned} (1 - \chi \nabla^2) & \left[\frac{4Eh}{15(1 + \nu)} \left(\frac{\partial \psi_y^2}{\partial y} + \frac{\partial \psi_x^2}{\partial x} + \frac{\partial^2 w^2}{\partial y^2} + \frac{\partial^2 w^2}{\partial x^2} \right) + \frac{4Eh^3}{315(1 - \nu^2)} \right. \\ & \left. \left(\frac{\partial^3 \psi_y^2}{\partial y^3} + \frac{\partial^3 \psi_x^2}{\partial x^3} + \frac{\partial^3 \psi_x^2}{\partial x \partial y^2} + \frac{\partial^3 \psi_x^2}{\partial y \partial x^2} \right) - \frac{Eh^3}{252(1 - \nu^2)} \left(\frac{\partial^4 w^2}{\partial y^4} + \frac{\partial^4 w^2}{\partial x^4} + 2 \frac{\partial^4 w^2}{\partial x^2 \partial y^2} \right) \right] \\ & = (1 - \mu \nabla^2) \left[I_0 \frac{\partial^2 w^2}{\partial t^2} + (K_w w^2 - K_G \nabla^2 w^2) - \left(\frac{E \mathfrak{F}}{1 - \nu} \int_{-h/2}^{h/2} \Delta T dz \nabla^2 w^2 \right) + \kappa (w^2 - w^1) \right. \\ & \left. - I_6 \frac{16}{9h^4} \left(\frac{\partial^4 w^2}{\partial t^2 \partial x^2} + \frac{\partial^4 w^2}{\partial t^2 \partial y^2} \right) + \hat{I}_4 \frac{4}{3h^2} \left(\frac{\partial^3 \psi_x^2}{\partial t^2 \partial x} + \frac{\partial^3 \psi_y^2}{\partial t^2 \partial y} \right) \right] \end{aligned} \quad (22a)$$

$$(1 - \chi \nabla^2) \left[-\frac{4Eh}{15(1+\nu)} \left(\psi_x^2 + \frac{\partial w^2}{\partial x} \right) + \frac{17Eh^3}{630(1+\nu)} \left(\frac{\partial^2 \psi_x^2}{\partial y^2} \right) + \frac{17Eh^3}{630(1-\nu^2)} \left(\frac{\partial^2 \psi_y^2}{\partial y \partial x} \right) \right. \\ \left. + \frac{17Eh^3}{315(1-\nu^2)} \left(\frac{\partial^2 \psi_x^2}{\partial x^2} \right) - \frac{4Eh^3}{315(1-\nu^2)} \left(\frac{\partial^3 w^2}{\partial x^3} + \frac{\partial^3 w^2}{\partial x \partial y^2} \right) \right] = (1 - \mu \nabla^2) \left(\hat{l}_2 \frac{\partial^2 \psi_x^2}{\partial t^2} - \hat{l}_4 \frac{4}{3h^2} \frac{\partial^3 w^2}{\partial t^2 \partial x} \right) \quad (22b)$$

$$(1 - \chi \nabla^2) \left[-\frac{4Eh}{15(1+\nu)} \left(\psi_y^2 + \frac{\partial w^2}{\partial y} \right) + \frac{17Eh^3}{630(1+\nu)} \left(\frac{\partial^2 \psi_y^2}{\partial x^2} \right) - \frac{4Eh^3}{315(1-\nu^2)} \left(\frac{\partial^3 w^2}{\partial y \partial x^2} \right) \right. \\ \left. + \frac{17Eh^3}{315(1-\nu^2)} \left(\frac{\partial^2 \psi_y^2}{\partial y^2} + \frac{1}{2} \frac{\partial^2 \psi_x^2}{\partial y \partial x} \right) - \frac{4Eh^3}{315(1-\nu^2)} \left(\frac{\partial^3 w^2}{\partial y^3} \right) \right] = (1 - \mu \nabla^2) \left(\hat{l}_2 \frac{\partial^2 \psi_y^2}{\partial t^2} - \hat{l}_4 \frac{4}{3h^2} \frac{\partial^3 w^2}{\partial t^2 \partial y} \right) \quad (22c)$$

where n_x, n_y are the direction cosines of the outward unit normal to the boundary of the mid-plane.

