MHD stagnation point flow of Williamson and Casson fluids past an extended cylinder: a new heat flux model

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Abstract

The impact of heat transfer in MHD flows along a stretching cylinder is playing a vital role for heat exchangers, fiber coating, transportation, etc. Currently, a lot of theoretical models are accessible for illustrating the thermal transfer impact of non-Newtonian liquid flows over a cylinder. Also, an external magnetic force is spot-on to deal with the physical features of the fluids to oversee the nature of thermal and momentum transfer in the system. Considering this fact, we inspect the heat transport behavior of two different non-Newtonian MHD flows due to stretching of a cylinder with heat generation, by taking the advantage of a new heat flux theory conceived by Christov–Cattaneo. The basic PDEs are converted into ODEs with the suitable similarity transformations. These ODEs are solved by fourth order Runge–Kutta based shooting system. Plots are drawn to discern the influence of sundry parameters on the flow fields (velocity and temperature). Along with them the rate of heat transfer and friction factor are bestowed in table. From the results, we notice that the influence of thermal stratification and curvature parameters have a propensity to increase both the velocity and thermal fields. Thermal relaxation parameter effectively lifts the friction factor in the flow of Williamson fluid than that of Casson fluid.

Keywords MHD · Heat transfer · Non-Newtonian fluid · Convection · Thermal stratification · Cylinder

List of symbols

B_0	Applied magnetic field strength
b	Radius of the cylinder
C_f	Friction factor
Gr	Grashof number
J_m	Wall shear stress
J_w	The measure of the heat transfer
k	Thermal conductivity
l_0	Characteristic length
M	Magnetic field parameter
m_1, m_2	Constants
Nu	Nusselt number
Pr	Prandtl number
p_0 & p_∞	Velocity components
c_p	Specific heat
Q_l	Component of heat source
Q	Heat source parameter
S	Thermal stratification parameter

T	Temperature of the fluid
T_0	Component of temperature
V_r	Velocity ratio parameter
u, v	Velocity components in x, y – directions
u_e	Free stream velocity
u_w	Stretching velocity of the cylinder

Greek letters

β	Thermal relaxation parameter
ν	Viscosity (kinematic)
σ	Electrical conductivity
β_T	Thermal expansion coefficient
δ	Relaxation time of heat flux
ρ	Fluid's density
Γ	Positive time constant
γ	Casson fluid parameter
λ	Williamson fluid parameter
ξ	Stream function

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μ Viscosity (dynamic)
 ζ Similarity variable

1 Introduction

In non-Newtonian liquids there is a non-linear interaction between change in warping and shear stress. The investigation on those fluids has allured so much attention due to their importance in industries, medicine and mechanical engineering. Fluids that do not fulfil the above said nonlinear correspondence are termed as non-Newtonian fluids. Some agreeable non-Newtonian fluids are starch, blood, ketchup and some oils etc. The most commonly known non-Newtonian fluids are Casson and Williamson fluids. Casson fluid is a shear thinning fluid. Honey, soups, thick juices and tomato sauce belong to Casson fluid. The analysis of the boundary layer flows of Williamson liquids have great importance in abundant branches of engineering and science, mainly in geophysics, bio-engineering, food processing and drilling operations. Owing to this significance, number of mathematical models can be found in the literature to examine their flow behaviour. Hayat et al. [1] bestowed an exact solution for boundary layer motion of non-Newtonian liquid across a stretched sheet. The impression of drag force on Newtonian fluid motion owing to an exponentially stretched sheet was deliberated by Nadeem et al. [2]. Further, the flows of non-Newtonian type were reported by Mukhopadhyay [3], with the aid of shooting technique. Malik et al. [4] inspected the impression of heat sink on Williamson liquid over a cylinder and conclude that non-Newtonian parameter has a propensity to reduce the velocity field. Recently, Kumar et al. [5, 6] presented dual solutions for MHD shear thickening liquid over a curved geometry.

