Research Article

Compound difference anti-synchronization between chaotic systems of integer and fractional order

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Abstract
In this paper the compound difference anti-synchronization between chaotic systems of integer order and fractional order has been studied. Numerical simulations have been performed considering Rikitake chaotic system of integer and fractional order, El-Nino chaotic system of integer and fractional order and generalized Lotka–Volterra chaotic system of integer order as master systems and generalized Lotka–Volterra chaotic system as slave system. Numerical simulations validate the effectiveness of the strategy.

Keywords Compound synchronization · Difference synchronization · Anti-synchronization · Fractional order · El-Nino system · Rikitake system · Generalized Lotka Volterra system

1 Introduction

The systems found in nature are non linear systems and they are generally found to be interacting in a non linear manner with other systems. Though these systems have their own unique identities but interactions between them lead to unpredictable behavior. A common method of this non linear interaction is the compound effect of interacting systems. While the difference synchronization can be considered as a replacement of combination synchronization, however not much work has been done in this direction. Also, applying such techniques to chaotic systems of different structures and orders increases the flexibility of the method thereby increasing its application by manifolds.

Chaos synchronization is a process of having different chaotic systems(non-identical or identical)following the same path, i.e. the dynamics of one system is locked into the another, thereby causing their synchronization in a way that the state of one asymptotically approaches to the other. The idea of synchronization was given by Pecora and Caroll [1], when two chaotic systems with different initial conditions were made stable by designing suitable controllers. To control the chaotic behavior of these systems many synchronization schemes have been developed such as complete synchronization, anti-synchronization, combination synchronization, compound synchronization, difference synchronization etc. [2–5] using various control techniques such as active control, sliding mode control, adaptive control, tracking control etc. These techniques are easily applicable on physical, chemical, biological, economical models. Also, work using these techniques have been extended using delay differential equations where the function's derivative at any time is dependent on the solution at previous time making them infinite dimensional and hence difficult to analyze analytically. These differential equations find applications in population dynamics, lasers, neuro-sciences, control systems [6]. Also, parameters have a significant role in chaotifying the systems and hence in chaos synchronization, thereby making parameter estimation a key point of Chaos Theory. Notable work has been done in this direction involving designing of an adaptive approach to estimate the uncertain parameters [7–11].

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SN Applied Sciences (2019) 1:757 | https://doi.org/10.1007/s42452-019-0776-x

Received: 6 February 2019 / Accepted: 12 June 2019 / Published online: 21 June 2019

With the leaping of cyber-crime cases, designing of techniques which involve combination of systems and further compound-combination of systems would help increase the anti-attack resistance of systems as predicting the unique way in which the systems have been added or multiplied is not an easy task. This would help increase the security of transmission of information which is the need of the hour.

Therefore designing of new synchronization techniques by various methods started to grow. Synchronization which initially was found to be difficult to achieve in case of one master and one slave chaotic system found its generalization in the form of combination synchronization [12] (two chaotic master systems are synchronized with one chaotic slave system or vice-versa), combination-combination synchronization [13] (involving synchronization of two or more chaotic master systems with two or more chaotic slave systems), compound synchronization [14, 15] (includes three chaotic master systems categorized as scaling and base master systems with one chaotic slave system), compound-combination synchronization [16] (comprising of both compound and combination synchronization) and so on.

Though difference synchronization (where difference of state variables of two chaotic systems synchronize with the third) is considered as a substitute for combination synchronization, it still adds to the diversity of the types of synchronization.

Synchronization techniques composed from many different synchronization techniques [16–19] produce complex signals as on splitting the signals onto different parts and loading into different chaotic master systems increases the anti-attack and anti-translated capability as compared to the usual transmitted signal. Motivated by the above discussion compound difference anti-synchronization promises to be a meaningful expansion in this direction.

In this article the compound difference anti-synchronization between the integer and fractional order Rikitake, integer and fractional order El-Nino and integer order generalized Lotka–Volterra chaotic system has been conducted. The numerical simulations and results of this article are shown graphically which show that the techniques involved is reliable and effective for the desired synchronization.

The rest of the article is organized as follows: Section 2: contains some preliminaries and problem formulation. Section 3: describes the synchronization theory. In Sect. 4: we have designed the compound difference anti-synchronization method using Active Control Method and its numerical simulations are displayed [1, 7, 20]. Section 5: consists of the compound difference anti-synchronization method using parameter estimation method and its numerical simulations, which have been displayed graphically. Section 6: gives a comparison of the error convergence rates for different fractional orders and for different synchronization schemes Sect. 7 shows the utilization of designed synchronization scheme in the field of secure communication. Section 8 concludes the article.

2 Preliminaries

2.1 Definition

CAPUTO DEFINITION: [21] As various definitions have been available for fractional order derivative, we have considered Caputo's definition.

$$ {}_aD_x^{\alpha} g(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{g^{(n)}(\tau)d\tau}{(x-\tau)^{\alpha-n+1}} $$

where n is integer, α is real number, $(n-1) \le \alpha < n$ and $\Gamma(.)$ is the Gamma function.

Throughout our studies Caputo's version of fractional derivative has been used.

2.2 Problem formulation

We now formulate the compound difference anti-synchronization method [22, 23]. We now consider three chaotic master systems and one chaotic slave system. We describe the scaling master system by

$$ D^q x = f(x) \tag{1} $$

and the base master systems by

$$ D^q y = h(y) \tag{2} $$

$$ D^q z = r(z) \tag{3} $$

Corresponding to the above, we describe the slave system by

$$ D^q w = g(w) + u \tag{4} $$

where $x = (x_1, x_2, ..., x_n)^T, y = (y_1, y_2, ..., y_n)^T, z = (z_1, z_2, ..., z_n)^T, w = (w_1, w_2, ..., w_n)^T$ are the state vectors of the respective systems. f,h,r,g are continuous functions. $u = (u_1, u_2, ..., u_n)^T : R^n \times R^n \times R^n \times R^n \to R^n$.

We define the compound difference anti-synchronization error as

$$ e = Dw + Ax(Cz - By) $$

where $A = diag(a_1, a_2, ..., a_n) B = diag(b_1, b_2, ..., b_n) C = diag(c_1, c_2, ..., c_n) D = diag(d_1, d_2, ..., d_n) and D \neq 0$

Definition The compound of the chaotic master systems (1–3) are said to be in compound difference anti-synchronization with slave system (4) if

$$ lim_{t\to\infty} ||e|| = lim_{t\to\infty} ||Dw + Ax(Cz - By)|| = 0. $$

Here, we assume $X = diag(x_1, x_2, ..., x_n)$, $Y = diag(y_1, y_2, ..., y_n)$, $Z = diag(z_1, z_2, ..., z_n)$, $W = diag(w_1, w_2, ..., w_n)$

3 Synchronization theory

In order to attain compound difference anti-synchronization among the systems (1–4), we design the controllers as

$$u_i = -\frac{\theta_i}{d_i} - g_i - \frac{K_i e_i}{d_i} \tag{5}$$

where

$\theta_i = a_i f_i(c_i z_i - b_i y_i) + a_i x_i(c_i r_i - b_i h_i)$ for $i = 1, 2, ..., n$

Theorem *The systems (1–4) will achieve the intended compound difference anti-synchronization if the controller are designed as given in (5).*

Proof The error is given by :

$e_i = d_i w_i + a_i x_i(c_i z_i - b_i y_i)$ for i = 1, 2, ..., n

The error dynamical system is given by:

$$\begin{aligned} D^q e_i &= d_i D^q w_i + a_i D^q x_i(c_i z_i - b_i y_i) + a_i x_i(c_i D^q z_i - b_i D^q y_i) \\ &= d_i(g_i + u_i) + a_i f_i(c_i z_i - b_i y_i) + a_i x_i(c_i r_i - b_i h_i) \\ &= d_i(g_i + u_i) + \theta_i \end{aligned}$$

where

$$\begin{aligned} \theta_i &= a_i f_i(c_i z_i - b_i y_i) + a_i x_i(c_i r_i - b_i h_i) \\ &= d_i\left(g_i - \frac{\theta_i}{d_i} - g_i - \frac{K_i e_i}{d_i}\right) + \theta_i \\ &= -K_i e_i \end{aligned}$$

