Construction and analysis of interdependent networks based on time series and horizontal visibility graphs

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Abstract

In this paper, the construction and statistical features of the directional complex networks with interdependencies (i.e. interdependent networks) are investigated, in which the intra-structure of each sub-network is not considered. Based on time series and horizontal visibility graph, two new visibility model methods are presented: many-to-many directional horizontal visibility algorithm and one-to-one directional horizontal visibility algorithm. Using R software, the interdependent structure of the directional interdependent network constructed by these two algorithms are analyzed and the statistical features are evaluated associated with two periodic time series, a periodic time series and a fractal time series, a periodic time series and a random series, separately. The results show that both algorithms ensure the simplicity of the interdependent structure of the networks and inherit the morphological characteristics of the time series.

Keywords Interdependent networks · Interdependent structure · Horizontal visibility graph · Time series

1 Introduction

Since the 21th century, with the rapid development of network science, people have a deeper understanding and cognition for many complex systems in the real world [1–3]. To adequately model most real-world systems, understanding the interdependence of networks and the effect of this interdependence on the structural and functional behavior of the coupled system is crucial. Therefore, the interdependent network is the inevitable product of science development. Compared with a single complex network, it is more complex and more influential, which brings a series of challenging problems for the current network science.

Interdependent networks are coupled by a number of complex systems with different characters and different scales. The structure of the interdependent network, which determines the function of the network, includes the intra-structure and the interdependent structure, which is also called interdependent relations between sub-networks.

Compared with the intra-structure, the interdependent structure of a network has a greater influence on the function or dynamic behavior of the network. In 2001, Rinaldi et al. [4] pointed out that the reason of widespread propagation of damage is the interdependence makes the failure of an infrastructure spread to other infrastructures. In 2004, Zimmerman [5] showed that the interdependent structure between the infrastructure systems allows the systems to cooperate with each other, and then make the whole system more fragile. Though the interdependence of systems has been studied for more than a decade, most of them are empirical research. In 2010, Buldyrev et al. [6] developed a mathematical framework to study percolation in a two-way system of two coupled interdependent networks which was subject to cascading failure for the first time. The interdependent structure of this model was a one-to-one connection, which opened a new chapter in the study of interdependent network models [7, 8].

In the past decade, scholars have presented many methods of constructing a single complex network

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Fig. 1 Horizontal visibility algorithm

using nonlinear time series. In 2006, Zhang and Small [9] constructed complex networks from pseudoperiodic time series, in which each cycle was represented by a single node. They found that time series with different dynamics exhibited distinct topological structures. In 2008, Lacasa [10] found a simple and fast visibility algorithm which converted a time series into a graph. The visibility algorithm was introduced as a mapping between time series and complex networks. The constructed graph inherited several properties of the series in its structure. That is, periodic series can be converted into regular graphs, and random series into random graphs. Moreover, fractal series were converted into scale-free networks. Since then, several results have been emerged for a single complex network constructed by time series and visibility graph [11–15].

Based on the above, in this paper, two kinds of directional interdependent algorithms are investigated based on the horizontal visibility graph method [12], which was simpler than the method in the literature [10]. Some remarkable examples are outlined to test the algorithms' reliability. Using these two algorithms and R software [16], two periodic time series, a periodic time series and a fractal time series, a periodic time series and a random series are taken as examples to construct interdependent networks. According to the statistical characteristics of the interdependent structure, these two algorithms proposed are evaluated. As far as I know, with respect to this problem, the existing literature has no relevant theoretical research results, so this will be a groundbreaking research work.

2 Preliminary

Horizontal visibility algorithm presented by Luque et al. [12] was a geometrically simpler and analytically solvable version of the visibility algorithm [10]. For example, it can be defined as follows (see Fig. 1). Let $X = \{x_i\}$, $i = 1, \dots, 20$ be a time series of 20 data. The algorithm assigns each datum of the series to a node in the horizontal visibility graph. Two nodes i and j in the graph are connected if one can draw a horizontal line between x_i and x_j in the time series and it does not intersect any intermediate data height. Therefore, the visibility criterion of the horizontal visibility algorithm is that two observed data (t_a, x_a) and (t_b, x_b) within the time series can be mutual visualized if $\forall (t_i, x_i) \in X$, we have

$$x_i < x_a, x_b, \quad (1)$$

where $t_a < t_i < t_b$.