Convective heat transfer shows an essential task in some physical practices like tumour cures and industrial processes to elucidate engineering problems. Normally, the performing ability of the machines will be contingent on strength of the fluid's heat transfer rate (water, oils, kerosene etc.). Revankar [7] discussed the impact of free convection on permeable vertical plate with suction or injection. The flow and heat transfer aspects of Newtonian fluid caused by a stretched geometry were reported by Chiam [8]. The similar type of investigation with velocity slip and heat transfer impact on third grade fluid flow was elucidated by Sahoo [9]. Hsiao [10] studied the action of drag force on non-Newtonian fluid owing to permeable wedge with natural convection.

The analysis of MHD deals with the motion of electrically accompanying liquids. This conception has related in numerous processes such as flow meters, power generators, pumps, magnetic drug treating, cooling of reactors,

plasma studies and the design of heat exchangers. The Lorentz force impact on Casson fluid due to stretched geometry was scrutinized by Nadeem et al. [11]. Sandeep and Sugunamma [12] considered a time dependent flow problem with heat source. The impact of frictional heat and Lorentz force on Newtonian fluid across a shrinking surface was reported by Kumar et al. [13]. Recently, Kumar et al. [14] and Kalyani et al. [15] discussed the flow and heat transfer characteristics of non-Newtonian fluid over a stretching surface in the presence of heat source.

Stagnation point flow plays a significant role in travelling management, hot rolling, production of paper, some extrusion production and wire drawing etc. Najib et al. [16] described the chemically reacting flow across a cylinder near the stagnation point. Further, this work was extended by Yasin et al. [17] with drag force and thermal transport. The influence of heat transport on nanofluid motion owing to stretched cylinder was deliberated by Omar et al. [18] and Raju et al. [19]. They found that suction constraint has propensity to shrink the thermal fields. Further Kumar et al. [20] deliberated the flow and thermal attributes of Newtonian liquid past a sheet near the neighbourhood of stagnation in the attendance of radiation.

The flows across a cylinder (stretched) has noble significance in manufacturing process, metal industries, plastic sheets extrusion, cooling of a large magnetite plates in a hot tub, making paper plates and spinning of fiber. Initially, in 1961, Rashko [21] designed and developed a model to examine the flow past a cylinder. In continuation of this, Proudman and Johnson [22] reported the stagnation point flow past a cylinder. Later on, Fornberg [23] had been given a numerical treatment to examine the flow owing to cylinder. The impact of magnetic force and constant heat flux on an unsteady flow through a cylinder was discussed by Ganesan and Loganathan [24] and they state that a growth in the magnitude of Prandtl number boosts the heat transfer performance. The characteristics of the flow caused by a circular cylinder were reported by Nobari and Naderan [25]. Recently, Authors [26, 27] inspected the flow of MHD non-Newtonian liquid past a melting surface with the attendance of heat source/sink. It was concluded that heat source/sink parameters plays a crucial role in the heat transfer performance.

The heat flux phenomenon was first initiated by Fourier [28] in 1900 century. Then after, so many researchers employed the Fourier's law to illustrate several engineering and industrial situations and found some drawbacks in this model. So Cattaneo [29], recommended a new model with thermal relaxation time. Further, Christov [30] had given an advanced heat flux derivative model by extending the earlier Cattaneo's model. Later on, sundry researchers focused their research on different kind of flows with Christov–Cattaneo heat flux. Han et al. [31] bestowed an

analytical treatment for Cattaneo–Christov motion of viscoelastic liquid over a stretched plate. Recently, Mustafa [32] and Waqas et al. [33] used the above said heat flux concept to analyse the thermal transport behaviour and concluded that thermal conductivity parameter helps to inflate the temperature distribution. The impact of variable heat generation or absorption on Cattaneo–Christov flow through a wedge was reported by Kumar et al. [34]. Ramadevi et al. [35] continued this work to scrutinize the influence of chemical response and heat source/sink on the flow past a variable thicked surface. They stated that the action of chemical reaction parameter is decreasing function of concentration field. Very recently, Sandeep and Reddy [36] focused on Cattaneo–Christov flow caused by cone or plate considering frictional heating and concluded that Eckert number escalates the momentum and thermal distributions. The impact of thermal stratification on Jeffery fluid flow due to stretched cylinder with modified heat flux was explained by Hayat et al. [37] and stated that thermal and momentum fields diminish with the action of thermal stratification.