We define the Lyapunov function as:

$V(t) = \frac{1}{2} e^T e$

$= \frac{1}{2} \sum_{i=1}^{n} {e_i}^2$

$\Rightarrow D^q V(t) = \sum e_i D^q e_i$

$= \sum e_i(-K_i e_i)$

$- \sum K_i {e_i}^2$

We choose $(K_1, K_2, ..., K_n)$ in such a way such that $D^q V(t)$ is negative definite.Therefore, by the Stability Theory of Lyapunov, we get $\lim_{t\to\infty} ||e|| = 0$. Hence, the compound of the master systems and slave system are anti-synchronized. □

4 Compound difference anti-synchronization between Rikitake, El-Nino, generalized Lotka Volterra chaotic systems via active control method

The scaling master system is the [24–26] chaotic Rikitake system of fractional order [27] given by:

$$\begin{aligned} \frac{d^q x_1}{dt^q} &= -A_2 x_1 + x_2 x_3 \\ \frac{d^q x_2}{dt^q} &= -A_2 x_2 + (x_3 - A_1)x_1 \\ \frac{d^q x_3}{dt^q} &= 1 - x_1 x_2 \end{aligned} \tag{6}$$

where $x = (x_1, x_2, x_3)$ are the state variables of the scaling master system, A_1, A_2 are parameters. For $A_1 = 5, A_2 = 2$, this system shows chaotic behavior for initial conditions (−4, 2.5, 2) and fractional order q = 0.987 displayed in Fig. 1a

The base master systems are the chaotic fractional order El-Nino system [28] and the chaotic integer order generalized Lotka–Volterra system [29] described respectively as follows:

$$\begin{aligned} \frac{d^q y_1}{dt^q} &= \mu'(y_2 - y_3) - by_1 \\ \frac{d^q y_2}{dt^q} &= y_1 y_3 - y_2 + cy_1 \\ \frac{d^q y_3}{dt^q} &= -y_1 y_2 - cy_1 - y_3 \end{aligned} \tag{7}$$

where $0 < q < 1$, $y = (y_1, y_2, y_3)$ are the state variables of the base master system I.

For parameter values $\mu' = 83.6, b = 10, c = 12$ and initial conditions (−2,3,5) the phase portrait shows chaotic behavior as displayed in Fig. 1b for q = 0.987.

$$\begin{aligned} \dot{z}_1 &= z_1 - z_1 z_2 + c{z_1}^2 - a{z_1}^2 z_3 \\ \dot{z}_2 &= -z_2 + z_1 z_2 \\ \dot{z}_3 &= -bz_3 + a{z_1}^2 z_3 \end{aligned} \tag{8}$$

where $z = (z_1, z_2, z_3)$ are the state variables of the system (8). For a = 2.9851, b = 3, c = 2 and initial conditions (1.2,1.2,1.2)this system shows chaotic behavior as displayed in Fig. 1c.

The slave system is considered as the integer order chaotic generalized Lotka–Volterra system given as :

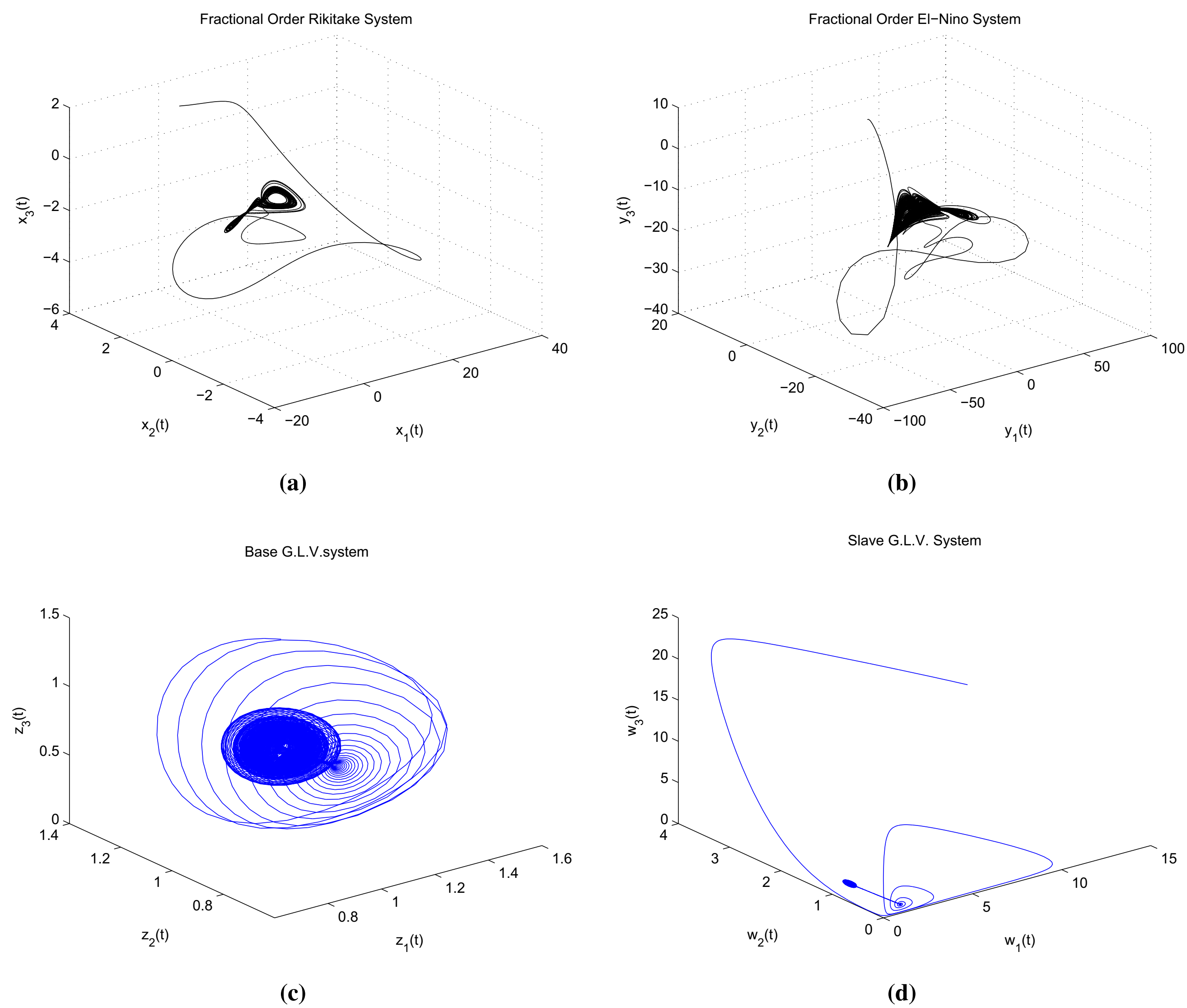

Fig. 1 Phase plots of the **a** fractional order Rikitake system, **b** fractional order El-Nino system, **c** integer order generalized Lotka–Volterra system, **d** slave generalized Lotka–Volterra system respectively

$$
\begin{aligned}
\dot{w_1} &= w_1 - w_1 w_2 + c' {w_1}^2 - a' {w_1}^2 w_3 + u_1 \\
\dot{w_2} &= -w_2 + w_1 w_2 + u_2 \\
\dot{w_3} &= -b' w_3 + a' {w_1}^2 w_3 + u_3
\end{aligned}
\tag{9}
$$

where $w = (w_1, w_2, w_3)$ are the state variables of the system (9) and $u = (u_1, u_2, u_3)$ is a control function to be designed. For a′ = 2.9851, b′ = 3, c′ = 2 and initial conditions (14.5, 3.4, 10.1)the system shows chaotic behavior as displayed in Fig. 1d.

