Effect of nano-porosity on postbuckling of non-uniform microbeams

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Abstract

This study is accomplished to investigate the effect of different types of porosity distribution on the static postbuckling response of non-uniform microbeams under the simply supported boundary conditions. The modified couple stress theory is used to capture the size effect in microscales, and the von-Karman strain–displacement relation is employed to describe the nonlinearity effect. The relative elasticity characteristics, e.g., Young's modulus and the material length scale parameter, are considered as a function of relative mass density. The nonlinear governing equations for Euler–Bernoulli porous microbeams with non-uniform cross section are derived by a variational energy method and solved using a closed-form solution. The critical buckling load and the static postbuckling response of porous non-uniform microbeams, non-porous non-uniform microbeams and porous uniform microbeams are compared with together. Therefore, effects of nano-porosity and non-uniformity on the postbuckling behavior of the microbeam are investigated.

Keywords Porosity · Postbuckling · MCST · Non-uniform microbeams

1 Introduction

There are some materials which have inherent porosities such as metal foams and body tissues. Also, some types of porosity are discovered in many structures due to the fabrication process. These porosities can be affected on the behaviors of the structures. In fact, the mechanical behaviors of the porous structures are different from their conventional counterparts. Therefore, it is important to investigate the influences of the porosities on different mechanical behaviors like free vibration, wave propagation, bending, buckling, etc. To this end, it is necessary to guess the porosity distributions that the material properties follow them. The other significant issues in the study of structures made of the porous materials are a correlation between the properties and porosity [1–3] and the determination of a relation between various properties [4, 5].

Even and uneven porosity distributions have been employed to study the different behaviors of porous structures [6–10] as two porosity patterns. Symmetric,

asymmetric and uniform porosity distributions along the thickness are other types of the porosity distributions which have been taken into account to model the metal foams [11–14]. Also, the effects of pore size and pore shape were studied in tissue engineering and biomechanics [15, 16].

In the literature, a wide range of studies have been done on the vibration, buckling and postbuckling analyses of beams especially micro- or nano-beams [17–33], but there are few studies on the postbuckling analysis of porous micro- or nanostructures [34–37]. She et al. [34] presented the thermal buckling and postbuckling of FG nanotubes with porosities. They used even and uneven distributions of porosities to gain the effective material properties. Akbas [35] investigated the nonlinear behaviors of FG porous beams under the thermal effects and presented the porosity and temperature effects, simultaneously. Also, Akbas [36] examined the porosity effect on the postbuckling response of FG porous beams. A postbuckling analysis of nonlocal nano-plates made of

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Fig. 1 **a** Uniform, **b** symmetric and **c** asymmetric distributions of the nano-porosity [13]

FG materials with even and uneven porosities was investigated by Barati and Zenkour [37]. The obtained results by the mentioned authors indicate that the porosity and its distribution can be affected on the nonlinear behaviors like postbuckling response. The present paper illuminates a research for the postbuckling of microbeams made of a nano-porous material based on the modified couple stress theory and considers a variable material length scale parameter for the porous microbeam, while the previous papers considered a constant value for this parameter. Also, this paper focuses on the non-uniformity of the beam cross section. This assumption makes our model more realistic in order to simulate the beam-like bio-structures or other microscale porous beams. In the previously published papers, the non-uniform beams were used for the energy harvesting analysis [38, 39], buckling analysis [40] and free vibration analysis [41, 42].

In this paper, a general study for postbuckling of porous non-uniform microbeams is presented. Three nano-porosity distribution models and the simply supported boundary conditions are considered. A variable material length scale parameter is assumed to describe the size effect of a porous medium. The governing equations of an Euler–Bernoulli microbeam are derived and solved, directly. Finally, the influences of different nano-porosity distributions on the static postbuckling response of porous non-uniform microbeams are studied as a main investigation. Linear and exponential cross-sectional non-uniformities are used as two examples, while the formulations and solution are

presented as a general form. There are some novelties for the present study. First, the material length scale parameter is not constant and can be varied as well as Young's modulus. Second, the effects of porosity, size and cross-sectional non-uniformity are compounded in a single paper.

