Dynamic analysis of crank mechanism with complex trigonometric nonlinearity: a comparative study of approximate analytical methods

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Abstract

The geometrically nonlinear crank (GNC) is a mechanism that is found in machines. Its vibration is governed by a differential equation with complex trigonometric nonlinearity. This paper presents two approximate analytical solutions for dynamic analysis of the GNC using a quasi-static cubication method (QCM) and the energy balance method (EBM). Comparison of the results of the QCM and EBM with standard numerical results shows that the QCM gives a better estimate of the GNC vibration and captures the qualitative response better than the EBM. The study shows that the GNC exhibits a strong nonlinear vibration response at all amplitudes. The dynamic analysis also reveals an asymptotic response in the relation between the frequency and a geometrical constant defined as the ratio of the crank length to the distance between the supports of the GNC.

Keywords Crank mechanism · Nonlinear vibration · Trigonometric nonlinearity · Quasi-static cubication method · Energy balance method

1 Introduction

The nonlinear world is gaining more attention as scientists and engineers seek to have deeper understanding of their physical world rather than a superficial interaction. To this end, many nonlinear models have been developed and analyzed.

An important component of many machines is a crank mechanism known as geometrically nonlinear crank [1]. It consists of a crank pivoted at one end and having a mass at the other end that is constrained by means of a coaxial elastic spring (see Fig. 1). Its vibration is governed by a restoring force that is characterized by a complex trigonometric nonlinearity. Hence, determining its dynamic response analytically is quite challenging and no dynamic solution seems to have been provided in the published literature.

A key consideration in the analysis of nonlinear vibration models is the dynamic solution for estimating the

natural frequency and displacement/velocity history. Numerical schemes to solve virtually all kinds of nonlinear vibration models exist but have the limitations of (i) lack of physical insight [2, 3] and (ii) sometimes producing incorrect results even when a convergent solution is achieved [4]. In the last three decades, a number of approximate analytical methods for the solution of nonlinear vibration models have been proposed and include asymptotic methods [2], homotopy perturbation method [5], variational iteration method [6], linearized harmonic balance method [7], quintification method [8], Adomain decomposition method [9], Chebyshev cubication method [10], energy balance method [11], quasi-static cubication method [12], continuous piecewise linearization method [13, 14], variational approach [15], homotopy analysis method [16], He-Chengtian interpolation method [2], Hamiltonian approach [17], coupled Chebyshev expansion-Mickens iteration method [18] and differential transform method [19]. Some of these methods [10, 16, 18] involve

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Fig. 1 Schematic representation of geometrically nonlinear crank

complicated mathematical processes when applied to solve models possessing trigonometric nonlinearity. For example, the simple pendulum model, which has sine nonlinearity, was solved using the Chebyserv cubication method [10] and the solution was derived in terms of Bessel function of the first kind. A model with a more complicated trigonometric nonlinearity than the simple pendulum model is that of the pendulum with rotating support [16, 18, 20]. Liao and Chwang [16] solved the model of the pendulum with rotating support using the homotopy analysis method. Their solution was derived in terms of Bessel function which makes derivation of the more accurate higher-order approximations complicated. Lai and co-workers [18] derived very accurate approximate analytical solutions for the large-amplitude oscillations of the pendulum with rotating support. Their solution was based on Chebyserv expansion of the trigonometric nonlinearity to obtain an equivalent cubic-quintic oscillator and then solving the latter using a modified Micken iteration method. The frequency-amplitude response obtained by this method produced slightly more accurate results compared to the second-order homotopy analysis method [16] but the analytical solution is complicated. Khan and co-workers also studied the oscillations of the pendulum with rotating support using Hamiltonian approach [20]. Their solution was derived in terms of Bessel function but the oscillation history proved to be accurate only for small amplitudes. The GNC model [1] possesses a far more complicated trigonometric nonlinearity compared to the pendulum with rotating support.