The properties of the complex networks constructed by the horizontal visibility algorithm were the same as the complex networks obtained by the classical visibility method [10]. However, the connection edges of each node with the horizontal visibility algorithm are less, and then the average degree of the whole network is smaller, and the degree distribution is relatively concentrated.

3 Algorithm description

In this paper, based on the horizontal visibility method, we will present two kinds of algorithms to model directional interdependent networks.

Fig. 2 Bar graphs of time series

Fig. 3 Node A₁ horizontal visibility

3.1 Many-to-many directional horizontal visibility method (Algorithm 1)

Step 1 Select two time series A and B with equal length and they are considered as two sub-networks of the interdependent networks (see Fig. 2). Figure 2 is a bar graph of time series A and B.

Step 2 Each node i in the time series A is selected and looks at the node j in the time series B horizontally (see

Fig. 3). If visible, the directional edge is generated from i to j . Then, the directional interdependent structure is obtained dependent upon the sequence A (see Fig. 4).

In particular, we have to follow two principles in modeling the networks. (i) We do not consider the intra-structure of each sub-network, so every two nodes connected by each edge comes from different time series. (ii) In order to keep the simplicity of the interdependent structure, nodes in the time series B no longer participate in the subsequent horizontal visual process if they have been connected (see Fig. 5). In Fig. 5, the nodes labeled '1' and '2' in the time series B have been removed.

Step 3 Each node j in the time series B is selected and looks at the node i in the time series A horizontally. The connection method is similar to that of Step 2, thus the interdependent structure is generated depending upon the sequence B (see Fig. 6). Specially, two principles (i) and (ii) in Step 2 should be followed. Similarly, the nodes in the time series A that have the connection edges are no longer involved in the subsequent horizontal visualization.

Step 4 Combining Step 2 with Step 3, and constructing an interdependent network with a directional many-to-many structure (see Fig. 7).

Remark 1 In Step 2 of Algorithm 1, when the nodes within the tail of the time series A looking at the nodes in the

Fig. 4 Directional structure relying on the series A

Fig. 5 Node A_2 horizontal visibility

sequence B horizontally, maybe there are no nodes in the sequence B to be horizontally visible because all the nodes have connected with the front nodes in the time series A (see Fig. 4). Figure 4 reveals that there are no edges coming out from the nodes labeled '6', '7', '8', '9', '10'. A similar situation also appears in Step 3. This leads to an unbalanced network, in which the outdegree and indegree of each node in the network are not always equal. For example, in Fig. 7, the node labeled '1' in the time series A has outdegree 3 and indegree 1. To avoid this situation, the following one-to-one directional horizontal visibility algorithm is investigated.

3.2 One-to-one directional horizontal visibility method (Algorithm 2)

In the Sect. 3.1 mentioned above, when the selected node is horizontally visible to the other time series, there may be multiple visible nodes and then can generate multiple edges. Therefore, in Algorithm 2, we give a selection criterion in which only one of the likely nodes is selected, and other nodes are put back. Therefore, a balanced interdependent structure is constructed. The algorithm is described as the following.

Step 1 This step is similar with Step 1 of Algorithm 1 (see Fig. 2).

Step 2 Each node i in the sequence A is selected and looks at the node j of the sequence B horizontally. Among the nodes satisfying the visible condition in the sequence B, the node with the biggest data value is selected to connect with the node i , and the remaining nodes are put back (see Fig. 8). The principles followed are the same as in Step 2 of Algorithm 1. Then a interdependent structure is obtained depending on the sequence A (see Fig. 9).

Step 3 Similarly, the node in the sequence A with the biggest data value is selected if satisfying the horizontally visible condition. The principles followed are the same as in Step 3 of Algorithm 1. The structure relying on the sequence B is given (see Fig. 10).

Step 4 When combining Steps 2 with Step 3, we develop a directional interdependent network with one-to-one structure (see Fig. 11).

Fig. 6 Directional structure relying on the series B

Fig. 7 Many-to-many directional interdependent network

Fig. 8 Node A_1 horizontal visibility

4 Model evaluation

In this section, firstly, with the Algorithm 1, Algorithm 2 and R software, some interdependent network models consisting of two time series are created, where the time series are two periodic sequences, a periodic sequence and a fractal sequence, a periodic sequence and a random sequence, separately. Secondly, statistical properties of these networks are evaluated associated with the interdependent structure (coupling matrix).