2 Nano-porous beams

In this study, three types of porosity distributions are considered [13]. In these distributions, the porosity changes the material properties along the beam thickness. Therefore, both maximum and minimum values are assumed for each property. This behavior is similar to behavior of the functionally graded materials. For uniform nano-porosity distribution, Young's modulus and the material length scale parameter are defined as

$$E(z) = E_1(1 - \xi p) \quad (1a)$$

$$l(z) = l_1(1 - \xi_1 p) \quad (1b)$$

As shown in Fig. 1a, the material properties of the uniform porosity distribution are constants but are not equal to the conventional non-porous material. Therefore, the coefficients ξ and ξ_1 can be defined as [13, 43]

$$\xi = \frac{1}{p} - \frac{1}{p} \left(1 - \frac{2}{\pi} + \frac{2}{\pi} \sqrt{1-p} \right)^2 \quad (2a)$$

Fig. 2 Schematic of a non-uniform beam

$$\xi_1 = \frac{1}{p_1} - \frac{1}{p_1} \left(1 - \frac{2}{\pi} + \frac{2}{\pi} \sqrt{1 - p_1} \right)^2 \quad (2b)$$

For symmetric nano-porosity distribution (Fig. 1b):

$$E(z) = E_1 \left(1 - p \cos \left(\frac{\pi z}{h} \right) \right) \quad (3a)$$

$$I(z) = I_1 \left(1 - p_1 \cos \left(\frac{\pi z}{h} \right) \right) \quad (3b)$$

For asymmetric nano-porosity distribution (Fig. 1c):

$$E(z) = E_1 \left(1 - p \cos \left(\frac{\pi z}{2h} + \frac{\pi}{4} \right) \right) \quad (4a)$$

$$I(z) = I_1 \left(1 - p_1 \cos \left(\frac{\pi z}{2h} + \frac{\pi}{4} \right) \right) \quad (4b)$$

where $p = 1 - E_0/E_1$ and $p_1 = 1 - I_0/I_1$ are the nano-porosity effects for Young's modulus and the material length scale parameter, respectively. Also, E_0 and I_0 are, respectively, the minimum values of Young's modulus and the material length scale parameter, and E_1 and I_1 denote the maximum values of Young's modulus and the material length scale parameter, respectively.

The relations between different material properties can be expressed as a function of the relative mass density [5, 13]. According to type of the study, the following relations are defined.

$$\frac{E_1}{E_0} = \left(\frac{\rho_1}{\rho_0} \right)^n \quad (5a)$$

$$\frac{I_1}{I_0} = \left(\frac{\rho_1}{\rho_0} \right)^m \quad (5b)$$

where n and m are constants to express the behavior of the porous materials which can be variable for different materials. For example, n is, respectively, equal to 2 and 3 for open-cell and closed-cell metal foams [5]. Equation (5b) is proposed for the first time; also, n in Eq. (5a) was not obtained for all porous materials. Therefore, these constants are defined as variable parameters in the present study. Using Eqs. (5a) and (5b), we have

$$\frac{I_1}{I_0} = \left(\frac{E_1}{E_0} \right)^{\frac{m}{n}} \quad (6)$$

It should be mentioned that Poisson's ratio is held constant because its variations are negligible against the other properties.

3 Governing equations

Consider a beam with length L and a non-uniform cross section (the cross section is assumed as a function of x) under an axial compressive load N at beam ends (Fig. 2). According to the modified couple stress theory and the first variation in the total strain energy in the beam, the total strain energy U in an isotropic linear elastic material occupying region Ω (with a volume element dV) can be written as

$$\delta U = \int_{\Omega} (\sigma_{ij} \delta \varepsilon_{ij} + m_{ij} \delta \chi_{ij}) dV - N \int_0^L w' \delta w' dx = 0 \quad (7)$$