The in-plane inertia terms can be neglected because the in-plane natural frequencies have small value than the natural frequencies of the transverse motion.

Equations (17a) and (17b) can be satisfied exactly by introducing the stress function $F(x, y, t)$ defined by

$$\begin{aligned} N_{xx} &= \frac{\partial^2 F}{\partial y^2} \\ N_{yy} &= \frac{\partial^2 F}{\partial x^2} \\ N_{xy} &= -\frac{\partial^2 F}{\partial x \partial y} \end{aligned} \quad (23)$$

Eliminating u_0 and v_0 from Eq. (4) yields

Fig. 2 Comparison of the numerical and multiple scale method

$$\frac{\partial^2 e_{xx}}{\partial y^2} - 2 \frac{\partial^2 e_{xy}}{\partial x \partial y} + \frac{\partial^2 e_{yy}}{\partial x^2} = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \quad (24)$$

Therefore, the compatibility equation in the term of displacement can be obtain as follow

$$\nabla^4 F = Eh \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] - \nabla^2 \left(E \mathfrak{F} \int_{-h/2}^{h/2} \Delta T dz \right) \quad (25)$$

3.3 SSSS and CCCC boundary conditions

The boundary conditions of a simply supported and clamped nanoplate are presented as [30]

Table 1 The natural nonlinear frequencies (THz) of DLP for IPM and APM ($\chi = 2 \text{ nm}^2$, $K_W = 0.1 \text{ Gpa/nm}$, $K_G = 2 \text{ pa}$, $\alpha = -1.6 \times 10^{-6} \text{ K}^{-1}$, $\Delta T = 100 \text{ K}$, $\kappa = -108 \text{ Gpa/nm}$)

$\mu \text{ (nm}^2\text{)}$	$a \times b \text{ (nm)}$	IPM		APM	
		SSSS1	SSSS2	SSSS1	SSSS2
0	10 × 10	0.8762	0.9583	15.22443	15.25015
	15 × 15	0.5751	0.6004	15.21008	15.22031
	20 × 20	0.4705	0.4804	15.20649	15.21200
	25 × 25	0.4207	0.4252	15.20503	15.20849
	30 × 30	0.3930	0.3953	15.20429	15.20666
1	10 × 10	0.8557	0.9396	15.22326	15.24899
	15 × 15	0.5726	0.5980	15.20999	15.22022
	20 × 20	0.4699	0.4798	15.20647	15.21199
	25 × 25	0.4205	0.4251	15.20502	15.20848
	30 × 30	0.3929	0.3953	15.20429	15.20666
2	10 × 10	0.8407	0.9259	15.22242	15.24815
	15 × 15	0.5705	0.5960	15.20991	15.22014
	20 × 20	0.4694	0.4794	15.20646	15.21197
	25 × 25	0.4204	0.4250	15.20502	15.20848
	30 × 30	0.3929	0.3952	15.20428	15.20666

Table 2 The natural nonlinear frequencies (THz) of DLP for IPM and APM ($a=b=15$ nm, $K_W=0.1$ Gpa/nm, $K_G=2$ pa, $\alpha=-1.6\times 10^{-6}$ K $^{-1}$, $\Delta T=100$ k, $\kappa=-108$ Gpa/nm)

μ (nm 2)	χ (nm 2)	IPM		APM	
		SSSS1	SSSS2	SSSS1	SSSS2
0	1	0.5728	0.5982	15.21001	15.22022
	1.5	0.5740	0.5993	15.21004	15.22027
	2	0.5751	0.6004	15.21008	15.22031
	2.5	0.5763	0.6015	15.21013	15.22036
	3	0.5774	0.6027	15.21017	15.22040
1	1	0.5705	0.5960	15.20991	15.22014
	1.5	0.5715	0.5970	15.20995	15.22018
	2	0.5726	0.5980	15.20999	15.22022
	2.5	0.5737	0.5990	15.21003	15.22026
	3	0.5748	0.6001	15.21007	15.22030
2	1	0.5685	0.5940	15.20983	15.22006
	1.5	0.5695	0.5950	15.20987	15.22010
	2	0.5705	0.5960	15.20991	15.22014
	2.5	0.5715	0.5969	15.20995	15.22017
	3	0.5724	0.5979	15.20998	15.22021

