Leapfrogging dynamics of interacting solitons in weakly coupled nonlinear transmission lines

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Abstract

The dynamics of two electrical pulses forming a boundstate, propagating along two nonlinear transmission lines weakly coupled by linear capacitors shunted with linear resistances, is considered from both analytical and numerical standpoints. The study rests on an analysis of time series of the amplitudes and phases of the two interacting electrical pulses, within the framework of the variational theory based on exact one-soliton solution to the Korteweg-de Vries equation. In the regime where the two pulses propagate at nearly equal velocities, their relative amplitude/phase evolutions can result in periodic quasi-harmonic oscillations so-called leapfrogging motion. In this specific regime of motion, it is found that besides the expected damping effect on the soliton amplitudes, the resistance can also sustain their leapfrogging motion. Analytical expression of the leapfrogging frequency is derived, providing a better understanding of the competing effects of the coupling capacitor and the resistive shunt on the leapfrogging motion. Leapfrogging motions of co-propagating pulses in electrical networks can be very useful in high-intensity signal transmissions involving least energy cost for the propagating signals.

Keywords Coupled nonlinear transmission lines · Soliton pairs · Leapfrogging · KdV equations · Numerical simulations

1 Introduction

Nonlinear transmission lines (NLTs) are dispersive media in which electric signals propagate in form of well-localized pulses [1–13], usually referred to as electrical solitons. Owing to their robustness that is their ability to cover long distances (thousands of kilometers) without change in profiles, electrical solitons have been actively investigated over the past fifty years starting from the pioneer model of Hirota and Suzuki [1, 2, 14–23]. In the microwave domain in particular NLTs have recently been shown [21] to represent the ideal source of stable high-intensity sharp pulses.

Typically a NLT is constructed by periodically loading a linear transmission line with reverse-biased semiconductor diodes (such as Schottky varactors), or by arranging inductors and varactors in a one-dimensional (1D)

lattice. In such structures nonlinearity originates from the varactors whose capacitance is designed to change with applied voltage, while line dispersion stems from the structural periodicity of the loaded elementary circuits composing the ladder transmission line. When nonlinearity and dispersion are balanced, electrical currents and all related physical parameters (such as the voltages) acquire solitonic shape profiles and thus long-lived high-intensity electric signals are generated [15]. In addition to maintaining their shapes during propagation, solitons possess other important properties [15, 24]. Namely they can survive collisions with other solitons or solitary waves, and in electrical transmission lines specifically it was recently established [25–27] that they can retain their profiles upon scatterings with localized as well as distributed structural defects (i.e. impurities) on the lines. Current applications

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involving NLTs are numerous, under large signal conditions they can serve as impulse compressors and frequency multipliers [21], they are utilized for transportation of electrical energy through overhead high voltage cables, they serve as metal strips on printed circuit boards to carry digital data or control signals, they are utilized as coupled microstrips in microwave filters, as ribbon cables to connect electronic systems and so on. Resistive NLTs are ideal devices for the study of short-pulse amplitude control [20, 28] and reshaping [21].

Nonlinear electrical networks, composed of at least two coupled nonlinear transmission lines, have been considered in the studies of simultaneous propagation of electrical soliton packets obeying coupled Korteweg-de Vries (KdV) equations [18, 29–32]. Theoretical as well as experimental results, in the specific context of two coupled NLTs, have revealed a rather complex dynamics of the interacting pulses as they propagate together. In particular when two electrical pulses propagate along two separate but coupled transmission lines, the strength of their interaction mediated by the coupling will usually depend on their speeds one relative to the other. It was observed experimentally [33] that when the difference in velocities of the two pulses is very small, their interaction is optimized thus favoring a bound state in which electrical energy will be alternately transferred from a leading soliton to a trailing soliton. In this process, when one of the two solitons in the bound state is at its maximum amplitude, it leads the pair motion while the trailing soliton is at its minimum amplitude. As the energy leaves the leading soliton to the trailing soliton, the amplitude of the leading soliton dies down while the amplitude of the trailing soliton grows. Given that the velocity of the KdV soliton increases with amplitude [24], the trailing soliton is expected to speed up and to eventually overtake the leading soliton such that the direction of energy transfer is reversed. Under certain conditions this overtaking will occur repeatedly, giving rise to relative oscillations