Here we assume $A = diag(a_1, a_2, a_3)$, $B = diag(b_1, b_2, b_3)$, $C = diag(c_1, c_2, c_3)$, $D = diag(d_1, d_2, d_3)$.The scaling factors $a_i, b_i, c_i, d_i (i = 1, 2, 3)$ are chosen as required and may assume different or same values.

We define the error (e_1, e_2, e_3) as:

$$
\begin{aligned}
e_1 &= d_1 w_1 + a_1 x_1 (c_1 z_1 - b_1 y_1) \\
e_2 &= d_2 w_2 + a_2 x_2 (c_2 z_2 - b_2 y_2) \\
e_3 &= d_3 w_3 + a_3 x_3 (c_3 z_3 - b_3 y_3)
\end{aligned}
\tag{10}
$$

Therefore,the error dynamical system so obtained is:

$$
\begin{aligned}
\dot{e_1} &= d_1 \dot{w_1} + a_1 \dot{x_1} (c_1 z_1 - b_1 y_1) + a_1 x_1 (c_1 \dot{z_1} - b_1 \dot{y_1}) \\
\dot{e_2} &= d_2 \dot{w_2} + a_2 \dot{x_2} (c_2 z_2 - b_2 y_2) + a_2 x_2 (c_2 \dot{z_2} - b_2 \dot{y_2}) \\
\dot{e_3} &= d_3 \dot{w_3} + a_3 \dot{x_3} (c_3 z_3 - b_3 y_3) + a_3 x_3 (c_3 \dot{z_3} - b_3 \dot{y_3})
\end{aligned}
\tag{11}
$$

Therefore,error dynamical system simplifies to

$$\begin{aligned}
\dot{e_1} &= d_1(w_1 - w_1w_2 + 2{w_1}^2 - 2.9851{w_1}^2w_3 + u_1) + a_1(-5x_1 + x_2x_3)(c_1z_1 - b_1y_1) \\
&\quad + a_1x_1(c_1(z_1 - z_1z_2 + 2{z_1}^2 - 2.9851{z_1}^2z_3) - b_1(83.6y_2 - 83.6y_3 - 10y_1)) \\
\dot{e_2} &= d_2(-w_2 + w_1w_2) + a_2(-5x_2 + x_1x_3 - 2x_1)(c_2z_2 - b_2y_2) \\
&\quad + a_2x_2(c_2(-z_2 + z_1z_2) - b_2(y_1y_3 - y_2 + 12y_1)) \\
\dot{e_3} &= d_3(-3w_3 + 2.9851{w_1}^2w_3) + a_3(1 - x_1x_2)(c_3z_3 - b_3y_3) \\
&\quad + a_3x_3(c_3(-3z_3 + 2.9851{z_1}^2z_3) - b_3(-y_1y_2 - y_3 - 12y_1))
\end{aligned} \tag{12}$$

We choose the control functions:

$$\begin{aligned}
u_1 &= -\frac{\theta_1}{d_1} - g_1 - \frac{K_1e_1}{d_1} \\
&\text{where } \theta_1 = a_1f_1(c_1z_1 - b_1y_1) + a_1x_1(c_1r_1 - b_1h_1) \\
u_2 &= -\frac{\theta_2}{d_2} - g_2 - \frac{K_2e_2}{d_2} \\
&\text{where } \theta_2 = a_2f_2(c_2z_2 - b_2y_2) + a_2x_2(c_2r_2 - b_2h_2) \\
u_3 &= -\frac{\theta_3}{d_3} - g_3 - \frac{K_3e_3}{d_3} \\
&\text{where } \theta_3 = a_3f_3(c_3z_3 - b_3y_3) + a_3x_3(c_3r_3 - b_3h_3)
\end{aligned} \tag{13}$$

Substituting (13) into(12),we get

$\dot{e_1} = -K_1e_1$
$\dot{e_2} = -K_2e_2$
$\dot{e_3} = -K_3e_3$

We now take the Lyapunov function as $V(e(t)) = \frac{1}{2}e(t)e(t)^T$
$= \frac{1}{2}({e_1}^2 + {e_2}^2 + {e_3}^2)$
$V(\dot{e}(t)) = e_1\dot{e_1} + e_2\dot{e_2} + e_3\dot{e_3}$
$= e_1(-K_1e_1) + e_2(-K_2e_2) + e_3(-K_3e_3)$
$= -K_1{e_1}^2 - K_2{e_2}^2 - K_3{e_3}^2$

where K_1, K_2, K_3 are positive constants.
$\Rightarrow V(\dot{e}(t))$ is negative definite.

Therefore by the Stability Theory of Lyapunov ,we have that error vanishes with time,i.e. $e_i \to 0$ for i $= 1, 2, 3$. Hence, the master systems (6–8) are now anti-synchronized with the slave system (9)

Note

The compound of the master systems so generated here from scaling master system and base master systems is also a chaotic system as displayed in Fig. 2, this may not be the case in general.

Numerical simulations and results

Numerical simulations have been done using MATLAB. We take here $a_i = b_i = c_i = d_i = 1 \forall i = 1..3$, which means the slave system will completely anti-synchronize with the compound of the multi-drive master systems. Also K_1, K_2, K_3 have been chosen to be 1 for fractional order 0.987. The trajectories of the master systems(6–8) and slave system (9) are shown to get anti-synchronized in Fig. 3. Also, the error plot of the system converges to zero and have been displayed in Fig. 3 for the initial conditions (8.1,–1.1,2.5).

5 Compound difference anti-synchronization between Rikitake, El-Nino, generalized Lotka Volterra chaotic systems via parameter estimation method

The scaling master system is the chaotic Rikitake system given by:

$$\begin{aligned}
\dot{x_1} &= -Ax_1 + x_2x_3 \\
\dot{x_2} &= -Ax_2 + (x_3 - B)x_1 \\
\dot{x_3} &= 1 - x_1x_2
\end{aligned} \tag{14}$$

where $x = (x_1, x_2, x_3)$ are the state variables of the scaling master system. For $A = 2$, $B = 5$, this system shows chaotic behavior for initial conditions $(2, -1, -2)$ as displayed in Fig. 4a.