Table 3 The natural nonlinear frequencies (THz) of DLP for IPM and APM ($a=b=15$ nm, $\mu=1$ nm 2 , $\chi=2$ nm 2 , $\alpha=-1.6\times 10^{-6}$ K $^{-1}$, $\Delta T=100$ k, $\kappa=-108$ Gpa/nm)

K_G (pa)	K_W (Gpa/nm)	IPM		APM	
		CCCC1	CCCC2	CCCC1	CCCC2
2	0.1	0.5869	0.6116	15.21052	15.21150
	0.2	0.6717	0.6935	15.52140	15.21501
	0.3	0.7471	0.7667	15.52175	15.21853
	0.4	0.8155	0.8335	15.22106	15.22204
	0.5	0.8786	0.8953	15.22458	15.22555
6	0.1	0.8482	0.8659	15.22285	15.22383
	0.2	0.9090	0.9253	15.22636	15.22734
	0.3	0.9661	0.9813	15.22988	15.23085
	0.4	1.0199	1.0344	15.23339	15.23436
	0.5	1.0710	1.0848	15.23690	15.23787
10	0.1	1.0463	1.0604	15.23517	15.23615
	0.2	1.09621	1.1097	15.23868	15.23966
	0.3	1.1439	1.1568	15.24219	15.24317
	0.4	1.1897	1.2022	15.24570	15.24668
	0.5	1.2339	1.2459	15.24921	15.25018

Fig. 3 The variations of normalized IPM frequency with nonlocal parameter for various amplitude of vibration and **a** SSSS1 condition, **b** SSSS2 condition ($\chi=2$ nm 2 , $a=b=1$ nm, $K_W=0.1$ Gpa/nm, $K_G=2$ pa, $\Delta T=50$ k)

(a) Simply supported with movable in-plane edge (SSSS1)

$$w = 0, \quad M_{xx} = \frac{\partial^2 F}{\partial x \partial y} = \int_0^b \frac{\partial^2 F}{\partial y^2} dy = 0 \quad \text{at } x = 0, x = a$$

$$w = 0, \quad M_{yy} = \frac{\partial^2 F}{\partial x \partial y} = \int_0^a \frac{\partial^2 F}{\partial x^2} dx = 0 \quad \text{at } y = 0, y = b$$

(26a)

Fig. 4 The variations of normalized APM frequency with nonlocal parameter for various amplitude of vibration and **a** SSSS1 condition, **b** SSSS2 condition ($\chi = 2 \text{ nm}^2$, $a = b = 1 \text{ nm}$, $K_W = 0.1 \text{ GPa/nm}$, $K_G = 2 \text{ pa}$, $\Delta T = 50 \text{ k}$, $\kappa = -108 \text{ GPa/nm}$)

Fig. 5 The variations of normalized IPM frequency with length scale parameter for various amplitude of vibration and **a** SSSS1 condition, **b** SSSS2 condition ($a = b = 5 \text{ nm}$, $K_W = 0.1 \text{ GPa/nm}$, $K_G = 2 \text{ pa}$, $\Delta T = 50 \text{ k}$)

(b) Simply supported with immovable in-plane edge (SSSS2)

$$w = 0, \quad M_{xx} = \frac{\partial^2 F}{\partial x \partial y} = u_0 = 0 \quad \text{at } x = 0, x = a$$

$$w = 0, \quad M_{yy} = \frac{\partial^2 F}{\partial x \partial y} = v_0 = 0 \quad \text{at } y = 0, x = b \quad (26b)$$

(c) Camped movable edge (CCCC1)

$$w = \frac{\partial w}{\partial x} = \frac{\partial^2 F}{\partial x \partial y} = \int_0^b \frac{\partial^2 F}{\partial y^2} dy = 0 \quad \text{at } x = 0, x = a$$

$$w = \frac{\partial w}{\partial x} = \frac{\partial^2 F}{\partial x \partial y} = \int_0^a \frac{\partial^2 F}{\partial x^2} dx = 0 \quad \text{at } y = 0, x = b \quad (26c)$$