in amplitudes and phases in which the two solitons continuously leap over each other periodically. Such motion, called “leapfrogging”, was discussed in ref. [33] in the context of capacitively coupled nonlinear transmission lines. Actually the leapfrogging of solitons is a quite common phenomenon in nature, it has been widely discussed in the past particularly in fluid dynamics [34–40]. The leapfrogging of electrical solitons may offer the possibility to quasi-resonantly transfer energy between coupled ladder lines in distributed electric networks, such process can be of great interest for space and time multiplexed ultrafast electronics.

In this work we are interested in the leapfrogging dynamics of a bound soliton pair propagating in two NLTs, weakly coupled by a capacitive component as considered in Ref. [33] but shunted with a linear resistance. Our main objective is to examine the influence of the resistive component on relative amplitude and phase oscillations of the two interacting solitons. As a matter of fact, if the damping of soliton amplitudes due to resistive components along NLTs is a well established fact, we are anticipating a control role of the resistance in the leapfrogging dynamics of the two interacting electrical pulses.

2 Model and coupled KdV equations

The model of two coupled nonlinear transmission lines we are interested in is depicted in Fig. 1. The two lines consist of periodically loaded identical LC elementary circuits, where sections of elementary circuits in line 1 and line 2 are connected by interfacial linear capacitors C_m shunted by linear resistors R_m . Dispersion in the system is achieved by the discrete nature of the lines, while nonlinearity is achieved by making use of the conductive nonlinearity of capacitive diodes (i.e. varactors) [41]. In our study we assume the case when the varactors in the two lines are connected with opposite polarities.

Fig. 1 Equivalent representation of two nonlinear LC transmission lines, coupled by a capacitor C_m with a resistance R_m in its shunt branch

Let V_n and I_n denote the voltage and current respectively, at the n th section of line 1, and W_n and J_n the voltage and current respectively at the n th section of line 2. The bias voltages in lines 1 and 2 are $-V_b$ and V_b respectively, reflecting the opposite polarities of the varactor diodes on the two lines. For these varactor diodes we assume the following capacitance-voltage characteristics [42]:

$$C_1(x) = C_0 \left(1 - \frac{x}{V_J}\right)^{-m}, \quad C_2(x) = C_0 \left(1 + \frac{x}{V_J}\right)^{-m}, \quad (1)$$

where C_0 is the zero-bias junction capacitance, V_J is the junction potential and m (a real number) is a grading index coefficient determined by the doping profile of the varactors [41, 42]. Note that except for the opposite polarities of the varactor diodes the lines are symmetrical, this symmetry turns out to be fundamental for it favors electrical pulses of almost equal velocities to travel on the two lines as a bound state.

Applying Kirchhoff's voltage law on the n th cell of lines 1 and 2, we obtain:

$$L_0 \frac{dI_n}{dt} = V_{n-1} - V_n, \quad L_0 \frac{dJ_n}{dt} = W_{n-1} - W_n. \quad (2)$$

Kirchhoff's current law on the n th cell of lines 1 and 2 results in the two coupled transmission line equations:

$$C_1(V_n) \frac{d^2 V_n}{dt^2} + C_m \frac{d^2}{dt^2} (V_n - W_n) + \frac{1}{R_m} \frac{d}{dt} (V_n - W_n) = L_0^{-1} (V_{n+1} - 2V_n + V_{n-1}), \quad (3)$$

$$C_2(W_n) \frac{d^2 W_n}{dt^2} + C_m \frac{d^2}{dt^2} (W_n - V_n) + \frac{1}{R_m} \frac{d}{dt} (W_n - V_n) = L_0^{-1} (W_{n+1} - 2W_n + W_{n-1}). \quad (4)$$