4.1 Two periodic time series

Considering two time series A and B, the series A is $y = \sin(\pi x)$ and the series B is $y = \sin(3\pi x)$ with sampling interval $\Delta x = 0.05$ and $n = 200$ data, respectively.

Fig. 9 Directional structure relying on the series A

Fig. 10 Directional structure relying on the series B

Fig. 11 One-to-one directional interdependent structure

Fig. 12 Many-to-many directional interdependent structure **Fig. 13** Degree distribution of many-to-many directional interdependent network

1. Many-to-many directional horizontal visibility method

According to Algorithm 1, the coupling matrix between sub-networks A and B and degree distribution are shown in Figs. 12 and 13.

In order to observe the statistical characteristics of the coupling matrix, according to the dependencies between two sub-networks A and B, the coupling matrix can be divided into two parts. One depends on the series A and the other on the series B. Figure 12a shows that most nodes in the sequence B only depend on the first 50 nodes of the sequence A. In contrast, Fig. 12b shows that the nodes in the sequence A are more evenly dependent on the nodes in the sequence B. In general, both of the two parts of the interdependent structure exhibit regular networks features, and the morphological characteristics of the periodic time series are well inherited.

The degree distribution of the network shows that there are a few peaks (Fig. 13a). Most nodes have small

degree and their degrees are $K = 1$. The average degree is relatively small, and the overall simplicity of the interdependent structure is well guaranteed. In Fig. 13b, the out-degree distribution displays the same shape as the degree distribution of the network. Moreover, the out-degrees of most nodes are $K_{out} = 0$, and all nodes in the interdependent structure have the same in-degree $K_{in} = 1$ in Fig. 13c.

Because the period of the time series A is bigger, there are more nodes with $K_{out} = 0$. At the same time, its maximum out-degree is $K_{out} = 8$ (Fig. 13d). But for the series B, fewer nodes have $K_{out} = 0$ and its maximum out-degree is $K_{out} = 5$ (Fig. 13e).

2. One-to-one directional horizontal visibility method

According to Algorithm 2, the interdependent structure and degree distributions between sub-networks A and B are exhibited in Figs. 14 and 15.

Figure 14a tells us that the nodes in the sequence B are evenly dependent on the nodes in the sequence A, vice versa. Moreover, at this time, the topology structure is

Fig. 14 One-to-one directional interdependent structure

Fig. 15 **a** Degree distribution, **b** out-degree distribution and **c** in-degree distribution of one-to-one directional interdependent networks

Fig. 16 Many-to-many directional interdependent structure

very simple. The two parts of the interdependent structure exhibit regular networks features, and display the morphological features of the periodic sequences very well. The degree distribution of the interdependent structure shown in Fig. 15 is constant 1, and the out-degree and the in-degree of each node are $K_{in} = K_{out} = 1$.

4.2 Periodic time series and fractal time series

Suppose that the series *A* is $y = \sin(\pi x)$ with sampling interval $\Delta x = 0.05$ and $n = 120$ data and the series *B* is Conway fractal time series [17] with length $n = 120$, we have the following results.

1. Many-to-many directional horizontal visibility method

Based on Algorithm 1, the coupling matrix and degree distribution between sub-networks *A* and *B* are shown in Figs. 16 and 17.

Figure 16a displays that the nodes in the sequence *B* are more evenly dependent on the nodes in the sequence *A*. In contrast, Fig. 16b shows that most nodes in the sequence *A* only depend on the first 20 nodes of the sequence *B*. The two parts of the interdependent structure exhibit regular networks features, inheriting the morphological features of the periodic time series and Conway sequence.

The degree distribution of the network shows the shape provided in Fig. 17a with a few peaks. Most nodes have small degree $K = 2$. The average degree is relatively small, and the overall simplicity of the interdependent structure is well guaranteed. Figure 17b shows the same shape as the degree distribution for the out-degree distribution. The out-degrees of most nodes are $K_{out} = 1$.