(d) Clamped immovable edge (CCCC2)

$$\begin{aligned} w = \frac{\partial w}{\partial x} = \frac{\partial^2 F}{\partial x \partial y} = u_0 = 0 \quad \text{at } x = 0, x = a \\ w = \frac{\partial w}{\partial y} = \frac{\partial^2 F}{\partial x \partial y} = v_0 = 0 \quad \text{at } y = 0, x = b \end{aligned} \quad (26d)$$

The nonlinear free vibration response of the simply supported and clamped nanoplate can be obtained by introducing the following transverse deflection

$$\begin{aligned} w(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} h W_{mn}(t) \left(\sin \left(\frac{m\pi x}{a} \right) \sin \left(\frac{n\pi y}{b} \right) \right) \\ w(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} h W_{mn}(t) \left[\sin^2 \left(\frac{m\pi x}{a} \right) \sin^2 \left(\frac{n\pi y}{b} \right) \right] \end{aligned} \quad (27)$$

where m and n are the numbers of half cosine waves in the x and y directions, respectively. Now the general solution for stress function F can be obtained as

$$\begin{aligned} \text{SSSS} : F &= \frac{W_{mn}^2(t)}{32(m^2n^2)(a^2b^2)} \left[(a^4n^4) \cos \left(\frac{2m\pi x}{a} \right) + (b^4m^4) \cos \left(\frac{2n\pi y}{b} \right) \right] Eh^3 + \Gamma x^2 + \Upsilon y^2 \\ \text{CCCC} : F &= -\frac{(m^2n^2\pi^2)W_{mn}^2(t)}{4(a^2b^2)} \left[\frac{1}{128m^4n^4\pi^4} \left(a^4n^4 \cos \left(\frac{4m\pi x}{a} \right) + b^4m^4 \cos \left(\frac{4n\pi y}{b} \right) \right) \right. \\ &\quad + 16a^4n^4 \cos \left(\frac{2m\pi x}{a} \right) + 16b^4m^4 \cos \left(\frac{2n\pi y}{b} \right) \left. + \frac{1}{16\pi^4(4n^2a^2 + m^2b^2)^4} \right. \\ &\quad \left(a^4b^4 \cos \left(\frac{-2m\pi x}{a} + \frac{4n\pi y}{b} \right) + a^4b^4 \cos \left(\frac{2m\pi x}{a} + \frac{4n\pi y}{b} \right) + \right. \\ &\quad a^4b^4 \cos \left(\frac{-4m\pi x}{a} + \frac{2n\pi y}{b} \right) + a^4b^4 \cos \left(\frac{4m\pi x}{a} + \frac{2n\pi y}{b} \right) + \\ &\quad \left. \left. 2a^4b^4 \cos \left(\frac{-2m\pi x}{a} + \frac{2n\pi y}{b} \right) + 2a^4b^4 \cos \left(\frac{2m\pi x}{a} + \frac{2n\pi y}{b} \right) \right) \right] Eh^3 + \Gamma x^2 + \Upsilon y^2 \end{aligned} \quad (28)$$

$$W(t; \varepsilon) = \varepsilon W_1(T_0, T_1, T_2, \dots) + \varepsilon^2 W_2(T_0, T_1, T_2, \dots) + \varepsilon^3 W_3(T_0, T_1, T_2, \dots) + \dots \quad (31)$$

where for (SSSS1) $\Gamma = \Upsilon = 0$ and for (SSSS2) $\Gamma = \frac{W_{mn}^2(t)Eh^3\pi^2(a^2 + vb^2)}{16(a^2b^2)(1 - v^2)}$, $\Upsilon = \frac{W_{mn}^2(t)Eh^3\pi^2(va^2 + b^2)}{16(a^2b^2)(1 - v^2)}$. Also, for (CCCC1) $\Gamma = \Upsilon = 0$ and for (CCCC2) $\Gamma = \frac{3W_{mn}^2(t)Eh^3\pi^2(a^2 + vb^2)}{64(a^2b^2)(1 - v^2)}$, $\Upsilon = \frac{3W_{mn}^2(t)Eh^3\pi^2(va^2 + b^2)}{64(a^2b^2)(1 - v^2)}$.

Now, the orthogonality of the vibration mode shapes of nanoplate must be used [31]. For this purpose, the resulting equation multiplying by $\sin(m\pi x/a) \sin(n\pi y/b)$ and

then integrating the obtained relation over the nanoplate's area [32] an nonlinear second-order differential equation called Duffing's type is derived.