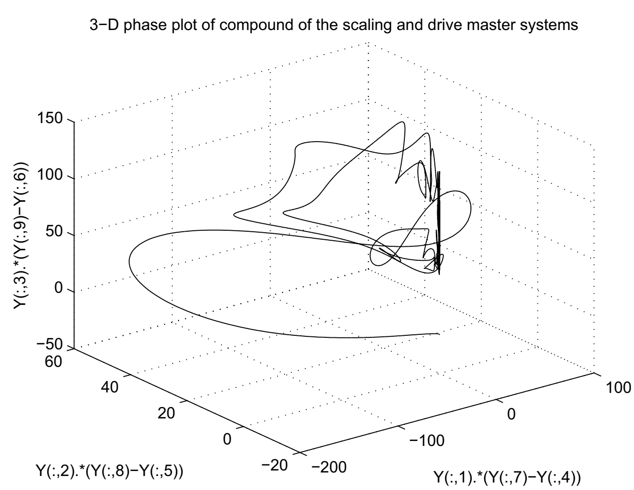

Fig. 2 Phase portrait of the compound of the master systems

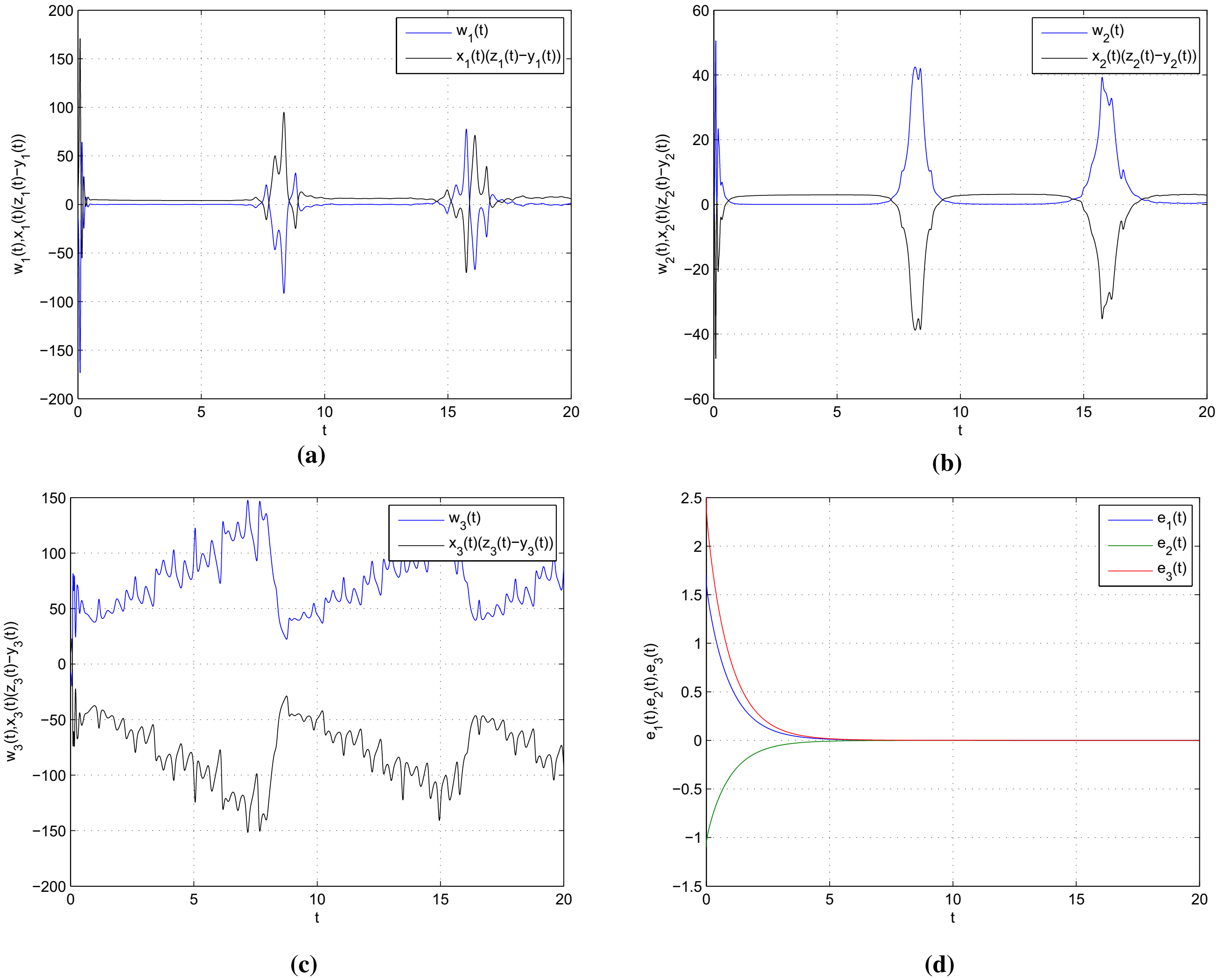

Fig. 3 **a**, **b**, **c** Trajectories of the anti-synchronized master and slave systems. **d** The simultaneous error plot of the system

The base master systems are the El-Nino chaotic system and generalized Lotka–Volterra chaotic system described respectively as follows:

$$\begin{aligned} \dot{y_1} &= A_1(y_2 - y_3) - B_1 y_1 \\ \dot{y_2} &= y_1 y_3 - y_2 + C_1 \\ \dot{y_3} &= -y_1 y_2 - y_3 + C_1 \end{aligned} \tag{15}$$

where $y = (y_1, y_2, y_3)$ are the state variables of the base master system I.

For parameter values $A_1 = 83.6$, $B_1 = 10$, $C_1 = 12$ and initial conditions (−2,3,5) the phase portrait shows chaotic behavior as displayed in Fig. 4b.

$$\begin{aligned} \dot{z_1} &= z_1 - z_1 z_2 + c{z_1}^2 - a{z_1}^2 z_3 \\ \dot{z_2} &= -z_2 + z_1 z_2 \\ \dot{z_3} &= -b z_3 + a{z_1}^2 z_3 \end{aligned} \tag{16}$$

where $z = (z_1, z_2, z_3)$ are the state variables of system (16). For a = 2.9851, b = 3, c = 2 and initial conditions (1,2,1.2,1.2) this system shows chaotic behavior as displayed in Fig. 4c.

The slave system is the generalized Lotka–Volterra system given as :

$$\begin{aligned} \dot{w_1} &= w_1 - w_1 w_2 + c_1 {w_1}^2 - a_1 {w_1}^2 w_3 + \sigma_1 \\ \dot{w_2} &= -w_2 + w_1 w_2 + \sigma_2 \\ \dot{w_3} &= -b_1 w_3 + a_1 {w_1}^2 w_3 + \sigma_3 \end{aligned} \tag{17}$$

where $w = (w_1, w_2, w_3)$ are the state variables of system (17) and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a control function to be designed. For parameter values $a_1 = 2.9851, b_1 = 3, c_1 = 2$ and initial conditions (14.5, 3.4, 10.1)the system shows chaotic behavior as displayed in Fig. 4d.

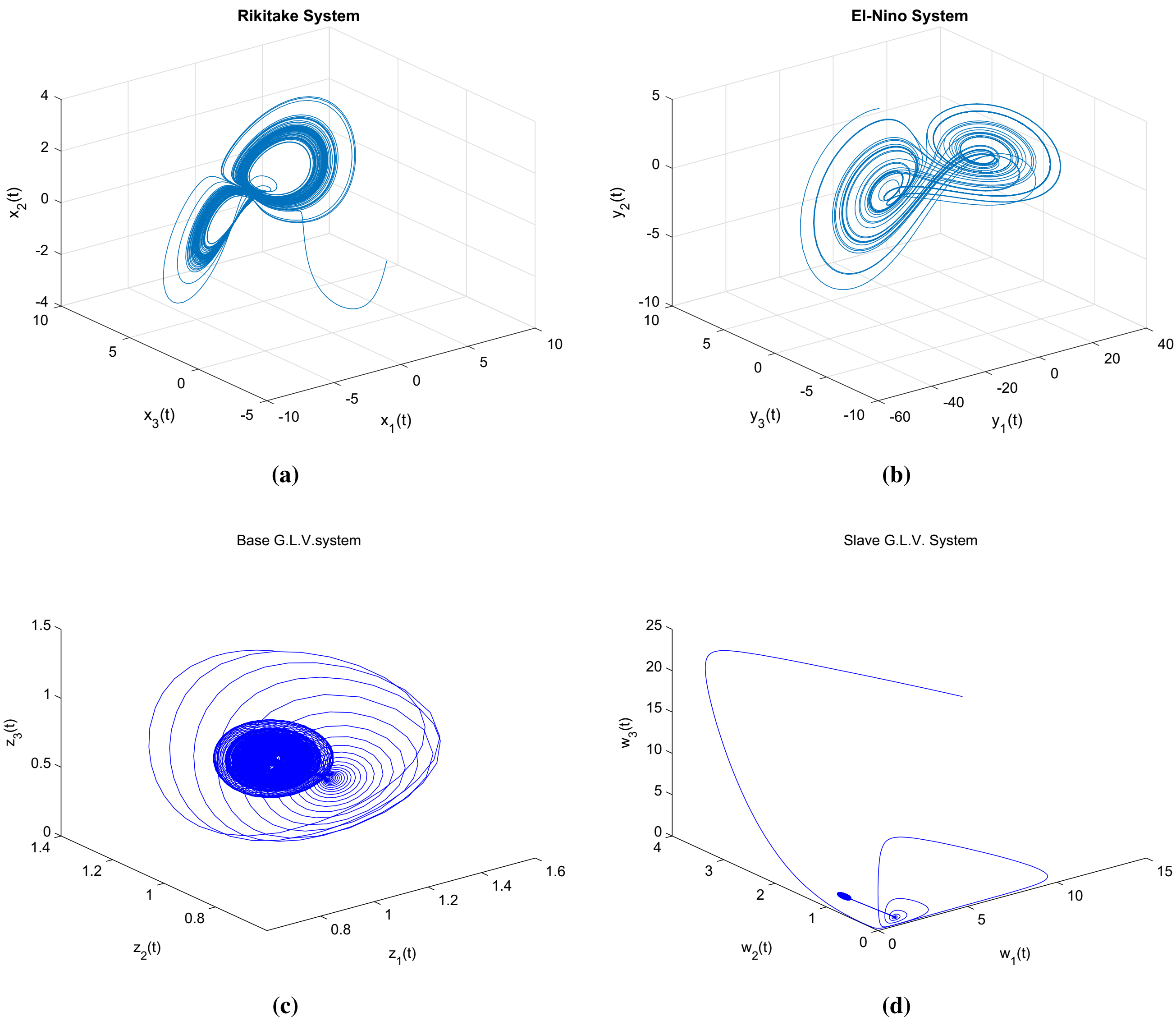

Fig. 4 Phase plots of the **a** fractional order Rikitake system, **b** fractional order El-Nino system, **c** generalized Lotka–Volterra system, **d** slave generalized Lotka–Volterra system respectively