To the best of our knowledge, it is clear that a few researchers concentrated on the stagnation point motion of fluids due to stretched cylinder. Also, to the extent of our belief no researcher examined the Cattaneo–Christov motion of non-Newtonian liquids via stretching cylinder near the stagnation point. Heat source and thermal stratification are accounted. Similarity transmutations are applied to alter the flow governing non-linear equations into coupled ODE's. Bvp5c technique is utilized to solve these ODE's. Impacts of numerous dimensionless parameters on the heat and velocity functions are represented through plots. Along with them measure of thermal transport and skin friction coefficient are presented in a separate table.

2 Formation of the model

Consider a two-dimensional electrically accompanying flow of non-Newtonian (Casson and Williamson) liquids across a stretched cylinder with heat absorption/generation and stratification. The fluid motion is time independent and laminar. Let us take $u_w = p_0 x l_0^{-1}$ as the velocity near the cylinder and $u_e = p_\infty x l_0^{-1}$ as the free stream

Fig. 1 Flow geometry

velocity. Here p_0 & p_∞ are velocity components and l_0 is the characteristic length. The following are some assumptions on the flow model.

- Casson and Williamson non-Newtonian fluid model.
- Ohmic heating is very small.
- The coefficient of electrical conductivity is regarded as scalar.
- Impact of frictional heat and induced magnetic field are ignored.
- Cattaneo–Christov heat flux model is adopted.

Let us consider the co-ordinate system (x, r) , where x axis is appraised in the way of axis of the geometry and r –axis is orthogonal to it. A constant magnetic field of strength B_0 is deployed in the radial way of the cylinder as portrayed in Fig. 1. The cylinder radius is b . The temperature adjacent to the cylinder is $T_w = T_0 + m_1 x l_0^{-1}$ and the ambient fluid temperature is $T_\infty = T_0 + m_2 x l_0^{-1}$, where m_1 & m_2 are constants.

With the afore taken assumptions, the flow equations are (see Malik et al. [4], Omar et al. [16] and Hayat et al. [37])

$$\frac{\partial^2 \xi}{\partial x \partial r} - \frac{\partial^2 \xi}{\partial r \partial x} = 0, \quad (1)$$

$$\frac{1}{r^2} \left(\frac{\partial \xi}{\partial r} \frac{\partial^2 \xi}{\partial x \partial r} - \frac{\partial \xi}{\partial x} \frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r} \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial r} \right) = u_e \frac{du_e}{dx} - \frac{\sigma B_0^2}{\rho r} \frac{\partial \xi}{\partial r} + g \beta_T (T - T_\infty) + v \left(1 + \frac{1}{\gamma} \right) \left\{ \left(\frac{1}{r^3} \frac{\partial \xi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r} \frac{\partial^3 \xi}{\partial r^3} \right) + \frac{\sqrt{2} r}{r^3} \left(r \frac{\partial^2 \xi}{\partial r^2} \frac{\partial^3 \xi}{\partial r^3} - \frac{\partial \xi}{\partial r} \frac{\partial^3 \xi}{\partial r^3} \right) \right\} + \frac{3r}{\sqrt{2} r^3} \left(\frac{2}{r} \frac{\partial \xi}{\partial r} \frac{\partial^2 \xi}{\partial r^2} - \frac{1}{r^2} \left(\frac{\partial \xi}{\partial r} \right)^2 - \left(\frac{\partial^2 \xi}{\partial r^2} \right)^2 \right) \quad (2)$$