In this paper, two simple approximate analytical solutions were derived to determine the dynamic response (i.e. frequency and oscillation history) of the GNC, which is governed by a complex trigonometric nonlinearity. The approximate analytical solutions are based on the energy balance method [11] and a recent cubication method [12]. Comparison of both methods with standard numerical solutions are presented and discussed. The aim of this paper is to derive reasonably accurate solutions for the

GNC vibration that can be applied without complicated mathematical processes.

2 Vibration model for the geometrically nonlinear crank

As depicted in Fig. 1, consider a GNC of crank length l having a mass m and a linear elastic spring with stiffness k . The spring free length in the coaxial position is x_0 . The equation of motion for the GNC can be obtained using Lagrangian approach. The linear displacement of the mass is $s = l\varphi$ and its linear velocity is $v = l\dot{\varphi}$. Therefore, the kinetic and potential energies are:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\varphi}^2 \quad (1)$$

$$V = \frac{1}{2}kx^2 = \frac{1}{2}k\left(\sqrt{L_0^2 + l^2 - 2lL_0\cos\varphi} - x_0\right)^2 \quad (2)$$

where x is the spring extension due to the angular displacement of the crank and $L_0 = l + x_0$. Then, the Lagrangian is:

$$L = T - V = \frac{1}{2}ml^2\dot{\varphi}^2 - \frac{1}{2}k\left(\sqrt{L_0^2 + l^2 - 2lL_0\cos\varphi} - x_0\right)^2 \quad (3)$$

The vibration model is derived from the Lagrange equation given as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{\partial L}{\partial \varphi} \quad (4)$$

Substituting Eqs. (3) in (4) and after algebraic simplification, the GNC vibration model was derived as:

$$\frac{d^2\varphi}{dt^2} + \frac{kL_0\left(\sqrt{L_0^2 + l^2 - 2lL_0\cos\varphi} - x_0\right)}{ml\left(\sqrt{L_0^2 + l^2 - 2lL_0\cos\varphi}\right)}\sin\varphi = 0 \quad (5)$$

with initial conditions $\varphi(0) = A$ and $\frac{d\varphi}{dt}(0) = 0$. Let $\Omega = \sqrt{k/m}$ and $r = l/L_0 < 1$; then Eq. (5) simplifies to:

$$\frac{d^2\varphi}{dt^2} + \frac{\Omega^2}{r} \frac{\left[\sqrt{1+r^2-2r\cos\varphi} + r - 1\right]}{\sqrt{1+r^2-2r\cos\varphi}}\sin\varphi = 0 \quad (6)$$

Therefore, the restoring force for the GNC is:

$$g(\varphi) = \frac{\Omega^2}{r} \frac{\left[\sqrt{1+r^2-2r\cos\varphi} + r - 1\right]}{\sqrt{1+r^2-2r\cos\varphi}}\sin\varphi \quad (7)$$

which is obviously characterized by a complex trigonometric nonlinearity.

Equation (6) shows that the motion of the GNC depends on two constants, Ω and r . The constant Ω captures the system's physical properties while r is a dimensionless geometrical constant.

For small-angle displacements, $\sin \varphi \cong \varphi$ and $\cos \varphi \cong 1 - \varphi^2/2$. Therefore, Eq. (6) reduces to:

$$\frac{d^2\varphi}{dt^2} + \frac{\Omega^2}{r} \frac{\left[\sqrt{(1-r)^2 + r\varphi^2} + r - 1 \right]}{\sqrt{(1-r)^2 + r\varphi^2}} \varphi = 0 \quad (8)$$

Assuming the small-angle approximation $\cos \varphi \cong 1$ was used, then the restoring force in Eq. (6) will vanish and there will be no vibration. Instead a rigid body response of the entire system is envisaged. Accordingly, $\cos \varphi \cong 1 - \varphi^2/2$ is the appropriate approximation for the small-angle vibrations of the GNC. This deduction was further confirmed by plotting Eq. (7) for $0 < \varphi \leq 0.05\pi$ radians (not shown here) and the plots clearly showed a nonlinear restoring force.