Here we assume $A = diag(\alpha_1, \alpha_2, \alpha_3), B = diag(\beta_1, \beta_2, \beta_3)$, $C = diag(\gamma_1, \gamma_2, \gamma_3), D = diag(\delta_1, \delta_2, \delta_3)$. The scaling factors $\alpha_i, \beta_i, \gamma_i, \delta_i (i = 1, 2, 3)$ are chosen as required and may assume different or same values.

We define the error(e_1, e_2, e_3) as:

$$\begin{aligned} e_1 &= \delta_1 w_1 + \alpha_1 x_1(\gamma_1 z_1 - \beta_1 y_1) \\ e_2 &= \delta_2 w_2 + \alpha_2 x_2(\gamma_2 z_2 - \beta_2 y_2) \\ e_3 &= \delta_3 w_3 + \alpha_3 x_3(\gamma_3 z_3 - \beta_3 y_3) \end{aligned} \tag{18}$$

Therefore,the error dynamical system so obtained is:

$$\begin{aligned} \dot{e_1} &= \delta_1 \dot{w_1} + \alpha_1 \dot{x_1}(\gamma_1 z_1 - \beta_1 y_1) + \alpha_1 x_1(\gamma_1 \dot{z_1} - \beta_1 \dot{y_1}) \\ \dot{e_2} &= \delta_2 \dot{w_2} + \alpha_2 \dot{x_2}(\gamma_2 z_2 - \beta_2 y_2) + \alpha_2 x_2(\gamma_2 \dot{z_2} - \beta_2 \dot{y_2}) \\ \dot{e_3} &= \delta_3 \dot{w_3} + \alpha_3 \dot{x_3}(\gamma_3 z_3 - \beta_3 y_3) + \alpha_3 x_3(\gamma_3 \dot{z_3} - \beta_3 \dot{y_3}) \end{aligned} \tag{19}$$

Taking $\alpha_i, \beta_i, \gamma_i, \delta_i = 1$ Therefore,error dynamical system simplifies to

$$\begin{aligned} \dot{e_1} &= (w_1 - w_1 w_2 + c_1 {w_1}^2 - a_1 {w_1}^2 w_3) + (-Ax_1 + x_2 x_3)(z_1 - y_1) \\ &\quad + x_1(z_1 - z_1 z_2 + Cz_1^2 - az_1^2 z_3 - A_1 y_2 + A_1 y_3 + B_1 y_1) + \sigma_1 \\ \dot{e_2} &= (-w_2 + w_1 w_2) + (-Ax_2 + x_1 x_3 - Bx_1)(z_2 - y_2) \\ &\quad + x_2(-z_2 + z_1 z_2 - y_1 y_3 + y_2 - C_1) + \sigma_2 \\ \dot{e_3} &= (-b_1 w_3 + a_1 w_1^2 w_3) + (1 - x_1 x_2)(z_3 - y_3) \\ &\quad + x_3(-bz_3 + az_1^2 z_3 + y_1 y_2 + y_3 - C_1) + \sigma_3 \end{aligned} \tag{20}$$

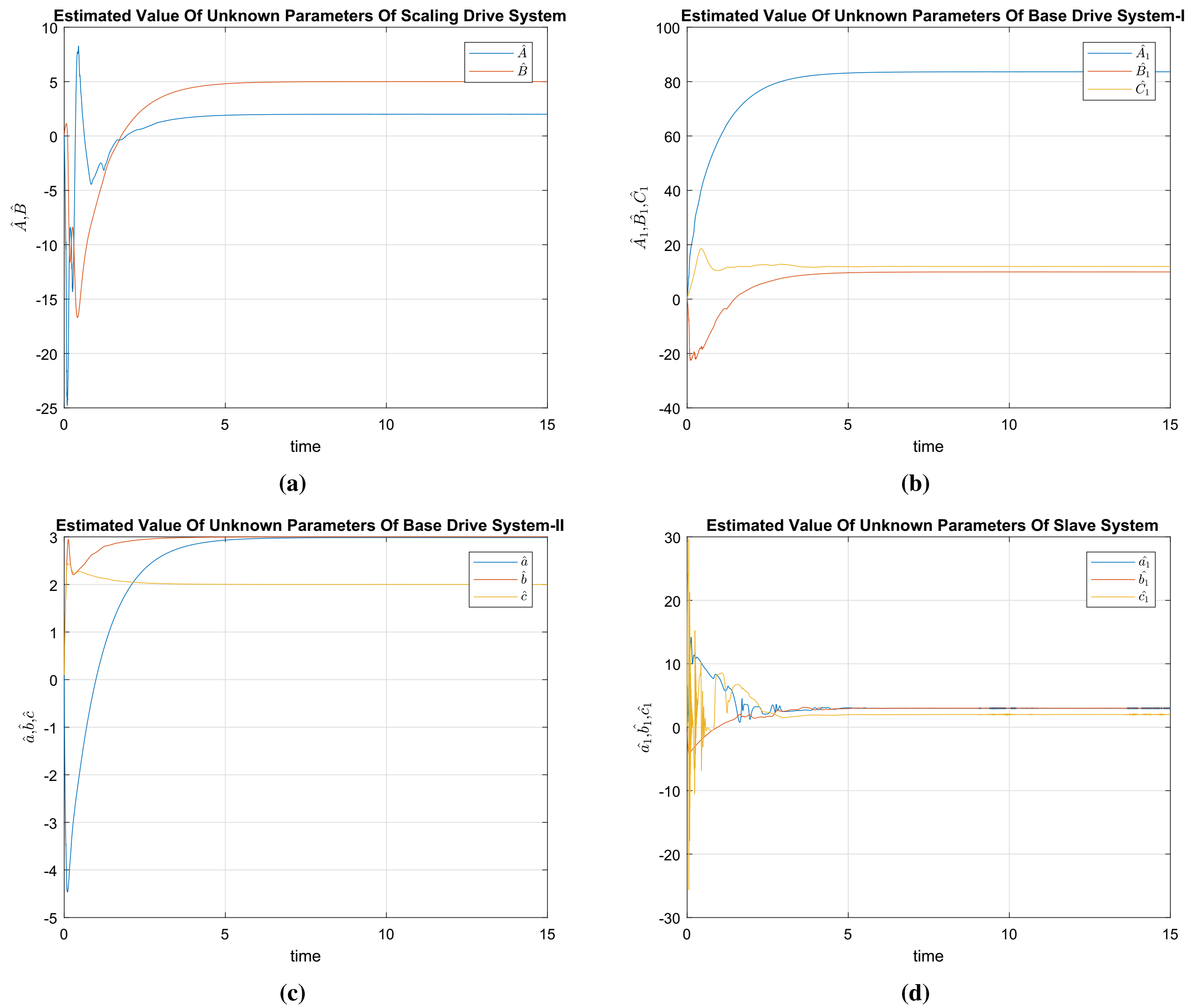

Fig. 5 Parameter estimate for **a** scaling drive system, **b** Base drive system-I, **c** base drive system-II, **d** response system