For an Euler–Bernoulli microbeam and using the von-Karman nonlinearity, we have

$$\begin{aligned} \varepsilon_{xx} &= u' - zw' + \frac{1}{2}(w')^2 \\ \sigma_{xx} &= (\lambda(z) + 2\mu(z))\varepsilon_{xx} \\ \chi_{xy} &= \chi_{yx} = -\frac{1}{2}w'' \\ m_{xy} &= m_{yx} = 2(I(z))^2 \mu(z) \chi_{xy} \end{aligned} \quad (8)$$

where ε , σ , χ and m are the strain, the Cauchy stress, the symmetric curvature and the deviatoric part of the couple stress, respectively. Also, $\lambda = \frac{Ev}{(1+\nu)(1-2\nu)}$ and $\mu = \frac{E}{2(1+\nu)}$ are Lamé's constants. Thus, the nonlinear governing equations are derived as

$$\int_0^L \left[I_1 w'''' + I_1' w''' - I_0 (u'' + w' w'') - I_0' \left(u' + \frac{1}{2} (w')^2 \right) \right] dx = 0 \quad (9a)$$

$$\begin{aligned} & \int_0^L \left[(I_2 + R) w^{(4)} + 2(I_2' + R') w'''' + (I_2'' + R'' + N) w'' \right. \\ & \quad \left. - I_1 u'''' - I_1' (2u'' + w' w'') - I_1'' \left(u' + \frac{1}{2} (w')^2 \right) \right. \\ & \quad \left. - I_0 (u' w'' + u'' w' + \frac{3}{2} (w')^2 w'') - I_0' \left(u' w' + \frac{1}{2} (w')^3 \right) \right] dx = 0 \end{aligned} \quad (9b)$$

where R , I_0 , I_1 and I_2 are as functions of x due to change in the cross section along the x -direction and are defined as

$$\begin{aligned} R &= \iint_A \mu(z) (I(z))^2 dA \\ I_0 &= \iint_A (\lambda(z) + 2\mu(z)) dA \\ I_1 &= \iint_A (\lambda(z) + 2\mu(z)) z dA \\ I_2 &= \iint_A (\lambda(z) + 2\mu(z)) z^2 dA \end{aligned} \quad (10)$$

From Eq. (9a), we have

$$(I_1 w'')' - \left(I_0 \left(u' + \frac{1}{2} (w')^2 \right) \right)' = 0 \quad (11)$$

Therefore,

$$u' = \frac{I_1}{I_0} w'' - \frac{1}{2} (w')^2 + c_1 \quad (12)$$

$$u = \int_0^x \frac{I_1}{I_0} w'' d\eta - \frac{1}{2} \int_0^x (w')^2 d\eta + c_1 x + c_2 \quad (13)$$

where c_1 and c_2 are integration constants and will be obtained using boundary conditions. In order that the

where

$$M_1 = I_2 + R - \frac{I_1^2}{I_0}, \quad M_2 = 2 \left(I_2' + R' - \left(I_1 \frac{I_1}{I_0} \right)' - I_1 \right), \\ M_3 = 2I_1' + \left(I_1 \left(\frac{I_1}{I_0} \right)' \right)' + \frac{I_1 I_1''}{I_0} - I_2'' - R'', \quad M_4 = I_0' + I_1''$$

Equation (15) represents the governing equation for post-buckling of porous non-uniform microbeams. I_1 becomes zero when the material properties are symmetric with respect to z -axis, e.g., the uniform porous, the symmetric porous and the homogeneous materials. Therefore, the governing equation of the non-porous non-uniform microbeams is derived as

$$\int_0^L \left[((\lambda + 2\mu)I + \mu I^2 A) w^{(4)} + 2((\lambda + 2\mu)I' + \mu I^2 A') w''' - ((\lambda + 2\mu)I'' - \mu I^2 A'' + (\lambda + 2\mu)A c_1 - N) w'' - (\lambda + 2\mu)A' c_1 \right] dx = 0 \quad (16)$$

midplane stretching to be significant, the beam ends must be restrained. Therefore, the boundary conditions for the axial displacement are assumed as $u=0$ at $x=0, L$. So, we have

$$c_1 = \frac{1}{2L} \int_0^L (w')^2 d\eta - \frac{1}{L} \int_0^L \frac{I_1}{I_0} w'' d\eta \quad (14)$$