Equation (8) proves that the GNC exhibits strong nonlinearity even for small-amplitude vibrations. Thus, the vibration of the GNC is contrasted from most nonlinear oscillators that exhibit a linear response for small-amplitude vibrations. The implication is that classical perturbation methods [23], which usually work well for small-amplitude vibrations of weakly nonlinear systems, cannot provide reasonably accurate solutions for the small-amplitude vibration of the GNC.

3 Approximate analytical solution

In Sect. 2, the GNC vibration model [see Eq. (6)] was derived by Lagrangian approach and it was demonstrated that the vibration of the GNC is highly nonlinear for small- and large-amplitudes. Although the small-amplitude oscillations can be modelled by a non-rational nonlinearity [Eq. (8)], the large-amplitude oscillations has a restoring force that is characterized by a complex trigonometric nonlinearity [see Eqs. (6) and (7)]. Since an exact analytical solution to Eq. (6) is not available, an approximate analytical solution would be required. Hence, two simple analytical methods are applied here to derived approximate dynamic solutions for Eq. (6).

3.1 Quasi-static cubication method

Cubication methods [10, 12, 20, 21] are based on the concept of representing a nonlinear oscillator by an equivalent

cubic oscillator so that the solution of the latter gives an approximate solution to the former. The main challenge in applying cubication methods is to determine the cubication constants that will accurately represent the original system. Different methods have been proposed to determine the cubication constants and they include Chebyshev series [10], weighted-mean square [21], harmonic balance principles [22] and most recently, quasi-static equilibrium principles [12]. The advantage of the QCM over the other cubication methods is that it is simple and always produces the cubication constants in terms of elementary functions while the other cubication methods [10, 20, 21] may sometimes produce the cubication constants in terms of special functions or complicated expressions. Since the QCM was only published recently, the present study explores its capability to handle more complex nonlinear oscillators.

For odd nonlinear Hamiltonian systems such as Eq. (6), the equivalent cubic oscillator can be expressed as [10, 12]:

$$\frac{d^2\varphi}{dt^2} + \alpha_1(A)\varphi + \alpha_3(A)\varphi^3 = 0 \quad (9)$$

where $\alpha_1(A)$ and $\alpha_3(A)$, which are functions of the amplitude, are the cubication constants to be determined by the cubication method. For convenience the cubication constants will be written as α_1 and α_3 henceforth. Once the cubication constants for a nonlinear oscillator have been derived, the exact solution to Eq. (9) can be used to provide approximate dynamic solution to the original nonlinear oscillator. The exact solution to Eq. (9) is given as [12]:

$$\varphi(t) = A \operatorname{cn}(\psi t, m) \quad (10)$$

where cn is the Jacobi elliptic cosine function, $\psi = \sqrt{\alpha_1 + \alpha_3 A^2}$ and $m = (\psi^2 - \alpha_1)/2\psi^2$. Therefore, the oscillation frequency of Eq. (10) can be determined as [12]:

$$\omega = 0.5\pi\psi/K(m) \quad (11)$$

where $K(m)$ is the complete elliptic integral of the first kind.

Based on the QCM, the cubication constants for an odd nonlinear Hamiltonian oscillator are derived as [12]:

$$\alpha_1 = \frac{4 \int_0^A g(\varphi) d\varphi - Ag(A)}{A^2} \quad (12a)$$

$$\alpha_3 = \frac{2Ag(A) - 4 \int_0^A g(\varphi) d\varphi}{A^4} \quad (12b)$$

Substituting Eq. (7) in Eqs. (12a, 12b) and after algebraic simplification, we get the cubication constants for Eq. (6) as:

$$\alpha_1 = \frac{\Omega^2}{rA^2} \left\{ 4 \left[(1 - \cos A) - (\Delta + r - 1) \left(\frac{1}{r} - 1 \right) \right] - \frac{(\Delta + r - 1)}{\Delta} A \sin A \right\} \quad (13a)$$

$$\alpha_3 = \frac{\Omega^2}{rA^4} \left\{ \frac{2(\Delta + r - 1)}{\Delta} A \sin A - 4 \left[(1 - \cos A) - (\Delta + r - 1) \left(\frac{1}{r} - 1 \right) \right] \right\} \quad (13b)$$

where $\Delta = \sqrt{1 + r^2 - 2r \cos A}$. Equations (13a, 13b) are applied together with Eqs. (10) and (11) to compute the vibration response of the GNC.