In the long-wavelength limit, we can approximate the discrete position n by a continuous variable $x = nh$ where h is the size of an elementary circuit. In this continuous regime the discrete voltages transform as $V_n(t) \rightarrow V(x, t)$ and $W_n(t) \rightarrow W(x, t)$, such that we can carry out a continuum-limit expansion in powers of h , for the two quantities $V_{n\pm 1}(t) = V(x \pm h, t)$ and $W_{n\pm 1}(t) = W(x \pm h, t)$. To the fourth order in h the expansion yields:

$$V_{n\pm 1}(t) = V(x, t) \pm h \frac{\partial V}{\partial x} + \frac{h^2}{2!} \frac{\partial^2 V}{\partial x^2} \pm \frac{h^3}{3!} \frac{\partial^3 V}{\partial x^3} + \frac{h^4}{4!} \frac{\partial^4 V}{\partial x^4} + O(h^5), \quad (5)$$

$$W_{n\pm 1}(t) = W(x, t) \pm h \frac{\partial W}{\partial x} + \frac{h^2}{2!} \frac{\partial^2 W}{\partial x^2} \pm \frac{h^3}{3!} \frac{\partial^3 W}{\partial x^3} + \frac{h^4}{4!} \frac{\partial^4 W}{\partial x^4} + O(h^5). \quad (6)$$

Setting $h \equiv 1$ for simplicity, Eqs. (3) and (4) become:

$$C_1(V) \frac{\partial^2 V}{\partial t^2} + C_m \frac{\partial^2}{\partial t^2} (V - W) + \frac{1}{R_m} \frac{\partial}{\partial t} (V - W) = L_0^{-1} \left(\frac{\partial^2 V}{\partial x^2} + \frac{1}{12} \frac{\partial^4 V}{\partial x^4} \right), \quad (7)$$

$$C_2(W) \frac{\partial^2 W}{\partial t^2} + C_m \frac{\partial^2}{\partial t^2} (W - V) + \frac{1}{R_m} \frac{\partial}{\partial t} (W - V) = L_0^{-1} \left(\frac{\partial^2 W}{\partial x^2} + \frac{1}{12} \frac{\partial^4 W}{\partial x^4} \right), \quad (8)$$

where $C_1(V)$ and $C_2(W)$ are defined in (1).

3 Adiabatic equations

As we are interested in electric signals with localized pulse shapes, we shall follow the reductive perturbation theory [35, 43] to obtain the equations governing spatiotemporal evolutions of such localized electric signals. In this respect, we introduce a small parameter ϵ and define the new variables:

$$s = \epsilon^{1/2} (x - v_0 t), \quad \tau = \epsilon^{3/2} t, \quad (9)$$

with $v_0 = 1/\sqrt{L_0 C_b}$. Next, express the voltages $V(s, \tau)$ and $W(s, \tau)$ as series in powers of ϵ i.e.:

$$V(s, \tau) = -V_b + \sum_{i=1}^{\infty} \epsilon^i v_i(s, \tau), \quad (10)$$

$$W(s, \tau) = V_b + \sum_{i=1}^{\infty} \epsilon^i w_i(s, \tau). \quad (11)$$

Consistently with the weak-coupling assumption, the coupling capacitance C_m and coupling resistance R_m will be taken of the order ϵ . In agreement with this assumption, we can rewrite the two coupling parameters as:

$$C_m = \epsilon C'_m, \quad (12)$$

$$R_m = \epsilon^{-3/2} R'_m. \quad (13)$$

Substituting Eqs. (9), (10), (11), (12) and (13) into Eqs. (7) and (8), and keeping only terms of order ϵ^3 for which non-linearity balances the dispersion, we obtain the following two coupled KdV equations at the third order in ϵ :

$$v'_{\tau'} - 6v'v'_{s'} + v'_{s's's'} = \eta(v' + w')_{s'} - \eta_1(w' + v'), \quad (14)$$

$$w'_{\tau'} - 6w'w'_{s'} + w'_{s's's'} = \eta(w' + v')_{s'} - \eta_1(v' + w'), \quad (15)$$

where we define:

$$\begin{aligned}\tau' &= \frac{\tau}{\beta_1}, & s' &= \sqrt[3]{12C_0L_0}S, \\ v' &= 6\beta_2\sqrt[3]{12C_0L_0}v_3, & w' &= -6\beta_2\sqrt[3]{12C_0L_0}w_3, \\ \eta &= \frac{C'_m v_0^2 \sqrt[3]{12C_0L_0}}{C_0}, & \eta_1 &= \frac{v_0 r_m}{C_0},\end{aligned}\quad (16)$$