We choose the control functions:

$$
\begin{aligned}
\sigma_1 &= -w_1 + w_1 w_2 - \hat{c}_1 w_1^2 + \hat{a}_1 w_1^2 w_3 + \hat{A} x_1 z_1 - \hat{A} x_1 y_1 - x_2 x_3 z_1 + x_2 x_3 y_1 - x_1 z_1 \\
&\quad + x_1 z_1 z_2 - \hat{c} z_1^2 x_1 + \hat{a} z_1^2 z_3 x_1 + \hat{A}_1 y_2 x_1 - \hat{A}_1 y_3 x_1 - \hat{B}_1 y_1 x_1 - K_1 e_1 \\
\sigma_2 &= w_2 - w_1 w_2 + \hat{A} x_2 z_2 - x_3 x_1 z_2 + \hat{B} x_1 z_2 - \hat{A} x_2 y_2 \\
&\quad + x_3 x_1 y_2 - \hat{B} x_1 y_2 + z_2 x_2 - x_2 z_1 z_2 + y_1 y_3 x_2 - x_2 y_2 + \hat{c}_1 x_2 - K_2 e_2 \\
\sigma_3 &= \hat{b}_1 w_3 - \hat{a}_1 w_1^2 w_3 - z_3 + y_3 + x_1 x_2 z_3 - x_1 x_2 y_3 + \hat{b} x_3 z_3 - \hat{a} z_1^2 z_3 x_3 \\
&\quad - x_3 y_1 y_2 - x_3 y_3 + \hat{c}_1 x_3 - K_3 e_3
\end{aligned}
\tag{21}
$$

and the parameter update laws are designed as follows:

$$\begin{aligned}
\dot{\hat{A}} &= -x_1 z_1 e_1 + x_1 y_1 e_1 - x_2 z_2 e_2 + x_2 y_2 e_2 - K_4 e_A \\
\dot{\hat{B}} &= -x_1 z_2 e_2 + x_1 y_2 e_2 - K_5 e_B \\
\dot{\hat{A}}_1 &= -y_2 x_1 e_1 + x_1 y_3 e_1 - K_6 e_{A_1} \\
\dot{\hat{B}}_1 &= x_1 y_1 e_1 - K_7 e_{B_1} \\
\dot{\hat{C}}_1 &= -x_2 e_2 - x_3 e_3 - K_8 e_{C_1} \\
\dot{\hat{a}} &= -z_1^2 z_3 x_1 e_1 + z_1^2 z_3 x_3 e_3 - K_9 e_a \\
\dot{\hat{b}} &= -x_3 z_3 e_3 - K_{10} e_b \\
\dot{\hat{c}} &= z_1^2 x_1 e_1 - K_{11} e_c \\
\dot{\hat{a}}_1 &= w_1^2 w_3 e_3 - w_1^2 w_3 e_1 - K_{12} e_{a_1} \\
\dot{\hat{b}}_1 &= -w_3 e_3 - K_{13} e_{b_1} \\
\dot{\hat{c}}_1 &= w_1^2 e_1 - K_{14} e_{c_1}
\end{aligned} \tag{22}$$

where $K_i > 0$ are constants, $e_A = \hat{A} - A, e_B = \hat{B} - B, e_{A_1} = \hat{A}_1 - A_1, e_{B_1} = \hat{B}_1 - B_1, e_{C_1} = \hat{C}_1 - C_1, e_a = \hat{a} - a, e_b = \hat{b} - b, e_c = \hat{c} - c, e_{a_1} = \hat{a}_1 - a_1, e_{b_1} = \hat{b}_1 - b_1, e_{c_1} = \hat{c}_1 - c_1$ and $\hat{A}, \hat{B}, \hat{A}_1, \hat{B}_1, \hat{C}_1, \hat{a}, \hat{b}, \hat{c}, \hat{a}_1, \hat{b}_1, \hat{c}_1$ are the estimated values of the parameters $A, B, A_1, B_1, C_1, a, b, c, a_1, b_1, c_1$ respectively.

The Lyapunov function V, being a positive definite function on R^{14} and having a negative definite derivative on R^{14}, satisfies the condition of the Stability Theory of Lyapunov, the slave system anti-synchronizes with the compound of the master systems with the designed controllers (21) and parameter update law (22).

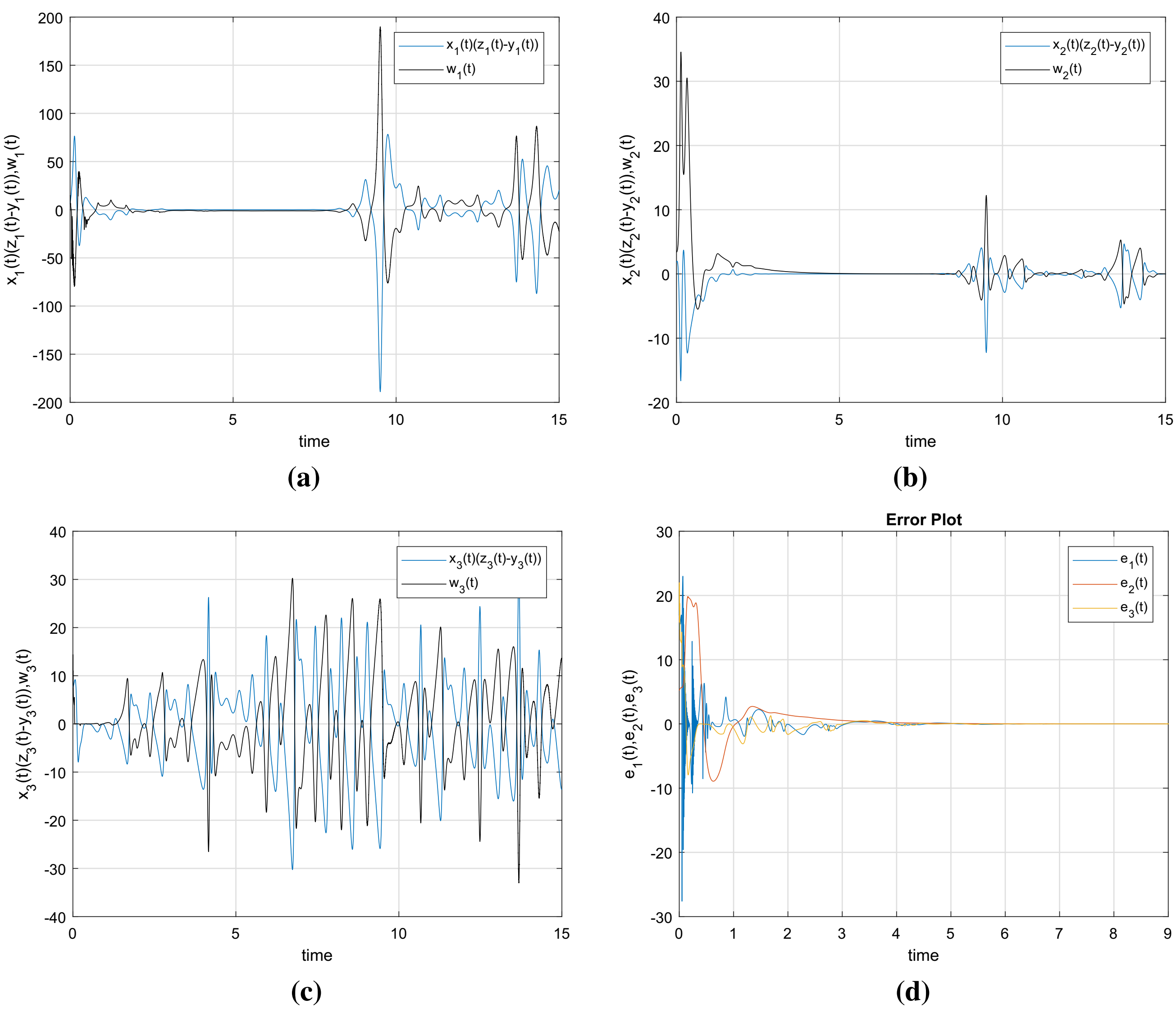

Fig. 6 **a**, **b**, **c** Trajectories of the anti-synchronized master and slave systems, **d** the simultaneous error plot of the system