Figure 17c and d are the out-degree distributions of the two sub-networks in the interdependent structure. The out-degree of each node of the sequence *A* is $K_{out} = 1$. The out-degree distribution of the Conway sequence *B* obeys the power-law distribution and most nodes have out-degree $K_{out} = 0$.

Fig. 17 Degree distribution of many-to-many directional interdependent network

Fig. 18 One-to-one directional interdependent structure

2. One-to-one directional horizontal visibility method

Based on Algorithm 2, the coupling matrix between sub-networks A and B is obtained shown in Fig. 18. The graphic of degree distribution is the same as that of Fig. 15. Figure 18a and b show that the nodes in the sequence B and A are evenly dependent on the nodes in the sequence A and B, separately. The two parts of the interdependent structure show the morphological characteristics of the periodic time series and Conway sequence.

With this method, the degree of each node is $K = 2$ and both of the out-degree and in-degree are $K_{in} = K_{out} = 1$. They ensure the overall simplicity of the interdependent structure.

4.3 Periodic time series and random time series

Provided that the series A is $y = \sin(\pi x)$ with sampling interval $\Delta x = 0.05$ and $n = 120$ data, and the time series B is white noise sequence with length 120 that follows a normal distribution and whose mean is zero, then we have the following results.

1. Many-to-many directional horizontal visibility method

Based on Algorithm 1, the coupling matrix and degree distribution between sub-networks A and B are shown in Figs. 19 and 20.

Figure 19a shows that most nodes in the sequence B only depend on the first 40 nodes of the sequence A. Figure 9b shows that the nodes in the sequence A are evenly dependent on the nodes in the sequence B. The two parts of the interdependent structure exhibit random network and regular network features, separately, and they maintain the morphological characteristics of periodic sequence and white noise sequence.

The degree distribution of the network is shown in Fig. 20a. The average degree of nodes is relatively small, and the degrees of most nodes are $K = 1$, which ensure the simplicity of interdependent structure. Figure 20b tells us that the out-degree distribution maintains the same morphology as the degree distribution, and the out-degree of the majority nodes is $K_{out} = 0$. Figure 20c and d are the distribution of the out-degree of two sub-networks. The degree distributions of periodic sequence A and white

Fig. 19 Many-to-many directional interdependent structure

Fig. 20 Degree distribution of many-to-many directional interdependent network

Fig. 21 One-to-one directional interdependent structure

noise sequence both have the characteristics of power-law distribution.

2. One-to-one directional horizontal visibility method

According to Algorithm 2, the coupling matrix between the series A and B is obtained shown in Fig. 21. The degree distribution diagram is the same as Fig. 15, which is omitted here.

Figure 21a shows that almost all of the nodes in the sequence B depend equally on the nodes in the sequence A, vice versa. The two parts of the interdependent structure display random graph network and regular network features, separately. At the same time, they maintain the morphological characteristics of periodic sequence and white noise sequence. In this interdependent structure, the degree of each node is $K = 2$, and the out-degree and the in-degree are $K_{in} = K_{out} = 1$.

5 Conclusions

With the development and maturity of society, science and technology, the connections among the natural environment, social environment and various man-made environment become more closely and intricate. These connections are reflected not only in the internal environment of all kinds, but also in the interdependence of all kinds of environments. Here, the interdependence is also called interdependent structure. Compared with the intra-structure of the sub-network, the interdependent structure of an interdependent network has a greater impact on the function or dynamics of the network. In this paper, based on time series and horizontal visibility method, the interdependent structure of interdependent networks is discussed. Two algorithms are proposed, which are many-to-many directional horizontal visibility method and one-to-one directional horizontal visibility method. Based on these two algorithms, some interdependent network models composed of two time series are built. And then, the statistical features of these models are analyzed. The results tell us that both algorithms can inherit the morphological characteristics of each time series well, and can ensure the simplicity of network interdependent structure. For example, the interdependent structure of the interdependent network generated by two periodic time series exhibit regular networks features. The interdependent structure of the interdependent network generated by random time series display random features. These two algorithms give new insight into the construction of interdependent networks, and may potentially be applied to real data or real systems. In the future, our research will be directed towards using the current results as a basis upon which to further improve the construction methods of interdependent networks.

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Compliance with ethical standards

Conflict of interest The authors have declared that no competing interests exist.

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