3.2 Energy balance method

The energy balance method [11] is based on the idea that when an approximate solution is substituted into the Hamiltonian, a residual function is obtained that can be minimized to derive an expression for the frequency-amplitude response. The EBM assumes an approximate solution given as $\varphi(t) = A \cos(\omega t)$ where ω is the natural frequency.

The Hamiltonian for the GNC vibration model (i.e. equation (6)) was derived as:

$$H = \frac{1}{2} \dot{\varphi}^2 - \frac{\Omega^2}{r} \left[\cos \varphi + \left(\frac{1}{r} - 1 \right) \sqrt{1 + r^2 - 2r \cos \varphi} \right] = -\frac{\Omega^2}{r} \left[\cos A + \left(\frac{1}{r} - 1 \right) \Delta \right] \quad (14)$$

Then, the residual was derived from the Hamiltonian as:

$$R(t) = \frac{1}{2} \dot{\varphi}^2 - \frac{\Omega^2}{r} \left[\cos \varphi + \left(\frac{1}{r} - 1 \right) \sqrt{1 + r^2 - 2r \cos \varphi} \right] + \frac{\Omega^2}{r} \left[\cos A + \left(\frac{1}{r} - 1 \right) \Delta \right] \quad (15)$$

Therefore, substituting $\varphi(t) = A \cos(\omega t)$ into Eq. (15) gives:

$$R(t) = \frac{1}{2} A^2 \omega^2 \sin^2(\omega t) - \frac{\Omega^2}{r} \left[\cos(A \cos(\omega t)) + \left(\frac{1}{r} - 1 \right) \sqrt{1 + r^2 - 2r \cos(A \cos(\omega t))} \right] + \frac{\Omega^2}{r} \left[\cos A + \left(\frac{1}{r} - 1 \right) \Delta \right] \quad (16)$$

A simple residual error minimization method that has proved very effective is the collocation method [11]. Using this method and collocating at $\omega t = \pi/4$, the frequency-amplitude response for Eq. (6) was derived as:

$$\omega = \frac{2\Omega}{A\sqrt{r}} \left[\cos\left(A/\sqrt{2}\right) - \cos A + \left(\frac{1}{r} - 1\right) \left(\sqrt{1 + r^2 - 2r \cos\left(A/\sqrt{2}\right)} - \Delta \right) \right]^{1/2} \quad (17)$$

Hence, the EBM solution to Eq. (6) can be expressed as $\varphi(t) = A \cos(\omega t)$ where ω is given as Eq. (17).

4 Discussion of results

4.1 Frequency-amplitude response

In order to conduct a dynamic analysis of the GNC based on the QCM and EBM solutions, the following inputs were used for simulation: $l = 1.0$ (m), $x_0 = 1.0$ (m), $m = 1.0$ (kg) and $k = 1.0$ (N/m). For verification purposes, the results of QCM and EBM solutions derived in Sect. 3 were compared with standard numerical solutions with an accuracy of the order of 10^{-8} . The numerical solutions were obtained by solving Eq. (6) using the NDSolve algorithm in Mathematica™. The results of the frequency-amplitude response for amplitudes in the range $0 < A \leq 0.50\pi$ are plotted in

Fig. 2a and the corresponding error analysis is plotted in Fig. 2b. It was observed that the frequency increases with amplitude and both solutions predict this trend. However,

the plots show that the QCM performs better than the EBM for most of the amplitudes considered, especially at small amplitudes.