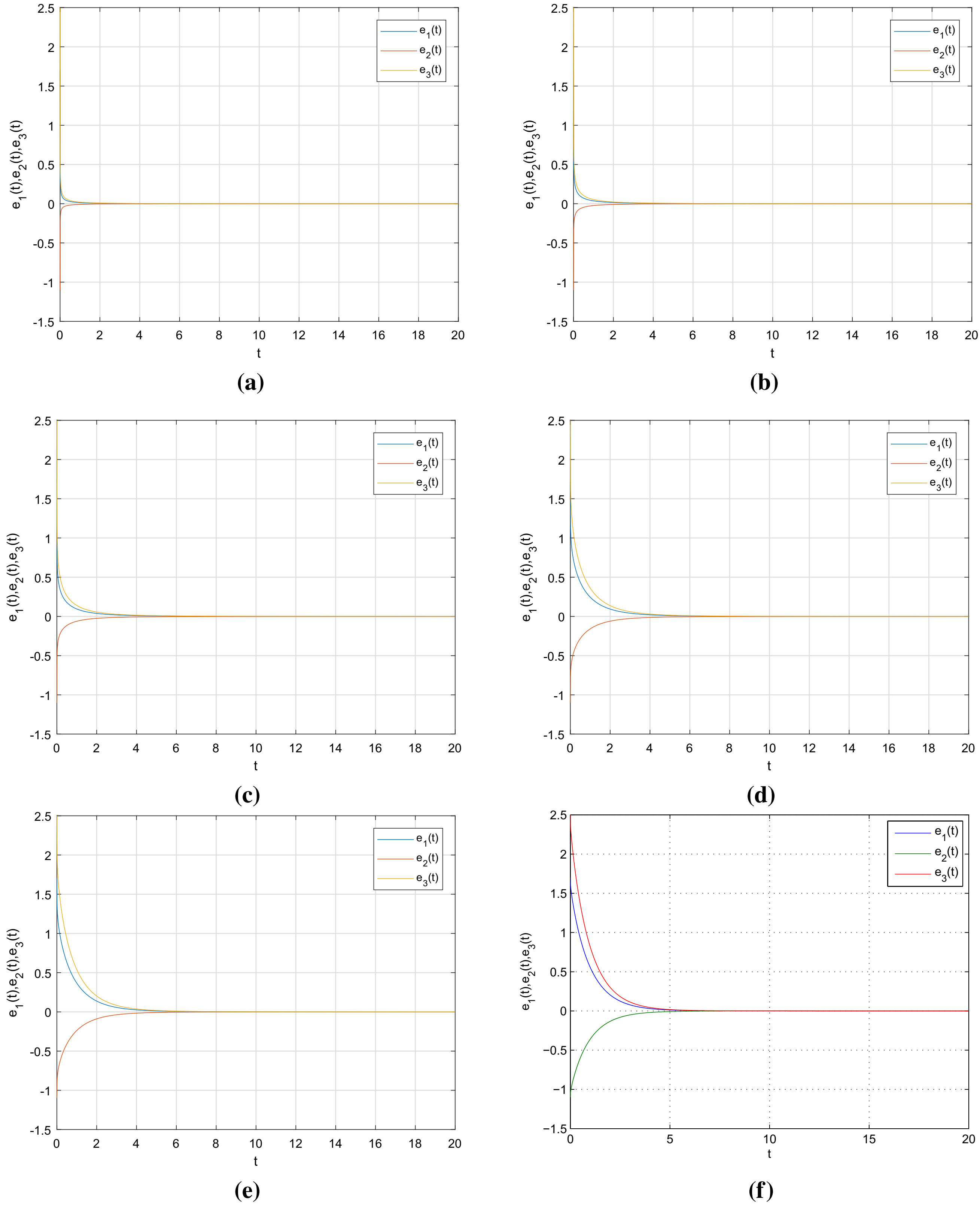

Fig. 7 Error convergence for fractional order **a** 0.6, **b** 0.7, **c** 0.8, **d**, 0.9, **e** 0.943, **f** 0.987

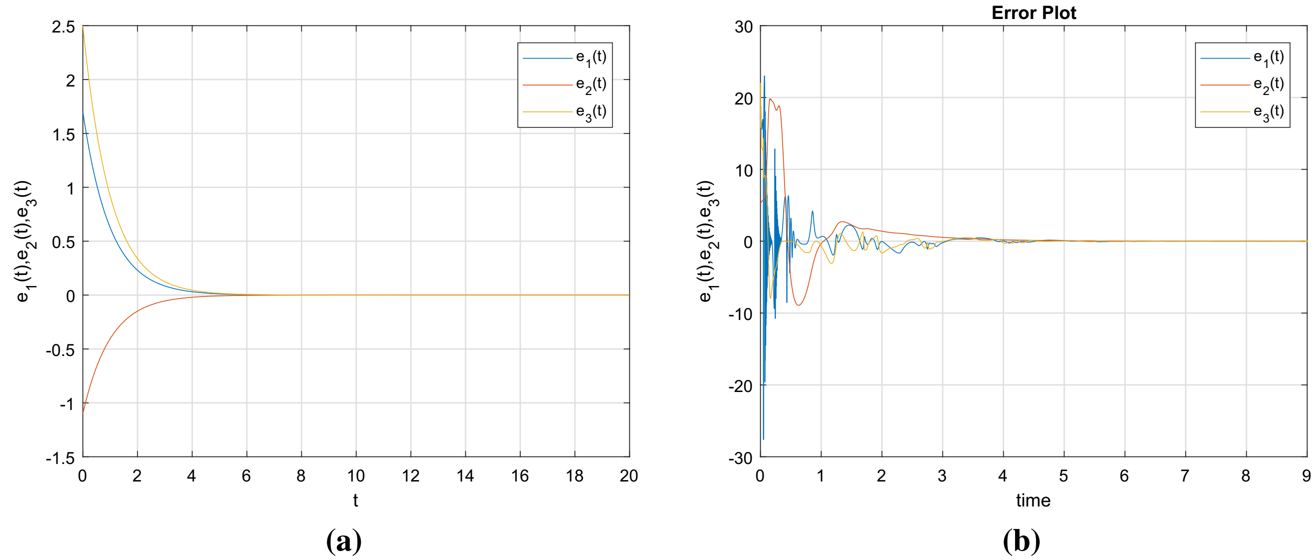

Fig. 8 Error convergence for order one by **a** active control method, **b** parameter estimation adaptive method

We now take the Lyapunov function as $V(e(t)) = \frac{1}{2}e(t)e(t)^T$

$$= \frac{1}{2}({e_1}^2 + {e_2}^2 + {e_3}^2 + {e_A}^2 + {e_B}^2 + {e_{A_1}}^2 + {e_{B_1}}^2 + {e_{C_1}}^2 + {e_a}^2 + {e_b}^2 + {e_c}^2 + {e_{a_1}}^2 + {e_{b_1}}^2 + {e_{c_1}}^2)$$

$$\begin{aligned} V(\dot{e}(t)) &= e_1\dot{e_1} + e_2\dot{e_2} + e_3\dot{e_3} + e_A\dot{e_A} + e_B\dot{e_B} + e_{A_1}\dot{e_{A_1}} \\ &+ e_{B_1}\dot{e_{B_1}} + e_{C_1}\dot{e_{C_1}} + e_a\dot{e_a} \\ &+ e_b\dot{e_b} + e_c\dot{e_c} + e_{a_1}\dot{e_{a_1}} + e_b\dot{e_{b_1}} + e_{c_1}\dot{e_{c_1}} \\ &= e_1(-K_1e_1) + e_2(-K_2e_2) + e_3(-K_3e_3) + e_A(-K_4e_A) \\ &+ e_B(-K_5e_B) + e_{A_1}(-K_6e_{A_1}) + e_{B_1}(-K_7e_{B_1}) \\ &+ e_{C_1}(-K_8e_{C_1}) + e_a(-K_9e_a) + e_b(-K_{10}e_b) \\ &+ e_c(-K_{11}e_c) + e_{a_1}(-K_{12}e_{a_1}) \\ &+ e_{b_1}(-K_{13}e_{b_1}) + e_{c_1}(-K_{14}e_{c_1}) < 0 \end{aligned}$$