3.4 Multiple scale technique

One of the advantages of the multiple scale method, with respect to other methods, is that, by using this method, the effects of system parameters on the system responses can be recognized accurately. One begins by introducing new independent variables according to [33, 34]

$$T_n = \varepsilon^n t \quad \text{for } n = 0, 1, 2, \dots \quad (29)$$

The first and the second derivatives are written as

$$\begin{aligned} \frac{d}{dt} &= D_0 + \varepsilon D_1 \\ \frac{d^2}{dt^2} &= D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) \end{aligned} \quad (30)$$

Now by considering the above equations, the expanded function can be seen as follow

By equating the coefficients of ε , ε^2 , ε^3 to zero, one obtains the following

$$D_0^2 W_1 + \omega_0^2 W_1 = 0 \quad (32a)$$

$$D_0^2 W_2 + \omega_0^2 W_2 = -2D_0 D_1 W_1 \quad (32b)$$

$$D_0^2 W_3 + \omega_0^2 W_3 = -2D_0 D_1 W_2 - D_1^2 W_1 - 2D_0 D_2 W_1 - \Psi W_1^3 \quad (32c)$$

It should be noted that $\Omega = \omega_0^2$. A general solution for Eq. (32a) can be written as

$$W_1 = A(T_1, T_2) \exp(i\omega_0 T_0) + \bar{A} \exp(-i\omega_0 T_0) \quad (33)$$

in which $A(T_1)$ is an undetermined function of T_1 and will be determined later. $\bar{A}(T_1)$ is the complex conjugate of $A(T_1)$.

Substituting Eq. (33) into Eq. (29) will result in

$$D_0^2 W_2 + \omega_0^2 W_2 = -2i\omega_0 D_1 A \exp(i\omega_0 T_0) + cc \quad (34)$$

where prime denotes a derivative, with respect to T_1 and cc means complex conjugates. Secular terms, which are the terms that produce non-periodic solutions, must be eliminated from Eq. (34).

Substituting for W_1 and W_2 into Eq. (32c), it can be stated that

$$D_0^2 W_3 + \omega_0^2 W_3 = -[2i\omega_0 D_2 A + 3\Psi(A^2 \bar{A})] \exp(i\omega_0 T_0) - (\Psi A^3) \exp(3i\omega_0 T_0) + cc \quad (35)$$

To eliminate secular terms from W_3 , it should be noted that

$$2i\omega_0 D_2 A + 3\Psi(A^2 \bar{A}) = 0 \quad (36)$$

In order to solve the above equation, the A in polar form can be stated as

$$\begin{aligned} A &= \frac{1}{2} \alpha \exp(i\beta) \\ \bar{A} &= \frac{1}{2} \alpha \exp(-i\beta) \end{aligned} \quad (37)$$

where α is the amplitude and β is the phase. Substituting Eq. (37) into Eq. (36) and separating the real and imaginary components equal to zero, the following are obtained

$$\begin{aligned} \omega_0 \alpha' &= 0 \\ \omega_0 \alpha \beta' - \frac{3}{8} \Psi \alpha^3 &= 0 \end{aligned} \quad (38)$$

in which the prime suggests the $(\dots)' = \partial(\dots)/\partial T_2$. Hence that

$$\alpha = \text{constant}, \quad \beta = \frac{3}{8\omega_0} \Psi \alpha^2 T_2 + \beta_0 \quad (39)$$

where β_0 is a constant. Now it can be found that

$$A = \frac{1}{2} \alpha \exp \left[i \frac{3}{8\omega_0} \Psi \epsilon^2 \alpha^2 t + i \beta_0 \right] \quad (40)$$

Finally, it can be mentioned that

$$\begin{aligned} W(t) &= \epsilon \alpha \cos \left(\omega_0 t + \frac{3\alpha^2}{8\omega_0} \Psi \epsilon^2 t + \beta_0 \right) \\ &\quad + \frac{\Psi}{32\omega_0^2} \alpha^3 \epsilon^3 \cos \left[3\omega_0 t + \frac{9\alpha^2}{8\omega_0} \Psi \epsilon^2 t + 3\beta_0 \right] + \dots \\ \omega^{nl} &= \omega_0 + \frac{3}{8\omega_0} \Psi \epsilon^2 \alpha^2 \end{aligned} \quad (41)$$