Fig. 2 **a** Frequency-amplitude response, **b** error in frequency estimates of QCM and EBM

Fig. 3 **a** Variation of frequency with geometric constant for $A = 0.05\pi$, **b** corresponding error of QCM and EBM

4.2 Effect of GNC geometry on frequency response

When $r = 0.5$, the crank length and free length of the spring are equal. For $r < 0.5$ the crank length is shorter while for $r > 0.5$ the crank length is longer. Hence, parametric analysis was conducted to determine the effect of r on the natural frequency. It was observed that the natural frequency increased with r and demonstrated an asymptotic behaviour when $r \rightarrow 1$. This happens because, when $r \rightarrow 1$, the geometric effect of the spring becomes negligible (i.e. $x_0/l \cong 0$). Consequently, the asymptotic response of the GNC is governed by a pendulum-like equation, $d^2\varphi/dt^2 + \Omega^2 \sin \varphi = 0$. It follows that the asymptotic natural frequency can be determined as $\omega_{r \rightarrow 1} = 0.5\pi\Omega/K(\sin^2(A/2))$. For small displacements, as considered in Fig. 3a, the governing equation for the asymptotic response simplifies to $d^2\varphi/dt^2 + \Omega^2\varphi = 0$ and $\omega_{r \rightarrow 1} = \Omega$. Therefore, $\omega_{r \rightarrow 1} = 1.0$ [rad/s] for $A = 0.05\pi$ as shown in Fig. 3a. The QCM and EBM solutions both

Fig. 4 Oscillation history of the GNC for $A = 0.05\pi$

captured this trend and their error estimates are plotted in Fig. 3b. Again, the QCM performs better than the EBM.

Fig. 5 Oscillation history of the GNC for $A = 0.25\pi$

Fig. 6 Oscillation history of the GNC for $A = 0.50\pi$

4.3 Oscillation plots

Existing studies [3, 24] have demonstrated that thorough justification of the accuracy of a solution method for nonlinear vibration models requires testing of both the frequency-amplitude response and the displacement/velocity history. Figures 4, 5, 6 show the displacement/velocity histories of the GNC from small-amplitude ($A = 0.05\pi$) to large-amplitude ($A = 0.50\pi$) vibrations. The QCM solution matches the numerical solution for $A = 0.05\pi$ while the EBM shows significant errors, especially in the velocity history. Also, the QCM captures the quasi-linear behaviour of the response history while the EBM could not. This is because of the inherent limitation of the EBM that it is based on an approximation that can only produce sinusoidal profiles and therefore cannot predict non-sinusoidal response histories as in the case when $A = 0.05\pi$.

The quasi-linear response of nonlinear oscillators has been reported [3] for the case of large-amplitude and strong nonlinear vibrations of the cubic-Duffing oscillator. Essentially, this response involves a period of time when

the displacement profile exhibits an almost linear relationship with time and therefore, the velocity profile is characterized by an almost constant velocity in that period. The observed quasi-linear response when $A = 0.05\pi$ shows that the GNC exhibits strong nonlinearity at small amplitudes. Additionally, we observe that the response history looks more like a sinusoidal profile when $A = 0.50\pi$ (see Fig. 6), resulting in EBM results that are comparable to QCM results. However, the QCM gives a better overall estimate of the vibration history compared to the EBM.

5 Conclusions

The dynamic response of the GNC has been investigated in this paper. First, the GNC vibration model [Eq. (6)] was formulated using Lagrangian approach. The model is in the form of a nonlinear ODE with complex trigonometric nonlinearity. Approximate solutions to Eq. (6) were derived using the QCM and EBM. The two solution schemes are simple to implement and do not involve complicated mathematical processes. To verify the accuracy of the derived solutions, the results of both methods were compared with standard numerical results. The QCM was found to give better estimates of the GNC vibration and captured important qualitative behaviour in the oscillation history that the EBM did not capture.

The oscillation plots revealed that the GNC exhibits strong nonlinearity during small-amplitude vibrations. This means that classical perturbation solutions cannot give reasonably accurate estimates of the GNC vibration response. Furthermore, investigation on the effect of the crank geometry (r) on the natural frequency (ω) showed that ω increased with r and an asymptotic frequency is obtained as $r \rightarrow 1$.

Finally, the study shows that the QCM has a good potential for deriving periodic solutions of Hamiltonian systems with complex nonlinearity and may perform better than the well-known EBM.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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