Substituting Eqs. (12) and (13) into Eq. (9b), the governing equation can be rewritten as

$$\int_0^L [M_1 w^{(4)} + M_2 w''' - (M_3 + I_0 c_1 - N) w'' - M_4 c_1] dx = 0 \quad (15)$$

Fig. 3 Linear and exponential non-uniformities of the microbeams

where $A=A(x)$ and $I=I(x)$ are the cross section area and the moment of inertia of the non-uniform microbeams which change along the x -axis. If we consider a porous uniform microbeam, Eqs. (9a) and (9b) are rewritten as

$$I_1 w''' - I_0 (u'' + w' w'') = 0 \quad (17a)$$

$$(I_2 + R) w^{(4)} - I_1 u''' - I_0 (u' w'' + u'' w' + \frac{3}{2} (w')^2 w'') + N w'' = 0 \quad (17b)$$

Substituting Eq. (17a) into Eq. (17b), the following linear equation is derived for porous microbeams with uniform cross section.

$$\left(I_2 + R - \frac{I_1^2}{I_0} \right) w^{(4)} - (I_0 c_1 - N) w'' = 0 \quad (18)$$

Equations (17a), (17b) and (18) can be validated with the governing equations of non-homogeneous microbeams obtained by Akbarzadeh Khorshidi et al. [19]. Also, this equation can be simplified for homogeneous microbeams as

$$((\lambda + 2\mu)I + \mu I^2) w^{(4)} - ((\lambda + 2\mu)A c_1 - N) w'' = 0 \quad (19)$$

4 Solution

In this section, the simply supported boundary conditions are applied for the microbeam. For the boundary conditions, a transverse displacement is assumed for the first buckling mode and the critical buckling load and the static postbuckling response are obtained based on this assumption.

In general form, the relation between the external applied load and the beam deflection is obtained as an integral form.

$$N = \frac{\int_0^L [M_1 w^{(4)} + M_2 w''' - M_3 w'' - I_0 c_1 w'' - M_4 c_1] dx}{-\int_0^L w'' dx} \quad (20)$$

By considering the simply supported boundary conditions, the transverse displacement for the first buckling mode of simply supported beam is assumed as

$$w = W \sin \alpha x \quad (21)$$

This function can satisfy the boundary conditions of simply supported beams where W is the postbuckling amplitude of microbeam and $\alpha = \pi/L$. Substituting Eq. (21) into Eq. (15), the postbuckling response of the porous non-uniform microbeam is determined as

$$N = \frac{\alpha^2}{2} \left[\int_0^L M_1 \sin \alpha x dx - \alpha \int_0^L M_2 \cos \alpha x dx + \int_0^L M_3 \sin \alpha x dx + c_1 \left(\int_0^L I_0 \sin \alpha x dx - \frac{1}{W \alpha^2} \int_0^L M_4 dx \right) \right] \quad (22)$$

Also, by substituting Eq. (21) into Eq. (14), we have

$$c_1 = \frac{W \alpha^2}{L} \left[\frac{WL}{4} + \int_0^L \frac{I_1}{I_0} \sin \alpha x dx \right] \quad (23)$$

For non-porous non-uniform microbeams:

$$N = \frac{\alpha^2}{2} \left[(\lambda + 2\mu) \alpha^2 \int_0^L I \sin \alpha x dx + \mu l^2 \alpha^2 \int_0^L A \sin \alpha x dx - 2(\lambda + 2\mu) \alpha \int_0^L I' \cos \alpha x dx - 2\mu \ell^2 \alpha \int_0^L A' \cos \alpha x dx - (\lambda + 2\mu) \int_0^L I'' \sin \alpha x dx - \mu \ell^2 \int_0^L A'' \sin \alpha x dx + (\lambda + 2\mu) \left(\frac{W \alpha}{2L} \right)^2 \left(\int_0^L A \sin \alpha x dx - \frac{1}{W \alpha^2} \int_0^L A' dx \right) \right] \quad (24)$$