$$\frac{1}{r} \frac{\partial \xi}{\partial r} \frac{\partial T}{\partial x} - \frac{1}{r} \frac{\partial \xi}{\partial x} \frac{\partial T}{\partial r} = \frac{k}{\rho C_p} \left(\frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} \right) + \frac{Q_l(T - T_\infty)}{\rho C_p} - \lambda_1 \left(\frac{1}{r^2} \left(\frac{\partial \xi}{\partial r} \right)^2 \frac{\partial^2 T}{\partial x^2} + \frac{1}{r^2} \left(\frac{\partial \xi}{\partial x} \right)^2 \frac{\partial^2 T}{\partial r^2} - \frac{2}{r^2} \frac{\partial \xi}{\partial r} \frac{\partial \xi}{\partial x} \frac{\partial^2 T}{\partial r \partial x} + \frac{1}{r^2} \frac{\partial \xi}{\partial r} \frac{\partial^2 \xi}{\partial x \partial r} \frac{\partial T}{\partial x} - \frac{1}{r^2} \frac{\partial \xi}{\partial r} \frac{\partial^2 \xi}{\partial x^2} \frac{\partial T}{\partial r} + \frac{1}{r^3} \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial r} \frac{\partial T}{\partial x} - \frac{1}{r^2} \frac{\partial \xi}{\partial x} \frac{\partial^2 \xi}{\partial r^2} \frac{\partial T}{\partial x} - \frac{1}{r^3} \left(\frac{\partial \xi}{\partial x} \right)^2 \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial \xi}{\partial x} \frac{\partial^2 \xi}{\partial r \partial x} \frac{\partial T}{\partial r} \right), \quad (3)$$

For the designed problem, the boundary conditions are taken as (see Malik et al. [4], Omar et al. [16] and Hayat et al. [37])

$$r^{-1} \xi_r = u_w = p_0 x l_0^{-1}, \quad -r^{-1} \xi_x = 0, \\ T = T_w = T_0 + m_1 x l_0^{-1}, \quad \text{at } r = b, \quad (4)$$

$$r^{-1} \xi_r \rightarrow u_e = p_\infty x l_0^{-1}, \quad T = T_\infty = T_0 + m_2 x l_0^{-1} \quad \text{as } r \rightarrow \infty, \quad (5)$$

The proposed problem shows two different fluid flows as explained below.

- Casson fluid flow: $\Gamma = 0$, $\gamma \neq 0$.
- Williamson fluid flow: $\Gamma \neq 0$, $\gamma \rightarrow \infty$.

It is possible to transfigure the Eqs. (1)–(3) as a group of dimensionless non-linear equations. We can do the same for the boundary conditions (4)–(5) also. So we first introduce the stream function $\xi(x, r)$ as

$$\xi = \sqrt{p_0 v l_0^{-1}} b x f(\zeta), \quad (6)$$

Here ζ is the similarity variable defined as

$$\delta = \left(\frac{v l_0}{p_0 b^2} \right)^{1/2}, \quad \lambda = \frac{\Gamma x}{\sqrt{2v}} \left(\frac{p_0}{l_0} \right)^{3/2}, \quad M = \frac{\sigma B_0^2 l_0}{p_0 \rho}, \quad Gr = \frac{m_1 g \beta_T l_0}{p_0^2}, \\ V_r = \frac{p_\infty}{p_0}, \quad Pr = \frac{\mu C_p}{k}, \quad \beta = \frac{\lambda_1 p_0}{l_0}, \quad S = \frac{m_2}{m_1}, \quad Q = \frac{Q_l l_0}{p_0 \rho C_p}, \quad (13)$$

$$\zeta = \sqrt{p_0 (v l_0)^{-1}} \frac{r^2 - b^2}{2b}, \quad (7)$$

So the velocity distributions u , v and the expression for temperature T will be, (see Hayat et al. [37])

$$u = r^{-1} \xi_r = p_0 x l_0^{-1} f'(\zeta), \quad v = -r^{-1} \xi_x \\ = -b r^{-1} \sqrt{p_0 v l_0^{-1}} f(\zeta), \quad T = T_\infty + m_1 x l_0^{-1} \theta(\zeta), \quad (8)$$