with $\beta_1 = 2v_0 - 4\beta_2 v_0 V_b$, $\beta_2 = \frac{m}{V_j}$, and for convenience we have set $r_m = R'_m$. Contributions from terms of order ε give $v_1 = 0$ and $w_1 = 0$, while contributions of order ε^2 (i.e. v_2 and w_2) are determined by the following equations:

$$0 = C_0 v_0^2 \frac{\partial^2 v_2}{\partial s^2} - 2C_0 v_0^2 \alpha_1 v_b \frac{\partial^2 v_2}{\partial s^2} + \frac{1}{12L_0} \frac{\partial^2 v_2}{\partial s^2}, \quad (17)$$

$$0 = C_0 v_0^2 \frac{\partial^2 w_2}{\partial s^2} - 2C_0 v_0^2 \alpha_1 v_b \frac{\partial^2 w_2}{\partial s^2} + \frac{1}{12L_0} \frac{\partial^2 w_2}{\partial s^2}, \quad (18)$$

suggesting that v_2 and w_2 are constants.

The two coupled Eqs. (14) and (15), that provide shape profiles of the nonlinear components of the voltage fields expressed as in series as (10), (11), are not tractable analytically because of the coupling terms in their right-hand side. However, if we assume the two coupling components to be very small, they can readily be treated as perturbations and therefore we can rewrite the set (14) and (15) formally with their right-hand side grouped to give a single perturbation function:

$$\varepsilon P(v', w') = \eta(v' + w')_{s'} - \eta_1(w' + v'). \quad (19)$$

In the absence of perturbation, i.e. when $\varepsilon P(v', w') = 0$, Eqs. (14) and (15) are two independent KdV equations whose exact single-pulse soliton solutions are [35]:

$$v' = -2\kappa_1^2 \text{sech}^2(z_1), \quad w' = -2\kappa_2^2 \text{sech}^2(z_2), \quad (20)$$

where the arguments $z_1 = \kappa_1(s' - \zeta_1)$ and $z_2 = \kappa_2(s' - \zeta_2)$ connect the pulse amplitudes $\kappa_{1,2}$ to their phases $\zeta_{1,2}$ via:

$$\zeta_1 = 4\kappa_1^3 \tau', \quad \zeta_2 = 4\kappa_2^3 \tau'. \quad (21)$$

It is worthwhile stressing that (20) are just parts of the general solutions to the two line equations, however they are most relevant in that they provide the soliton components in the series representation of these solutions. Hence we shall be mainly interested in the dynamics of these soliton components in the presence of the perturbation $\varepsilon P(v', w') = 0$. To determine the temporal evolution of characteristic parameters κ_i and ζ_i of the two pulse solitons for small $\varepsilon P(v', w')$, we use the adiabatic perturbation theory [35] which enables us obtain the following set of first-order ordinary differential equations:

$$\frac{d\kappa_i}{d\tau'} = -\frac{\eta}{4\kappa_i} \int_{-\infty}^{\infty} \varepsilon P(v', w') \text{sech}^2(z_i) dz_i, \quad (22)$$

$$\begin{aligned}\frac{d\zeta_i}{d\tau'} &= 4\kappa_i^2 - \frac{\eta}{4\kappa_i^3} \int_{-\infty}^{\infty} \varepsilon P(v', w') \text{sech}^2(z_i) \\ &\times \left(z_i + \frac{1}{2} \sinh(2z_i) \right) dz_i,\end{aligned}\quad (23)$$

with $i = 1, 2$. Note that the interaction between the two solitons will be optimal when their velocities coincide, i.e. $\zeta_1 = \zeta_2$. In terms of (20) this can also be formulated in terms of the equality of their amplitudes, i.e.:

$$\kappa_1 = \kappa_2. \quad (24)$$

Let us consider a very small deviation from this optimal state of interaction, due to the soliton velocities not being exactly equal but remaining nevertheless always very close. This can be described by small fluctuations in amplitudes of the two pulses from their unperturbed values $\kappa_1^0 = \kappa_2^0 = \kappa$. It follows that since the two lines are identical and symmetric, we can write:

$$\kappa_{1,2} = \kappa + \lambda_{1,2}, \quad (25)$$

where κ is constant and the variables λ_1 and λ_2 are very small compared to κ . Similarly the phase difference $\Delta\zeta \equiv \zeta_1 - \zeta_2$ can be assumed so small that we have:

$$z_2 \simeq z_1 + \kappa \Delta\zeta. \quad (26)$$

Substituting Eqs. (20), (25) and (26) in (22) and (23) and integrating, we obtain:

$$\begin{aligned}\frac{d\lambda_1}{d\tau'} &= -\frac{\eta\kappa_2^3}{15\kappa_1} \text{sech}^2(y) [8\tanh(y) + 8\tanh^3(y)] \\ &- \frac{2\eta_1\kappa_2^2}{3\kappa_1} \text{sech}^2(y) - \frac{2\eta_1\kappa_1}{3},\end{aligned}\quad (27)$$

$$\begin{aligned}\frac{d\lambda_2}{d\tau'} &= \frac{\eta\kappa_1^3}{15\kappa_2} \text{sech}^2(y) [8\tanh(y) + 8\tanh^3(y)] \\ &- \frac{2\eta_1\kappa_1^2}{3\kappa_2} \text{sech}^2(y) - \frac{2\eta_1\kappa_2}{3},\end{aligned}\quad (28)$$

$$\begin{aligned}\frac{d\zeta_1}{d\tau'} &= 4\kappa_1^2 - \eta - \frac{\eta\kappa_2^3}{3\kappa_1^3} \text{sech}^2(y) \left[3 - \frac{17}{3} \tanh^2(y) \right] \\ &+ \frac{\eta_1\kappa_2^2}{\kappa_1^3} \text{sech}^2(y) \tanh(y),\end{aligned}\quad (29)$$

$$\begin{aligned}\frac{d\zeta_2}{d\tau'} &= 4\kappa_2^2 - \eta - \frac{\eta\kappa_1^3}{3\kappa_2^3} \text{sech}^2(y) \left[3 - \frac{17}{3} \tanh^2(y) \right] \\ &- \frac{\eta_1\kappa_1^2}{\kappa_2^3} \text{sech}^2(y) \tanh(y),\end{aligned}\quad (30)$$

where we have set $y = \kappa \Delta \zeta$. Eqs. (27)–(30), that describe the motion of the two weakly interacting electrical solitons in the specific context when their velocities are very close, are solved numerically using a sixth-order Runge-Kutta scheme [44] with fixed step. Numerical results are discussed in the next section.

4 Numerical results

To investigate the leapfrogging dynamics of the two electrical solitons we carried out numerical simulations of the four coupled ordinary differential equations Eqs. (27)–(30). Since it is also relevant to see the effects the coupling resistance would have on amplitudes of the individual solitons, in Fig. 2a, b we started with plots of the time series of λ_1 and λ_2 , for a fixed value of the capacitive coupling η ($\eta = 0.025$), for different values of the resistive coupling parameter i.e. $\eta_1 = 0.001, 0.005$ and 0.009 . One sees that time series of λ_1 and λ_2 are periodic oscillations, with an exponential decrease of their maxima. It is quite

remarkable that the exponential damping is more and more pronounced as η_1 increases.

To highlight the anti-phase oscillations of the two amplitudes, we plot λ_1 and λ_2 on the same graph for large values of η_1 ($\eta_1 = 0.005$ and $\eta_1 = 0.009$) keeping $\eta = 0.025$ and $\kappa = 0.5$. The leapfrogging motions observed in Fig. 2c, d, occur more exactly when initial values of λ_1 and λ_2 used in the simulations are very close. Figure 3 illustrates the absence of leapfrogging, when the difference between initial values of the two parameters in the simulations is relatively large. In fact, the signature of leapfrogging is to be observed both in the oscillating phase difference $\Delta \zeta = \zeta_1 - \zeta_2$, and in the oscillating amplitude difference $\Delta \lambda = \lambda_1 - \lambda_2$ of the two solitons. Therefore, to gain a consistent picture of this signature in the two parameters, we solved numerically the amplitude and phase difference equations derived from the coupled set Eqs. (27)–(30), by subtracting Eq. (29) from Eq. (30) on one hand and Eq. (27) from Eq. (28) on the other hand. Figure 4 suggests a dynamics of the soliton pair which we can summarize as follows: initially (i.e. at $t = 0$) the amplitude difference of the two solitons is at some finite initial value, while