For porous uniform microbeams, Eq. (22) is simplified to the following equation.

$$N = M_1 \alpha^2 + \frac{2I_1 \alpha}{L} W + \frac{I_0 \alpha^2}{4} W^2 \quad (25)$$

Equation (25) can also be derived using Eq. (18), individually. By neglecting the porosity effect, Eq. (25) can be converted as (for non-porous uniform microbeams)

$$N = [(\lambda + 2\mu)I + \mu l^2 A] \alpha^2 + \frac{1}{4} (\lambda + 2\mu) A \alpha^2 W^2 \quad (26)$$

The postbuckling loads obtained in Eqs. (22), (24), (25) and (26) can lead to the critical buckling loads N_{cr} when the postbuckling amplitude is set to zero $W = 0$.

N_{cr} expressed in Eq. (26) can be validated with obtained results in Ref. [22].

Equation (22) represents a general form for the postbuckling load of porous microbeams with non-uniform cross section. In this study, the results of tapered, exponential and uniform beams are presented.

5 Results and discussion

This study is designated for general conditions of porous non-uniform microbeams, but three types of porosity distributions and two types of non-uniformity are assumed in this section. The uniform, symmetric and asymmetric distributions are employed to describe the porosity of material; also, the linear (tapered beam) and exponential functions for beam thickness $z(x)$ and a constant value for beam width $b = 2h$ are used to consider the non-uniformity of the cross section. These non-uniformities are shown in Fig. 3.

Table 1 shows a comparison between the non-dimensional critical buckling loads obtained by present study and Ref. [22]. These non-dimensional critical buckling loads were acquired for the uniform microbeams without porosity.

As an example, the microbeam made of nickel is chosen with $E_1 = 207$ GPa, $\nu = 0.31$, $h = 5.2$ μm , $l_1 = 2.304$ μm , $L = 20h$ and $b = 2h$ [44]. Table 2 shows the critical buckling loads of

uniform microbeams for different porosity distributions, and the critical buckling loads of non-uniform microbeams for different non-uniformity distributions are presented in Tables 3 and 4. As expected, the nano-porosity leads to a decrease in the critical buckling loads except when the porosity distribution is asymmetric. This asymmetric

Table 1 Non-dimensional critical buckling load ($P_{cr} L^2/EI$) of non-porous uniform microbeams

l/h	1	0.333	0.2	0.142
Present	52.78	14.637	11.586	10.745
Ref [22]	52.78	14.637	11.586	10.745

Table 2 Critical buckling loads (N) of tapered microbeams for uniform, symmetric and asymmetric porosity distributions

	Non-porous	Uniform porosity	Symmetric porosity	Asymmetric porosity
$p = p_1 = 0$ ($m = n = 0$)	0.0262	0.0262	0.0262	0.0262
$p_1 = 0, p = 0.5$ ($n = 0$)	0.0262	0.0173	0.0197	0.0288
$p_1 = 0.75, p = 0.5$ ($m = 2n$)	0.0262	0.0121	0.0153	0.0245
$p_1 = 0.29, p = 0.5$ ($n = 2m$)	0.0262	0.0150	0.0176	0.0268
$p = p_1 = 0.5$ ($m = n$)	0.0262	0.0136	0.0164	0.0256

Table 3 Critical buckling loads (N) of uniform, tapered and exponential microbeams for the symmetric porosity distribution

	Uniform beam	Tapered beam	Exponential beam
$p = p_1 = 0$ ($m = n = 0$)	0.0480	0.0262	0.0262
$p_1 = 0, p = 0.5$ ($n = 0$)	0.0371	0.0197	0.0197
$p_1 = 0.75, p = 0.5$ ($m = 2n$)	0.0303	0.0153	0.0153
$p_1 = 0.29, p = 0.5$ ($n = 2m$)	0.0339	0.0176	0.0176
$p = p_1 = 0.5$ ($m = n$)	0.0320	0.0164	0.0164