Using Eqs. (6)–(8), the Eqs. (2)–(5) can be transformed as

$$(1 + \gamma^{-1}) \left\{ (1 + 2\delta \zeta) \frac{d^3 f}{d\zeta^3} + 2\delta \frac{d^2 f}{d\zeta^2} + 3\lambda \delta \left(\frac{d^2 f}{d\zeta^2} \right)^2 + 2\lambda (1 + 2\delta \zeta) \frac{d^2 f}{d\zeta^2} \frac{d^3 f}{d\zeta^3} \right\} \\ - M \frac{df}{d\zeta} + Gr \theta + f \frac{d^2 f}{d\zeta^2} - \left(\frac{df}{d\zeta} \right)^2 + V_r^2 = 0, \quad (9)$$

$$(1 + 2\delta \zeta) \frac{d^2 \theta}{d\zeta^2} + 2\delta \frac{d\theta}{d\zeta} + Pr f \frac{d\theta}{d\zeta} - Pr (S + \theta) \frac{df}{d\zeta} \\ + Pr \beta \left\{ (S + \theta) f \frac{d^2 f}{d\zeta^2} - f \frac{df}{d\zeta} \frac{d\theta}{d\zeta} - (S + \theta) \left(\frac{df}{d\zeta} \right)^2 - f^2 \frac{d\theta}{d\zeta} \right\} + Q \theta = 0, \quad (10)$$

$$\frac{df}{d\zeta} = 1, \quad f = 0, \quad \theta = 1 - S, \quad \text{at } \zeta = 0, \quad (11)$$

$$\frac{df}{d\zeta} = V_r, \quad \theta = 0, \quad \text{as } \zeta \rightarrow \infty, \quad (12)$$

In Eqs. (9)–(12), $\delta, \lambda, M, Gr, V_r, Pr, \beta, S$ and Q are dimensionless physical parameters. These can be defined as,

The physical quantities in view of industrial, bio-medical and scientific needs are skin friction coefficient and heat transfer coefficient. There can be defined as (see Ref. [4] and [37])

$$C_f = \frac{J_m}{1/2 \rho p_0^2}, \quad Nu = \frac{x J_w}{\alpha (T_w - T_\infty)}, \quad (14)$$

For Casson and Williamson fluid model shear stress (J_m) at the surface of cylinder is

$$J_m = \mu (1 + \gamma^{-1}) \left(\frac{\partial u}{\partial r} + \frac{\Gamma}{\sqrt{2}} \left(\frac{\partial u}{\partial r} \right)^2 \right)_{r=b}, \quad (15)$$

Fig. 2 Effect of curvature parameter on fluid velocity

Fig. 3 Effect of curvature parameter on fluid temperature

The heat flux (J_w) can be taken as

$$J_w = -k \left(\frac{\partial T}{\partial r} \right)_{r=b}, \quad (16)$$

Using Eqs. (15) and (16) in (14), we get the required expressions for C_f and Nu is

$$C_f Re_x^{1/2} = \left(1 + \frac{1}{\gamma} \right) \left(f''(0) + \frac{\lambda}{2} (f''(0))^2 \right), \quad (17)$$

boundary value problem denoted by Eqs. (9)–(11) are converted to initial value problem by making use of shooting technique. Since the governing dimensionless equations are nonlinear, they cannot be solved by Runge–Kutta method. We solve them by reducing to a set of first order partial differential equations. The procedure is as follows.

First we assume that

$$f = y_1, f' = y_2, f'' = y_3, f''' = y_3', \theta = y_4, \theta' = y_5, \theta'' = y_5', \quad (19)$$

Substituting Eq. (19) in the Eqs. (9) and (10), we get the following set of simultaneous first order ODE's.

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ y_3' &= \left(\frac{1}{(1 + \gamma^{-1})(1 + 2\delta\zeta)(1 + 2\lambda f'')} \right) \left(-(1 + \gamma^{-1})(2\delta f'' + 3\lambda\delta(f'')^2) + Mf' - Gr\theta - ff'' + (f')^2 - V_r^2 \right) \\ y_4' &= y_5, \\ y_5' &= \left(\frac{1}{(1 + 2\delta\zeta)} \right) \left(-2\delta\theta' - Pr f\theta' + Pr(S + \theta)(f' - \beta(ff'' + (f')^2)) + Pr\beta\{ff'\theta' + f^2\theta'\} - Q\theta \right), \end{aligned}$$

$$Nu Re_x^{-1/2} = -\theta'(0), \quad (18)$$

Here $Re_x = \frac{xu_w}{\nu}$ (Reynolds number).