Fig. 2 (Color online) Top graphs: Temporal evolutions of λ_1 (a) and λ_2 (b), for $\eta = 0.23$ and $\kappa = 0.6$. The solid, dashed and dotted curves correspond respectively to $\eta_1 = 0.001, 0.005$ and 0.009 . Bottom

graphs: Leapfrogging dynamics of λ_1 (solid curve) and λ_2 (dashed curve), for $\eta_1 = 0.005$ (c) and $\eta_1 = 0.009$ (d)

Fig. 3 (Color online) Suppression of leapfrogging for large initial values of λ_1 (i.e. 0.1) and λ_2 (i.e. 0.2), with $\eta = 0.25$, $\eta_1 = 0.0005$ and $\kappa = 0.5$

their phase difference is zero. As the two solitons propagate they exchange energy, this causes them to slowly approach each other. Their amplitude difference thus decreases gradually to zero at the time when the two electrical pulses coincide and have an equal velocity. After hopping pass each other, their amplitude and phase differences begin to rise again and the cycle continues, for as long as the leapfrogging motion goes on resulting in oscillating amplitude and phase differences.

To further emphasize the effects of the resistance on the leapfrogging dynamics of the bound solitons, we have plotted three sets of curves to illustrate the behaviour of the oscillating amplitude difference and phase difference, with increasing resistance (Fig. 5). Using fairly large values for the resistance, the curves clearly show that as the resistance increases the number of oscillations in $\Delta\lambda$ and $\Delta\zeta$ decreases until the leapfrogging motion stops. Since

the oscillating evolutions of the amplitude and phase differences are here assumed to characterize the leapfrogging of the soliton pair, the decrease in the number of oscillations readily indicates a weakening and eventually a relaxation of the leapfrogging motion.

We can understand the origin of the harmonic oscillations observed in numerical solutions to the phase and amplitude difference equations, by remarking that the arguments of the hyperbolic functions in Eqs. (27)–(30) are proportional to $\Delta\zeta$. Thus for leapfrogging to occur in numerical simulations, this later parameter should remain very small which justifies our choice of very small initial values for the phase and amplitude differences. Analytically, small amplitude and phase differences imply linearizing Eqs. (27)–(30), which yields:

$$\frac{d\lambda_1}{d\tau'} = -\frac{8}{15}\eta\kappa^2y - \frac{4}{3}\eta_1k - \frac{4}{3}\eta_1\lambda_2, \quad (31)$$

$$\frac{d\lambda_2}{d\tau'} = \frac{8}{15}\eta\kappa^2y - \frac{4}{3}\eta_1k - \frac{4}{3}\eta_1\lambda_1, \quad (32)$$

$$\frac{d\zeta_1}{d\tau'} = 4\kappa_1^2 - \eta - \frac{\eta\kappa_2^3}{4\kappa_1^3} + \frac{\eta_1\kappa_2^2y}{\kappa_1^3}, \quad (33)$$

$$\frac{d\zeta_2}{d\tau'} = 4\kappa_2^2 - \eta - \frac{\eta\kappa_1^3}{4\kappa_2^3} - \frac{\eta_1\kappa_1^2y}{\kappa_2^3}. \quad (34)$$

Subtracting Eq. (34) from (33) and replacing in (31) and (28), we obtain:

$$\frac{d^2\Delta\zeta}{d\tau'^2} = \left(\frac{128}{15}\kappa^4\eta + \frac{96}{15}\kappa^2\eta^2 - 4\eta_1^2 \right) \Delta\zeta. \quad (35)$$

Fig. 4 (Color online) Temporal oscillations of the amplitude difference $\Delta\lambda$ and phase difference $\Delta\zeta$ are shown in the left and right graphs respectively, for $\eta = 0.25$, $\eta_1 = 0.0005$, and $\kappa = 0.63$