Employing the initial conditions $W(0) = -x_0$, $\dot{W}(0) = 0$, $\alpha = 1/\epsilon$, it follows that $\alpha^2 = x_0^2$ because $\beta_0 = 0$ for $\dot{W}(0) = 0$. Hence, it can be mentioned that

$$\begin{aligned} W(t) &= \cos \left[\omega_0 t + \frac{3}{8\omega_0} \Psi t \right] + \frac{\Psi}{32\omega_0^2} \cos \left[3\omega_0 t + \frac{9}{8\omega_0} \Psi t \right] + \dots \\ \omega^{nl} &= \omega_0 \left[1 + \frac{3}{8\omega_0^2} \Psi \right] \end{aligned} \quad (42)$$

4 Results and discussions

One can see from Fig. 2 that the numerical results are in good agreement with the multiple scale method. However, the multiple scale method solution has the advantage that it provides a closed-form solution with a good physical insight, whereas the numerical methods do not provide a closed form solution and lack this type of physical insight.

In the Table 1 based on SSSS1, SSSS2 boundary conditions, for APM and IPM different nonlinear frequencies are presented. Eringen parameter with the plate's length are stated as 0, 1, 2, 3 nm², 10–30 nm, respectively. As can be seen increasing the small scale parameter led to decrease the nonlinear frequencies for both APM and IPM.

The other important parameter to study in this paper is the length scale parameter which has been picked in the range of 1–3 nm². Also, the boundary condition is considered SSSS. It can be observed that increasing the length scale parameter doesn't have a significant effect on both nonlinear frequencies of APM due to the van der Waals forces considered between two layers (Table 2).

In Table 3, nonlinear frequencies for CCCC1, CCCC2 boundary conditions and various elastic foundation are shown. As it can be seen both spring and shear foundations have inevitable effect on natural frequency and the nonlinear frequency increases with increasing the elastic parameters.

Figures 3 and 4 depict the variations of normalized IPM and APM frequency with deflection of nanoplate for various nonlocal parameter and boundary conditions. It can be seen that for both conditions, normalized frequency increases with increasing the amplitude of vibration from 0 to 2. It is remarkable that IPM and APM frequency rise by an increase in the deflection magnitude, regardless of boundary conditions. This means that the normalized

natural frequency is strongly affected by the deflection magnitude. So, it is clearly observed that the small scale effect plays a significant role in the vibration behavior of nanoplate.

The variation of the normalized natural frequency of DLP with the length scale changes for IPM and SSSS1 and SSSS2 boundary conditions is plotted in Fig. 5. The length scale parameter is taken in the range of 0–5 nm², while the amplitude of vibration is considered between 1 and 1.75. It has been found that the length scale effects on DLP aren't negligible because of the van der Waals interaction between the layers. Moreover, it is demonstrated in Fig. 5 that a rise of the deflection change has a minor reduction in the stiffness of nanostructures in IPM and hence slightly increase the normalized frequency.

5 Conclusions

Nonlinear large-amplitude free vibrations of a double-layered plate subjected to uniform temperature rise with various simply supported boundary conditions is investigated analytically based on third order shear deformation theory and nonlocal strain gradient theory. Employing the von Kármán large deformation theory and Hamilton's principle the nonlinear governing equations of DLP for both APM and IPM are derived. An improved perturbation technique called the method of multiple scales as well as the Galerkin method are used to study the nonlinear free vibration response of DLP resting on elastic foundation. The influence of various parameters such as small scales coefficient, material length scale parameter and plate deflections with different boundary conditions on the vibration behavior of DLP are also discussed. It is shown that the analytical results are in good agreement with the numerical results. The advantage of the present solution is that the effects of nonlinearities can be determined accurately. Numerical result based on the method of multiple scales illustrated that the nonlinear frequencies are much higher than the linear ones. Based on the proposed results the natural frequency for the case of APM vibration is higher than case of IPM vibration due to the van der Waals interaction between the layers. It can be seen that normalized frequencies for APM condition are weakly sensitive to the nonlocal and the length scale parameters.

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Compliance with ethical standards

Conflict interests The authors declare that there is no conflict of interest.

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