The subjected boundary conditions in new variables are

$$\begin{aligned} y_1(0) &= 0, y_2(0) = 1, y_4(0) = 1 - S, \\ y_2(\infty) &= V_r, y_4(\infty) = 0, \end{aligned}$$

3 Solution procedure

The system of nonlinear ODEs (9) and (10) with the restrictions of boundary (11) and (12) have solved by Runge–Kutta fourth order method. Afore that the

4 Expansion of results

The transmuted nonlinear ODE's (9) and (10) in view of the conditions (11) and (12) are solved by shooting and fourth order R.K. methods. For calculation drive, we assumed

Fig. 4 Effect of magnetic field parameter on fluid velocity

Fig. 6 Effect of Grashof number on fluid velocity

Fig. 5 Effect of magnetic field parameter on fluid temperature

Fig. 7 Effect of Grashof number on fluid temperature

sundry parameter values as $S = 0.5$, $\delta = 1$, $\beta = 1.2$, $Q = 0.5$, $M = 0.2$, $Gr = 0.1$, $Pr = 0.71$, $\xi = 2$ and $V_r = 0.1$. In the entire analysis the above standards are believed as perpetual excluding the variants display in the graphs and table. In graphs dashed and solid line specify the curves of Williamson and Casson liquids correspondingly. The impact of dimensionless sundry constraints on skin friction coefficient, measure of thermal transport, velocity and temperature were reported through tables and plots, which are obtained by bvb5c MATLAB package.

The impact of increase in curvature parameter (δ) on velocity and heat functions can be shown in Figs. 2 and 3. It is perceived that a growth in δ boosts the curves of velocity filed. Generally, the radius of the cylinder will be minimized when δ is maximized. So the velocity enhances

due to low resistance in the fluid motion. Meanwhile, we detect that higher value of δ give rise to fluid temperature. Because, increase in δ causes the transport of heat from cylinder to the flow.

The impact of M on the distribution of momentum and temperature can be revealed in Figs. 4 and 5. An uplifting the values of M shrinks the velocity distribution and its layer thickness. But a conflicting consequence is perceived in fluid temperature distribution via Fig. 5. A resistive type of force (Lorentz force) will be developed in the flow due to higher values of M . This force is capable of preventing the flow speed. The same force delivers some heat energy to the flow.

Fig. 8 Effect of velocity ratio parameter on fluid velocity

Fig. 10 Effect of Prandtl number on fluid velocity

Fig. 9 Effect of velocity ratio parameter on fluid temperature

Fig. 11 Effect of Prandtl number on fluid temperature

Figure 6 renders the graph of Grashof number (Gr) versus fluid velocity. We see that ascending values of Gr improves the fluid motion. Figure 7 allows one to say that rise in Gr dwindles the fluid temperature. The ratio between the buoyancy and viscous forces is known as Grashof number. Also it is curious to remark that the velocity and temperature fields of the Williamson fluid fruitfully inflated when likened to the other fluid (Casson).

Figures 8 and 9 are drawn to know the essence of V_r on velocity and thermal fields. Increase in V_r maximizes the velocity field but minimizes the fluid temperature. The consequence of Pr on the curves of temperature and velocity are unveiled in Figs. 10 and 11. For intensifying Pr

there is decay in the fluid velocity and temperature. It is also explicit that fluids having low Pr moves faster when compared to the fluid with high Pr . This helps to maximize the cooling rate in the flow. So, the thickness of both boundary layers shrinks with high Pr .

Figures 12 and 13 are delineated to know the spirit of thermal relaxation parameter (β) on the profiles of velocity, temperature. We found that β helps to diminish the velocity and temperature curves. The gap between the fluid molecules will occur with the presence of relaxation time. This gap causes a decrement in both fluid speed and thermal energy in the region.