Numerical simulations Numerical simulations have been done using MATLAB. We take here $\alpha_i = \beta_i = \gamma_i = \delta_i = 1 \forall i = 1..3$, which means the slave system will completely anti-synchronize with the compound of the multi-drive master systems. Also K_1, K_2, K_3 have been chosen to be 1. The initial conditions of the parameter estimates are chosen as $\hat{A} = \hat{B} = \hat{A_1} = \hat{B_1} = \hat{C_1} = \hat{a} = \hat{b} = \hat{c} = \hat{a_1} = \hat{b_1} = \hat{c_1} = 0.1$ The parameter estimates of the unknown parameters have been displayed graphically in Fig. 5. The trajectories of the master systems(6)-(8) and slave system (9) are shown to get anti-synchronized in Fig. 6. Also,the error plot of the system converges to zero and have been displayed in Fig. 6 for the initial conditions (8.1, −1.1, 2.5).

6 Comparison of the error convergence rates for different fractional orders and for different synchronization schemes

A comparison between the rate of error convergence using Active Control Method for different fractional orders brings to notice an interesting property which has been displayed in Fig. 7. On increasing the fractional order from 0.6 to 0.7 the time of convergence increases from 2 units to 3 units, for fractional order 0.8 it reaches 4 units, for fractional order 0.9 it increases by a unit to approx. 5.6 units and eventually becomes constant for fractional orders 0.943, 0.987 and 1.

Also, a comparison of rate of error convergence for order 1 using active control method and parameter estimate adaptive method has been made which has been displayed in Fig. 8. For active control method error converges to 0 at 5.6 units approx. and for parameter estimation adaptive method error converges at 5.9 units approx.

7 Application in secure communication

Owing to the sensitive dependence of chaotic systems on initial conditions and parameter values, the application of chaos synchronization in the field of secure communication, image encryption, control systems etc. is growing, leading to designing of new synchronization techniques using various control methods.

We now display the application of the above designed novel synchronization technique in the field of secure communication. The main idea is to hide the information

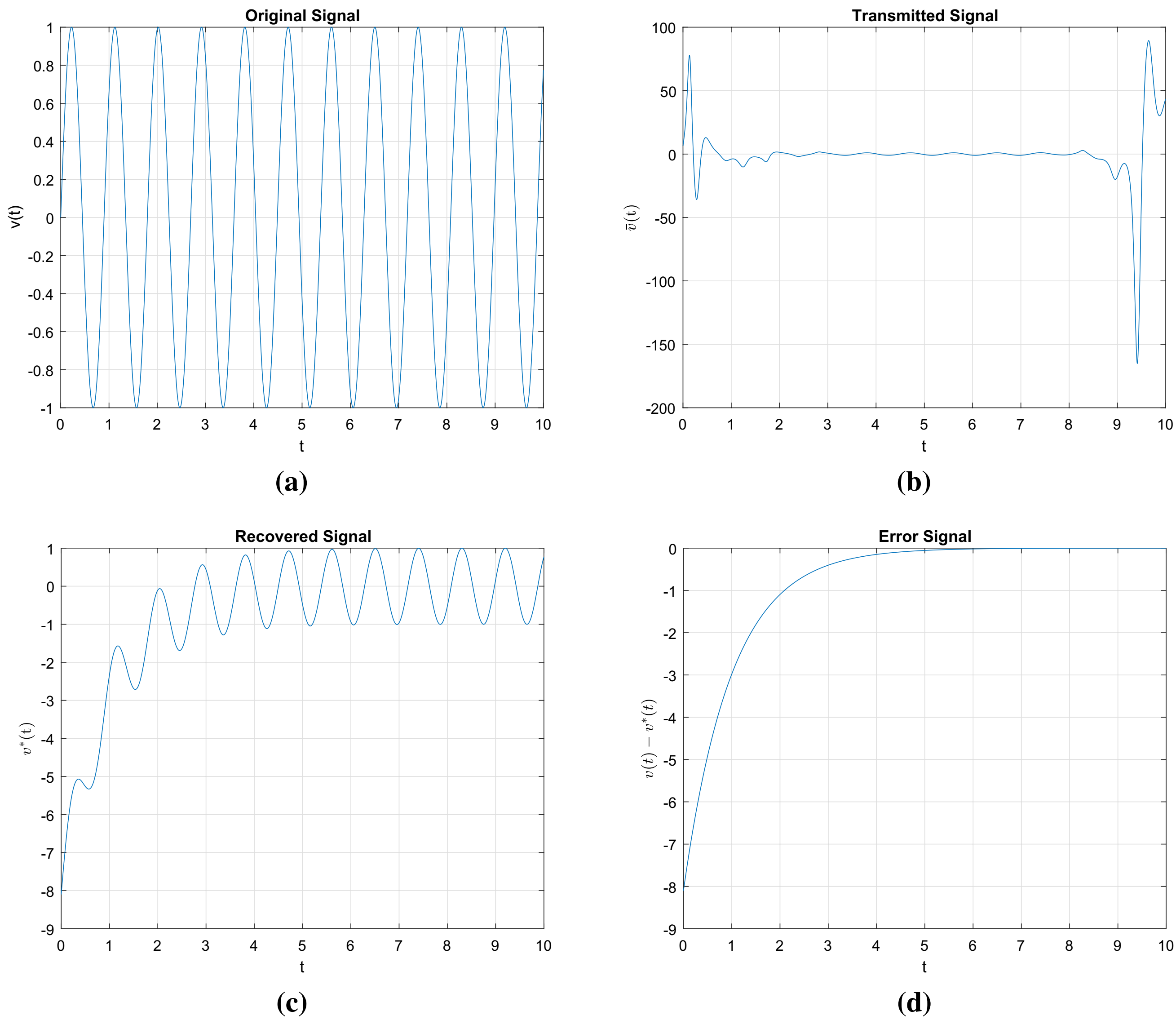

Fig. 9 **a** The original signal, **b** the transmitted signal, **c** the recovered signal, **d** error signal

signal masking it with the chaotic signals and transmit to be recovered later only by the authorized receiver after carrying out the required synchronization. The following example illustrates this.

Let the information signal be $v(t) = sin(7t)$. We suitably mask it with chaotic signal $x_1(z_1 - y_1)$ and transmit the encrypted signal $\bar{v}(t)$. By performing the required synchronization with the chaotic system at the receiving end using the controller u_1 as designed above, the original information signal is recovered, $v^*(t)$. The results have been displayed in Fig. 9.

8 Conclusion

In this paper, compound difference anti-synchronization results have been computed using active control method and parameter estimate adaptive method. The results obtained from both the techniques have been compared. Also, a comparison between the error convergence rates for different fractional orders has been made.

Here we have considered the fractional and integer ordered Rikitake system, fractional and integer ordered El-Nino system and the generalized Lotka Volterra system for the numerical simulations. The computational results carried out in this manuscript validate the theoretical results. Such techniques can be used to study the effect of the Earth's magnetic field (Rikitake system), weather(El-Nino System) and other co-existing species (G.L.V-biological System) on a particular species (represented by the slave G.L.V. System).

Also, with the increasing demand of security of transmission of information, the designed technique would find suitable application in field of secure communication, chaotic encryption and so on.

Further, in this direction we can extend these studies on systems interrupted by model uncertainties and external disturbances. Also, we can conduct the study of compound difference anti-synchronization on discrete time interval in comparison to the continuous time as taken in this manuscript.

Acknowledgements The second author is funded by the J.R.F. of C.S.I.R., India HRDG(CSIR)sanction letter no. 09/466(0189)/2017-EMR-I.

Compliance with ethical standards

Conflict of interest The corresponding author confirms no conflict of interest among the authors.

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