Fig. 5 (Color online) Oscillating amplitude and phase differences for increasing resistance. The top, middle and bottom set of curves are for $\eta_1 = 0.01$, 0.05 and 0.09 respectively. Here $\eta = 0.25$ and $\kappa = 0.63$. Leapfrogging ceases in the last set of curves due to very high value of resistance

This is the equation of motion for an harmonic oscillator, the frequency ω_{lf} of which is given by:

$$\omega_{lf}^2 = \left(\frac{128}{15} \kappa^4 \eta + \frac{96}{15} \kappa^2 \eta^2 - 4\eta_1^2 \right). \quad (36)$$

Formula (36) shows that the leapfrogging frequency is decreased by the presence of the resistive element in the coupling branch. From the expression of ω_{lf} it is also apparent that as η_1 increases the leapfrogging frequency decreases, vanishing at some finite characteristic value of the resistance. This is in agreement with the behaviour observed in Figs. 2a, b, c and 4a, b. In Fig. 6, we plot the leapfrogging frequency against the resistance for different values of κ . The frequency decreases to zero, indicating a decay of the leapfrogging motion as the resistance increases.

In the above analysis it emerged that a choice of small initial values for the amplitude and phase differences was necessary for the observation of leapfrogging in numerical simulations of the coupled set Eqs. (27)–(30). In Fig. 7, we plot the time series of λ_1 and λ_2 for relatively large values

Fig. 6 (Color online) Influence of the resistance on the leapfrogging frequency for different values of κ . $\eta = 0.25$

of the initial phase and amplitude differences, considering different values of η_1 and $\kappa = 0.6$. One sees that the time series of soliton characteristic parameters are now strongly

Fig. 7 (Color online) Anharmonic oscillations of λ_1 (solid curves) and λ_2 (dashed curves), for large initial values: $r_m = 5 \times 10^{-10}$ (a), $r_m = 5 \times 10^{-8}$ (b), $r_m = 5 \times 10^{-6}$ (c) and $r_m = 5 \times 10^{-4}$ (d). $\kappa = 0.6$

anharmonic, and more generally an enhancement of the anharmonicity by relatively large initial values will cause a relaxation of leapfrogging after a finite propagation time. Quite remarkably this later behaviour is valid irrespective of the value of the unperturbed amplitude κ , which indeed remains fix in time. To end, an explicit picture of the leapfrogging dynamics of the two solitons is summarized in the phase-space representation Fig. 8, where λ_1 is plotted as a function of λ_2 .

5 Conclusion

The leapfrogging of soliton pairs was proposed in the past to describe the particular motion of two weakly interacting solitons, characterized by their anti-phase oscillations and oppositely varying amplitudes [34–39, 45]. In fluid dynamics the leapfrogging of co-propagating KdV pulses have been widely discussed, and shown to provide an interesting configuration of the system dynamics in which solitons in the pair oscillate periodically one with respect to the other such as to mutually sustain their propagation at very small velocity [34]. This concept was recently extended to

Fig. 8 (Color online) Phase-space representation of the leapfrogging dynamics. Here, $\eta_1 = 0.23$, $\eta = 0.001$ and $\kappa = 0.6$

nonlinear electrical transmission lines by Narahara [33], who considered two capacitively coupled nonlinear transmission lines and determined characteristic properties for the leapfrogging motion of electrical soliton pairs.

In the present work we examined the possible leapfrogging motion of a pair of solitons along two nonlinear LC transmission lines, weakly coupled by a linear

capacitor shunted by a linear resistor. We established the leapfrogging motion by solving numerically the coupled variational equations associated with the two interacting solitons for very small initial values of the amplitude and phase differences, and obtained analytically that the resistive coupling component would cause a decrease in the leapfrogging frequency. Note that numerical curves also suggest a damping of amplitudes of the two solitons, also caused by the resistive coupling.

Although the results obtained in this work reveal a rich dynamics of the system, it would also be interesting to look at the competing effects of the coupling capacitance and resistance on the leapfrogging motion when the coupling parameters are not too small. Given that adiabatic considerations in this context cannot be applied, only a full numerical treatment of the discrete line equations would be consistent. This later study requires a specific context since several characteristic parameters are involved in the model and their contributions need to be well emphasized both quantitatively and qualitatively. This aspect is considered in a separate work.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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