Table 4 Critical buckling loads (N) of uniform, tapered and exponential microbeams for the asymmetric porosity distribution

	Uniform beam	Tapered beam	Exponential beam
$p = p_1 = 0$ ($m = n = 0$)	0.0480	0.0262	0.0262
$p_1 = 0, p = 0.5$ ($n = 0$)	0.0325	0.0288	0.0257
$p_1 = 0.75, p = 0.5$ ($m = 2n$)	0.0258	0.0245	0.0213
$p_1 = 0.29, p = 0.5$ ($n = 2m$)	0.0293	0.0268	0.0236
$p = p_1 = 0.5$ ($m = n$)	0.0274	0.0256	0.0224

distribution keeps an extra resultant I_1 that makes the microbeam stiffer; therefore, a porous microbeam with asymmetric distribution can be stiffer than non-porous microbeams, depending on values of p and p_1 . Also, it can be found that the tapered beams and exponential beams may have the same buckling load, while they have a different postbuckling response (Fig. 4b). Another point worth mentioning is that the most values for critical buckling loads of porous microbeams occur when $p_1 = 0$. Note that $p = p_1 = 0$ denotes a non-porous microbeam. By comparison of Tables 2 and 3, it is found that the critical buckling loads of uniform microbeams with symmetric porosity distribution are greater than the asymmetric one, this is due to the fact that the buckling load of uniform microbeams depends only on M_1 [Eq. (25)] and in symmetric distribution, I_1 is equal to zero so $M_{1,\text{symmetric}} > M_{1,\text{asymmetric}}$ while the buckling load depends on M_1, M_2 and M_3 for the linear and exponential non-uniformities.

The static postbuckling responses of non-uniform porous microbeams are shown in Figs. 4 and 5. These figures illustrate nonlinear behaviors of postbuckling for different porosity distributions (uniform, symmetric and asymmetric) and different cross-sectional non-uniformities (uniform, linear and exponential microbeams). Also, effects of the nano-porosity parameters p and p_1 on the

postbuckling load are investigated in Figs. 6 and 7, respectively. It can be found from these figures that the nano-porosity effect for Young's modulus p is stronger than the nano-porosity effect for the material length scale parameter p_1 .

6 Conclusion

This study deals with nano-porosity and non-uniformity of microbeams and investigates the effects of these non-homogeneities (both in material and geometry) on the static postbuckling response. Three special cases of nano-porosity are considered for this study. However, the present nano-porosity was proposed for metal foams, and we can use different types of nano-porositities for any other case studies. Also, it is assumed that the cross section of beam changes in the direction of beam length, so, linear and exponential functions are employed as two examples and their results are compared with the uniform microbeam.

The results indicate that considering the nano-porosity effect only on Young's modulus can be led to considerable errors. Therefore, the nano-porosity effect is also taken into account for the material length scale parameter which

Fig. 4 Effects of nano-porosity on the postbuckling response of non-uniform microbeams $p=p_1=0.5$, **a** uniform, **b** symmetric and **c** asymmetric porosity distributions

Fig. 5 Effects of non-uniformity on the postbuckling response of porous microbeams $p=p_1=0.5$, **a** uniform, **b** Linear and **c** exponential non-uniformities

Fig. 6 Effect of the nano-porosity parameter of Young's modulus on the postbuckling response of porous exponential microbeams ($W/h = 1$ and $p_1 = 0.5$)

Fig. 7 Effect of the nano-porosity parameter of the material length scale parameter on the postbuckling response of porous exponential microbeams ($W/h = 1$ and $p = 0.5$)

describes the size effect of the modified couple stress theory.

Since porous materials in nature are not usually found as a structure with the uniform cross section, the non-uniform beams are considered for this study. As can be understood from the results, the non-uniform beams may have different behaviors for buckling and postbuckling.

Compliance with ethical standards

Conflict of interest The author declares that they have no conflict of interest.

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