Fig. 12 Effect of thermal relaxation parameter on fluid velocity

Fig. 14 Effect of heat source/sink on fluid velocity

Fig. 13 Effect of thermal relaxation parameter on fluid temperature

Fig. 15 Effect of heat source/sink on fluid temperature

The influence of heat source (Q) on the velocity and heat functions are shown in Figs. 14 and 15. The temperature and velocity fields enhance with higher values of Q . Because, a growth in the values of Q leads to the making of heat to the flow. Also it is worth to note the heat and velocity curves of the Williamson liquid fruitfully inflated when likened to the Casson liquid.

Figures 16 and 17 render the flow attributes for distinct values of thermal stratification parameter (S). Inflation in S results an upgrade in momentum and thermal fields. For increasing values of S , the fluid density will be more

in higher region than the lower region. Owing to this, we found an enhancement in both the layers thickness.

Table 1 unveils the impact of numerous parameters on the friction factor ($f''(0)$) and heat transfer coefficient ($-\theta'(0)$) for both non-Newtonian liquid flows. It is understood that a raise in the values of Gr , S and V_r , maximizes $f''(0)$ and $-\theta'(0)$ for both Williamson and Casson fluids but a conflicting tendency is apparent with M . An elevation in δ and Q enhances the skin friction coefficient but

Fig. 16 Effect of thermal stratification on fluid velocity

Fig. 17 Effect of thermal stratification on fluid temperature

suppresses the rate of thermal transport for both the non-Newtonian fluids. Boosting values of Pr and β minimizes the skin friction coefficient but maximizes the local Nusselt number.

5 Conclusions

The current work delineates the thermal transport and flow features of Casson and Williamson fluids owing to

Table 1 Variation in $f''(0)$ and $-\theta'(0)$ for Casson and Williamson fluids

δ	M	Gr	V_r	Pr	β	Q	S	$f''(0)$		$-\theta'(0)$	
								Casson fluid	Williamson fluid	Casson fluid	Williamson fluid
1.5								1.0167	1.0341	2.7159	2.6797
3.0								1.3752	1.5255	2.1661	2.1392
5.0								1.6274	1.9205	1.8349	1.1851
	1.5							-0.5308	-0.6696	1.7526	1.7066
	3.0							-1.0377	-1.1356	1.6291	1.5930
	5.0							-1.5813	-1.6124	1.5105	1.4844
		1.5						0.3158	0.2446	1.7278	1.6863
		3.0						1.4054	1.6257	1.9605	1.9414
		5.0						2.6625	3.4646	2.1837	2.1950
			1.5					-0.9535	-1.0250	2.7062	2.6098
			3.0					-0.9124	-0.9818	2.8319	2.7572
			5.0					-0.8539	-0.9210	2.9676	2.9100
				1.5				0.6233	0.6108	3.8565	3.8056
				3.0				0.2742	0.2172	5.2251	5.1576
				5.0				-0.0064	-0.0768	6.5828	6.5074
					1.5			2.2891	2.7735	2.8182	2.8186
					3.0			2.0350	2.3761	3.3145	3.3130
					5.0			1.7869	2.0174	3.8312	3.8247
						1.5		3.3579	4.6893	2.7744	2.7991
						3.0		3.5420	5.1093	2.4460	2.4806
						5.0		3.8508	5.8726	1.9339	1.9919
							1.5	-0.1330	-0.2354	1.5496	1.5294
							3.0	0.8616	0.9269	2.7413	2.7112
							5.0	2.1022	2.6424	4.5076	4.4801

stretching of a cylinder. The Lorentz force, thermal stratification and heat source effects is also considered. A non-Fourier heat flux model is considered to scrutinise the better thermal transport performance. Shooting and fourth order R.K. method is utilized to attain the solution. The principal outcomes are listed below:

- Thermal stratification and curvature parameters have a propensity to boost up both velocity and thermal fields where as the result is reversed with the action of Prandtl number.
- Lorentz force has a tendency to diminish the friction factor.
- Prandtl number displays higher impact on heat transfer rate when compared to the friction factor.
- In the attendance of heat source parameter Casson fluid is slightly more affected when compared with the Williamson fluid.
- Thermal relaxation parameter has proclivity to diminish the curves of velocity and temperature.
- Local heat transfer rate and friction are escalating functions of Grashof number.

Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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